Chapter 1

Multistate Coherent Systems

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1 Introduction

A basic problem in the study of reliability systems is to explain relationships among the operating performances of systems and the components consisting the systems. Using Boolean functions, Mine [14] introduced the concept of monotone systems, in which all the state spaces of components and the systems were assumed to be $\{0, 1\}$, so were also called binary state systems, where 0 and 1 denote the failure and the functioning states, respectively. Monotone system means that the more the number of functioning components is, the higher the performance level of the system consisted of the components is.

Mathematical aspects of these binary state monotone systems were explained [3, 4, 8]. Barlow and Proschan [1] have summarized the reliability studies of the binary state monotone systems. Pham [22] has edited the recent work about reliability engineering, and in this handbook, we can find out formulae useful for solving practical reliability problems.

In many practical situations, however, systems and their components could take many other performance levels, from the perfectly functioning state to the complete failure state. Thus, reliability models of multistate systems and components are required for more practical treatment of real reliability systems.

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Such multistate systems were introduced in the context of cannibalization [10,11], but these works were not concerned with mathematical aspects of the systems. More mathematical studies of multistate systems were carried out by [2, 7]: Barlow and Wu [2] defined multistate coherent systems based on the minimal path and cut sets of binary state systems, and discussed some properties of the multistate systems. El-Neweihi, Proschan and Sethuraman [7] defined the multistate systems assuming that all the state spaces of the systems and their components could be expressed as $\{0, 1, \cdots, M\}$. Their results were very analogous to those of binary systems. Huang, Zuo and Fang [12] introduced the multistate consecutive k-out-of-n systems and provided algorithms to evaluate the performance probabilities of the systems. Zuo, Hang and Kuo [28] defined a multistate coherent systems assuming all the state spaces of the systems and components were the same finite totally ordered sets as [7], and also they presented a definition of multistate k-out-of-n systems in the context. The definitions were technical and then applicable to the real situations.

This chapter is concerned with a mathematical generalization of the concepts of binary state monotone systems mainly based on the work of [17]. Section 2 presents a definition of multistate systems, assuming that state spaces of systems and their components need not to be the same, and are mathematically finite totally ordered sets. We discuss series and parallel coherent systems and obtain an existence theorem which justifies the usual formulae of the series and parallel systems, *i.e.,* $\min_{1\leq i\leq n} x_i$ and $\max_{1\leq i\leq n} x_i$ in the theory of binary state systems. A definition of dual systems is also presented in this section, using the concept of dual ordered sets.

In Section 3, we present newly a definition of multistate k-out-of-n:G systems and show some properties of them. In the theory of binary state systems, the dual systems of k-out-of-n:G systems are well known to be $n - k + 1$-out-of-n:G systems. But, in our context, the similar proposition generally no longer holds, since the state spaces of components and the system are arbitrarily finite ordered sets and have less restriction than those of binary state systems. But, the duality holds generally for the maximum and minimum k-out-of-n:G systems.

In Section 4, we treat modules of multistate systems. The modules are practically familiar concept for us to follow when constructing a large system. A system is generally composed of many systems of smaller size each of which is called a module and is also composed of many systems. In other words, practical systems have a hierarchic structure and each layer of the hierarchy consists of modules, and each module also consists of modules of smaller size. From the reliability point of view, we are interested in algebraic and probabilistic relations between systems and modules. In other words, how the reliability of the system is determined by the reliabilities of the modules.

The examination of the concept of modules occupies mathematically and practically important part in the theory of binary state systems, where an elegant theorem holds, called three modules theorem [6], but the similar proposition can no longer hold in our multistate cases.

In Section 5, we examine stochastic aspects of multistate systems. The IFR(increasing failure rate), IFRA(increasing failure rate average) and NBU(new better than used) stochastic processes are defined, and IFRA and NBU closure theorems of [23] are then proved in a slightly different situation.

In Section 6, a generalization of the concept of hazard transforms is presented and, using it, we prove that the preservation of IFR property determines the structure of a multistate system as a series system.

Throughout this work, we may recognize that a basic theory of reliability systems treats the algebraic and stochastic relationship between a product partially ordered set and a partially ordered set through an increasing mapping from the former to the latter. There practically exist some examples in which a component has two deteriorating states for which we cannot say the one state is better/worse than the other [27]. So, a model of reliability systems including partially ordered sets as the state spaces is useful, and Yu, Koren and Guo [27] presented more general multistate systems and some properties of them, but which were not a thorough treatment.

This article considers only totally ordered finite sets except for the probabilistic examination, but the basic concepts of reliability systems defined for totally ordered case, as series and parallel systems, k-out-of-n:G systems, modules and stochastic properties, are thought to be easily extended to the partially ordered case. The thorough generalization to the case of arbitral ordered sets, though, is remained to be an open problem.

Notations

We use the following notations: A finite set $C = \{1, 2, \cdots, n\}$ is the set of the components consisting a system and Ω_i $(i \in C)$ is the state space of the ith component defined to be a finite totally ordered set in Definition 2.1 and is not necessarily the binary set $\{0, 1\}$. The state space of a system composed of the components is also defined to be a finite totally ordered set.

1) For $A \subset C$, the product set of Ω_i $(i \in A)$ is denoted by $\Omega_A = \prod_{i \in A} \Omega_i$. When $A = \{i\}$, $\Omega_A = \Omega_i$.
2) An element of Ω_A $(A \subset C)$ is denoted by $\boldsymbol{x}^A$ and also simply $\boldsymbol{x}^A = \boldsymbol{x}$ if there is no confusion. When $A = C$, $\boldsymbol{x} \in \Omega_C$ is precisely written as $\boldsymbol{x} = (x_1, \cdots, x_n),\ x_i \in \Omega_i\ (i = 1, \cdots n)$.
3) For a subset $A \subset C$, $A' = C \backslash A = \{\ i \mid i \in C,\ i \notin A\ \}$.
4) For $B \subset A \subset C$, P_B is the projection mapping from Ω_A to Ω_B. For $\boldsymbol{x} \in \Omega_C$, $x_i = P_{\{i\}}(\boldsymbol{x})$.
5) For $B \subset A \subset C$ and $V \subset \Omega_A$, $P_{\Omega_B} V = \{\ P_{\Omega_B} \boldsymbol{x} \mid \boldsymbol{x} \in V\ \}$.
6) Let $\{\ B_j \mid 1 \le j \le m\ \}$ be a partition of $A \subset C$. Then, for $\boldsymbol{x}_j \in \prod_{i \in B_j} \Omega_i$ $(1 \le j \le m)$, $\boldsymbol{x} = (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m)$ is an element of Ω_A such that $P_{\Omega_{B_j}} \boldsymbol{x} = \boldsymbol{x}_j$. Then for every $\boldsymbol{x} \in \Omega_A$ $(A \subset C)$, $\boldsymbol{x} = (\boldsymbol{x}^{B_1}, \cdots, \boldsymbol{x}^{B_m})$, where $\boldsymbol{x}^{B_i} = P_{B_i}(\boldsymbol{x})$ $(i = 1, \cdots, m)$.
7) $(k_i, \boldsymbol{x}) \in \Omega_A$ $(A \subset C,\ i \in A)$ is an element of Ω_A such that $k \in \Omega_i$ and $\boldsymbol{x} \in \prod_{j \in A \backslash \{i\}} \Omega_j$.
8) $|A|$ is the cardinal number of a set A.

2 Coherent Systems

Definition 2.1. A system composed of n components (a system of order n) is a triplet (Ω_C, S, φ) satisfying the following conditions:

1) $C = \{1, \cdots, n\}$ is the set of the components.
2) Ω_i $(i \in C)$ and S are finite totally ordered sets.
3) φ is a surjection from $\Omega_C = \prod_{i=1}^{n} \Omega_i$ to S.

Ω_i is the state space of the i-th component and S is the one of the system composed of the n components. The orders on Ω_i $(i \in C)$ and S are denoted by a common symbol $\le$. m_i and M_i denote the minimum and maximum elements of Ω_i, respectively, and we also use m and M to show the minimum and maximum elements of S, respectively. The order $\le$ on the product sets $\Omega_C = \prod_{i=1}^{n} \Omega_i$ is defined as the followings ; for $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, \cdots, y_n)$ of Ω_C, $\boldsymbol{x} \le \boldsymbol{y}$ means $x_i \le y_i$ $(\forall i \in C)$, and $\boldsymbol{x} < \boldsymbol{y}$ means $x_i < y_i$ $(\forall i \in C)$.

Throughout this chapter, we assume $\Omega_i = \{0, 1, \cdots, N_i\}$ $(i \in C)$ and $S = \{0, 1, \cdots, N\}$, since any finite totally ordered set is isomorphic to some finite set $\{0, 1, \cdots, L\}$ of nonnegative integers $0, 1, \cdots, L$.

We use the notation for a system (Ω_C, S, φ):

$$V_s(\varphi) = \{ \boldsymbol{x} \mid \varphi(\boldsymbol{x}) = s,\ \boldsymbol{x} \in \Omega_C\},\ s \in S,$$

which is the inverse image of $s \in S$ respect to φ, then for $s \neq t$, $V_s(\varphi) \cap V_t(\varphi) = \phi$ holds. From the surjective property of φ, we have $V_s(\varphi) \neq \phi$ for every $s \in S$. The symbol $MIV_s(\varphi)$ means the set of the minimal elements of $V_s(\varphi)$ $(s \in S)$.

When there is no confusion, we write V_s in place of $V_s(\varphi)$ and a system (Ω_C, S, φ) is simply called a system φ.

Definition 2.2. A system φ is called increasing iff for $\boldsymbol{x}$ and $\boldsymbol{y}$ of Ω_C, $\boldsymbol{x} \leq \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$.

For $\boldsymbol{x}_i \in \Omega_C$ $(1 \leq i \leq m)$, $\vee_{i=1}^k \boldsymbol{x}_i = \boldsymbol{x}_1 \vee \cdots \vee \boldsymbol{x}_k$ and $\wedge_{i=1}^k \boldsymbol{x}_i = \boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_k$ mean the supremum and the infimum of $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k\} \subseteq \Omega_C$, respectively. The same symbols $\vee$ and $\wedge$ are also used for every subset of S with the same meanings. For an increasing system φ,

$$\varphi(\wedge_{i=1}^k \boldsymbol{x}_i) \leq \wedge_{i=1}^k \varphi(\boldsymbol{x}_i), \quad \vee_{i=1}^k \varphi(\boldsymbol{x}_i) \leq \varphi(\vee_{i=1}^k \boldsymbol{x}_i),$$

and furthermore,

$$\varphi(m_1, \cdots, m_n) = m, \quad \varphi(M_1, \cdots, M_n) = M,$$

because of the surjective and increasing properties of φ.

Definition 2.3.

1) Ω_i (or the component i) is said to be relevant to the system φ iff for every r and s of S such that $r \neq s$, there exist k and l of Ω_i and $\boldsymbol{x} \in \prod_{j=1, j\neq i}^n \Omega_j$ such that $(k_i, \boldsymbol{x}) \in V_r$ and $(l_i, \boldsymbol{x}) \in V_s$. In this case, Ω_i is simply called relevant
2) A system φ is said to be relevant iff every Ω_i $(i \in C)$ is relevant to the system.

Definition 2.4. A system φ is said to be a coherent system iff the system φ is increasing and relevant.

In the sequel of this section, we examine series and parallel systems and show the existence theorem of series and parallel systems. We start with a definition of series and parallel systems.

Definition 2.5.

