Chapter 1

Positive Solutions for Systems of Two Equations

1.1 Introduction

In this chapter, we consider a system of two partial differential equations describing two interacting population species. Each species diffuse from location of higher to lower concentration, and they interact with each other in a prey-predator, competing or cooperating relationship. We emphasize the situation when the species must have zero concentration at the boundary of the environment. These are known as reaction-diffusion equations with homogeneous Dirichlet boundary condition. The boundary condition is known as "hostile" in some ecological studies. We first consider the possibility of positive coexistence equilibrium for the case of prey-predator in Section 1.2, competing species in Section 1.3, and cooperating species in Section 1.4. They are thus systems of elliptic partial differential equations of the form:

$$(1.1) \qquad \begin{cases} \sigma_1 \Delta u + u(a_1 + f_1(u,v)) = 0 & \\ & \text{in } \Omega, \\ \sigma_2 \Delta v + v(a_2 + f_2(u,v)) = 0 & \\ & \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

In this chapter, we always assume that Ω is a bounded domain in $R^N, N \geq 2$, unless otherwise stated. If $N > 1$, we assume that the boundary $\partial\Omega \in C^{2+\alpha}, 0 < \alpha < 1$; that is, the boundary has local representation whose second order partial derivatives are Hölder continuous with exponent α. The symbol Δ denotes the Laplacian operator:

$$\sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j}.$$

The constants a_1, a_2 are respectively the intrinsic growth rates of the species whose population concentrations at the position x are denoted by $u = u(x)$ and $v = v(x)$. The parameters σ_1, σ_2 are positive diffusion coefficient constants. The reaction functions $f_1(u, v), f_2(u, v)$ involve many other parameters reflecting interaction rates and self-crowding effects of the species. We shall investigate the ranges of these parameters and their sizes relative to that of the size of the environment domain Ω so that coexistence states are possible. We shall use various methods of nonlinear analysis to study these problems, including upper-lower solutions, monotone schemes, bifurcation, degree theory and their generalizations. We usually begin with the simplest cases in order to illustrate how the various methods are used in obtaining the results.

In Section 1.5, we consider the time dependent parabolic system associated with system (1.1):

$$(1.2) \qquad \begin{cases} u_t = \sigma_1 \Delta u + u(a_1 + f_1(u, v)) & \text{in } \Omega \times (0, \infty), \\ v_t = \sigma_2 \Delta v + v(a_2 + f_2(u, v)) & \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

The main emphasis is to analyze the long time behavior of the system, and to find whether the solutions are tending to the equilibria described in the previous sections.

We now proceed to introduce some symbols which will be used repeatedly in this and later chapters. For any real $q(x)$ in $C^\alpha(\bar{\Omega})$ and $\sigma > 0$, the linear eigenvalue problem:

$$(1.3) \qquad -\sigma \Delta u + q(x) u = \rho u \ \text{ in } \Omega, \ \ u = 0 \ \text{ on } \partial\Omega$$

has an infinite sequence of eigenvalues, $\rho_1 < \rho_2 < \rho_3 < \dots$, which are bounded below. It is also known that the first eigenvalue:

$$(1.4) \qquad \rho = \rho_1 = \rho_1(-\sigma\Delta + q(x))$$

is simple, and all solutions of (1.3) with $\rho = \rho_1(-\sigma\Delta + q(x))$ are multiples of a particular eigenfunction, which does not change sign on Ω and has its normal derivatives never vanish on the boundary $\partial\Omega$.

For convenience, we define

$$(1.5) \qquad \lambda_1 := \rho_1(-\Delta),$$

and denote by $\omega(x)$, a positive eigenfunction of the operator $-\Delta$ on Ω with boundary condition $u = 0$ on $\partial\Omega$. Similarly, for any real $\hat{q}(x)$ in $C^\alpha(\bar{\Omega})$ and $\sigma > 0$, the linear eigenvalue problem:

$$(1.6) \qquad \sigma \Delta u + \hat{q}(x) u = \hat{\rho} u \ \text{ in } \Omega, \ \ u = 0 \ \text{ on } \partial\Omega$$

has an infinite sequence of eigenvalues, $\hat{\rho}_1 > \hat{\rho}_2 > \hat{\rho}_3 > \ldots$, which are bounded above. We denote the largest eigenvalue by:

$$\hat{\rho} = \hat{\rho}_1 = \hat{\rho}_1(\sigma\Delta + \hat{q}(x)), \tag{1.7}$$

which is simple. As shown in Sections 1.4 and 1.5 below, most of the results in this chapter are valid if the Laplacian operator is replaced with the uniform elliptic operator:

$$L \equiv \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j=1}^{N} b_i(x)\frac{\partial}{\partial x_i} + c(x),$$

where $a_{ij}(x)$, $b_i(x), c(x)$ are in $C^\alpha(\bar{\Omega}), 0 < \alpha < 1$, $c(x) \leq 0$ in $\bar{\Omega}$, and

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \mu_0 \sum_{i=1}^{N} \xi_i^2, \; \mu_0 > 0,$$

for all $x \in \bar{\Omega}$, all $(\xi_1, ..., \xi_N) \in R^N$. For simplicity, we present most of the results using the Laplacian.

For convenience, we state a simple direct consequence of the maximum principle, which will be used repeatedly to assert that in many instances non-negative non-constant solutions in $\bar{\Omega}$ are actually strictly positive in Ω.

Lemma 1.1. *Let $u \in C^2(\bar{\Omega})$ be a non-negative non-constant solution of:*

$$Lu + h(x)u = 0 \;\; in \; \Omega, \quad u = 0 \;\; on \;\; \partial\Omega,$$

where L is the operator described above and $h(x)$ is bounded, then u must satisfy $u(x) > 0$ for all $x \in \Omega$.

Proof. Let P be a positive constant such that $h(x) - P \leq 0$ for all $x \in \Omega$, and define $v = -u$. Then we have

$$Lv + [h(x) - P]v = -Pv \geq 0 \;\; \text{in } \Omega, \; v = 0 \;\; \text{on } \; \partial\Omega.$$

From the maximum principle, we obtain $v(x) < 0$ in Ω, since $v(x)$ is not a constant function. This means $u(x) > 0$ in Ω.

In this chapter, we avoid the consideration of zero outward normal derivative:

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Such homogeneous Neumann boundary condition, which represents no flux of species across the boundary, has been studied more extensively in other books in the literature, e.g. Smoller [209] and Leung [125].

1.2 Strictly Positive Coexistence for Diffusive Prey-Predator Systems

Let $u(x)$ and $v(x)$ be respectively the density of prey and predator at the point x in a bounded domain Ω. We first consider an earliest result concerning coexistence equilibrium when both species are restricted to vanish on the boundary. We consider the following homogeneous Dirichlet boundary value problem for the coupled Volterra-Lotka type reaction-diffusive system.

$$(2.1)\qquad \begin{cases} \sigma_1 \Delta u + u(a - bu - cv) = 0 & \\ & \text{in } \Omega, \\ \sigma_2 \Delta v + v(e + fu - gv) = 0 & \\ & \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, $\sigma_1, \sigma_2, a, b, c, e, f, g$ are positive constants. The parameters a, e are the intrinsic growth rates and b, c, f, g are interaction rates. Note that the prey-predator relation is reflected by the signs of $-c$ and $+f$.

Part A: Early Results via Upper-Lower Solutions and Bifurcation.

The following theorem concerning the coexistence of both species can be readily deduced by means of upper-lower solutions method for a system of elliptic equations.

Theorem 2.1. *The boundary value problem (2.1) under hypotheses:*

$$(2.2)\qquad \begin{cases} a > \sigma_1 \lambda_1, \; e > \sigma_2 \lambda_1, & \\ cf < gb, & \text{and} \\ a > \frac{\sigma_1 gb}{gb - cf}[\lambda_1 + \frac{ce}{g\sigma_1}] & \end{cases}$$

has a solution with each component strictly positive in Ω. Here λ_1 is defined in (1.5).

Proof. The last two inequalities of hypotheses (2.2) imply that $a(1 - \frac{cf}{gb}) > \sigma_1(\lambda_1 + \frac{ce}{g\sigma_1})$; hence $a > \sigma_1 \lambda_1 + \frac{c}{g}(e + f\frac{a}{b})$. It follows that for each fixed v, $0 \le v \le \frac{1}{g}(e + f\frac{a}{b})$, the function $u_1 := \delta\omega(x) > 0$ is a lower solution of the first equation in (2.1), for $\delta > 0$ sufficiently small. (Here $\omega(x)$ is described in Section 1.1.) That is, we have for each such v:

$$(2.3)\qquad \begin{aligned} &\sigma_1 \Delta u_1 + u_1(a - bu_1 - cv) \ge 0 && \text{in } \Omega, \text{ and} \\ &u_1 \le 0 && \text{on } \partial\Omega. \end{aligned}$$

On the other hand, the function $u_2(x) :\equiv a/b$ is an upper solution for the first equation in (2.1). That is,

$$\sigma_1 \Delta u_2 + u_2(a - bu_2 - cv) \leq 0 \qquad \text{in } \Omega, \text{ and}$$

$$u_1 \geq 0 \qquad \text{on } \partial\Omega.$$

Similarly, for each fixed u, $0 \leq u \leq a/b$, the functions $v_1 := \delta\omega(x)$, for sufficiently small positive δ and $v_2(x) :\equiv \frac{1}{g}(e + f\frac{a}{b})$ are respectively lower and upper solutions for the second equation in (2.1). By means of an intermediate-value type theorem (see Tsai [221] or Theorem 1.4-2 in Leung [125]), we assert that there exists a solution $(u^*(x), v^*(x))$ of (2.1) satisfying $u_1(x) \leq u^*(x) \leq u_2, v_1(x) \leq v^*(x) \leq v_2$ for all $x \in \bar{\Omega}$. Note that since $v_1 > 0$, we have (2.3) valid for all v satisfying $v_1 < v < v_2$.

Remark 2.1. The proof of Theorem 2.1 is simple. However, it uses an intermediate-value type theorem, whose proof requires Leray-Schauder degree theory. Observe also that the inequalities in (2.2) are more readily satisfied for large domains, because λ_1 will then be small.

We next use a more sophisticated procedure to see how the various sizes of the parameters a and e lead to different results of existence and non-existence of positive solutions. We first consider the boundary value problem:

$$\text{(2.4)} \qquad -\sigma\Delta u + q(x)u = u(a - bu) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\sigma > 0, q(x), a$ and b are as described above. Suppose that $a \leq \rho_1(-\sigma\Delta + q(x))$. Let $\phi(x) > 0$ be an eigenfunction for (2.4) with $\rho = \rho_1(-\sigma\Delta + q(x))$. Using the family of upper solutions $\epsilon\phi(x), \epsilon > 0$, for (2.4) and the sweeping principle described in Theorem 1.4-3 [125] one readily deduces that $u = 0$ is the only non-negative solution of (2.4) if $a \leq \rho_1(-\sigma\Delta + q(x))$. On the other hand, suppose $a > \rho_1(-\sigma\Delta + q(x))$. We use large constant as upper solution and small multiple of $\phi(x)$ as lower solution for (2.4) to deduce the existence of a solution which is positive in Ω. Furthermore, such positive solution is unique, when $a > \rho_1(-\sigma\Delta + q(x))$. (See Lemma 5.2-2 in [125].) We will state the above observation in a slightly more general situation, which will be used repeatedly in many chapters.

Lemma 2.1. *Let $q(x)$ be in $C^\alpha(\bar{\Omega})$, $0 < \alpha < 1$; $G \in C^1([0,\infty)), G' < 0$ in $(0,\infty)$ and there exists some $c_0 > 0$ such that $G(c_0) < 0$. Consider the boundary value problem*

$$\text{(2.5)} \qquad -\sigma\Delta u + q(x)u = uG(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

(i) If $G(0) \leq \rho_1(-\sigma\Delta + q(x))$, then $u \equiv 0$ is the only non-negative solution of the problem.

(ii) If $G(0) > \rho_1(-\sigma\Delta + q(x))$, then the problem has a unique strictly positive solution in Ω.

We consider the problem (2.1) under the simplest situation when the growth rate a of the prey is small. In such situation, no prey population can survive as described below.

