

# Preface

Existence and/or nonexistence of roots of functions have been the subjects of numerous investigations. In some cases, the function under investigation may have one or more parameters, to which the corresponding existence problem then depends on. A simple example is the following. We are given a quadratic polynomial of the form  $p(\lambda|\alpha, \beta) = \lambda^2 + \alpha\lambda + \beta$ , where  $\alpha, \beta$  are real numbers, and we are required to find the exact conditions on the parameters  $\alpha$  and  $\beta$  such that the quadratic polynomial  $p$  does not have any real roots. Such a problem is easy since a necessary and sufficient condition for all roots of  $p$  to be nonreal numbers is that its discriminant is negative, that is,  $\alpha^2 - 4\beta < 0$ .

More generally, given a nonlinear function  $F(\lambda|\alpha_1, \alpha_2, \dots, \alpha_n)$  with  $n$  parameters  $\alpha_1, \dots, \alpha_n$ , it is desired to determine the exact region containing these parameters such that for each  $(\alpha_1, \dots, \alpha_n)$  in this region, none, some or all of the roots of  $F$  lie in a specified subregion of the domain of  $F$ . Unfortunately a general answer to this problem is unknown.

In some situations, however, the corresponding problems can be transformed into the existence and nonexistence of tangents of curves associated with the function  $F$ . Such an approach has been exploited by the authors and collaborators recently for characteristic functions associated with difference and differential equations. A formal theory, however, has only been developed by the first author in the last two years. In this book, we present this Cheng-Lin envelope method in a systematic manner, introduce sufficiently many technical tools related to this method, and show how they are used in handling characteristic functions involving a reasonable number of parameters.

- The book begins with an elementary example involving the quadratic polynomials. Basic definitions, symbols and results are then introduced which will be used throughout the book.
- In Chapter 2, envelopes of families of straight lines and dual points of order  $m$  of envelopes are introduced.
- In Chapter 3, quasi-tangent lines of curves are introduced. Then sweeping functions are used to obtain information on the number of tangent lines of

curves and the distribution of dual points.

- In Chapter 4, quasi-polynomials associated with ordinary difference equations and ordinary functional differential equations are introduced.
- In Chapter 5, necessary and sufficient conditions are established for real quintic polynomials to have nonpositive roots.
- In Chapter 6, quasi-polynomials related to difference equations are considered, and necessary as well as sufficient conditions for the nonexistence of positive roots are found.
- In Chapter 7, quasi-polynomials related to functional differential equations are considered, and necessary as well as sufficient conditions for the existence of real roots are found.
- In the appendix, we collect some useful distribution maps of dual sets of order 0 of curves made up of several pieces.

This book is self-contained in the sense that only basic Calculus and elementary properties of convex functions are needed. Therefore this book will be useful to college students who want to see immediate applications of first semester analysis they have just learned. On the other hand, the Cheng-Lin envelope method announced here is new, therefore this book will also be useful to graduate students who wish to pursue research in the area of equations involving several parameters. Since we have derived numerous necessary and sufficient conditions for the existence and/or nonexistence of roots of characteristic functions, our book can also be used as a reference for scientists who are concerned with qualitative properties of functional equations.

Our thanks go to Shao Yuan Huang who helped in the preparation of the graphs and in suggesting some of the results in this book, and also to the National Science Council of R. O. China for the financial supports of the first author during the last 30 years.

We tried our best to eliminate any errors. If there are any that have escaped our attention, your comments and corrections will be much appreciated.

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