1) A system φ is called a series system iff $\inf V_s \in V_s$ holds for every $s \in S$. In other words, every V_s has the minimum element.
2) A system φ is called a parallel system iff $\sup V_s \in V_s$ holds for every $s \in S$. In other words, every V_s has the maximum element.

Lemma 2.1.

(i) For a coherent series system φ, $\inf V_s < \inf V_t$ for every s and t of S such that $s < t$.
(ii) For a coherent parallel system φ, $\sup V_s < \sup V_t$ for every s and t of S such that $s < t$

Proof. We prove only (i), since the proof of (ii) is similar to (i). Letting s and t of S be $s < t$, from the relevant property of φ, there exist $(k_i, \boldsymbol{x})$ and $(l_i, \boldsymbol{x})$ such that $(k_i, \boldsymbol{x}) \in V_s$ and $(l_i, \boldsymbol{x}) \in V_t$.

If $(\inf V_t)_i \leq k$ holds, $\inf V_t \leq (k_i, \boldsymbol{x})$ since $\inf V_t \leq (l_i, \boldsymbol{x})$ and $(\inf V_t)^{C\setminus\{i\}} \leq \boldsymbol{x}$, and hence, $\varphi(\inf V_t) = t \leq \varphi(k_i, \boldsymbol{x}) = s$, which contradicts to the assumption $s < t$. Then, $k < (\inf V_t)_i$.

Thus $(\inf V_s)_i < (\inf V_t)_i$ holds for every $i \in C$. □

The following proposition gives us characterizations of series and parallel coherent systems:

Proposition 2.1. For a coherent system φ, we have the following equivalent relations:

(i) φ is a series system iff $\varphi(\boldsymbol{x} \wedge \boldsymbol{y}) = \varphi(\boldsymbol{x}) \wedge \varphi(\boldsymbol{y})$ holds for every $\boldsymbol{x}$ and $\boldsymbol{y}$ of Ω_C.
(ii) φ is a parallel system iff $\varphi(\boldsymbol{x} \vee \boldsymbol{y}) = \varphi(\boldsymbol{x}) \vee \varphi(\boldsymbol{y})$ holds for every $\boldsymbol{x}$ and $\boldsymbol{y}$ of Ω_C.

Proof. We prove only (ii), since (i) is similarly proved.
Sufficiency: $\sup V_s \in V_s$ holds for every $s \in S$, because

$$\varphi(\sup V_s) = \sup\{\varphi(\boldsymbol{x}) \mid \boldsymbol{x} \in V_s\} = s,$$

from the sufficiency of the equality and that V_s is a finite set.
Necessity: Without loss of generality, we assume $\boldsymbol{x} \in V_s$, $\boldsymbol{y} \in V_t$ and $s \leq t$. $\boldsymbol{y} \leq \boldsymbol{x} \vee \boldsymbol{y} \leq (\sup V_s) \vee (\sup V_t)$ follows from $\boldsymbol{x} \leq \sup V_s$ and $\boldsymbol{y} \leq \sup V_t$. On the other hand, $\sup V_s \vee \sup V_t = \sup V_t$ from Lemma 2.1. Then, $\boldsymbol{y} \leq \boldsymbol{x} \vee \boldsymbol{y} \leq \sup V_t$ holds. Now the parallel system φ implies $\sup V_t \in V_t$ and

$\boldsymbol{y} \in V_t$ from the assumption. Then, the increasing property of φ leads to $\varphi(\boldsymbol{x} \vee \boldsymbol{y}) = t$. Noticing $\varphi(\boldsymbol{x}) \vee \varphi(\boldsymbol{y}) = t$, we conclude the proof. □

Table 1 Coherent series

		Ω_2		
		0	1	2
	0	0	0	0
Ω_1	1	0	1	1
	2	0	1	2

Table 2 Increasing series

		Ω_2		
		0	1	2
	0	0	0	0
Ω_1	1	0	1	1
	2	2	2	2

Example 2.1. Tables 1 and 2 are examples of series coherent and series increasing systems, respectively. Table 1 gives us an intuitive explanation of the equality in (i) of Proposition 2.1. Table 2 shows that the equalities in Proposition 2.1 do not necessarily hold when the coherent property is not assumed. □

Theorem 2.1. (Existence theorem of series and parallel coherent systems) Let Ω_i $(i \in C)$ and S be totally ordered finite sets.

(i) A series coherent system (Ω_C, S, φ) exists iff $|S| \leq \min_{i \in C} |\Omega_i|$.

(ii) A parallel coherent system (Ω_C, S, φ) exists iff $|S| \leq \min_{i \in C} |\Omega_i|$.

Proof. The necessity of the conditions (i) and (ii) are evident from Lemma 2.1, then we prove the sufficiency part of each case.

Sufficiency in (i): Let $\inf V_s$ and V_s $(s \in S)$ be

$$\inf V_s = (s, \cdots, s),$$
$$V_s = \{\boldsymbol{x} \mid \inf V_s \leq \boldsymbol{x},\ \inf V_t \not\leq \boldsymbol{x}\ (t > s),\ \boldsymbol{x} \in \Omega_C\},$$

and we construct $\varphi : \Omega_C \to S$ as $\varphi(\boldsymbol{x}) = s$ for $\boldsymbol{x} \in V_s$. Then, this system φ is easily shown to be a series increasing system. Noticing that $(t, \cdots, t) \in V_t$ and $(t, \cdots, t, s, t, \cdots, t) \in V_s$ holds for every s and t of S such that $s < t$, the relevant property of the system φ is evident,

Sufficiency in (ii): Letting $\sup V_s$ and V_s $(s \in S)$ be

$$\sup V_s = (s, \cdots, s),\ \text{for } s < N,$$
$$\sup V_N = (N_1, \cdots, N_n),$$
$$V_s = \{\boldsymbol{x} \mid \boldsymbol{x} \leq \sup V_s,\ \boldsymbol{x} \not\leq \sup V_r\ (r < s),\ \boldsymbol{x} \in \Omega_C\},$$

we assume φ as $\varphi(\boldsymbol{x}) = s$ for $\boldsymbol{x} \in V_s$. Then, this system φ is a coherent parallel system. □

Theorem 2.1 tells us a necessary and sufficient condition for us to construct series and parallel coherent systems but not their uniqueness. In fact, we can easily construct several series coherent systems for given Ω_i $(i \in C)$ and S satisfying the condition of Theorem 2.1. The following proposition, however, shows us the existence of the maximum and minimum series and parallel coherent systems.

Proposition 2.2. Suppose that $|S| \leq \min_{i \in C} |\Omega_i|$ holds.

(i) There exist the minimum series coherent system $(\Omega_C, S, \varphi_{smin})$ and the maximum series coherent system $(\Omega_C, S, \varphi_{smax})$ satisfying

$$\forall \boldsymbol{x} \in \Omega_C, \quad \varphi_{smin}(\boldsymbol{x}) \leq \psi(\boldsymbol{x}) \leq \varphi_{smax}(\boldsymbol{x}).$$

for every series coherent system (Ω_C, S, ψ).

(ii) There exist the minimum parallel coherent system $(\Omega_C, S, \varphi_{pmin})$ and the maximum parallel coherent system $(\Omega_C, S, \varphi_{pmax})$ satisfying

$$\forall \boldsymbol{x} \in \Omega_C, \quad \varphi_{pmin}(\boldsymbol{x}) \leq \psi(\boldsymbol{x}) \leq \varphi_{pmax}(\boldsymbol{x}).$$

for every parallel coherent system (Ω_C, S, ψ).

Proof.

(i) *Existence of* φ_{smax}: We prove that φ_{smax} is the series coherent system constructed in the proof of Theorem 2.1. Let ψ be a series coherent system satisfying $\varphi_{smax}(\boldsymbol{x}) < \psi(\boldsymbol{x})$ for some $\boldsymbol{x} \in \Omega_C$, and without loss of generality, we assume $\varphi_{smax}(\boldsymbol{x}) = s$ and $\psi(\boldsymbol{x}) = t$, where $s < t$. For some $i \in C$, $(\boldsymbol{x})_i = s$ must hold from the construction of φ_{smax}. Then, $(\inf V_t(\psi))_i \leq s < t$ holds. On the other hand, from Lemma 2.1, we have $(\inf V_p(\psi))_i < p$ for any $p \leq t$, then $(\inf V_0(\psi))_i < 0$, which contradicts to $(\inf V_0(\psi))_i = 0$. Hence, $\varphi_{smax}(\boldsymbol{x}) \geq \psi(\boldsymbol{x})$ holds for every $\boldsymbol{x} \in \Omega_C$.

Existence of φ_{smin}: An argument similar to the proof of the existence of φ_{smax} easily takes us to that the coherent series system constructed by the following is the φ_{smin}:

$$\begin{aligned}
&\inf V_s = (N_1 - (N - s), \cdots, N_n - (N - s)) \quad s \in S,\\
&\inf V_0 = (0, \cdots, 0),\\
&V_s = \{\ \boldsymbol{x} \mid \boldsymbol{x} \leq \sup V_s,\ \boldsymbol{x} \not\leq \sup V_r\ (r < s),\ \boldsymbol{x} \in \Omega_C\ \},\\
&\varphi_{smin}(\boldsymbol{x}) = s \quad \text{for } \boldsymbol{x} \in V_s.
\end{aligned}$$

(ii) It is easily shown by the argument similar to (i) that φ_{pmax} is as constructed in the proof of Theorem 2.1, and φ_{pmin} is as the following:

$$\sup V_s = (N_1 - (N - s), \cdots, N_n - (N - s)) \quad s \in S,$$
$$V_s = \{ \boldsymbol{x} \mid \boldsymbol{x} \leq V_s,\ \boldsymbol{x} \not\leq \sup V_r\ (r < s),\ \boldsymbol{x} \in \Omega_C \},$$
$$\varphi_{pmin}(\boldsymbol{x}) = s, \quad \text{for } \boldsymbol{x} \in V_s. \ \square$$

From Proposition 2.2, when $|S| = |\Omega_i|$ $(i \in C)$, series and parallel coherent systems are uniquely determined, and we may express each of them as

$$\varphi_{smin}(\boldsymbol{x}) = \varphi_{smax}(\boldsymbol{x}) = \min\{x_1, \cdots, x_n\} = \wedge_{i=1}^n x_i,$$
$$\varphi_{pmin}(\boldsymbol{x}) = \varphi_{smax}(\boldsymbol{x}) = \max\{x_1, \cdots, x_n\} = \vee_{i=1}^n x_i.$$

These are the usual formulae of series and parallel systems [1], [7].

Example 2.2. The following Tables 2.3–2.8 are examples of φ_{smax}, φ_{smin}, φ_{pmax}, φ_{pmin}, and coherent systems φ_1 and φ_2:

Table 2.3. φ_{smin}

		Ω_2			
		0	1	2	3
Ω_1	0	0	0	0	0
	1	0	0	1	1
	2	0	0	1	2

Table 2.4. φ_{smax}

		Ω_2			
		0	1	2	3
Ω_1	0	0	0	0	0
	1	0	1	1	1
	2	0	1	2	2

Table 2.5. φ_{pmin}

		Ω_2			
		0	1	2	3
Ω_1	0	0	0	1	2
	1	1	1	1	2
	2	2	2	2	2

Table 2.6. φ_{pmax}

		Ω_2			
		0	1	2	3
Ω_1	0	0	1	2	2
	1	1	1	2	2
	2	2	2	2	2

Table 2.7. φ_1

		Ω_2			
		0	1	2	3
Ω_1	0	0	0	0	0
	1	0	0	0	1
	2	0	0	1	2

Table 2.8. φ_2

		Ω_2			
		0	1	2	3
Ω_1	0	0	1	2	2
	1	1	2	2	2
	2	2	2	2	2

Relating to these examples, we note that $\varphi_1(\boldsymbol{x}) \leq \varphi_{smin}(\boldsymbol{x})$ and $\varphi_{pmax}(\boldsymbol{x}) \leq \varphi_2(\boldsymbol{x})$ for every $\boldsymbol{x} \in \Omega_C$.