Theorem 2.2. *Suppose $a \le \rho_1(-\sigma\Delta)$ and (u, v) is a non-negative solution of (2.1). Then the following are true:*
(i) $u \equiv 0$ in $\bar{\Omega}$.
(ii) If $e \le \rho_1(-\sigma_2\Delta)$, then we also have $v \equiv 0$ in $\bar{\Omega}$; if $e > \rho_1(-\sigma_2\Delta)$, then either $v \equiv 0$ in $\bar{\Omega}$ or v is the unique positive solution of

$$\sigma_2\Delta v + v(e - gv) = 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega. \tag{2.6}$$

Proof. Multiplying the first equation of (2.1) by u, and integrating over Ω, we obtain

$$-\sigma_1 \int_\Omega u\Delta u dx \le a \int_\Omega u^2 dx - b \int_\Omega u^3 dx. \tag{2.7}$$

On the other hand, the characterization of the first eigenvalue gives

$$\rho_1(-\sigma_1\Delta) \int_\Omega u^2 dx \le \int_\Omega \sigma_1 |\nabla u|^2 dx = -\sigma_1 \int_\Omega u\Delta u dx. \tag{2.8}$$

Inequalities (2.7) and (2.8) imply that $\int_\Omega u^2 dx < \frac{a}{\rho_1(-\sigma_1\Delta)} \int_\Omega u^2 dx$ if $u \not\equiv 0$. Thus we must have $u \equiv 0$ in $\bar{\Omega}$. Consequently, assertion (ii) follows from the discussion for single equation above analogous to (2.4), with $q \equiv 0$, and σ, a, b respectively replaced by σ_2, e, f.

We next use bifurcation technique to analyze problem (2.1) as the parameters e or a varies. This will eventually lead to Theorem 2.3 and Theorem 2.4. The approach involves decoupling the two equations in (2.1). We write the first equation in (2.1) in the form;

$$-\sigma_1\Delta u + cvu = u(a - bu) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{2.9}$$

which can be regarded as a special case of (2.4) with $\sigma = \sigma_1, q(x) = cv(x)$. Thus, if $a \le \rho_1(-\sigma_1\Delta + cv(x))$, then (2.9) has no positive solution; while if $a > \rho_1(-\sigma_1\Delta + cv(x))$, then (2.9) has a unique positive solution in Ω. Let v be an arbitrary function in $C^1(\bar{\Omega})$, we define $u(v)$ as a function on $\bar{\Omega}$ by:

$$u(v) = \begin{cases} 0 \text{ if } a \le \rho_1(-\sigma_1\Delta + cv), \\ \\ \text{unique solution of problem (2.9) if } a > \rho_1(-\sigma_1\Delta + cv). \end{cases} \tag{2.10}$$

Clearly, if v satisfies the single equation:

$$-\sigma_2\Delta v = v(e - gv + fu(v)) \text{ in } \Omega, \ \ v = 0 \text{ on } \partial\Omega, \tag{2.11}$$

then the pair $(u(v), v)$ will be a solution of (2.1). To analyze (2.11), we first obtain the following properties of the mapping $v \to u(v)$.

Lemma 2.2. *(i) The mapping: $v \to u(v)$ defined by (2.10) considered as a function from $C^1(\bar{\Omega})$ to $C^1(\bar{\Omega})$ is continuous;*
(ii) if $v_1 \geq v_2$ in $\bar{\Omega}$, then $u(v_1) \leq u(v_2)$ in $\bar{\Omega}$.

The proof of this lemma can be found in Brown [15] or p. 360 in Leung [125].

To study more interesting situations, we now suppose that

$$a > \rho_1(-\sigma_1\Delta). \tag{2.12}$$

In the following Theorem 2.3, we let the parameter e varies, while all other parameters are held fixed. Problem (2.1) has two non-negative solutions $(0, 0)$ and $(u(0), 0)$ for all values of e. We consider the global bifurcations as e varies in the decoupled equation (2.11). This leads to bifurcation from the line of solution $(u(0), 0)$ to solution of (2.1) with both components positive in Ω. Let L be the operator defined by

$$Lv = -\sigma_2\Delta v - fu(0)v.$$

Without loss of generality, we may assume that $\rho_1(-\sigma_2\Delta - fu(0)) \neq 0$. Otherwise, we replace L by $L+k$ for an appropriate constant k. For each h in $C^1(\bar{\Omega})$, let Kh denote the unique solution of the problem: $Lu = h$ in Ω, $u = 0$ on $\partial\Omega$. The map $K : C^1(\bar{\Omega}) \to C^1(\bar{\Omega})$ is a compact linear operator. Let $F : C^1(\bar{\Omega}) \to C^1(\bar{\Omega})$ be defined by

$$F(v) = -gv^2 + f[u(v) - u(0)]v.$$

By Lemma 2.2, F is continuous and $||F(v)|| = o(||v||)$ as $v \to 0$ in $C^1(\bar{\Omega})$, where $|| \cdot ||$ denotes the norm in $C^1(\bar{\Omega})$. We now write (2.11) in the form:

$$v - eKv - KF(v) = 0. \tag{2.13}$$

Since $||KF(v)|| = o(||v||)$ as $v \to 0$ in $C^1(\bar{\Omega})$, we can apply the global bifurcation results of Rabinowitz [190] as the parameter e varies. We can also apply results concerning bifurcation from simple eigenvalues described in Crandall and Rabinowitz [33], Blat and Brown [11] or [125] to obtain properties concerning the local behavior of the bifurcation solutions. It is shown that in a neighborhood of the bifurcation point $(\rho_1(-\sigma_2\Delta - fu(0)), 0)$, all non-trivial solutions (e, v) of (2.13) lie on a curve of the form $\{(\bar{e}(\alpha), \phi(\alpha)) : -\delta \leq \alpha \leq \delta\}$ in $R \times C^1(\bar{\Omega})$, where $\bar{e}(0) = \rho_1(-\sigma_2\Delta - fu(0))$ and $\phi(\alpha) = \alpha\phi_1 +$ terms of higher "order" in α. Here, ϕ_1 is a positive principal eigenfunction for the eigenvalue $\rho_1(-\sigma_2\Delta - fu(0))$.

From the fact that $\frac{\partial \phi_1}{\partial \nu} < 0$ on $\partial\Omega$ where ν is the outward unit normal at the boundary, we thus conclude that for α sufficiently small and positive, the corresponding non-trivial solution v lies in the cone

$$P = \{v \in C^1(\bar{\Omega}) : v(x) > 0 \ for \ x \in \Omega, \ \frac{\partial v}{\partial \nu}(x) < 0 \ for \ x \in \partial\Omega\}.$$

Moreover, the closure of the set of non-trivial solutions (e, v) of (2.13) contains a component S (i.e. a maximal connected subset) such that either S joins $(\rho_1(-\sigma_2\Delta - fu(0)), 0)$ to ∞ in $R \times C^1(\bar{\Omega})$ or S joins $(\rho_1(-\sigma_2\Delta - fu(0)), 0)$ to $(\bar{\rho}, 0)$, where $\bar{\rho}$ is some other eigenvalue of L. More precisely, we can further deduce (see [125] or [11]) the following properties for the set S.

Lemma 2.3. *The component S contains a connected subset $S^+ \subset S - \{(\bar{e}(\alpha), \phi(\alpha)) : -\delta \le \alpha \le 0\}$ with the following properties:*
(i) S^+ is contained in $R \times P$;
(ii) $\{\rho \in R : (\rho, v) \in S^+\} = (\rho_1(-\sigma_2\Delta - fu(0)), +\infty)$.

Let (u, v) be any solution of (2.1) with each component non-negative in $\bar{\Omega}$. Suppose that v is not the trivial function, then v is the unique positive solution of the equation

$$-\sigma_2\Delta v - fuv = v(e - gv) \text{ in } \Omega, \ v = 0 \text{ on } \partial\Omega. \tag{2.14}$$

Let λ_1 and ω_1 be the principal eigenvalue and the corresponding eigenfunction with $\max\{\omega_1(x) : x \in \Omega\} = 1$. It is readily checked that if $e > \sigma_2\lambda_1$, then the function $g^{-1}(e - \sigma_2\lambda_1)\omega_1$ is a lower solution of the problem (2.14), and that any sufficiently large positive constant is an upper solution. Since v must be between the upper and lower solutions, we conclude that if $e > \sigma_2\lambda_1$ we have $v \ge g^{-1}(e - \sigma_2\lambda_1)\omega_1 := k(e)\omega_1$ in $\bar{\Omega}$, where $k(e) := g^{-1}(e - \sigma_2\lambda_1) \to \infty$ as $e \to \infty$. Now, consider the eigenvalue problem

$$-\sigma_1\Delta u + ck(e)\omega_1 u = \lambda u \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega. \tag{2.15}$$

The least eigenvalue $\lambda = \hat{\lambda}_1(e)$ has the characterization:

$$\hat{\lambda}_1(e) = inf.\{\int_\Omega \sigma_1|\nabla u|^2 + ck(e)\omega_1 u^2 dx : u \in W_0^{1,2}(\Omega), \int_\Omega u^2 dx = 1\}.$$

Thus from the limit of $k(e)$, we can deduce that $\hat{\lambda}_1(e) \to \infty$ as $e \to \infty$. Consequently, we have $\hat{\lambda}_1(e) > a$, if e is large enough.

Next, from the characterization of first eigenvalue and comparing with (2.15), we find that the first eigenvalue of

$$-\sigma_1\Delta w + cvw = \lambda w \text{ in } \Omega, \ w = 0 \text{ on } \partial\Omega$$

is greater that a. Hence the only non-negative solution of

$$-\sigma_1\Delta u + cvu = u(a - bu) \text{ in } \Omega, \;\; u = 0, \;\text{ on } \partial\Omega$$

is the zero function. We have proved that if e is large enough and v is not the trivial function, then $u \equiv 0$. From Lemma 2.3, we see that the only way the continuum of solutions S^+ can join the bifurcation point $(\rho_1(-\sigma_2\Delta - fu(0)), 0)$ on the (e, v) plane to ∞ is by $u(v)$ becoming equal to zero for e sufficiently large. However, when $u(v) \equiv 0$, then clearly v satisfies (2.6). If we consider the bifurcation diagram on the $e-(u, v)$ plane, the continuum of solutions $\{(e, u(v), v) : (e, v) \in S^+\}$ for (2.1) must join up with the continuum of solutions $\{(e, 0, v) : (e, v)$ is a solution of (2.6)$\}$, Solutions of (2.6) are discussed in Theorem 2.2(ii). From the above arguments, we obtain the following theorem.

Theorem 2.3. *(i) Suppose:*

$$a > \sigma_1\lambda_1. \tag{2.16}$$

Then there exists $\lambda^* > \sigma_2\lambda_1 = \rho_1(-\sigma_2\Delta)$ such that if e satisfies: $\rho_1(-\sigma_2\Delta - fu(0)) < e < \lambda^*$, that is:

$$\hat\rho_1(\sigma_2\Delta + e + fu(0)) > 0 \qquad \textit{and} \quad e < \lambda^*, \tag{2.17}$$

the boundary value problem (2.1) has a solution with each component strictly positive in Ω. Moreover, there exists $\tilde\lambda \geq \lambda^*$ such that if $e > \tilde\lambda$, then any non-negative solution (u,v) of problem (2.1) with $v \not\equiv 0$ must have $u \equiv 0$. (Recall the definition of $\hat\rho_1$ in (1.7).)
(ii) Suppose:

$$a < \sigma_1\lambda_1. \tag{2.18}$$

Then any non-negative solution (u,v) of problem (2.1) must have $u \equiv 0$.

In the following theorem, we let the parameter a varies, while all other parameters are held fixed. We write the second equation in (2.1) in the form of (2.14). Analogous to Lemma 2.2, we define a map from $C^1(\bar\Omega)$ to $C^1(\bar\Omega)$ by:

$$v(u) = \begin{cases} 0 \text{ if } e \leq \rho_1(-\sigma_2\Delta - fu), \\ \text{unique solution of problem (2.14) if } e > \rho_1(-\sigma_2\Delta - fu). \end{cases} \tag{2.19}$$

We can show as in Lemma 2.2 that $u \to v(u)$ is a continuous function from $C^1(\bar\Omega)$ to $C^1(\bar\Omega)$ and that $u \to v(u)$ is an increasing function.