In the theory of binary state systems, any coherent system is bounded from the below by series system and from the above by parallel system.

These examples, however, show us that the similar results no longer hold in our theory of multistate systems.

Table 2.9. φ_1

		Ω_2 0	1	2
Ω_1	0	0	0	0
	1	0	0	1
	2	0	1	2

Table 2.10. φ_2

		Ω_2 0	1	2
Ω_1	0	0	1	2
	1	1	2	2
	2	2	2	2

Furthermore, noticing that $\varphi_1(x_1, 0) = \varphi_1(x_1, 1)$ for every $x_1 \in \Omega_1$ and $\varphi_2(x_1, 2) = \varphi_2(x_1, 3)$ for every $x_1 \in \Omega_1$, we may merge some states of a component to one state. For the case of the system φ_1, the state space of the second component is essentially $\{1, 2, 3\}$, and the state space of the second component is essentially $\{0, 1, 2\}$ for the system φ_2. Then, we have transformed coherent systems which are equivalent to the original coherent systems. □

In this chapter we present the following propositions.

Proposition 2.3. Let φ be an increasing system satisfying $(s, \cdots, s) \in V_s(\varphi)$ for every $s \in S$. Then, using the φ_{smax} and φ_{pmax} of Proposition 2.2,

$$\forall \boldsymbol{x} \in \Omega_C, \quad \varphi_{smax}(\boldsymbol{x}) \le \varphi(\boldsymbol{x}) \le \varphi_{pmax}(\boldsymbol{x}).$$

Proof. Let $\varphi(\boldsymbol{x}) < \varphi_{smax}(\boldsymbol{x})$ hold for some $\boldsymbol{x} \in \Omega_C$, and assume $\varphi_{smax}(\boldsymbol{x}) = t$. From the construction of φ_{smax}, $\boldsymbol{x} \ge \boldsymbol{t} = (t, \cdots, t)$. Then, $\varphi(\boldsymbol{t}) \le \varphi(\boldsymbol{x}) < \varphi_{smax}(\boldsymbol{x}) = t$ and $\varphi(\boldsymbol{t}) \ne t$, which contradicts to $\boldsymbol{t} \in V_t(\varphi)$. Hence, $\varphi_{smax}(\boldsymbol{x}) \le \varphi(\boldsymbol{x})$ for every $\boldsymbol{x} \in \Omega_C$.

A similar argument proves $\varphi(\boldsymbol{x}) \le \varphi_{pmax}(\boldsymbol{x})$ for every $\boldsymbol{x} \in \Omega_C$. □

We close this section with an alternative definition of systems and a definition of dual systems, which will be used in Sections 4 and 3.

Definition 2.6. (An alternative definition of systems) A system composed of n components is a triplet $(\Omega_C, S, \mathcal{V})$ satisfying the following conditions:

1) $C = \{1, \cdots, n\}$.
2) Ω_i $(i \in C)$ and S are totally ordered finite sets.
3) $\mathcal{V} = \{V_s; s \in S\}$ is a partition of Ω_C.

The equivalence of Definitions 2.1 and 2.6 is evident. Using Definition 2.6, we may express increasing and relevant properties of systems.

Proposition 2.4.

(i) A system $(\Omega_C, S, \mathcal{V})$ is increasing iff $\boldsymbol{y} \not\leq \boldsymbol{x}$ holds for every $\boldsymbol{x} \in V_s$ and $\boldsymbol{y} \in V_t$, whenever $s < t$.
(ii) A system $(\Omega_C, S, \mathcal{V})$ is relevant iff $(P_{\Omega_C \setminus \{i\}} V_s) \cap (P_{\Omega_C \setminus \{i\}} V_t) \neq \phi$ for every $i \in C$ and $s, t \in S$ such that $s \neq t$.

Next, we present a definition of dual systems and some remarks.

Definition 2.7. The dual system of a system $(\prod_{i=1}^n \Omega_i, S, \varphi)$ is the system $(\prod_{i=1}^n \Omega_i^D, S^D, \varphi^D)$ defined as the following:

1) Ω_i^D $(i = 1, \cdots, n)$ and S^D are the dual ordered sets of Ω_i $(i = 1, \cdots, n)$ and S, respectively.
2) $\varphi^D : \prod_{i=1}^n \Omega_i^D \to S^D$ is defined as

$$\boldsymbol{x} \in \prod_{i=1}^n \Omega_i^D,\ \varphi^D(\boldsymbol{x}) = \varphi(\boldsymbol{x}).$$

Denoting the dual orders commonly by $\leq_D$,

$$k, l \in \Omega_i^D,\ k \leq_D l \iff k \geq l, \quad (i = 1, \cdots, n),$$
$$\boldsymbol{x}, \boldsymbol{y} \in \prod_{i=1}^n \Omega_i^D,\ \boldsymbol{x} \leq_D \boldsymbol{y} \iff \boldsymbol{x} \geq \boldsymbol{y},$$
$$s, t \in S,\ s \leq_D \iff t \geq s.$$

The state spaces, say, S and S^D differ only by the order and the elements consisting the sets are the same. For example, the maximum element of S is the minimum element of S^D. It is easily verified that if a system is increasing, then the dual system is also increasing. The similar proposition holds for the coherent properties of the dual systems.

In the theory of binary state systems, it is well known that the dual systems of series(parallel) systems are parallel(series) systems. The same proposition holds in our coherent context, but, we treat these properties in a wider context in the next section of k-out-of-n systems.

3 k-out-of-n systems

The treatment of k-out-of-n systems in this section is newly presented here. For a system (Ω_C, S, φ) and $A \subset C$, we define $\varphi_A : \prod_{i \in A} \Omega_i \to S$ as the following:

$$\boldsymbol{x} \in \prod_{i\in A} \Omega_i, \ \varphi_A(\boldsymbol{x}) = \varphi(\boldsymbol{x}, \mathbf{0}^{A'}).$$

Then, we have a system (Ω_A, S, φ_A).

Definition 3.1. A coherent system (Ω_C, S, φ) of order n, *i.e.*, $C = \{1, \cdots, n\}$, is called:

1) A coherent k-out-of-n:G system, when the following conditions hold:
   (1-i) $\forall A \subset C$ such that $|A| = k$, (Ω_A, S, φ_A) is coherent series.
   (1-ii) $\forall \boldsymbol{x} \in \Omega_C, \ \varphi(\boldsymbol{x}) = \max_{A: A\subset C, \ |A|=k} \varphi_A(\boldsymbol{x}^A)$.
2) A coherent k-out-of-n:F system, when the following conditions hold:
   (2-i) $\forall A \subset C$ such that $|A| = k$, (Ω_A, S, φ_A) is coherent parallel.
   (2-ii) $\forall \boldsymbol{x} \in \Omega_C, \ \varphi(\boldsymbol{x}) = \min_{A: A\subset C, \ |A|=k} \varphi_A(\boldsymbol{x}^A)$.

Reminding Proposition 2.2, we notice that the condition about the state spaces, $|S| \leq \min_{1\leq i\leq n} |\Omega_i|$, should be hold, when we consider the coherent k-out-of-n:G(F) systems.

Proposition 3.1.

(i) φ is a coherent series system of order n iff φ is a coherent n-out-of-n:G system.
(ii) φ is a coherent series system of order n iff φ is a coherent 1-out-of-n:F system.
(iii) φ is a coherent parallel system of order n iff φ is a coherent 1-out-of-n:G system.
(iv) φ is a coherent parallel system of order n iff φ is a coherent n-out-of-n:F system.

Proof. The equivalent relationship of (i) and (ii) is clear from Definition 3.1 1) and 2), respectively. We prove only (iii), since (ii) is similarly proved.

For a 1-out-of-n:G system φ, we have from Definition 3.1 1),

$$\begin{aligned}
\varphi(\boldsymbol{x} \vee \boldsymbol{y}) &= \max_{1\leq i\leq n} \varphi(x_i \vee y_i, \mathbf{0}^{\{i\}'}) \\
&= \max_{1\leq i\leq n} \left(\varphi(x_i, \mathbf{0}^{\{i\}'}) \vee \varphi(y_i, \mathbf{0}^{\{i\}'})\right) \\
&= \max_{1\leq i\leq n} \varphi(x_i, \mathbf{0}^{\{i\}'}) \vee \max_{1\leq i\leq n} \varphi(y_i, \mathbf{0}^{\{i\}'}) \\
&= \varphi(\boldsymbol{x}) \vee \varphi(\boldsymbol{y}),
\end{aligned}$$

because Ω_i and S are totally ordered sets. Then, φ is a parallel system.

Let φ be a coherent parallel system. From Lemma 2.1, for $s < t$, $\sup V_s < \sup V_t$, so to say,

$$\forall i \in C,\ (\sup V_0)_i < (\sup V_1)_i < \cdots < (\sup V_N)_i,$$

reminding that we assumed $S = \{0, 1, \cdots, N\}$ and $\Omega_i = \{0, 1, \cdots, N_i\}$ ($i \in C$). Then,

$$\begin{aligned}
&\text{the minimum element of } V_0 = (0, \cdots, 0),\\
&\text{the set of the minimal elements of } V_t\\
&\qquad = \left\{ \left((\sup V_{t-1})_i + 1, \mathbf{0}^{\{i\}'} \right) \middle|\, i = 1, 2, \cdots n \right\},\ 1 \le t \le N,
\end{aligned}$$

and $\varphi(\boldsymbol{x}) = \max_{1 \le i \le n} \varphi_{\{i\}}(x_i)$. Hence, the coherent parallel system φ is 1-out-of-n:G system. □

Theorem 3.1. (Minimal elements of k-put-of-n:G systems) Let (Ω_C, S, φ) be a coherent k-out-of-n:G system of order n, and

$$\forall \boldsymbol{x} \in \Omega_C,\ \varphi(\boldsymbol{x}) = \max_{A: A \subset C,\ |A| = k} \varphi_A(\boldsymbol{x}^A),$$

where (Ω_A, S, φ_A) is a coherent series system. Then, for the minimal elements of $V_s(\varphi)$,

$$MIV_s(\varphi) = \bigcup_{A: A \subset C,\ |A| = k} \{\ (\boldsymbol{x}, \mathbf{0}^{A'}) \mid \boldsymbol{x} \in MIV_s(\varphi_A)\ \}.$$

We notice that $MIV_s(\varphi_A)$ consists of only one element, that is, the minimum element of $V_s(\varphi_A)$, since the system φ_A is a coherent series system.

Proof. For an element $\boldsymbol{x} \in MIV_s(\varphi)$ ($s \neq 0$), there exists uniquely a subset A of C such that $|A| = k$, $\boldsymbol{x} = (\boldsymbol{x}^A, \mathbf{0}^{A'})$, $\boldsymbol{x}^A > \mathbf{0}$ and $\boldsymbol{x}^A \in MIV_s(\varphi_A)$, by Lemma 2.1 and that each φ_A is a coherent series system.