Theorem 2.4. *(i) Suppose:*

$$\begin{cases} e > \sigma_2\lambda_1 \qquad \text{and} \\ \hat\rho_1(\sigma_1\Delta + a - cv(0)) > 0, \end{cases} \tag{2.20}$$

then the boundary value problem (2.1) has a solution with each component strictly positive in Ω.
(ii) Suppose that $e \leq \sigma_2\lambda_1$. *Then, provided that* a *is sufficiently large, the problem (2.1) has a solution with each component strictly positive in* Ω.

Proof. Let $e > \sigma_2\lambda_1$, then problem (2.1) has a solution $(u, v) = (0, v(0))$ with $v(0)$ non-trivial. We write the first equation of (2.1) as:

$$(2.21) \qquad -\sigma_1\Delta u + cv(0)u = au - bu^2 - c[v(u) - v(0)]u \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega,$$

and bifurcate with the parameter a at $a = \rho_1(-\sigma_1\Delta + cv(0))$ when $(u, v) = (0, v(0))$. As in Lemma 2.3, we can show that there exists a continuum of solutions S^+ of (2.21) contained in $R \times P$, i.e. $u \geq 0$ whenever $(a, u) \in S^+$, and that $\{a : (a, u) \in S^+\} = (\rho_1(-\sigma_1\Delta + cv(0)), \infty)$. If $(a, u) \in S^+$, then $u \geq 0$ and so $v(u) \geq v(0)$, i.e. v(u) is not the trivial function. Consequently, the continuum of solutions $\{(a, u, v(u)) : (a, u) \in S^+\}$ for the system (2.1) cannot connect with the continuum of solutions $\{(a, u(0), 0) : a > \sigma_1\lambda_1\}$. This leads to the assertion of part (i).

For part (ii), suppose that $e \leq \sigma_1\lambda_1$. We have $u(0)$ satisfies

$$-\sigma_1\Delta u = au - bu^2 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega.$$

Let λ_1 and ω_1 be the principal eigenvalue and the corresponding eigenfunction with $\max\{\omega_1(x) : x \in \Omega\} = 1$. Using a/b and $b^{-1}(a - \sigma_1\lambda_1)\omega_1$ for a large enough as upper and lower solutions respectively, we find that $b^{-1}(a - \sigma_1\lambda_1)\omega_1 \leq u(0) \leq ab^{-1}$. Comparing the least eigenvalue of

$$(2.22) \qquad -\sigma_2\Delta w - fu(0)w = \lambda w \text{ in } \Omega, \ w = 0 \text{ on } \partial\Omega,$$

with that of

$$-\sigma_2\Delta v - fb^{-1}(a - \sigma_1\lambda_1)\omega_1 v = \lambda v \text{ in } \Omega, \ v = 0 \text{ on } \partial\Omega,$$

by means of Rayleigh's quotient, we conclude that the first eigenvalue $\rho_1(-\sigma_2\Delta - fu(0))$ of (2.22) tends to $-\infty$ as $a \to +\infty$. That is we have $\rho_1(-\sigma_2\Delta - fu(0)) \leq e \leq \sigma_2\lambda_1$ for a sufficiently large. Thus by Theorem 2.3(i), we assert that problem (2.1) has a solution which is positive in both components.

Part B: General Results via Degree Theory.

We next consider a prey-predator system with more general type of interaction than quadratic (or Lotka-Volterra type). Moreover, we will obtain somewhat sharper results, and find necessary and sufficient conditions for the existence of positive solutions. We shall use degree theory method of cone index to prove that the conditions in parts (ii) and (iii) of the following Theorem 2.5 is sufficient for the existence of positive solution. More precisely, for a constant $d > 0$, we will consider the boundary value problem:

$$(2.23)\qquad \begin{cases} \Delta u + uM(u,v) = 0 & \\ & \text{in } \Omega, \\ d\Delta v + v(h(u) - m(v)) = 0 & \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $M(u,v)$ and its first partial derivatives are continuous in the first closed quadrant. Moreover, it satisfies

$$(2.24)\qquad M_v(u,v) < 0 \text{ for } u,v \geq 0;\;\; M_u(u,0) < 0 \text{ for } u \geq 0,$$

(2.25)
$M(0,0) > 0$; there exists a constant $C_0 > 0$ such that $M(u,0) < 0$ for $u > C_0$.

(2.26) The functions h and m belong to $C^1([0,\infty))$, with each function strictly increasing in $[0,\infty)$.

A solution (u,v) of problem (2.23) is called a positive solution if both components are ≥ 0 and $\not\equiv 0$ on Ω. A common assumption in ecological studies is to set the rate $M(u,v) = \sigma_1^{-1}(a - bu - \frac{vc}{k+u})$ where σ_1, a, b, c, k are positive constants or other rates involving ratios of u and v. Such type of growth rate is called Holling's type. From the smoothness of M, h and m, the positive solutions of (2.23) are classical solutions with components in $C^2(\bar{\Omega})$, if they exist.

Theorem 2.5. *Assume hypotheses (2.24) to (2.26) and there exists a positive number B_0 such that*

$$(2.27)\qquad m(B_0) > h(C_0).$$

Then all positive solutions (u,v) of (2.23) must satisfy $0 \leq u \leq C_0, 0 \leq v \leq B_0$. Moreover:
(i) Suppose $M(0,0) \leq \lambda_1$, and $h(0) \leq \lambda_1 d + m(0)$, then (0,0) is the only non-negative solution of (2.23).
(ii) Suppose $h(0) < \lambda_1 d + m(0)$, then problem (2.23) has a positive solution (u,v) iff:

$$(2.28)\qquad M(0,0) > \lambda_1; \;\; \textit{and} \;\; \hat{\rho}_1(d\Delta + (h(u_0) - m(0))) > 0.$$

(iii) If $h(0) > \lambda_1 d + m(0)$, and $M(0,v) \geq M(u,v)$ for $u,v \geq 0$. Then (2.23) has a positive solution (u,v) iff

$$(2.29)\qquad M(0,0) > \lambda_1; \;\; \textit{and} \;\; \hat{\rho}_1(\Delta + M(0,v_0)) > 0.$$

(Note that the assumption on $h(0)$ in case (iii) already implies that the second inequality in (2.28) is true.) Furthermore, from the boundedness of the positive solution of (2.23), we have each component of the positive solution strictly positive in Ω, by Lemma 1.1.

Remark 2.2. In (ii) above, the function u_0 is the unique positive solution of $\Delta u + uM(u,0) = 0$ in Ω, $u = 0$ on $\partial\Omega$. Such solution exists provided $M(0,0) > \lambda_1$, by Lemma 2.1. In (iii) above, the function v_0 is the unique positive solution of $d\Delta v + v(h(0) - m(v)) = 0$ in Ω, $v = 0$ on $\partial\Omega$. Such solution exists provided $h(0) > \lambda_1 d + m(0)$, by Lemma 2.1.

Example 2.1. In the usual Volterra-Lotka model, we let

$$M(u,v) = \sigma_1^{-1}(a - bu - cv),\ d = \sigma_2,\ h(u) = e + fu,\ m(v) = gv, \tag{2.30}$$

where $\sigma_1, \sigma_2, a, b, c, f, g$ are positive constants and e is any constant, then Theorem 2.5 readily leads to the following corollary.

Corollary 2.6. *Consider problem (2.23) with $M(u,v), d, h(u), m(v)$ as given in (2.30).*

(i) If $a \le \sigma_1\lambda_1$, $e \le \lambda_1\sigma_2$, then (0,0) is the only non-negative solution of (2.23).

(ii) If $e < \sigma_2\lambda_1$, then problem (2,23) has a positive solution iff

$$a > \sigma_1\lambda_1 \ \textit{ and } \ \hat{\rho}_1(\sigma_2\Delta + e + fu_0) > 0. \tag{2.31}$$

(iii) If $e > \sigma_2\lambda_1$, then (2.23) has a positive solution iff

$$\hat{\rho}_1(\sigma_1\Delta + a - cv_0) > 0. \tag{2.32}$$

(Note that part (ii) is closely related to Theorem 2.4(ii). When a is sufficiently large, the second inequality in (2.31) will be satisfied. Note also part (iii) above is closely related to Theorem 2.4(i) and Theorem 2.3(i); when e is large enough, (2.32) cannot hold.)

The following lemma is very useful for proving Theorem 2.5.

Lemma 2.4. *Assume $a(x) \in L^\infty(\Omega)$. Let u be an arbitrary function satisfying $u \ge 0, \not\equiv 0$ in Ω with $u = 0$ on $\partial\Omega$.*
(i) If $0 \not\equiv (\Delta + a(x))u \ge 0$ in Ω, then $\hat{\rho}_1(\Delta + a(x)) > 0$.
(ii) If $0 \not\equiv (\Delta + a(x))u \le 0$ in Ω, then $\hat{\rho}_1(\Delta + a(x)) < 0$.
(iii) If $(\Delta + a(x))u \equiv 0$ in Ω, then $\hat{\rho}_1(\Delta + a(x)) = 0$.

Proof. (i) Let $\theta(x) > 0$ in Ω be an eigenfunction corresponding to the principal eigenvalue $\bar{\rho} = \hat{\rho}_1(\Delta + a(x))$. Then $0 < \int_\Omega (\Delta + a(x))u\theta\, dx = \bar{\rho}\int_\Omega u\theta\, dx$, hence we must have $\bar{\rho} > 0$.
(ii) We simply reverse the sign in the argument in part (i) involving the integral.

(iii) We have $0 = \bar{\rho}\int_{\Omega} u\theta\, dx$ with $u \geq 0, u \not\equiv 0$, $\theta > 0$ in Ω. This implies that $\bar{\rho} = \rho_1(\Delta + a(x)) = 0$.

Proof (of Part (i) and necessity part of (ii), (iii) of Theorem 2.5 for the existence of positive solution). We first prove the existence of an a-priori bound for all non-negative solutions of (2.23). For any given $v \geq 0$ in $\bar{\Omega}$, we have by (2.24) and (2.25) a family of upper solutions $w \equiv \bar{C}$ in Ω, $\bar{C} \geq C_0$ for the first equation in (2.23), i.e.

$$\Delta w + wM(w, v) < 0 \text{ in } \Omega, \;\; w \geq 0 \text{ on } \partial\Omega.$$

By the sweeping principle,any positive solution of problem (2.23) must have $0 \leq u \leq C_0$. Let $(\tilde{u}, \tilde{v})$ be a positive solution of (2.23). Suppose $x_0 \in \Omega$ such that $\tilde{v}(x_0) = max_{x\in\bar{\Omega}}\tilde{v}(x) > 0$. Then from the second equation in (2.23), $\tilde{v}(x_0)[h(\tilde{u}(x_0)) - m(\tilde{v}(x_0))] = -d\Delta\tilde{v}(x_0) \geq 0$ at the interior maximum point. Thus we must have

$$m(\tilde{v}(x_0)) \leq h(\tilde{u}(x_0)) \leq h(C_0). \tag{2.33}$$

By the increasing property of the function m, we must have $\tilde{v}(x_0) < B_0$.

We now prove the necessity assertion for part (ii) and (iii) of Theorem 2.5. Suppose $h(0) < \lambda_1 d + m(0)$, and $(\tilde{u}, \tilde{v})$ is a positive solution of (2.23). Since $\tilde{v} \geq 0$ in Ω, and $d\Delta\tilde{v} + \tilde{v}(h(\tilde{u}) - m(\tilde{v})) = 0$ in Ω, $\tilde{v} = 0$ on $\partial\Omega$, we can obtain by maximum principle that $\tilde{v}$ cannot have a nonpositive minimum in Ω. Consequently, we must have $\tilde{v} > 0$ in Ω (cf. Lemma 1.1). Next, consider the scalar problem: $\Delta w + wM(w, 0) = 0$ in Ω, $w = 0$ on $\partial\Omega$. The constant function $C_0 + \epsilon$ is a upper solution. On the other hand, the fact that $\Delta\tilde{u} + \tilde{u}M(\tilde{u}, 0) \geq \Delta\tilde{u} + \tilde{u}M(\tilde{u}, \tilde{v}) = 0$ in Ω, implies the function $\tilde{u}$ is a lower solution. We conclude that $\tilde{u} \leq u_0 \leq C_0 + \epsilon$. By Lemma 1.1, we have $u_0 > 0$ in Ω, and Lemma 2.1(i) implies $M(0, 0) > 0$. The fact that $u_0 \geq \tilde{u}$ implies that

$$d\Delta\tilde{v} + \tilde{v}(h(u_0) - m(0)) \geq d\Delta\tilde{v} + \tilde{v}(h(\tilde{u}) - m(\tilde{v})) = 0$$

in Ω. By Lemma 2.4(i), the above inequalities implies the second inequality in (2.28).