Let

$$\begin{aligned}
&A \subset C,\ |A| = k, \quad B \subset C,\ |B| = k, \quad A \neq B,\\
&\boldsymbol{x} \in MIV_s(\varphi_A), \quad \boldsymbol{y} \in MIV_t(\varphi_B), \quad s \neq t, s \neq 0, t \neq 0.
\end{aligned}$$

We have $(A \backslash B) \cup (B \backslash A) \neq \phi$, $\boldsymbol{x} > \mathbf{0}$, $\boldsymbol{y} > \mathbf{0}$, and then, $(\boldsymbol{x}, \mathbf{0}^{A'})^B \not\geq \boldsymbol{y}$. Hence, $\varphi_B(\boldsymbol{x}, \mathbf{0}^{A'})^B = 0$, since $\boldsymbol{y}$ is the minimum element of $V_t(\varphi_B)$. Finally, have $\varphi(\boldsymbol{x}, \mathbf{0}^{A'}) = \varphi_A(\boldsymbol{x}) = s$ and $(\boldsymbol{x}, \mathbf{0}^{A'}) \in MIV_s(\varphi)$. □

Theorem 3.1 tells us that the type of minimal elements of $V_s(\varphi)$ is restricted to be $(\boldsymbol{x}, \mathbf{0}^{A'})$, $|A| = k$, $\boldsymbol{x} \in \Omega_A$, for the case of k-out-of-n:G system. Noticing this theorem, the next example shows us that the dual system of a k-out-of-n:G system is not necessarily $n-k+1$-out-of-n:G system.

Example 3.1. Let $\Omega_i = \{0,1,2,3\},\ S = \{0,1,2\}$. A 2-out-of-3:G system φ is defined by specifying the minimal elements as

$$MIV_0(\varphi) = \{(0,0,0)\},$$
$$MIV_1(\varphi) = \{(1,2,0),(2,0,1),(0,1,2)\},$$
$$MIV_2(\varphi) = \{(2,3,0),(3,0,2),(0,2,3)\},$$

$(2,2,2)$ is easily verified a maximal element of $V_1(\varphi)$, and then, the dual system of φ is not a $n-k+1$-out-of-n:G, in this example a 2-out-of-3:G system. □

Example 3.1 shows us that the dual system of a k-out-of-n:G system is not generally a $n-k+1$-out-of-n:G system. But, as a special case, the dual system of a coherent n-out-of-n:G (series) system is a coherent 1-out-of-n:G (parallel) system, and vice versa.

Following Proposition 2.2 which asserts the existence of the minimum and maximum coherent series systems, there exist the minimum and maximum coherent k-out-of-n:G systems for given state spaces Ω_i $(i \in C)$ and S. The following Corollary 3.1 shows that the dual system of the minimum (maximum) coherent k-out-of-n:G system is the maximum (minimum) coherent $n-k+1$-out-of-n:G system.

Definition 3.2.

1) A coherent k-out-of-n:G system is said to be maximum (minimum) when the series systems of Definition 3.1 (1) are all the maximum (minimum).
2) A coherent k-out-of-n:F system is said to be maximum (minimum) when the parallel systems of Definition 3.1 (2) are all the maximum (minimum).

The following Theorem 3.2 shows us the pattern of the maximal elements of the minimum (maximum) coherent k-out-of-n:G systems, of which proof needs the following lemma:

Lemma 3.1. Let C be a set with cardinal number n. For a subset $B \subset C$,

$$\forall A \subset C \text{ such that } \ |A| = k,\ A \cap B \neq \phi \tag{3.1}$$

iff

$$|B| \geq n-k+1.$$

Then, the minimum number of the cardinal number of B satisfying (3.1) is $n-k+1$.

Proof. If $|B| \le n-k$, then

$$|C\backslash B| = |C| - |B| \ge n-(n-k) = k, \quad B \cup (C\backslash B) = \phi,$$

which contradicts to (3.1). Then, (3.1) implies $|B| \ge n-k+1$.

Suppose $|B| \ge n-k+1$. For $A \subset C$ such that $|A| = k$, if $A \cap B = \phi$,

$$A \subset C\backslash B, \quad |A| \le |C\backslash B| \le n-(n-k+1) = k-1,$$

which contradicts to $|A| = k$. Thus $A \cap B \ne \phi$. □

Theorem 3.2. (Maximal elements of the minimum (maximum) coherent k-out-of-n:G systems) Let (Ω_C, S, φ) be the minimum (maximum) coherent k-out-of-n:G system. Then, for every $s \in S\backslash\{N\}$ and every maximum element $\boldsymbol{x}$ of $V_s(\varphi)$, there exists $B \subset C$ such that $|B| = n-k+1$ and

$$x_i = \begin{cases} \min\limits_{A:i\in A,|A|=k,A\subset C} (\inf V_{s+1}(\varphi_A))_i - 1\ , & i \in B, \\ N_i\ , & i \notin B. \end{cases} \tag{3.2}$$

By contraries, every $\boldsymbol{x} \in \Omega_C$ constructed by the above formulae (3.2) by using every $B \subset C$ such that $|B| = n-k+1$ is a maximal element of $V_s(\varphi)$.

Proof. Let $A_i \subset C$, $|A_i| = k$ $(i = 1, 2)$, $A_1 \cap A_2 \ne \phi$. Noticing how to construct the maximum (minimum) series system of Proposition 2.2, we have for every $i \in A_1 \cap A_2$ and every $s \in S$, the i-th coordinate state of every minimal element of $V_s(\varphi_{A_1})$ is equal to the i-th coordinate state of every minimal element of $V_s(\varphi_{A_2})$. Then, using Lemma 3.1, this theorem is proved. □

The next corollary is evident from Theorem 3.2.

Corollary 3.1. The dual system of the minimum (maximum) coherent k-out-of-n:G system is the coherent maximum (minimum) $n-k+1$-out-of-n:G system.

For k-out-of-n:F systems, we also have theorem and corollary similar to Theorem 3.2 and Corollary 3.1.

4 Modules of Coherent Systems

In this section, we examine the concepts of modules of multistate systems which are practically important, since systems in real situations have a hierarchic structure and each layer of the hierarchy consists of modules, and each module also consists of modules of smaller size. In other words, a system is constructed of systems of smaller size. Most part of the theory of binary coherent systems is devoted to an examination of modules.

We start with a definition of a modular decomposition, and then, a definition of a module is presented. Our argument is in reverse order to that of [1], but presents a unified approach to a module and a modular decomposition.

Definition 4.1. A partition $\mathcal{A} = \{A_j, 1 \le j \le m\}$ of C is called a modular decomposition of a system (Ω_C, S, φ) iff there exist systems $(\Omega_{A_j}, S_j, \chi_j)$ $(1 \le j \le m)$ and $(\prod_{j=1}^m S_j, S, \psi)$ satisfying

$$\forall \boldsymbol{x} \in \Omega_C, \quad \varphi(\boldsymbol{x}) = \psi\left(\chi_1\left(\boldsymbol{x}^{A_1}\right), \cdots, \chi_m\left(\boldsymbol{x}^{A_m}\right)\right).$$

Each $A_j \in \mathcal{A}$ is called a module of the system φ.

If systems χ_j $(1 \le j \le m)$ and ψ are increasing (coherent), then $\mathcal{A}$ is called an increasing (a coherent) modular decomposition and each $A_j \in \mathcal{A}$ is called an increasing (a coherent) module.

We can easily prove that any partition of C of any coherent series (parallel) systems is a coherent modular decomposition.

Proposition 4.1. If $\mathcal{A}$ is a modular decomposition of a coherent system φ, then ψ is relevant.

The proof of the proposition is easy and then omitted.

Proposition 4.1 tells us that we only examine the properties of each element of A_j $(1 \le j \le m)$ of a modular decomposition $\mathcal{A}$ to characterize a coherent modular decomposition. With a characterization of increasing and coherent modular decomposition, we prove a theorem almost similar to Three Modules Theorem of [6].

We define a pseudo-order $\overset{\varphi}{\le}$ on Ω_A $(A \subset C)$ as the following: For $\boldsymbol{x}$ and $\boldsymbol{y}$ in Ω_A, $\boldsymbol{x} \overset{\varphi}{\le} \boldsymbol{y}$ means $\varphi(\boldsymbol{x}, \boldsymbol{z}) \le \varphi(\boldsymbol{y}, \boldsymbol{z})$ for every $\boldsymbol{z} \in \Omega_{A'}$. When $\boldsymbol{x} \overset{\varphi}{\le} \boldsymbol{y}$ and $\boldsymbol{y} \overset{\varphi}{\le} \boldsymbol{x}$, we define $\boldsymbol{x} \overset{\varphi}{=} \boldsymbol{y}$. The relation $\overset{\varphi}{=}$ is an equivalent relation on Ω_A. $(\Omega_A, \overset{\varphi}{\le})$ is generally a partially pseudo-ordered set, and if $(\Omega_A, \overset{\varphi}{\le})$ is a totally pseudo-ordered set, then Ω_A can be partitioned by the equivalent relation $\overset{\varphi}{=}$.

Proposition 4.2.

(i) Let φ be an increasing system. Then, a partition $\mathcal{A} = \{A_j, 1 \le j \le m\}$ of C is an increasing modular decomposition iff each pseudo-ordered set $(\Omega_{A_j}, \overset{\varphi}{\le})$ $(1 \le j \le m)$ is a totally pseudo-ordered set.

(ii) Let φ be a coherent system. A partition $\mathcal{A} = \{A_j, 1 \leq j \leq m\}$ of C is a coherent modular decomposition iff the following two conditions are hold:

  (ii-i) Each $(\Omega_{A_j}, \overset{\varphi}{\leq})$ $(i \leq j \leq m)$ is a totally pseudo-ordered set.

  (ii-ii) For $\boldsymbol{x}$ and $\boldsymbol{y}$ of Ω_{A_j} satisfying $\boldsymbol{x} \overset{\varphi}{\leq} \boldsymbol{y}$ and $\boldsymbol{x} \overset{\varphi}{\neq} \boldsymbol{y}$, there exist $(k_i, \boldsymbol{z})$ and $(l_i, \boldsymbol{z})$ of Ω_{A_j} which satisfy $(k_i, \boldsymbol{z}) \overset{\varphi}{=} \boldsymbol{x}$ and $(l_i, \boldsymbol{z}) \overset{\varphi}{=} \boldsymbol{y}$ for $i \in A_j$ and $j = 1, \cdots, m$

Proof. From Proposition 2.4, (ii) of this proposition is easily proved by using (i), and then, we prove only (i). The necessity of the condition is clear from the definition of an increasing modular decomposition. Thus, we prove the sufficiency of the condition.

From our assumption, the equivalent relation $\overset{\varphi}{=}$ determines a partition $\mathcal{V}_{A_j} = \{V_i^{A_j}, 1 \leq j \leq m_{A_j}\}$ of Ω_{A_j} for every j $(1 \leq j \leq m)$, where we may assume $\boldsymbol{x} \overset{\varphi}{\leq} \boldsymbol{y}$ if $\boldsymbol{x} \in V_k^{A_j}$, $\boldsymbol{y} \in V_l^{A_j}$ for $k < l$. Then for $\boldsymbol{x} \in V_k^{A_j}$ and $\boldsymbol{y} \in V_l^{A_j}$, where $k < l$, we have $\boldsymbol{y} \not\leq \boldsymbol{x}$ by the increasing property of φ, and then, systems $(\Omega_{A_j}, S_j, \mathcal{V}_{A_j})$ or $(\Omega_{A_j}, S_j, \chi_j)$ $(1 \leq j \leq m)$ are determined, where $S_j = \{1, \cdots, m_j\}$ and $\chi_j(\boldsymbol{x}) = s_j$ for $\boldsymbol{x} \in V_{s_j} \in \mathcal{V}_{A_j}$. We now construct a system $(\prod_{j=1}^m S_j, S, \psi)$ as $\psi(s_1, \cdots, s_m) = \varphi(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m)$, where $\boldsymbol{x}_j$ is any element of $V_j^{A_j} \in \mathcal{V}_{A_j}$ $(1 \leq j \leq m)$, which is an increasing system and satisfies $\varphi(\boldsymbol{x}) = \psi(\chi_1(\boldsymbol{x}^{A_i}), \cdots, \chi_m(\boldsymbol{x}))$ for any $\boldsymbol{x} \in \Omega_C$. Coherent property is clear. □

Example 4.1. $\mathcal{A} = \{\{i\},\ i \in C\}$ and $\mathcal{A} = \{C\}$ are modular decomposition of a system φ, and an increasing (coherent) modular decomposition if the system φ is increasing (coherent). □

Corollary 4.1.