Next, suppose $h(0) > \lambda_1 d + m(0)$. Consider the scalar problem: $d\Delta w + w(h(0) - m(w)) = 0$ in Ω, $w = 0$ on $\partial\Omega$. We can verify readily as above that $\tilde{v}$ and $\delta\omega$ are respectively upper and lower solutions, where $\delta > 0$ is sufficiently small and moreover $\tilde{v} > \delta\omega$ in Ω. (Recall the definition of ω in (1.5).) The uniqueness of positive solution of this scalar problem leads to $\delta\omega \leq v_0 \leq \tilde{v}$ in Ω. Consequently, we have $\Delta\tilde{u} + \tilde{u}M(0, v_0) \geq \Delta\tilde{u} + \tilde{u}M(\tilde{u}, v_0) \geq \Delta\tilde{u} + \tilde{u}M(\tilde{u}, \tilde{v}) = 0$. By Lemma 2.4(i) again, we conclude that the second inequality of (2.29) is valid.

Before we begin to prove the sufficiency part of Theorem 2.5, we need to introduce some concepts in cone index method. Roughly speaking, we will apply the theory to compact operators on the cone of positive vector functions. Let E be a Banach space, $W \subset E$ is called a wedge in E if W is a closed convex set and $\alpha W \subseteq W$ for every real $\alpha \geq 0$. A wedge is called a cone if $W \cap \{-W\} = \{0\}$. For $y \in W$, define

$$W_y = \{x \in E : y + \theta x \in W, \text{ for } 0 \leq \theta \leq \gamma, \text{ for some } \gamma > 0\}.$$

One readily verifies that W_y is convex, and $W_y \supseteq \{y\} \cup \{-y\} \cup \{W\}$. Moreover, the set $\bar{W}_y$,(the closure of W_y), is also a wedge. Let $S_y = \{x \in \bar{W}_y : -x \in \bar{W}_y\}$; we easily see that S_y is a linear subspace of E.

A nonempty subset A of a metric space X is called a retract of X if there exists a continuous map $r : X \to A$ (called a retraction), such that $r|A = id_A$. By a theorem of Dugundji [53], [54], every nonempty closed convex subset of a Banach space E is a retract of E. Let X be a retract of a Banach space E. For every open subset U of X and every compact map $f : \bar{U} \to X$ which has no fixed points on ∂U, there exists an integer $i_X(f, U)$ defined by

$$i_X(f, U) = i_E(f \circ r, r^{-1}(U)) = deg(id - f \circ r, r^{-1}(U), 0),$$

where $i_E(f \circ r, r^{-1}(U))$ is the well-known Leray-Schauder degree. This definition is independent of the choice of the retraction. The integer $i_X(f, U)$ is called the fixed point index of f (over U with respect to X). This index satisfies the normalization, additivity, homotopic invariance and permanence properties as the Leray-Schauder degree (cf. Theorem A2-1 in Chapter 6). If $x_0 \in U$ is an isolated fixed point of f, and x_0 is the only fixed point of f in $x_0 + \rho B, \rho > 0$, where B is the open unit ball of E. We define the fixed point index by

$$index_X(f, x_0) := i_X(f, x_0 + \rho B).$$

Definition 2.1. *Let $L : E \to E$ be a compact linear operator such that $L(\bar{W}_y) \subseteq \bar{W}_y$. L is said to have property (α) on $\bar{W}_y$ if the following holds:*

(α) There exists $t \in (0, 1)$ and a $w \in \bar{W}_y \backslash S_y$ such that $w - tLw \in S_y$.

Remark 2.3. If $I - L$ is invertible in $\bar{W}_y$, an important consequence of property (α) on $\bar{W}_y$ is given below in Lemma 2.5, asserting that there exists $z \in \bar{W}_y$ such that the equation $x - Lx = z$ has no solution for x in $\bar{W}_y$. Under appropriate circumstances, this in turn leads to $index_W(A, y_0) = 0$, by Lemma 2.6(i), where L is the Fréchet derivative of A at y_0 in W.

Lemma 2.5. *Let $L : E \to E$ be a compact linear operator such that $L(\bar{W}_y) \subseteq \bar{W}_y$. Assume $I - L$ is invertible in $\bar{W}_y$ (in the sense that $h \neq Lh$ if $h \in \bar{W}_y \backslash \{0\}$). If L has property (α) on $\bar{W}_y$, then there exists $z \in \bar{W}_y$ such that the equation $x - Lx = z$ has no solution for x in $\bar{W}_y$.*

Proof. Since L has property (α), there exists $v \in \bar{W}_y \backslash S_y$ and $t \in (0,1)$ such that $v - tLv = h \in S_y$. Thus $-v \notin \bar{W}_y, -h \in S_y$ and $-v - L(-v) = -v + Lv = -v + tLv + (1-t)Lv = -h + (1-t)Lv \in \bar{W}_y$. Let $z := -h + (1-t)Lv = -(v - Lv)$. If there exists $q \in \bar{W}_y$ such that $q - Lq = z$. Then we have $v + q - L(v+q) = v - Lv + q - Lq = -z + z = 0$. This implies $v + q = 0$, and $-v = q \in \bar{W}_y$. Thus we have $v \in S_y$, which is a contradiction.

Lemma 2.6. *Let W be a wedge in Banach space E, $E_W := W - W$ is dense in E, and D is an open set in W. Suppose that $A : \bar{D} \to W$ is a compact map with fixed point $y_0 = Ay_0 \in D$, and the Fréchet derivative of A at y_0 in W, denoted by $L = A'_+(y_0)$, is compact on E. Then L maps $\bar{W}_{y_0}$ into itself. Moreover:*
(i) Assume that $I - L$ is invertible in $\bar{W}_{y_0}$(in the sense that $h \neq Lh$ if $h \in \bar{W}_{y_0} \backslash \{0\}$). If there exists an element $z \in \bar{W}_{y_0}$ such that the equation $x - Lx = z$ has no solution for x in $\bar{W}_{y_0}$, then $index_W(A, y_0) = 0$.
(ii) Assume $I - L$ is invertible in $\bar{W}_{y_0}$. If L does not have property (α) on $\bar{W}_{y_0}$, then $index_W(A, y_0) = index_{S_{y_0}}(L, 0) = (-1)^{\sigma(y_0)} = \pm 1$. Here, $\sigma(y_0)$ is the sum of multiplicities of the eigenvalues of L in S_{y_0} which are greater than one.

Proof. The proof can be found in Dancer [37] and Li [148]. More explanations are found in Remark 2.1(i) and (ii) in Ruan and Feng [194]. Remark 2.1(ii) in [194] is same as Theorem A2-3 in Chapter 6 (Appendices).

From Lemmas 2.5 and 2.6, we obtain the following lemma.

Lemma 2.7. *Under the hypotheses of Lemma 2.6, let $I - L$ be invertible on $\bar{W}_{y_0}$ as described in Lemma 2.6.*
(i) If L has property (α) on $\bar{W}_{y_0}$, then $index_W(A, y_0) = 0$.
(ii) If L does not have property (α) on $\bar{W}_{y_0}$, then $index_W(A, y_0) = index_{S_{y_0}}(L, 0)$ $= \pm 1$.

We are now ready to prove the sufficiency part (ii) and (iii) of Theorem 2.5. Let $[C_0(\bar{\Omega})]^2 := \{(u_1, u_2) : u_i \in C(\bar{\Omega}), \, and \, u_i = 0 \, on \, \partial\Omega, \;\; for \; i = 1, 2\}$. Let $[C_0^+(\bar{\Omega})]^2 := \{(u_1, u_2) : u_i \in C_0(\bar{\Omega}), u_i \geq 0 \; in \; \Omega, \;\; for \; i = 1, 2\}$, $B_1 = max.\{C_0, B_0\}$, and $[E(B_1)]^2 := \{(u_1, u_2) : u_i \in C(\bar{\Omega}), \, |u_i| < B_1 \; in \; \Omega, \;\; for \; i = 1, 2\}$, with closure $[\bar{E}(B_1)]^2$. For each $(u_1, u_2) \in [C(\bar{\Omega})]^2, \theta \in [0, 1]$, define the operator $A_\theta : [C_0(\bar{\Omega})]^2 \cap [\bar{E}(B_1)]^2 \to [C_0(\bar{\Omega})]^2$ by $A_\theta(u_1, u_2) = (v_1, v_2)$ where

$$\begin{cases} v_1 = (-\Delta + P)^{-1}[\theta u_1 M(u_1, u_2) + Pu_1] \\ v_2 = (-\Delta + P)^{-1}[\theta d^{-1} u_2(h(u_1) - m(u_2)) + Pu_2]. \end{cases} \tag{2.34}$$

Here, the inverse operator is taken with homogeneous Dirichlet boundary condition on $\partial\Omega$, and $P > 0$ is a large enough constant such that the operator A_θ is positive, compact and Fréchet differentiable on $[C_0^+(\bar{\Omega})]^2 \cap [\bar{E}(B_1)]^2$. For convenience, let K denotes the cone $K := [C_0^+(\bar{\Omega})]^2$, as described above, and

$D := [C_0^+(\bar{\Omega})]^2 \cap [E(B_1)]^2$. The bound on the solution implies that the operators A_θ has no fixed point on the boundary ∂D in the relative topology, i.e. on the intersection of boundary of $[E(B_1)]^2$ with K. We can further use a familiar cut-off procedure (see Li [148]) to extend A_θ to be defined outside D as a compact positive mapping from the cone K into itself. For convenience, we will denote $A := A_1$. We will denote the fixed point index of A_θ over D with respect to the cone K by $i_K(A_\theta, D)$. By homotopy invariance principle, we obtain $i_K(A, D) = i_K(A_1, D) = i_K(A_0, D)$. From definition, the i-th component of $A_0(u_1, u_2)$ is $(-\Delta + P)^{-1}(Pu_i)$. One readily verifies by maximum principle that $A_0(u) \neq \lambda u$ for every $u = (u_1, u_2) \in \partial D$ and $\lambda \geq 1$. Hence, by Theorem A2-4 in Chapter 6 (Appendices), we conclude by contraction argument that $i_K(A, D) = i_K(A_0, D) = 1$.

Let y be an isolated fixed point of the map A_θ in K, we denote the local index of A_θ at y with respect to K by $index_K(A_\theta, y)$.We now show that $index_K(A, (0, 0)) = 0$ for both cases (ii) and (iii). For $y \in K$, define

$$K_y := \{p \in [C(\bar{\Omega})]^2 : y + sp \in K \ \text{ for some } s > 0\}, \text{ and}$$

$$S_y := \{p \in \bar{K}_y : -p \in \bar{K}_y\}.$$

Here $\bar{K}_y$ denotes the closure of K_y. We have $\bar{K}_{(0,0)} = K, S_{(0,0)} = \{(0, 0)\}$. Let $A'_+((0, 0))$ be the Fréchet derivative of A at $(0, 0)$ in K. The first component of $A'_+((0, 0))(u_1, u_2)$ is $(-\Delta + P)^{-1}(M(0, 0) + P)u_1$. Hence $[I - A'_+((0, 0))]u = 0$ for $u = (u_1, u_2) \in K$ implies that $[\Delta + M(0, 0)]u_1 = 0$, $u_1 \in C_0^+(\bar{\Omega})$. Thus the assumption $M(0, 0) > \lambda_1$ in (2.28) or (2.29) implies that $u_1 = 0$. Similarly, we have for the second component $[d_1\Delta + h(0) - m(0)]u_2 = 0$, $u_2 \in C_0^+(\bar{\Omega})$. Thus the assumption $h(0) \neq \lambda_1 d + m(0)$ in (2.28) or (2.29) implies that $u_2 = 0$. We thus conclude $I - A'_+((0, 0))$ is invertible in $\bar{K}_{(0,0)}$. Further, the assumption $M(0, 0) > \lambda_1$ implies that $\hat{\rho}_1(\Delta + tM(0, 0) + (t - 1)P)$ is positive when $t = 1$ and negative when $t = 0$. From the continuity in $t \in [0, 1]$ for the eigenvalue $\hat{\rho}_1(\Delta + tM(0, 0) + (t - 1)P)$, there must exist some $t \in (0, 1)$ and a nontrivial function $\bar{u} \in C_0^+(\bar{\Omega})$ such that $(-\Delta + P)\bar{u} = t(M(0, 0) + P)\bar{u}$ or $\bar{u} - t(-\Delta + P)^{-1}(M(0, 0) + P)\bar{u} = 0$ in Ω. We thus have $[I - tA'_+((0, 0))](\bar{u}, 0) = (0, 0) \in S_{(0,0)}$, with $(\bar{u}, 0) \in \bar{K}_{(0,0)} \backslash S_{(0,0)}$. We thus conclude by Lemma 2.7(i), with $\bar{W}_{y_0}$ replaced by $\bar{K}_{(0,0)}$, that $index_K(A, (0, 0)) = 0$.