(i) Let φ be an increasing system. Then, a subset A of C is an increasing module of the system iff $(\Omega_A, \overset{\varphi}{\leq})$ is a totally pseudo-ordered set, *i.e.*, A is an element of some increasing modular decomposition $\mathcal{A}$ of the system φ.

(ii) Let φ be a coherent system. Then, a subset A of C is coherent module of the system φ iff Ω_A satisfies the conditions of Proposition 4.2 (ii), *i.e.*, A is an element of some coherent modular decomposition $\mathcal{A}$ of the system φ.

Proof. Since the increasing property of φ means that $(\Omega_i, \overset{\varphi}{\leq})$ is a totally pseudo-ordered set and a system (Ω_i, Ω_i, I), where I is the identity mapping of Ω_i, *i.e.*, evidently a coherent system, then the corollary is obvious by considering the partition $\mathcal{A} = \{A, \{i\},\ i \in A'\}$ of C. □

Corollary 4.1 shows us that a subset A of C is an increasing (a coherent) module iff Ω_A satisfies the conditions of Proposition 4.2(i) (Proposition 4.2(ii)), and then, we may treat the increasing (coherent) modules of increasing (coherent) systems without being conscious of increasing (coherent) modular decomposition.

Theorem 4.1. Let φ be a coherent system. If A and B are coherent modules of the system φ such that $A\backslash B$, $B\backslash A$ and $A \cap B$ are non-empty, then $A\backslash B$, $B\backslash A$ and $A \cup B$ are increasing modules of the system φ.

Proof. (Proof of that $A\backslash B$ is an increasing module) Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be arbitrarily given elements of $\Omega_{A\backslash B}$. If $\varphi(\boldsymbol{x}, \boldsymbol{z}) < \varphi(\boldsymbol{y}, \boldsymbol{z})$ holds for some $\boldsymbol{z} \in \Omega_{(A\backslash B)'}$, then we have $\varphi(\boldsymbol{x}, \boldsymbol{w}) \leq \varphi(\boldsymbol{y}, \boldsymbol{w})$ for every $\boldsymbol{w} \in \Omega_{(A\backslash B)'}$. Since B is a coherent module, for arbitrarily given $i \in B\backslash A$ there exist $(k_i, \boldsymbol{u})$ and $(l_i, \boldsymbol{u})$ of Ω_B such that $(k_i, \boldsymbol{u}) \overset{\varphi}{=} \boldsymbol{z}^B$ and $(l_i, \boldsymbol{u}) \overset{\varphi}{=} \boldsymbol{w}^B$.

Since A is an increasing module,

$$\varphi(\boldsymbol{x}, (k_i, \boldsymbol{u}), \boldsymbol{z}^{(A\cup B)'}) < \varphi(\boldsymbol{y}, (k_i, \boldsymbol{u}), \boldsymbol{z}^{(A\cup B)'}),$$
$$\varphi(\boldsymbol{x}, (l_i, \boldsymbol{u}), \boldsymbol{w}^{(A\cup B)'}) \leq \varphi(\boldsymbol{y}, (l_i, \boldsymbol{u}), \boldsymbol{w}^{(A\cup B)'}).$$

Noting that $(l_i, \boldsymbol{u}) \overset{\varphi}{=} \boldsymbol{w}^B$, $\varphi(\boldsymbol{x}, \boldsymbol{w}) \leq \varphi(\boldsymbol{y}, \boldsymbol{w})$ holds, and hence, $A\backslash B$ is an increasing module by Corollary 4.1. A similar examination shows us that $B\backslash A$ is an increasing module.

(Proof of that $A \cup B$ is an increasing module) Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be arbitrarily given elements of $\Omega_{A\cup B}$. We assume $\varphi(\boldsymbol{x}, \boldsymbol{z}) < \varphi(\boldsymbol{y}, \boldsymbol{z})$ for some $\boldsymbol{z} \in \Omega_{(A\cup B)'}$. Since A is a coherent module, for any $i \in A \cap B$ there exist $(k_i, \boldsymbol{u})$ and $(l_i, \boldsymbol{u})$ of Ω_A satisfying $(k_i, \boldsymbol{u}) \overset{\varphi}{=} \boldsymbol{x}^A$ and $(l_i, \boldsymbol{u}) \overset{\varphi}{=} \boldsymbol{y}^A$, and then

$$\varphi((k_i, \boldsymbol{u}), \boldsymbol{x}^{B\backslash A}, \boldsymbol{z}) < \varphi((l_i, \boldsymbol{u}), \boldsymbol{y}^{B\backslash A}, \boldsymbol{z}).$$

Since B is an increasing module,

$$\forall \boldsymbol{w} \in \Omega_{(A\cup B)'},\ \varphi((k_i, \boldsymbol{u}), \boldsymbol{x}^{B\backslash A}, \boldsymbol{w}) \leq \varphi((l_i, \boldsymbol{u}), \boldsymbol{y}^{B\backslash A}, \boldsymbol{w}).$$

Using $(k_i, \boldsymbol{u}) \overset{\varphi}{=} \boldsymbol{x}^A$ and $(l_i, \boldsymbol{u}) \overset{\varphi}{=} \boldsymbol{y}^A$, we have $\varphi(\boldsymbol{x}, \boldsymbol{w}) \leq \varphi(\boldsymbol{y}, \boldsymbol{w})$ for every $\boldsymbol{w} \in \Omega_{(A\cup B)'}$, and then the proof is completed. □

Remark 4.1. It is easy to construct an example to show that $A \cap B$ is not an increasing module, even if A and B are coherent modules of a coherent system satisfying the condition that $A\backslash B, B\backslash$ and $A \cap B$ are nonempty.

5 Probabilistic Aspect of Coherent Systems

In this section we consider probabilistic properties of coherent systems, particularly we discuss the comparison of probability measures, and IFRA and NBU closure theorems in a situation slightly different from [23].

Section 2 supposes that the state spaces of the components and the systems are finite totally ordered sets, but in this section we assume more generally that Ω_i $(i \in C)$ and S are at most countable partially ordered sets. As basic measurable spaces we set $(\Omega_i, \mathcal{A}_i)$, where $\mathcal{A}_i$ is the power set of Ω_i. $(\prod_{i=1}^n \Omega_i, \prod_{i=1}^n \mathcal{A}_i) = (\Omega_C, \mathcal{A}_C)$ is the product measurable set of $(\Omega_i, \mathcal{A}_i)$ $(1 \leq i \leq n)$. The order on Ω_C is defined similarly to that of Section 2. Throughout this section, we assume φ to be an increasing measurable function from $(\Omega_C, \mathcal{A}_C)$ to $(S, \mathcal{S})$, where $\mathcal{S}$ is the power set of S and the definition of an increasing function is similar to that of Section 2. Furthermore, we assume that Ω_i $(1 \leq i \leq n)$ and S are endowed with discrete topology. Then, φ is a continuous function.

Generally a subset W of an ordered set Ω is called increasing iff $\boldsymbol{x} \in W$ and $\boldsymbol{x} \leq \boldsymbol{y}$ imply $\boldsymbol{y} \in W$. The concept of increasing set plays an important role in the sequel.

Before proving the IFRA and NBU closure theorem, we present some remarks:

Remark 5.1.

(i) If $P(W) \geq Q(W)$ holds for every increasing set $W \in \mathcal{A}_C$, then we have $P(\varphi \geq s) \geq Q(\varphi \geq s)$ for every $s \in S$, where P and Q are probability measures on $(\Omega_C, \mathcal{A}_C)$.

(ii) If Ω_i $(1 \leq i \leq n)$ and S are finite totally ordered sets and $(s, \cdots, s) \in V_s(\varphi)$ for every $s \in S$, then by Proposition 2.3 $P\{\ \boldsymbol{x} \mid \boldsymbol{x} \geq (s, \cdots, s)\ \} \leq P\{\ \varphi \geq s\ \} \leq 1 - P\{\ \boldsymbol{x} \mid \boldsymbol{x} \leq (s-1, \cdots, s-1)\ \}$, where P is a probability measure on $(\Omega_C, \mathcal{A}_C)$.

(iii) Let $(\Omega_C, \mathcal{A}_C, P) = (\prod_{i=1}^n \Omega_i, \prod_{i=1}^n \mathcal{A}_i, \prod_{i=1}^n P_i)$ and $(\Omega_C, \mathcal{A}_C, Q) = (\prod_{i=1}^n \Omega_i, \prod_{i=1}^n \mathcal{A}_i, \prod_{i=1}^n Q_i)$, both of which imply that the performances of the components are stochastically independent.

If $P_i(W_i) \geq Q_i(W_i)$ for every increasing set $W_i \in \mathcal{A}_i$ $(1 \leq i \leq n)$, then $P(W) \geq Q(W)$ for every increasing set $W \in \mathcal{A}_C$. This is easily proved by using the indicator function of W and Fubini's Theorem. Hence, it is shown that (i) and (ii) of this remark generalize Theorems 4.2 and 4.4 of [7], respectively.

From now on we focus on proving the IFRA and NBU closure theorems in a situation slightly different from that of Ross [23]. Though he assumed that Ω_i $(1 \leq i \leq n)$ and S were subsets of $\mathbf{R}^+ = [0, +\infty)$, which means that they are totally ordered sets, our requirement for Ω_i $(1 \leq i \leq n)$ and S is that they are at most countable partially ordered sets.

We will use the following Proposition 5.1 to prove the IFRA and NBU closure theorems.

Lemma 5.1. Let P_i, Q_i and U_i be probability measures on $(\Omega_i, \mathcal{A}_i)$.

(i) If $P_i(W_i) \geq [Q_i(W_i)]^\alpha$ holds for every increasing set $W_i \in \mathcal{A}_i$ and $0 < \alpha < 1$, then $\int_{\Omega_i} f^\alpha dP_i \geq \left[\int_{\Omega_i} f dQ_i\right]^\alpha$ holds for $0 < \alpha < 1$, where f is an increasing measurable function from $(\Omega_i, \mathcal{A}_i)$ to $(\mathbf{R}^+, \mathcal{B}^+)$, where $\mathcal{B}^+$ is the class of Borel subset of $\mathbf{R}^+$.

(ii) If $U_i(W_i) \leq P_i(W_i)Q_i(W_i)$ holds for every increasing set $W_i \in \mathcal{A}_i$, then $\int_{\Omega_i} fg dU_i \leq \int_{\Omega_i} f dP_i \int_{\Omega_i} g dQ_i$, where f and g are increasing measurable functions from $(\Omega_i, \mathcal{A}_i)$ to $(\mathbf{R}^+, \mathcal{B}^+)$.

Proof.

(i) From the assumption, f is approximated by a step function of the form $\sum_{j=1}^m x_j I_{S_j}$, where S_j is an increasing set of $\mathcal{A}_i$, $S_1 \supset \cdots \supset S_m$, $x_j \geq 0$ and I_{S_j} is the indicator function of S_j $(1 \leq j \leq m)$. Then, using an argument similar to that of Lemma 1 of [23] and taking the limit, (i) is evident.