We will show that for both cases (ii) and (iii), we have $index_K(A, (u_0, 0)) = index_K(A, (0, v_0)) = 0$.

Consider case (ii) when $h(0) < \lambda_1 d + m(0)$, and assume (2.28). Let $L = A'_+((u_0, 0))$ be the Fréchet derivative of A at $(u_0, 0)$ in K. Suppose that

$(I-L)(u_1, u_2) = 0$, for some $u_1 \geq 0, u_2 \geq 0$ in $\bar{\Omega}$, and $u_1 = u_2 = 0$ on $\partial\Omega$. Then

$$(2.35) \quad \begin{cases} \Delta u_1 + [M(u_0,0) + u_0 M_u(u_0,0)]u_1 + u_0 M_v(u_0,0)u_2 = 0 & \\ & \text{in } \Omega, \\ d\Delta u_2 + [h(u_0) - m(0)]u_2 = 0 & \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus the second assumption in (2.28) and the second equation above imply that $u_2 \equiv 0$. We then consider the first equation above again. Since $\hat{\rho}_1(\Delta + M(u_0, 0)) = 0$, and $u_0 M_u(u_0, 0) < 0$ by (2.24), we have $\hat{\rho}_1(\Delta + M(u_0, 0) + u_0 M_u(u_0, 0)) < 0$. Hence, all the eigenvalues ρ of the problem:

$$\Delta u + [M(u_0, 0) + u_0 M_u(u_0, 0)]u = \rho u \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega,$$

satisfy $\rho < 0$. However, u_1 satisfies this problem with $\rho = 0$. Thus $u_1 \equiv 0$. That is the operator $(I - L)$ is invertible on $\bar{K}_{(u_0,0)}$.

We next show that the operator L has property (α) on $\bar{K}_{(u_0,0)}$. Let $P > 0$; observe that the eigenvalue $\hat{\rho}_1(d\Delta - dP + t[h(u_0) - m(0) + dP])$ is negative when $t = 0$, and is positive when $t = 1$. By continuity, there exists some $t^* \in (0, 1)$, such that $\hat{\rho}_1(d\Delta - dP - t^*[h(u_0) - m(0) + dP]) = 0$. There exists $u_2^* > 0$ in Ω, vanishing on $\partial\Omega$ such that $(-d\Delta + dP)u_2^* - t^*[h(u_0) - m(0) + dP]u_2^* = 0$ in Ω. Since $S_{(u_0,0)} = C_0(\bar{\Omega}) \times \{0\}$, we can readily verify that if we let $w = (0, u_2^*)$, then we have $w - t^* Lw \in S_{(u_0,0)}$ with $w \in \bar{K}_{(u_0,0)} \backslash S_{(u_0,0)}$. Consequently, by Lemma 2.7(i), with $\bar{W}_{y_0}$ replaced by $\bar{K}_{(u_0,0)}$, we conclude that $index_K(A, (u_0, 0)) = 0$.

Next, consider the case (iii) when $h(0) > \lambda_1 d + m(0)$, and assume (2.29). Let $L = A'_+((u_0, 0))$ and (u_1, u_2) be as described above for case (ii), and thus obtain (2.35) again. The assumption $h(0) > \lambda_1 d + m(0)$ and increasing property of the function h imply the validity of the second assumption in (2.28). Thus we obtain $u_2 \equiv 0$ as before. We then follow the same argument as before to conclude that $I - L$ is invertible in $\bar{K}_{(u_0,0)}$. We then prove the operator L has property (α) on $\bar{K}_{(u_0,0)}$ exactly as in case (ii) above. Thus we conclude that $index_K(A, (u_0, 0)) = 0$.

We now consider the point $(0, v_0)$. For case (ii), that is, $h(0) < \lambda_1 d + m(0)$, we must have $v_0 \equiv 0$. Thus $(0, v_0) = (0, 0)$, and the index of A at this fixed point has been shown to be 0.

For case (iii) when $h(0) > \lambda_1 d + m(0)$, let $\tilde{L} = A'_+((0, v_0))$ be the Fréchet derivative of A at $(0, v_0)$.Suppose that $(I - \tilde{L})(u_1, u_2) = 0$, for some $u_1 \geq 0, u_2 \geq 0$ in $\bar{\Omega}$, and $u_1 = u_2 = 0$ on $\partial\Omega$. Then

$$\begin{cases} \Delta u_1 + M(0, v_0)u_1 = 0 & \\ & \text{in } \Omega, \\ d\Delta u_2 + v_0 h'(0)u_1 + [h(0) - v_0 m'(v_0) - m(v_0)]u_2 = 0 & \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

The second assumption in (2.29) implies that $\hat{\rho}_1(\Delta + M(0, v_0)) \neq 0$; thus we conclude from the first equation above that $u_1 \equiv 0$. Since $\hat{\rho}_1(d\Delta + h(0) - m(v_0)) = 0$ and $m'(v_0) \geq 0$ is not the trivial function, we have $\hat{\rho}_1(d\Delta + h(0) - v_0 m'(v_0) - m(v_0)) < 0$. We thus deduce from the second equation above that $u_2 \equiv 0$. We thus conclude that the operator $I - \tilde{L}$ is invertible in $\bar{K}_{(0,v_0)}$.

From the second assumption in (2.29), we deduce that for $P > 0$, the eigenvalue $\hat{\rho}_1(\Delta - P + t[M(0, v_0) + P])$ is negative if $t = 0$ and is positive if $t = 1$. Hence, there exist $t^* \in (0, 1)$ and a non-trivial, non-negative function u_1^* vanishing on $\partial\Omega$, such that

$$-\Delta u_1^* + P u_1^* - t^*(M(0, v_0) + P)u_1^* = 0 \quad \text{in } \Omega.$$

Since $S_{(0,v_0)} = \{0\} \times C_0(\bar{\Omega})$, we can readily verify that if we let $\tilde{w} = (u_1^*, 0)$, then we have $\tilde{w} - t^* \tilde{L}\tilde{w} \in S_{(0,v_0)}$ with $\tilde{w} \in \bar{K}_{(0,v_0)} \backslash S_{(0,v_0)}$. Consequently, by Lemma 2.7(i), we conclude that $index_K(A, (0, v_0)) = 0$.

From the above paragraphs, we thus have

$$i_K(A, D) = 1, \;\; index_K(A, (0, 0)) = index_K(A, (u_0, 0)) = index_K(A, (0, v_0)) = 0.$$

If one component of a solution of (2.23) in K is identically zero, there are at most three solutions $(0, 0), (u_0, 0)$ and $(0, v_0)$ in K. In order to avoid contradicting the additive property of the indices of the map on disjoint open subsets, there must be at least another fixed point of A in D. (See Theorem A2-1(ii) in Chapter 6.) This complete the proof of Theorem 2.5(ii) and (iii).

In some interesting applications, the predator v may have no crowding effect on its own growth rates. This lead to the following theorem.

Theorem 2.7. *Let $N = 1, 2$ or 3. Assume hypotheses (2.24) to (2.26) except that here $m \equiv 0$. Moreover*

$$h(0) < 0; \text{ and there exists } \theta > 0, \text{ such that } M_v(u, v) < -\theta \text{ for } 0 \leq u \leq C_0, v > 0.$$

Then all positive solutions (u, v) of (2.23) must satisfy $0 \leq u \leq B_1, 0 \leq v \leq B_2$, for some positive constants B_1, B_2. Moreover

(i) If $M(0, 0) \leq \lambda_1$, then (0,0) is the only non-negative solution of (2.23).

(ii) Problem (2.23) has a positive solution iff

$$M(0, 0) > \lambda_1; \;\; \text{and} \;\; \hat{\rho}_1(d\Delta + h(u_0)) > 0. \tag{2.36}$$

Proof. We first prove the existence of an a-priori bound for all non-negative solutions of (2.23). For any given $v \geq 0$ in $\bar{\Omega}$, we have by (2.24) and (2.25) a family of upper solutions $w \equiv \bar{C}$ in Ω, $\bar{C} \geq C_0$ for the first equation in (2.23), i.e.

$$\Delta w + wM(w,v) < 0 \text{ in } \Omega, \;\; w \geq 0 \text{ on } \partial\Omega.$$

By the sweeping principle, any positive solution of problem (2.23) must have $0 \leq u \leq C_0$. Suppose there is no a-priori bound for v. Then there exists a sequence of positive solutions (u_n, v_n) for (2.23) satisfying:

$$||v_n||_{L^\infty} \to \infty, \;\; \text{as } n \to \infty, \;\; 0 \leq u_n \leq C_0 \text{ in } \bar{\Omega}.$$

Let $\bar{v}_n = v_n/||v_n||_{L^\infty}$; it satisfies $0 \leq \bar{v}_n \leq 1$ and:

$$d\Delta \bar{v}_n + h(0)\bar{v}_n = -[h(u_n) - h(0)]\bar{v}_n \;\text{ in } \Omega. \tag{2.37}$$

By $W^{2,p}(\Omega)$ estimates and appropriate embedding, we can find subsequence, again denoted as $\{\bar{v}_n\}$ such that $\bar{v}_n \to \bar{v}_0 \in C^{1,\alpha}(\bar{\Omega})$ uniformly for some $\alpha \in (0,1)$, and $v_0(x) \geq 0, \not\equiv 0$ in $\bar{\Omega}$.

Next, let $\tilde{u}_n = u_n/||u_n||_{L^2} \geq 0$ in Ω. Divide the first equation satisfied by (u_n, v_n), multiply by $\tilde{u_n}$, and integrate over Ω, we obtain:
(2.38)

$$-M(0,0) < \int_\Omega |\nabla \tilde{u}_n|^2 dx - \int_\Omega M(0,0)\tilde{u}_n^2 dx = \int_\Omega [M(u_n,v_n) - M(0,0)]\tilde{u}_n^2 dx \leq 0.$$

The last inequality above is due to assumption (2.24) on M_v, M_u. From (2.38), we obtain a uniform bound on the $W^{1,2}(\Omega)$ norm on $\tilde{u_n}$. We can select subsequence, denoted the same way, such that $\tilde{u_n}$ converge weakly in $W^{1,2}(\Omega)$ and strongly in $L^q(\Omega)$, $q < 2N(N-2)^{-1}$ to a non-negative function $\tilde{u}_0 \in W^{1,2}(\Omega)$, if $N > 2$ (by Rellich-Kondrachov Compactness Theorem, see e.g. p. 272 in Evans [57]). If the space dimension $N = 3$, the inequality $q < 2N(N-2)^{-1}$ is satisfied if we choose $q = N = 3$. If the space dimension $N = 2$, using $pN(N-p)^{-1} \to \infty$ as $p \to N^-$, we can also assume $\tilde{u}_n$ converge to $\tilde{u}_0$ in $L^q(\Omega), q = 2$. We also have $||\tilde{u}_0||_{L^2} = lim_{n\to\infty}||\tilde{u}_n||_{L^2} = 1$, and we may assume $||u_n||_{L^2} \to k \geq 0$, as $n \to \infty$. Taking limit in (2.37), and using $W^{2,q}$ theory, we find $\bar{v}_0 \in W_0^{2,q} = W_0^{2,N}$ is a strong solution of

$$d\Delta \bar{v}_0 + h(0)\bar{v}_0 = -[h(k\tilde{u}_0) - h(0)]\bar{v}_0 \tag{2.39}$$

for $N = 3$ or 2. Moreover the increasing property of $h(u)$ and equation (2.37) implies that

$$d\Delta \bar{v}_0 + h(0)\bar{v}_0 \leq 0 \quad \text{in } \Omega. \tag{2.40}$$

Since $h(0) < 0$, we obtain by maximum principle for the strong solution that $\bar{v}_0(x) > 0$ for all $x \in \Omega$. (See e.g. Gilbarg and Trudinger [71] or Theorem A3-1 in Chapter 6.) For the case $N = 1$, the convergence on the right of (2.37) to

(2.39) is valid in $C^{0,\gamma}(\bar{\Omega}), \gamma = 1 - \frac{N}{2}$, by Morrey's inequality (see e.g. p. 266 in [57]). Thus the solution $\bar{v}_0$ of (2.39) is in $C^2(\bar{\Omega})$ by Schauder's theory. Therefore we can also conclude that $\bar{v}_0(x) > 0$ for all $x \in \Omega$ by means of (2.40).