(ii) From the assumption, f and g are approximated by step functions of the form $\sum_{j=1}^m x_j I_{S_j}$ and $\sum_{k=1}^n y_k I_{V_k}$, respectively, where S_j (V_k) is an increasing set of $\mathcal{A}_i$, $S_1 \supset \cdots \supset S_m$ $(V_1 \supset \cdots \supset V_n)$, $x_j \geq 0$ $(y_k \geq 0)$ and I_{S_j} (I_{V_k}) is the indicator function of S_j (V_k) $(1 \leq j \leq m,\ 1 \leq k \leq n)$. Then, using the assumption,

$$\int_{\Omega_i} \left(\sum_{j=1}^m x_j I_{S_j}\right) \left(\sum_{k=1}^n y_k I_{V_k}\right) dU_i \leq \int_{\Omega_i} \left(\sum_{j=1}^m x_j I_{S_j}\right) dP_i \int_{\Omega_i} \left(\sum_{k=1}^n y_k I_{V_k}\right) dQ_i,$$

noticing that the intersection of increasing sets is also an increasing set. Hence (ii) is clear by taking the limit. □

Proposition 5.1. Let P_i, Q_i and U_i be probability measures on $(\Omega_i, \mathcal{A}_i)$ $(1 \leq i \leq n)$.

(i) If $P_i(W_i) \geq [Q_i(W_i)]^\alpha$ holds for every increasing set $W_i \in \mathcal{A}_i$ and $0 < \alpha < 1$ $(1 \leq i \leq n)$, then for every $0 < \alpha < 1$ and increasing set $W \in \mathcal{A}_C$,

$$\left(\prod_{i=1}^{n} P_i\right)(W) \geq \left[\left(\prod_{i=1}^{n} Q_i\right)(W)\right]^\alpha.$$

(ii) If $U_i(W_i) \leq P_i(W_i)Q_i(W_i)$ holds for every increasing set $W_i \in \mathcal{A}_i$ $(1 \leq i \leq n)$, then for every increasing set $W \in \mathcal{A}_C$,

$$\left(\prod_{i=1}^{n} U_i\right)(W) \leq \left[\left(\prod_{i=1}^{n} P_i\right)(W)\right]\left[\left(\prod_{i=1}^{n} Q_i\right)(W)\right].$$

Proof. Mathematical induction on n proves the proposition.

(i) When $n = 1$, (i) of this proposition is evident from the assumption. Suppose that (i) holds for $n = n$. Letting I_W be the indicator function of W, we have by Fubini's theorem,

$$\begin{aligned}\left(\prod_{i=1}^{n+1} P_i\right)(W) &= \int_{\prod_{i=1}^{n+1}\Omega_i} I_W d\prod_{i=1}^{n+1} P_i \\ &= \int_{\Omega_{n+1}} dP_{n+1}\int_{\prod_{i=1}^{n}\Omega_i}(I_W)_{x_{n+1}} d\prod_{i=1}^{n} P_i.\end{aligned}$$

The section $(I_W)_{x_{n+1}}$ $(x_{n+1} \in \Omega_{n+1})$ which is an increasing binary function denotes an increasing set in $\prod_{i=1}^{n} \mathcal{A}_i$. Then, using the induction hypothesis, Lemma 5.1(i) and Fubini's theorem,

$$\begin{aligned}\left(\prod_{i=1}^{n+1} P_i\right)(W) &\geq \int_{\Omega_{n+1}} dP_{n+1}\left[\int_{\prod_{i=1}^{n}\Omega_i}(I_W)_{x_{n+1}} d\prod_{i=1}^{n} Q_i\right]^\alpha \\ &\geq \left[\int_{\prod_{i=1}^{n+1}\Omega_i} I_W d\prod_{i=1}^{n+1} Q_i\right]^\alpha \\ &= \left[\left(\prod_{i=1}^{n+1} Q_i\right)(W)\right]^\alpha.\end{aligned}$$

(ii) When $n = 1$, (ii) of this proposition is clearly holds by the assumption.

Suppose that (ii) holds for $n = n$. Then,

$$\begin{aligned}\left(\prod_{i=1}^{n+1} U_i\right)(W) &= \int_{\Omega_{n+1}} dU_{n+1} \int_{\prod_{i=1}^{n}\Omega_i} (I_W)_{x_{n+1}} d\prod_{i=1}^{n} dU_i \\ &\le \int_{\Omega_{n+1}} dU_{n+1} \int_{\prod_{i=1}^{n}\Omega_i} (I_W)_{x_{n+1}} d\prod_{i=1}^{n} dP_i \int_{\prod_{i=1}^{n}\Omega_i} (I_W)_{x_{n+1}} d\prod_{i=1}^{n} dQ_i \\ &\le \int_{\prod_{i=1}^{n+1}\Omega_i} I_W d\prod_{i=1}^{n+1} dP_i \int_{\prod_{i=1}^{n+1}\Omega_i} I_W d\prod_{i=1}^{n+1} dQ_i \\ &= \left[\left(\prod_{i=1}^{n+1} P_i\right)(W)\right]\left[\left(\prod_{i=1}^{n+1} Q_i\right)(W)\right],\end{aligned}$$

where the first inequality comes from the inductive hypothesis, and the second inequality from Lemma 5.1(ii). □

Let $(\Omega, \mathcal{A}, P)$ be a given probability space and T be a subinterval of $\mathbf{R}^+ = [0, \infty)$. We suppose $X_i(t)$ $(t \in T)$ to be a measurable function from $(\Omega, \mathcal{A})$ to $(\Omega_i, \mathcal{A}_i)$ $(1 \le i \le n)$, which is a stochastic process denoting the state of the i-th component at time t, and then, $\boldsymbol{X}(t) = (X_1(t), \cdots, X_n)$ $(t \in T)$ is a measurable function from $(\Omega, \mathcal{A})$ to $(\Omega_C, \mathcal{A}_C)$.

Let μ_t be the probability measure induced by $\boldsymbol{X}(t)$ from $(\Omega, \mathcal{A}, P)$ and $\mu_{i,t}$ be the restriction of μ_t to the measurable space $(\Omega_i, \mathcal{A}_i)$ which is also the probability measure induced by $X_i(t)$ from $(\Omega, \mathcal{A}, P)$. If $\{X_i(t),\ t \in T\}$ $(1 \le i \le n)$ are mutually independent stochastic processes, then $\mu_t = \prod_{i=1}^{n} \mu_{i,t}$.

Definition 5.1. Let $T_W = \inf\{\ t \mid X_i(t) \notin W\ \}$ $(W \in \mathcal{A}_i)$.

1) A stochastic process $\{X_i(t),\ t \in T\}$ is called IFR iff T_W is an IFR random variable for any increasing set $W \in \mathcal{A}_i$.
2) A stochastic process $\{X_i(t),\ t \in T\}$ is called IFRA iff T_W is an IFRA random variable for any increasing set $W \in \mathcal{A}_i$.
3) A stochastic process $\{X_i(t),\ t \in T\}$ is called NBU iff T_W is an NBU random variable for any increasing set $W \in \mathcal{A}_i$.

For the definitions of IFR, IFRA and NBU random variables, see [1].

Theorem 5.1. Let $\{X_i(t),\ t \in T\}$ $(1 \le i \le n)$ be decreasing and right continuous with probability 1, and be mutually independent.

(i) If $\{X_i(t),\ t \in T\}$ $(1 \le i \le n)$ are IFRA processes, then $\{\varphi(\boldsymbol{X}(t)),\ t \in T\}$ is an IFRA process.

(ii) If $\{X_i(t),\ t \in T\}$ $(1 \le i \le n)$ are NBU processes, then $\{\varphi(\boldsymbol{X}(t)),\ t \in T\}$ is an NBU process.

Proof. For any increasing set $W \in \mathcal{S}$, $\{\ \boldsymbol{x} \mid \varphi(\boldsymbol{x}) \in W\ \}$ is an increasing set of $\mathcal{A}_C$. Then, since $\{X_i(t),\ t \in T\}$ $(1 \le i \le n)$ are decreasing and right continuous, it is sufficient to prove that for every increasing set $W \in \mathcal{A}_C$,

$$\mu_{\alpha t}(W) \ge [\mu_t(W)]^\alpha \ (0 < \alpha < 1) \quad (\ \mu_{s+t}(W) \le \mu_s(W)\mu_t(W)\ ).$$

Since for every increasing set $W_i \in \mathcal{A}_i$ $(1 \le i \le n)$,

$$\mu_{i,\alpha t}(W_i) \ge [\mu_{i,t}(W_i)]^\alpha \ (0 < \alpha < 1) \quad (\ \mu_{i,s+t}(W_i) \le \mu_{i,s}(W_i)\mu_{i,t}(W_i)\ ),$$

and $\{X_i(t),\ t \in T\}$ $(1 \le i \le n)$ are mutually independent, Proposition 5.1 proves the theorem. □

6 Hazard Transform of Multistate Coherent Systems

In the theory of binary state coherent systems, a hazard transform plays a useful role for the proof of IFRA and NBU closure theorems and also the proof of that the preservation of IFR property determines the structure of binary state coherent systems as series systems [9]. This section is about a generalization of the concept of hazard transform to multistate systems and shows the similar results. In this section, we again assume Ω_i $(1 \le i \le n)$ and S to be finite totally ordered sets. First, we prove several lemmas and propositions for the proof of our main theorems.

Lemma 6.1.

(i) Suppose that W is an increasing subset of $\Omega_1 \times \Omega_2$. Then we have $W = \cup_{j=1}^m (A_j \times B_j)$, where A_j $(1 \le j \le m)$ are nonempty subsets of Ω_1 such that $A_1 \subset \cdots \subset A_m$ and $A_i \ne A_j$ $(i \ne j)$ hold, and B_j $(1 \le j \le m)$ are nonempty subsets of Ω_2 such that $\cup_{k=j}^m B_k$ $(1 \le j \le m)$ are increasing subsets of Ω_2. Then, $W = (P_{\Omega_1} W) \times (P_{\Omega_2} W)$ holds iff $m = 1$ holds.

(ii) Suppose that W is an increasing subset of $\prod_{i=1}^n \Omega_i$. Then, $W = \prod_{i=1}^n (P_{\Omega_i} W)$ holds iff W has the minimal element.

Lemma 6.2.

(i) $a_0^\alpha + a_1^\alpha - b_1^\alpha > [a_0 + a_1 - b_1]^\alpha$ holds for $0 < \alpha < 1,\ a_0 \ge a_1 > b_1 > 0$.

(ii) $\sum_{i=1}^{n-1} a_i^\alpha (b_i^\alpha - b_{i+1}^\alpha) + a_n^\alpha b_n^\alpha > \left[\sum_{i=1}^{n-1} a_i(b_i - b_{i+1}) + a_n b_n\right]^\alpha$ holds for $0 < \alpha < 1,\ n \ge 2,\ 0 < a_1 < \cdots < a_n,\ b_1 > \cdots > b_n > 0$.

Lemma 6.1 is obvious and for the proof of Lemma 6.2, see [5] and [23].