Consider the integral on the right side of (2.38). We use (2.24) and the assumption concerning M_v in the statement of Theorem 2.7 to obtain

$$\begin{aligned} \int_\Omega [M(u_n, v_n) - M(0,0)]\tilde{u}_n^2\, dx &\le \int_\Omega -\theta v_n \tilde{u}_n^2\, dx \\ &= -\theta ||v_n||_{L^\infty} \int_\Omega \bar{v}_n \tilde{u}_n^2\, dx. \end{aligned} \tag{2.41}$$

However, we have

$$\int_\Omega \bar{v}_n \tilde{u}_n^2\, dx \to \int_\Omega \bar{v}_0 \tilde{u}_0^2\, dx \ > 0, \text{ as } n \to \infty. \tag{2.42}$$

Taking limit as $n \to \infty$ in (2.38) and using (2.41) and (2.42), we obtain the contradiction $-M(0,0) \le -\infty$ if $||v_n||_{L^\infty} \to \infty$. Consequently, we must have an a-priori bound for all positive solutions of problem (2.23).

Parts (i) and (ii) of this Theorem follow readily from parts (i) and (ii) of Theorem 2.5 respectively, with the role of C_0 and B_0 respectively replaced by B_1 and B_2.

Example 2.2. Let $M(u,v) = \sigma_1^{-1}(a - bu - cv)$, $d = \sigma_2$, $h(u) = u - \gamma. m \equiv 0$, where $\sigma_1, \sigma_2, a, b, c, \gamma$ are positive constants. Note that $h(0) = -\gamma < 0$. This is a very common model, when the predator has negative intrinsic growth rate, and there is no crowding effect of the population of predator on itself. Here, we can apply Theorem 2.7. The conditions in (2.36) becomes

$$a > \sigma_1 \lambda_1; \ \text{ and } \ \hat{\rho}_1(\sigma_2 \Delta + u_0 - \gamma) > 0. \tag{2.43}$$

Example 2.3. We can also apply Theorem 2.7 to models not of Volterra-Lotka type reaction. The popular Holling's type of growth rate assumption may be assumed. Let $M(u,v) = \sigma_1^{-1}(a - bu - \frac{vc}{\delta+u})$, $d = \sigma_2$, $h(u) = ku\frac{c}{\delta+u} - \gamma$, $m \equiv 0$, where $\sigma_1, \sigma_2, a, b, c, d,\ k, \gamma, \delta$ are positive constants.

Part C: Coexistence Regions in Parameter Space.

We now return to the diffusive Volterra-Lotka model, and describe a region on the (a, e) intrinsic growth rate parameter plane when positive solutions always exist while the other parameters are fixed. This leads to Theorem 2.8, Fig. 1.2.1 and Fig. 1.2.2. More precisely, consider problem (2.1) with $\sigma_1 = \sigma_2 = 1$ for simplicity. Let the parameters b, c, f and g be fixed positive constants. We can readily use Theorem 2.5(i) and (ii) to obtain a region in the (a, e) plane between certain lines or curves so that positive solutions will always exist. Here, we have

(2.23) with $\sigma_1 = \sigma_2 = d = 1$. The conditions $h(0) < \lambda_1 d + m(0)$ and (2.28) in Theorem 2.5(ii) are the same as

$$\lambda_1 > e > \rho_1(-\Delta - fu_0), \quad a > \rho_1(-\Delta + cv_0), \; v_0 \equiv 0. \tag{2.44}$$

The conditions $h(0) > \lambda_1 d + m(0)$ and (2.29) in Theorem 2.5(iii) are the same as

$$e > \lambda_1 := \rho_1(-\Delta) > \rho_1(-\Delta - fu_0), \quad a > \rho_1(-\Delta + cv_0). \tag{2.45}$$

Consequently, if $\sigma_1 = \sigma_2 = 1, \; a \neq \lambda_1, \; e \neq \lambda_1$, and

$$a > \rho_1(-\Delta + cv_0), \quad e > \rho_1(-\Delta - fu_0), \tag{2.46}$$

then Theorem 2.5(ii) and (iii) imply that problem (2.1) has a solution with each component strictly positive in Ω. We next use (2.44) to (2.46) and the characterization of principal eigenvalue to obtain very simple description, in terms of the interaction parameters and the size of Ω, of a region on the (a, e) plane where positive solutions always exist. By avoiding the reference to u_0 and v_0, the description is easier to use.

Theorem 2.8. *Consider the boundary value problem (2.1), under the assumptions $\sigma_1 = \sigma_2 = 1, \; a \neq \lambda_1, \; e \neq \lambda_1$,*
(i) Suppose $c \geq g$ and

$$a > \lambda_1 + (e - \lambda_1)\frac{c}{g}, \quad e > \lambda_1 - (a - \lambda_1)\frac{f}{b}. \tag{2.47}$$

Then (2.1) has a solution with each component strictly positive in Ω.
(ii) Suppose $c < g$ and

$$\begin{cases} e > \lambda_1, \quad a > min\{\lambda_1 + \frac{ec}{g}, \, e[1 - (1 - \frac{c}{g})(1 - \frac{\lambda_1}{e})^3 K]\}; \quad or \\ \\ \lambda_1 > e > \lambda_1 - (a - \lambda_1)\frac{f}{b}, \quad a > \lambda_1, \end{cases} \tag{2.48}$$

where $K := |\Omega|^{-1} \int_\Omega \phi^3 dx, |\Omega| = measure\, of\, \Omega$ and ϕ is the positive principal eigenfunction of $-\Delta$ with $max_\Omega \phi = 1$. Then problem (2.1) has a positive solution with each component strictly positive in Ω.

Proof. (i) Suppose $c \geq g$. First, assume $e > \lambda_1$. We now show that in this case

$$\lambda_1 + (e - \lambda_1)\frac{c}{g} \geq \rho_1(-\Delta + cv_0). \tag{2.49}$$

By the characterization of principal eigenvalue, we have

$$\rho_1(-\Delta + cv_0) = inf_{u \in S}\{\int_\Omega |\nabla u|^2 dx + cv_0 u^2 dx\},$$

where $S := \{u \in H_0^1(\Omega),\ \int_\Omega u^2 dx = 1\}$. Consequently,

$$\rho_1(-\Delta + cv_0) \le ||v_0||^{-2}_{L^2(\Omega)}\{\int_\Omega |\nabla v_0|^2 dx + c\int_\Omega v_0^3 dx\}. \tag{2.50}$$

From the equation satisfied by v_0, we have

$$g\int_\Omega v_0^3 dx = e\int_\Omega v_0^2 dx - \int_\Omega |\nabla v_0|^2 dx. \tag{2.51}$$

From (2.50) and (2.51), we obtain

$$\rho_1(-\Delta + cv_0) \le ||v_0||^{-2}_{L^2(\Omega)}\{(1 - \frac{c}{g})\int_\Omega |\nabla v_0|^2 dx + e\frac{c}{g}\int_\Omega v_0^2 dx\}. \tag{2.52}$$

Since $1 - \frac{c}{g} \le 0$, we obtain (2.49) as follows:

$$\rho_1(-\Delta + cv_0) \le (1 - \frac{c}{g})\lambda_1 + e\frac{c}{g} = \lambda_1 + (e - \lambda_1)\frac{c}{g}.$$

Thus by (2.45) and (2.49), we find that strictly positive solutions must exist for (a, e) satisfying:

$$e > \lambda_1 \text{ and } a > \lambda_1 + (e - \lambda_1)\frac{c}{g}. \tag{2.53}$$

Next, let $e < \lambda_1$. We obtain from Theorem 2.5(ii) above that (2.44) is sufficient for existence of positive solutions. By the characterization of principal eigenvalue, we find

(2.54)

$$\begin{aligned}\rho_1(-\Delta - fu_0) &= inf_{u\in S}\{(1 + \tfrac{f}{b})\textstyle\int_\Omega |\nabla u|^2 dx - \tfrac{f}{b}[\int_\Omega |\nabla u|^2 dx + b\int_\Omega u_0 u^2 dx]\} \\ &\le inf_{u\in S}\{(1 + \tfrac{f}{b})\textstyle\int_\Omega |\nabla u|^2 dx\} - \tfrac{f}{b}\, inf_{u\in S}\{\int_\Omega |\nabla u|^2 dx + b\int_\Omega u_0 u^2 dx\} \\ &= (1 + \tfrac{f}{b})\lambda_1 - \tfrac{f}{b}\rho_1(-\Delta + bu_0) \\ &= (1 + \tfrac{f}{b})\lambda_1 - \tfrac{f}{b}a = \lambda_1 - \tfrac{f}{b}(a - \lambda_1).\end{aligned}$$

Thus by (2.44) and (2.54), if $e < \lambda_1$, we find that strictly positive solutions must exist for (a, e) satisfying

$$\lambda_1 > e > \lambda_1 - \frac{f}{b}(a - \lambda_1),\ \ a > \lambda_1. \tag{2.55}$$

The assertion of part (i) follows from (2.53) and (2.55). (See Fig. 1.2.1.)

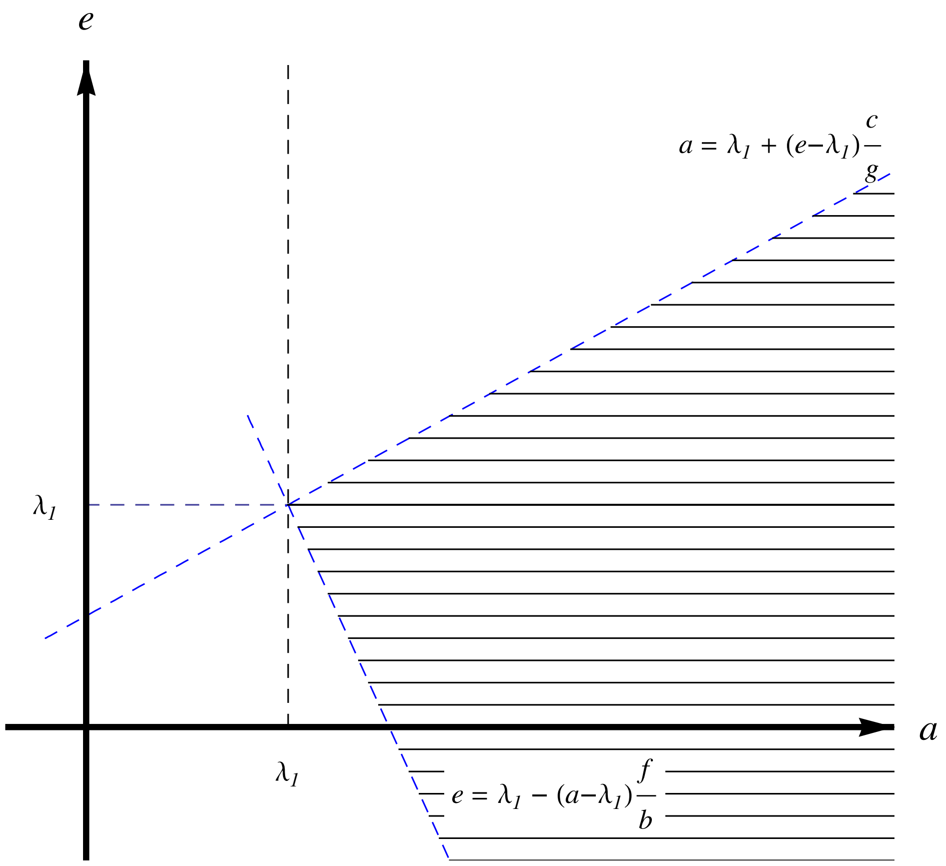

Figure 1.2.1: Coexistence Region in (a, e) Parameter Space, for case $c \geq g$.

We next consider part (ii) and assume $c < g$. Suppose $e > \lambda_1$, then Theorem 2.5(iii) asserts that if a satisfies (2.45) then problem (2.1) has positive solutions. It suffices to show that

$$(2.56) \qquad \rho_1(-\Delta + cv_0) \leq min\{\lambda_1 + \frac{ec}{g}, e[1 - (1 - \frac{c}{g})(1 - \frac{\lambda_1}{e})^3 K]\}.$$

By the characterization of the principal eigenvalue and the fact that $v_0 \leq e/g$ we find

$$\rho_1(-\Delta + cv_0) = inf_{u \in S}\{\int_\Omega |\nabla u|^2 dx + \int_\Omega cv_0 u^2 dx\}$$

$$(2.57) \qquad \leq inf_{u \in S}\{\int_\Omega |\nabla u|^2 dx + \tfrac{ec}{g}\int_\Omega u^2 dx\}$$

$$= \lambda_1 + \tfrac{ec}{g}.$$

Next, observe that (2.52) remains valid in the present case $c < g$, i.e. $1 - cg^{-1} > 0$. Thus using (2.51) and (2.52) we obtain

$$\rho_1(-\Delta + cv_0) \le [1 - \frac{c}{g}][e - (g\int_\Omega v_0^3 dx)(\int_\Omega v_0^2 dx)^{-1}] + \frac{ec}{g}. \tag{2.58}$$

Using the fact that $g^{-1}(e - \lambda_1)\phi$ is a lower solution for the problem satisfied by v_0, we have

$$(\frac{e - \lambda_1}{g})^3 \int_\Omega \phi^3 dx \le \int_\Omega v_0^3 dx. \tag{2.59}$$

From $v_0 \le eg^{-1}$, we also have

$$\int_\Omega v_0^2\, dx \le (\frac{e}{g})^2|\Omega|. \tag{2.60}$$

Letting $K = |\Omega|^{-1}\int_\Omega \phi^3 dx$, we deduce readily from (2.58) to (2.60) that

$$\begin{aligned} \rho_1(-\Delta + cv_0) &\le [1 - \tfrac{c}{g}][e - (e - \lambda_1)^3 e^{-2} K] + \tfrac{ec}{g} \\ &= e[1 - (1 - \tfrac{c}{g})(1 - \tfrac{\lambda_1}{e})^3 K]. \end{aligned} \tag{2.61}$$

Thus, from (2.57) and (2.61), we conclude that if (a, e) satisfy the first line of the inequalities in (2.48), there must exist positive solutions to problem (2.1).

Next, assume $e < \lambda_1$, we obtain the inequalities in the second line of (2.48) as sufficient condition for the existence for positive solution of (2.1) in exactly the same way as obtaining the second inequality of (2.47) as sufficient condition in part (i).

Remark 2.4. If we define

$$\hat{a}(e) = e[1 - (1 - \frac{c}{g})(1 - \frac{\lambda_1}{e})^3 K],$$

where K is defined in Theorem 2.8. It can be shown by calculus that
(1) The graph of $(\hat{a}(e), e)$ and $(\lambda_1 + (e - \lambda_1)\frac{c}{g}, e)$ do not intersect when $e > \lambda_1$.
(2) The graphs of $(\hat{a}(e), e)$ and $(\lambda_1 + \frac{ec}{g}, e)$ intersect at one point $(a_0(\frac{c}{g}), e_0(\frac{c}{g}))$, when $e > \lambda_1$, and

$$\lim_{c/g\to 1} a_0(\frac{c}{g}) = \infty, \quad \lim_{c/g\to 1} e_0(\frac{c}{g}) = \infty;$$

$$\lim_{c/g\to 0} a_0(\frac{c}{g}) = \lambda_1, \quad \lim_{c/g\to 0} e_0(\frac{c}{g}) = \lambda_1.$$

(3) $\lim_{c/g\to 1}[\hat{a}(e) - (\lambda_1 + (e - \lambda_1)\frac{c}{g})] = 0$ uniformly on compact subsets of $e \in [\lambda_1, \infty)$.

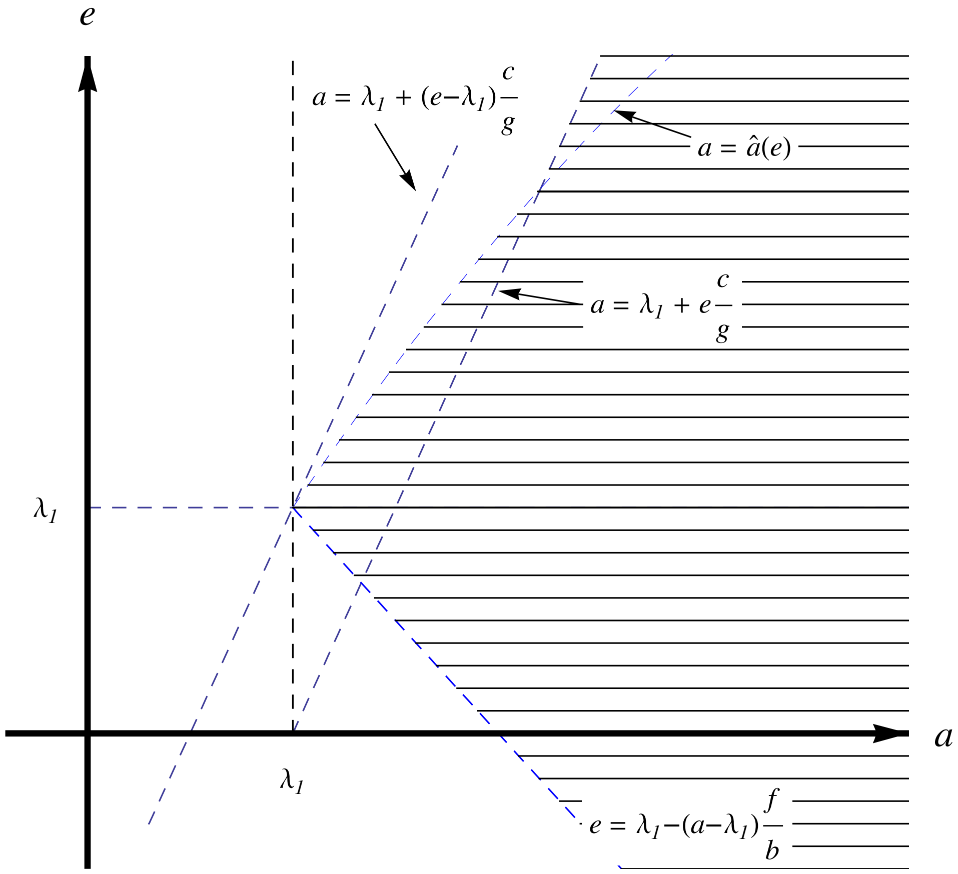

Figure 1.2.2: Coexistence Region in (a, e) Parameter Space, for case $c < g$.

The situation is illustrated in Fig 1.2.2 above. The details can be found in J. López-Gómez and R. Pardo San Gil [162].

Notes.

Theorem 2.1 was first proved in Leung [123], and Theorem 2.2 to Theorem 2.4 are due to Blat and Brown [11]. Theorem 2.5 and Corollary 2.6 are found in Li [148]. Lemmas 2.5 to 2.7 concerning indices of fixed point are obtained from Dancer [37] and Li [148]. Theorem 2.7 is a modification of a theorem in [148]. Theorem 2.8 is obtained from López-Gómez and Pardo San Gil [162]. More recent extension of the theory in this section for the case of ratio-dependent interaction rates can be found in Ryu and Ahn [197].

1.3 Strictly Positive Coexistence for Diffusive Competing Systems

In this section we study problem (1.1) when the functions $f_1(u,v)$ and $f_2(u,v)$ simulate competition between the two species populations $u(x)$ and $v(x)$ in a bounded domain Ω, with conditions as described in Section 1.1. More precisely, we first assume:

$$(3.1) \qquad \begin{cases} f_i(0,0) = 0, \;\; i = 1,2, \\ \frac{\partial f_i}{\partial u}, \frac{\partial f_i}{\partial v} < 0, \;\; \text{for } u \geq 0, v \geq 0, \;\; i = 1,2, \;\; \text{and} \\ a_i > \sigma_i \lambda_1, \;\; i = 1,2. \end{cases}$$

Part A: General Results.

The following theorem can be readily obtained by the method of upper and lower solution for a system of semilinear elliptic equations.

Theorem 3.1. *Assume that f_i, $i = 1,2$, are in $C^1([0,\infty) \times [0,\infty))$ and hypotheses (3.1) are valid. Suppose there exist positive constants k_1, k_2 such that the following inequalities:*

$$(3.2) \qquad \begin{cases} a_1 - \sigma_1 \lambda_1 + f_1(0,k_2) > 0, \\ a_2 + f_2(0,k_2) < 0, \\ a_2 - \sigma_2 \lambda_1 + f_2(k_1,0) > 0, \\ a_1 + f_1(k_1,0) < 0 \end{cases}$$

are satisfied. Then problem (1.1) has a positive solution $(\bar{u}(x), \bar{v}(x))$ with $\bar{u}(x) > 0, \bar{v}(x) > 0$ in Ω.

Proof. Let $\omega(x)$ be the positive principal eigenfunction in Ω for the eigenvalue problem $\Delta u + \lambda u = 0$ in Ω, $u = 0$ on $\partial\Omega$. For $r_1 > 0$ small enough, we have $\sigma_1 \Delta(r_1\omega) + r_1\omega[a_1 + f_1(r_1\omega, v)] = r_1\omega[a_1 - \sigma_1\lambda_1 + f_1(r_1\omega, v)] > 0$, and $\sigma_1 \Delta k_1 + k_1[a_1 + f_1(k_1, v)] < 0$ for $0 \leq v \leq k_2$. Also, for $r_2 > 0$ small enough, we have $\sigma_2\Delta(r_2\omega) + r_2\omega[a_2 + f_2(u, r_2\omega)] = r_2\omega[a_2 - \sigma_2\lambda_1 + f_2(u, r_2\omega)] > 0$, and $\sigma_2 \Delta k_2 + k_2[a_2 + f_2(u, k_2)] < 0$ for $0 \leq u \leq k_1$. The pair of functions $(r_1\omega(x), k_1), (r_2\omega(x), k_2)$ form a coupled upper-lower solution for the system (1.1). Thus by e.g. Theorem 1.4-2 in Leung [125], there exists a solution $(\bar{u}(x), \bar{v}(x))$ to (1.1) with $r_1\omega(x) \leq \bar{u}(x) \leq k_1, r_2\omega(x) \leq \bar{v}(x) \leq k_2$ in $\bar{\Omega}$.

The above theorem applies immediately to the following Volterra-Lotka competition system.

$$\text{(3.3)} \qquad \begin{cases} \sigma_1 \Delta u + u(a - bu - cv) = 0 & \\ & \text{in } \Omega, \\ \sigma_2 \Delta v + v(e - fu - gv) = 0 & \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\sigma_1, \sigma_2, a, b, c, e, f$ and g are positive constants.