Proposition 6.1. Let P_i and Q_i be probability measures on $(\Omega_i, \mathcal{A}_i)$ $(1 \le i \le n)$ and $0 < \alpha < 1$. Suppose that $P_i(W_i) = [Q_i(W_i)]^\alpha$ holds for every increasing set $W_i \in \mathcal{A}_i$, and $P_i(W_i) > P_i(W_i')$ holds for every increasing sets W_i and W_i' in $\mathcal{A}_i$ such that $W_i' \subset W_i$ and $W_i \ne W_i'$. Then, for each increasing set $W \in \mathcal{A}_C$,

$$\left(\prod_{i=1}^n P_i\right)(W) = \left[\left(\prod_{i=1}^n Q_i\right)(W)\right]^\alpha \quad \text{holds iff} \quad W = \prod_{i=1}^n P_{\Omega_i} W \quad \text{holds.}$$

Proof. "if" part is obvious. We prove "only if" part by the mathematical induction on n.

(The case of $n = 2$) If $W = (P_{\Omega_1} W) \times (P_{\Omega_2} W)$ does not hold, then using the same symbols of Lemma 6.1, we have $W = \cup_{j=1}^m (A_j \times B_j)$ $(m \ge 2)$, where $A_j \times B_j$ $(1 \le j \le m)$ are assumed to be mutually exclusive without loss of generality. Noticing

$$\left(\prod_{i=1}^2 P_i\right)(W) = \sum_{j=1}^m P_1(A_j) P_2(B_j)$$

$$= \sum_{j=1}^{m-1} P_1(A_j) \left\{ P_2\left(\bigcup_{k=j}^m B_k\right) - P_2\left(\bigcup_{k=j+1}^m B_k\right)\right\} + P_1(A_m)P_2(B_m),$$

we have by the assumption and Lemma 6.2,

$$\left(\prod_{i=1}^2 P_i\right)(W) = \sum_{j=1}^{m-1} [Q_1(A_j)]^\alpha \left\{\left[Q_2\left(\cup_{k=j}^m B_k\right)\right]^\alpha - \left[Q_2\left(\cup_{k=j+1}^m B_k\right)\right]^\alpha\right\}$$

$$+ [Q_1(A_m)]^\alpha [Q_2(B_m)]^\alpha$$

$$> \left[\left(\prod_{i=1}^2 Q_i\right)(W)\right]^\alpha \qquad \text{(by Lemma 6.2),}$$

and then, "only if" part is proved for the case of $n = 2$.

Now assuming "only if" part to hold for $n = n$, we prove the proposition for $n = n + 1$.

Let I_W be an indicator function of an increasing subset $W \subset \prod_{i=1}^{n+1} \Omega_i$. If $W = \prod_{i=1}^{n+1}(P_{\Omega_i} W)$ does not hold, then for some $x_j \in \Omega_j$ an increasing subset of $\prod_{i=1, i\ne j}^{n+1} \Omega_i$ defines by the section $(I_W)_{x_j}$ is not a product set of increasing subsets of Ω_i $(1 \le i \le n+1,\ i \ne j)$. Thus, the inductive hypothesis gives us

$$\int_{\prod_{i=1,i\ne j}^{n+1} \Omega_i} (I_W)_{x_j} d\left(\prod_{i=1,i\ne j}^{n+1} P_i\right) > \left[\int_{\prod_{i=1,i\ne j}^{n+1} \Omega_i} (I_W)_{x_j} d\left(\prod_{i=1,i\ne j}^{n+1} Q_i\right)\right]^\alpha.$$

From the assumption of P_j, we have $P_j(\{x_j\}) > 0$. Hence, using Lemma 5.1 and Fubini's theorem, $\left(\prod_{i=1}^{n+1} P_i\right)(W) > \left[\left(\prod_{i=1}^{n+1} Q_i\right)(W)\right]^{\alpha}$ holds. □

Lemma 6.3. For $0 < a_1 < \cdots < a_n,\ b_1 > \cdots > b_n > 0,\ 0 < \alpha_1 < \cdots < \alpha_n,\ \beta_1 > \cdots > \beta_n,\ n \geq 2$, we have the following inequality:

$$\left\{\sum_{j=1}^{n-1} a_j(b_j - b_{j+1}) + a_n b_n\right\}\left\{\sum_{j=1}^{n-1} \alpha_j(\beta_j - \beta_{j+1}) + \alpha_n \beta_n\right\} > \sum_{j=1}^{n-1} a_j \alpha_j (b_j \beta_j - b_{j+1}\beta_{j+1}) + a_n \alpha_n b_n \beta_n.$$

Proof. Mathematical induction on n proves the lemma. □

Proposition 6.2. Let P_i, Q_i and U_i be probability measures on $(\Omega_i, \mathcal{A}_i)$ $(1 \leq i \leq n)$. Suppose that $U_i(W_i) = P_i(W_i)Q_i(W_i)$ holds for every increasing set $W_i \in \mathcal{A}_i$, and $P_i(W_i) > P_i(W_i') > 0$ and $Q_i(W_i) > Q_i(W_i') > 0$ hold for every increasing sets W_i and W_i' of $\mathcal{A}_i$ such that $W_i' \subset W_i$ and $W_i' \neq W_i$ hold. Then, for every increasing set $W \in \mathcal{A}_C$,

$$\left(\prod_{i=1}^{n} U_i(W)\right)(W) = \left[\left(\prod_{i=1}^{n} P_i(W)\right)(W)\right]\left[\left(\prod_{i=1}^{n} Q_i(W)\right)(W)\right]$$

holds iff $W = \prod_{i=1}^{n}(P_{\Omega_i} W)$ holds.

Proof. "if" part is obvious. We prove "only if" part by the mathematical induction on n.

(The case of $n = 2$) If $W = (P_{\Omega_1} W) \times (P_{\Omega_2} W)$ does not hold, then using the same symbols of Lemma 6.1, $W = \cup_{i=1}^{m}(A_j \times B_j)$ $(m \geq 2)$, where $A_j \times B_j$ $(1 \leq j \leq m)$ are mutually exclusive. Then, setting $P = \prod_{i=1}^{2} P_i$, $Q = \prod_{i=1}^{2} Q_i$ and $U = \prod_{i=1}^{2} U_i$ and by Lemma 6.3,

$$\begin{aligned}
U(W) &= \sum_{j=1}^{m} U_1(A_j)U_2(B_j) \\
&= \sum_{j=1}^{m-1} U_1(A_j)\left\{U_2\left(\cup_{k=j}^{m} B_k\right) - U_2\left(\cup_{k=j+1}^{m} B_k\right)\right\} + U_1(A_m)U_2(B_m)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{m-1} P_i(A_j)Q_1(A_j)\left\{P_2\left(\cup_{k=j}^m B_k\right)Q_2\left(\cup_{k=j}^m B_k\right)\right.\\
&\qquad \left. -P_2\left(\cup_{k=j+1}^m B_k\right)Q_2\left(\cup_{k=j+1}^m B_k\right)\right\} + P_1(A_m)Q_1(A_m)P_2(B_m)Q_2(B_m)\\
&< \left[\sum_{j=1}^{m-1} P_1(A_j)\left\{P_2\left(\cup_{k=j}^m B_k\right) - P_2\left(\cup_{k=j+1}^m B_k\right)\right\} + P_1(A_m)P_2(B_m)\right]\\
&\quad \times \left[\sum_{j=1}^{m-1} Q_1(A_j)\left\{Q_2\left(\cup_{k=j}^m B_k\right) - Q_2\left(\cup_{k=j+1}^m B_k\right)\right\} + Q_1(A_m)Q_2(B_m)\right]\\
&= P(W)Q(W).
\end{aligned}
$$

Now assuming "only if" part to hold for $n = n$, we prove the proposition for $n = n + 1$. Let I_W be the indicator function of an increasing subset $W \subset \prod_{i=1}^{n+1} \Omega_i$. If $W = \prod_{i=1}^{n+1} P_{\Omega_i}(W)$ does not hold, then for some $x_j \in \Omega_j$, an increasing subset of $\prod_{i=1, i\neq j}^{n+1} \Omega_i$ defined by the section $(I_W)_{x_j}$ is not a product set of increasing subsets of Ω_i $(1 \le i \le n+1,\ i \neq j)$. Thus, by the inductive hypothesis,

$$
\begin{aligned}
&\int_{\prod_{i=1,i\neq j}^{n+1} \Omega_i} (I_W)_{x_j} d \prod_{i=1,i\neq j}^{n+1} U_i\\
&< \left[\int_{\prod_{i=1,i\neq j}^{n+1} \Omega_i} (I_W)_{x_j} d \prod_{i=1,i\neq j}^{n+1} P_i\right] \times \left[\int_{\prod_{i=1,i\neq j}^{n+1} \Omega_i} (I_W)_{x_j} d \prod_{i=1,i\neq j}^{n+1} Q_i\right].
\end{aligned}
$$

From the assumption on P_j and Q_j, we have $P_j(\{x_j\}) > 0$ and $Q_j(\{x_j\}) > 0$. Hence, using Lemma 7.1 and Fubini's theorem,

$$
\left(\prod_{i=1}^{n+1} U_i\right)(W) < \left[\left(\prod_{i=1}^{n+1} P_i\right)(W)\right] \times \left[\left(\prod_{i=1}^{n+1} Q_i\right)(W)\right]. \square
$$

Now we define a hazard transform of a multistate system as a mapping from $\prod_{i=1}^{n} \overline{\mathbf{R}}_{\le}^{N_i}$ to $\overline{\mathbf{R}}_{\le}^{N}$, where

$$
\overline{\mathbf{R}}_{\le}^{m} = \{\ (x_1, \cdots, x_m) \mid 0 \le x_1 \le \cdots \le x_m \le +\infty\ \}.
$$

Our definition is a straight extension of hazard transforms of binary state case [9] to multistate case.

Definition 6.1. A hazard transform of a system $(\Omega_{i=1}^n \Omega_i, S, \mathcal{V})$ is a mapping η from $\prod_{i=1}^n \overline{\mathbf{R}}_{\le}^{N_i}$ to $\overline{\mathbf{R}}_{\le}^{N}$ defined by the following procedure, where

$$\begin{aligned}
&\Omega_i = \{0, 1, \cdots, N_i\} \ (1 \le i \le n),\\
&S = \{0, 1, \cdots, N\},\\
&\mathcal{V} = \{V_s,\ s \in S\},\\
&W_j^i = \{j, j+1, \cdots, N_i\} \ (1 \le j \le N_i,\ 1 \le i \le n),\\
&W_j = \cup_{k=j}^N V_k \ (1 \le j \le N).
\end{aligned}$$

(the first step) For every i $(1 \le i \le n)$ and any given $\boldsymbol{x}^i = (x_1^i, \cdots, x_{N_i}^i) \in \overline{\mathbf{R}}_{\le}^{N_i}$, determine the probability measure P_i on $(\Omega_i, \mathcal{A}_i)$ such that

$$P_i(W_j^i) = e^{-x_j^i} \ (1 \le j \le N_i),$$

which is uniquely determined.

(the second step) Determine a probability measure on $(\Omega_C, \mathcal{A}_C)$ as $P = \prod_{i=1}^n P_i$, and then

$$(-\log P(W_1), -\log P(W_2), \cdots, -\log P(W_N)) \in \overline{\mathbf{R}}_{\le}^{N}$$

is determined.

From now on we use the following operation rules for vectors:

$$\begin{aligned}
&\boldsymbol{x} + \boldsymbol{y} = (x_1, \cdots, x_m) + (y_1, \cdots, y_m) = (x_1 + y_1, \cdots, x_m + y_m),\\
&\alpha\boldsymbol{x} = \alpha(x_1, \cdots, x_m) = (\alpha x_1, \cdots, \alpha x_m) \quad (\alpha : \text{a real number}),\\
&\boldsymbol{x} = (x_1, \cdots, x_m) \ge \boldsymbol{y} = (y_1, \cdots, y_m) \iff x_i \ge y_i \quad (1 \le i \le m),\\
&\boldsymbol{x} = (x_1, \cdots, x_m) = \boldsymbol{y} = (y_1, \cdots, y_m) \iff x_i = y_i \quad (1 \le i \le m),
\end{aligned}$$

Proposition 6.3.

(i) Let η be the hazard transform of an increasing system $(\prod_{i=1}^n \Omega_i, S, \mathcal{V})$, then for $\boldsymbol{x}^i,\ \boldsymbol{y}^i \in \overline{\mathbf{R}}_{\le}^{N_i}$ $(1 \le i \le m)$,

$$\eta(\boldsymbol{x}^1 + \boldsymbol{y}^1, \cdots, \boldsymbol{x}^m + \boldsymbol{y}^m) \ge \eta(\boldsymbol{x}^1, \cdots, \boldsymbol{x}^m) + \eta(\boldsymbol{y}^1, \cdots, \boldsymbol{y}^m). \quad (6.1)$$

(ii) Suppose that a system $(\prod_{i=1}^n \Omega_i, S, \mathcal{V})$ is a coherent system. Then, the equality in (6.1) holds iff the system is a series system.

Proof. Equation (6.1) is obvious from Proposition 5.1(ii) and the definition of hazard transforms.

"if" part of (ii) is immediate. If the system $(\prod_{i=1}^n \Omega_i, S, \mathcal{V})$ is a series coherent system, then each W_j $(1 \le j \le N)$ has the minimal element.

Thus, Lemma 6.1(ii) provides $W_j = \prod_{i=1}^n (P_{\Omega_i} W_j)$ $(1 \le j \le N)$, and thus, the equality in (6.1) follows by the definition of hazard transforms.

"only if" part of (ii) is also immediate. If the equality in (6.1) holds, we have $W_j = \prod_{i=1}^n P_{\Omega_i} W_i$ $(1 \le j \le N)$ by Proposition 6.2 and the definition of hazard transforms. Thus, W_j $(1 \le j \le N)$ has the minimal element by Lemma 6.1, and so does each V_j $(1 \le j \le N)$. Hence, the system $(\prod_{i=1}^n \Omega_i, S, \mathcal{V})$ is a series system. □

Proposition 6.4.

(i) Let η be the hazard transform of an increasing system $(\prod_{i=1}^n \Omega_i, S, \mathcal{V})$. Then, for $\boldsymbol{x}^i \in \overline{\mathbf{R}}_{\le}^{N_i}$ $(1 \le i \le n)$ and $0 < \alpha < 1$,

$$\eta(\alpha \boldsymbol{x}^1, \cdots, \alpha \boldsymbol{x}_m) \le \alpha \eta(\boldsymbol{x}^1, \cdots, \boldsymbol{x}_m). \tag{6.2}$$

(ii) Suppose that a system $(\prod_{i=1}^n \Omega_i, S, \mathcal{V})$ is a coherent system. Thus, the equality in (6.2) holds iff the system is a series system.

Proof. (i) is obvious from Proposition 5.1(i) and the definition of hazard transforms. (ii) is immediate from Lemma 6.1 and Proposition 6.1. □

In the sequel of this section, we examine some stochastic aspects of a system by using the hazard transform of the system. We use the symbols same as those in Section 5. Letting

$$\boldsymbol{H}^i(t) = \left(-\log \mu_{i,t}(W_1^i), \cdots, -\log \mu_{i,t}(W_{N_i}^i)\right),$$

we have the following proposition immediately, and thus, the proof is omitted:

Proposition 6.5. Suppose that $\{X_i(t),\ t \ge 0\}$ is decreasing and right continuous with probability 1. Then,

(i) $\{X_i(t),\ t \ge 0\}$ is IFR iff $\alpha \boldsymbol{H}^i(t_1) + \beta \boldsymbol{H}^i(t_2) \ge \boldsymbol{H}^i(\alpha t_1 + \beta t_2)$ holds for $t_1 \ge 0,\ t_2 \ge 0,\ \alpha \ge 0,\ \beta \ge 0,\ \alpha + \beta = 1$.
(ii) $\{X_i(t),\ t \ge 0\}$ is IFRA iff $\alpha \boldsymbol{H}^i(t) \ge \boldsymbol{H}^i(\alpha t)$ holds for $t \ge 0,\ 0 < \alpha < 1$.
(iii) $\{X_i(t),\ t \ge 0\}$ is NBU iff $\boldsymbol{H}^i(t_1 + t_2) \ge \boldsymbol{H}^i(t_1) + \boldsymbol{H}^i(t_2)$ holds for $t_1 \ge 0,\ t_2 \ge 0$.

Let η be the hazard transform of a system $(\prod_{i=1}^n \Omega_i, S, \mathcal{V})$. If $\{X_i(t),\ t \in T\}$ $(1 \le i \le n)$ are mutually independent, $\mu_t = \prod_{i=1}^n \mu_{i,t}$ holds. Then,

$$\boldsymbol{H}(t) = \eta(\boldsymbol{H}^1(t), \cdots, \boldsymbol{H}^n(t)), \tag{6.3}$$

where $\boldsymbol{H}(t) = (-\log \mu_t(W_1), \cdots, -\log \mu_t(W_N))$. Using Propositions 6.3, 6.4 and 6.5, and (6.3), IFRA and NUB closure theorems are immediately obtained.

Next, we prove that the preservation of IFR property determines the structure of multistate coherent system as a series system. For the proof of this proposition, we need the following lemma which is easily proved:

Lemma 6.4. For $a > \beta > 0,\ \alpha > a > 0$ and $\alpha - b \neq a - \beta$,

$$f(t) = \log[\exp\{-(a+b)t\} + \exp\{-(\alpha+\beta)t\} - \exp\{-(\alpha+a)t\}]$$

is neither convex nor concave in t.

Proposition 6.6. Let $\{X_i(t),\ t \in T\}$ be mutually independent, decreasing and right continuous with probability one, and let a system $(\prod_{i=1}^n \Omega_i, S, \mathcal{V})$ (or equivalently a system $(\prod_{i=1}^n \Omega_i, S, \varphi)$) be a coherent system. Then, $\{\varphi(\boldsymbol{X}(t)), t \in T\}$ is IFR whenever $\{X_i(t),\ t \in T\}$ $(1 \leq i \leq n)$ are IFR iff the system is a series system.

Proof. ("if" part) Since $W_j = \prod_{i=1}^n P_{\Omega_i} W_j$ $(1 \leq j \leq N)$ hold, $\mu_t(W_j) = \prod_{i=1}^n \mu_{i,t}(P_{\Omega_i} W_j)$ $(1 \leq j \leq N)$ follows, and then,

$$\boldsymbol{H}(t) = \left(\sum_{i=1}^n -\log \mu_{i,t}(P_{\Omega_i} W_1), \cdots, \sum_{i=1}^n -\log \mu_{i,t}(P_{\Omega_i} W_N) \right).$$

Using $\alpha \boldsymbol{H}^i(t_1) + \beta \boldsymbol{H}^i(t_2) \geq \boldsymbol{H}^i(\alpha t_1 + \beta t_2)$ $(\alpha \geq 0, \beta \geq 0, \alpha + \beta = 1, 1 \leq i \leq n)$, we have $\alpha \boldsymbol{H}(t_1) + \beta \boldsymbol{H}(t_2) \geq \boldsymbol{H}(\alpha t_1 + \beta t_2)$.

("only if" part) Let us consider probability measures

$$\mu_{i,t}(W_j^i) = \exp\{-t\alpha_j^i\} \quad \left(1 \leq j \leq N_i,\ \alpha_j^i \leq \alpha_{j+1}^i,\ 1 \leq i \leq n\right).$$

If the system is not series, then there exist $W_j \left(= \cup_{k=j}^N V_k,\ V_k \in \mathcal{V}\right)$ and $(x_{i_1}, \cdots, x_{i_{n-2}}) \in \prod_{k=1, k \neq i_{n-1}, k \neq i_n}^n \Omega_k$ such that the increasing set defined by the section $(I_{W_j})_{(x_{i_1}, \cdots, x_{i_{n-2}})}$ of the indicator function I_{W_j} is not a product set. Letting $\alpha_j^{i_k} \to 0$ $(j < x_{i_k})$ and $\alpha_j^{i_k} \to \infty$ $(j \geq x_{i_k}$ $(k = 1, \cdots, n-2)$,

$$\mu_t(W_j) \to \int_{\Omega_{i_{n-1}} \times \Omega_{i_{n-2}}} (I_{W_j})_{(x_{i_1}, \cdots, x_{i_{n-2}})} d(\mu_{i_{n-1},t} \times \mu_{i_{n-2},t}),$$

where the right hand side is log concave in t by the assumption. We may set $i_{n-1} = 1$ and $i_{n-2} = 2$ without loss of generality. Noticing that the increasing set defined by $(I_{W_j})_{(x_{i_1},\cdots,x_{i_{n-2}})}$ is expressed as $\cup_{j=1}^{m}(A_j \times B_j)$ $(m \geq 2)$ from Lemma 6.1. We here use the symbols same as those of the Lemma.

$$\int_{\Omega_1\times\Omega_2}(I_{W_j})_{(x_3,\cdots,x_n)}d(\mu_{1,t}\times\mu_{2,t})$$
$$=\sum_{i=1}^{m}\left[\mu_{1,t}(A_i)\left\{\mu_{2,t}\left(\bigcup_{k=i}^{m}B_k\right)-\mu_{2,t}\left(\bigcup_{k=i+1}^{m}B_k\right)\right\}\right]+\mu_{1,t}(A_m)\mu_{2,t}(B_m)$$

is log concave in t. We may set

$$\mu_{1,t}(A_i)=\exp\{-\alpha_i t\},\ \mu_{2,t}\left(\bigcup_{k=i}^{m}B_k\right)=\exp\{-\beta_i t\},$$
$$\alpha_1>\cdots>\alpha_m,\ \beta_1<\cdots<\beta_m,\ \beta_2-\beta_1\neq\alpha_1-\alpha_2.$$

Letting $\beta_3\to\infty$ in the above equation, we have the limit function

$$\exp\{-(\alpha_1+\beta_1)t\}+\exp\{-(\alpha_2+\beta_2)t\}-\exp\{-(\alpha_1+\beta_2)t\}$$

which is to be log concave. On the other hand, this function is not log concave by Lemma 6.4, which is a contradiction. Therefore, the system is series. □

7 Concluding Remarks

We have, in this chapter, examined a generalization of the concepts of binary state coherent systems like k-out-of-n systems, modules, IFR, IFRA and NBU processes, and so on. These are concerned with relations among the operating performances of systems and their components.

Throughout this work, we may recognize that a basic theory of reliability systems should be about algebraic and stochastic relations between two ordered sets through an increasing mapping from the one to the other. Some works have been done by [16, 20], but not sufficient. This chapter has considered only totally ordered finite sets except for the probabilistic examination, and the thorough generalization to arbitral ordered sets is remained to be an open problem.

Yu, Koren and Guo [27] defined multistate monotone coherent systems using partially ordered sets as state spaces of systems and components. But they did not define basic concepts as series systems, parallel systems, k-out-of-n systems and modules, and stochastic concepts as IFR, IFRA and NBU, and so on.

There are several definitions of multistate systems which are summarized in [18,19]. One of the important classes of multistate systems is EBW systems, which is an extension of Barlow and Wu [2] 's multistate systems. These multistate coherent systems are very close to binary state coherent systems, since the definition is based on the minimal cut and path sets of binary state coherent systems. Then, many properties of binary state systems are taken over to multistate case. For example, when restricted to this class, three modules theorem is perfectly held. Precise examinations have been seen in [24,25].

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