Corollary 3.2. *Suppose that*

$$\text{(3.4)} \qquad a > \sigma_1 \lambda_1 + c\frac{e}{g}, \quad \textit{and} \quad e > \sigma_2 \lambda_1 + f\frac{a}{b}.$$

Then the boundary value problem (3.3) has a positive solution $(\bar{u}(x), \bar{v}(x))$ *with* $\bar{u}(x) > 0, \bar{v}(x) > 0$ *in* Ω*.*

Proof. Identify a, e respectively with a_1, a_2 and let $f_1(u, v) = -bu - cv$, $f_2(u, v) = -fu - gv$. Consider (3.3) as a special case of (1.1) under hypotheses (3.1) and (3.2). Choose $k_1 = \frac{a}{b} - \epsilon$ and $k_2 = \frac{e}{g} - \epsilon$, for $\epsilon > 0$ sufficiently small, then we can verify that (3.2) is satisfied. The results follows from Theorem 3.1.

Remark 3.1. If $a > \sigma_1 \lambda_1, e > \sigma_2 \lambda_1$, then the inequalities in (3.4) are readily satisfied when competition between the two species are relatively weak, in the sense of small c and f. The conditions in (3.4) are very easy to verify.

By using bifurcation method, we can follow the procedures as in Theorem 2.3(i) to obtain positive solutions as the growth rate of the second species e varies.

Theorem 3.3. *Suppose*

$$\text{(3.5)} \qquad a > \sigma_1 \lambda_1.$$

Then there exist μ_1, μ_2, μ_3 *satisfying* $\sigma_1 \lambda_1 < \mu_1 \le \mu_2 < \mu_3$ *with the following properties:*

(i) If $\mu_1 \le e \le \mu_2$*, then the boundary value problem (3.3) has at least one solution with each component strictly positive in* Ω*.*

(ii) If $e > \mu_3$*, then every non-negative solution of the boundary value problem (3.3) has at least one component identically equal to zero.*

Proof. The proof is analogous to that of Theorem 2.3. Let (3.5) be satisfied. For any $v \in C^1(\Omega)$, define $u(v)$ as in (2.9) and (2.10). Then, for all values e, problem (3.3) has the solution $(u, v) = (u(0), 0)$. We consider the bifurcation of solution as the parameter e varies in the problem:

$$(3.6) \qquad -\sigma_2 \Delta v = ev - gv^2 - fu(v)v \;\text{ in}\,\Omega, \quad v = 0 \text{ on } \partial\Omega.$$

Bifurcation occurs when $e = \rho_1(-\sigma_2\Delta + fu(0))$. As in the proof of Theorem 2.3, we can show that there exists a continuum S^+ of solutions (e, v) of (3.6) such that $v > 0$ for all $v \in S^+$ and S^+ intersects with the curve corresponding to the zero solution only when $e = \rho_1(-\sigma_2\Delta + fu(0))$. Multiplying (3.6) by v and integrating by parts, we find $e > \rho_1(-\sigma_2\Delta)$ for all e such that $(e, v) \in S^+$. By means of sweeping principle argument, we find $v \leq e/f$ for all v with $(e, v) \in S^+$. Thus for all v such that $(e, v) \in S^+$, there exists a bound in $C^1(\bar{\Omega})$ which is dependent on e. Since S^+ connects $(\rho_1(-\sigma_2\Delta + fu(0)), 0)$ with ∞ in $C^1(\bar{\Omega})$, it follows that $\{e : (e, v) \in S^+\} \supseteq (\rho_1(-\sigma_2\Delta + fu(0)), \infty)$. (More details can be found in Blat and Brown [11].)

As in Theorem 2.3, we can show that there exists a constant $\mu_3 > 0$ such that if $e > \mu_3$, then all solutions of (3.3) have at least one component identically equal to zero. As in the proof of Theorem 2.3, we will need to obtain a lower bound for v in terms of e. For this purpose, we need a slight change in the definition of $k(e)$ for analyzing (2.15). The eigenvalue λ_1 and eigenfunction ω_1 will be respectively replaced with the least eigenvalue and principal eigenfunction of

$$-\sigma_2\Delta\phi + fu(0)\phi = \lambda\phi \;\text{ in } \Omega, \quad \phi = 0 \text{ on } \partial\Omega.$$

The remaining part of the proof follows the argument in the last part for the proof of Theorem 2.3(i).

One can obtain similar bifurcation results as above, when the growth rate parameter a varies. For competitive system more general than Volterra-Lotka type, we can obtain the following existence theorem by cone-index method. The conditions are in terms of the signs of principal eigenvalues of appropriate related scalar equations. The results are analogous to the sufficiency part of Theorem 2.5 in the last section.

Theorem 3.4. *Assume that $f_i, i = 1, 2$, are in $C^1([0, \infty) \times [0, \infty))$. Consider the boundary value problem (1.1), under assumptions:*

$$(3.7) \qquad \begin{cases} (i)\; f_i(0,0) = 0,\; lim_{u\to\infty} f_1(u,0) = -\infty,\; lim_{v\to\infty} f_2(0,v) = -\infty, \\ \\ (ii)\; \partial f_i/\partial u < 0, \partial f_i/\partial v < 0 \; for \;\; u \geq 0, v \geq 0, i = 1, 2. \end{cases}$$

Suppose that

$$a_1 > \rho_1(-\sigma_1\Delta),\ a_2 > \rho_1(-\sigma_2\Delta), \tag{3.8}$$

and one of the following two situations hold:

(i) $\hat{\rho}_1(\sigma_1\Delta + a_1 + f_1(0, v_0)) > 0$, *and* $\hat{\rho}_1(\sigma_2\Delta + a_2 + f_2(u_0, 0)) > 0$*;*

(ii) $\hat{\rho}_1(\sigma_1\Delta + a_1 + f_1(0, v_0)) < 0$, *and* $\hat{\rho}_1(\sigma_2\Delta + a_2 + f_2(u_0, 0)) < 0$*;*

Then the boundary value problem (1.1), has a positive solution with each component strictly positive in Ω*.*

Remark 3.2. In Theorem 3.4, u_0 is the unique positive solution of $\sigma_1 u + u[a_1 + f_1(u, 0)] = 0$ in Ω, $u = 0$ on $\partial\Omega$; v_0 is the unique positive solution of $\sigma_2 v + v[a_2 + f_2(0, v)] = 0$ in Ω, $v = 0$ on $\partial\Omega$.

Proof. Let B_1, B_2 be positive numbers such that $f_1(B_1, 0) = -a_1, f_2(0, B_2) = -a_2$. Using assumption (ii) in (3.7), we can show by applying sweeping principle to each scalar equation of (1.1) for a fixed positive function assigned to the other component to conclude that all positive solutions of (1.1) satisfy $0 \le u \le B_1$ and $0 \le v \le B_2$ in $\bar{\Omega}$. Let $[C_0^+(\bar{\Omega})]^2 := \{(u_1, u_2) : u_i \in C(\bar{\Omega}), u_i \ge 0\ in\ \Omega,\ and\ = 0\ on\ \partial\Omega,\ for\ i = 1, 2\}$, $B = max.\{B_1, B_2\}$, and $[E(B)]^2 := \{(u_1, u_2) : u_i \in C(\bar{\Omega}),\ |u_i| < B\ in\ \Omega,\ for\ i = 1, 2\}$, with closure $[\bar{E}(B)]^2$. For each $(u_1, u_2) \in [C(\bar{\Omega})]^2, \theta \in [0, 1]$, define the operator $A_\theta : [C_0(\bar{\Omega})]^2 \cap [\bar{E}(B)]^2 \to [C_0(\bar{\Omega})]^2$ by $A_\theta(u_1, u_2) = (v_1, v_2)$ where

$$\begin{cases} v_1 = (-\sigma_1\Delta + P)^{-1}[\theta u_1(a_1 + f_1(u_1, u_2)) + P u_1] \\ v_2 = (-\sigma_2\Delta + P)^{-1}[\theta u_2(a_2 + f_2(u_1, u_2)) + P u_2]. \end{cases} \tag{3.9}$$

Here, the inverse operator is taken with homogeneous Dirichlet boundary condition on $\partial\Omega$, and $P > 0$ is a large enough constant such that the operator A_θ is positive, compact and Fréchet differentiable on $[C_0^+(\bar{\Omega})]^2 \cap [\bar{E}(B)]^2$. Let K be the cone $K := [C_0^+(\bar{\Omega})]^2 := \{(u_1, u_2) : u_i \in C(\bar{\Omega}), u_i \ge 0\ in\ \Omega,\ and\ = 0\ on\ \partial\Omega,\ for\ i = 1, 2\}$, and $D := [C_0^+(\bar{\Omega})]^2 \cap [E(B)]^2$. The bound on the solution implies that the operators A_θ has no fixed point on the boundary ∂D in the relative topology, i.e. on the intersection of boundary of $[E(B)]^2$ with K. We can further use a familiar cut-off procedure to extend A_θ to be defined outside D as a compact positive mapping from the cone K into itself. For convenience, we will denote $A := A_1$. We will denote the fixed point index of A_θ over D with respect to the cone K by $i_K(A_\theta, D)$. As in the proof of Theorem 2.5(ii), (iii), we obtain $i_K(A, D) = i_K(A_0, D) = 1$.

Let y be an isolated fixed point of the map A_θ in K, we denote the local index of A_θ at y with respect to K by $index_K(A_\theta, y)$. We now show that $index_K(A,(0,0)) = 0$ for each case (i) and (ii). For $y \in K$, define

$$K_y := \{p \in [C(\bar{\Omega})]^2 : y + sp \in K \text{ for some } s > 0\}, \text{ and}$$

$$S_y := \{p \in \bar{K}_y : -p \in \bar{K}_y\}.$$

Here $\bar{K}_y$ denotes the closure of K_y. We have $\bar{K}_{(0,0)} = K, S_{(0,0)} = \{(0,0)\}$. Let $A'_+((0,0))$ be the Fréchet derivative of A at $(0,0)$ in K. The first component of $A'_+((0,0))(u_1,u_2)$ is $(-\sigma_1\Delta + P)^{-1}(a_1+P)u_1$. Hence $[I - A'_+((0,0))]u = 0$ for $u = (u_1,u_2) \in K$ implies that $[\sigma_1\Delta + a_1]u_1 = 0$, $u_1 \in C_0^+(\bar{\Omega})$. Thus the assumption $a_1 > \rho_1(-\sigma_1\Delta)$ implies that $u_1 = 0$. Similarly, we have for the second component $[\sigma_2\Delta + a_2]u_2 = 0$, $u_2 \in C_0^+(\bar{\Omega})$. Thus the assumption $a_2 > \rho_1(-\sigma_2\Delta)$ implies that $u_2 = 0$. We thus conclude $I - A'_+((0,0))$ is invertible in $W_{(0,0)}$. Further, the assumption $a_1 > \rho_1(-\sigma_1\Delta)$ and the continuity in $t \in [0,1]$ for the eigenvalue $\rho_1(\sigma_1\Delta + ta_1 + (t-1)P)$ imply that there exists some $t \in (0,1)$ and a nontrivial function $\bar{u} \in C_0^+(\bar{\Omega})$ such that $(-\sigma_1\Delta + P)\bar{u} = t(a_1+P)\bar{u}$ or $\bar{u} - t(-\sigma_1\Delta + P)^{-1}(a_1+P)\bar{u} = 0$ in Ω. We thus have $[I - tA'_+((0,0))](\bar{u},0) = (0,0) \in S_{(0,0)}$, with $(\bar{u},0) \in \bar{K}_{(0,0)} \backslash S_{(0,0)}$. We thus conclude by Lemma 2.7(i) that $index_K(A,(0,0)) = 0$.

We next show that for case (i), we have

$$index_K(A,(u_0,0)) = index_K(A,(0,v_0)) = 0.$$

Let $L = A'_+((u_0,0))$ be the Fréchet derivative of A at $(u_0,0)$. Suppose that $(I-L)(u_1,u_2) = 0$, for some $u_1 \geq 0, u_2 \geq 0$ in $\bar{\Omega}$, and $u_1 = u_2 = 0$ on $\partial\Omega$. Then

(3.10)
$$\begin{cases} \sigma_1\Delta u_1 + [a_1 + f_1(u_0,0) + u_0\frac{\partial f_1}{\partial u}(u_0,0)]u_1 + u_0\frac{\partial f_1}{\partial v}(u_0,0)u_2 = 0 & \\ & \text{in } \Omega, \\ \sigma_2\Delta u_2 + [a_2 + f_2(u_0,0)]u_2 = 0 & \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus the second equation above and the second assumption in situation (i) implies that $u_2 \equiv 0$. We then consider the first equation above again. Since $\hat{\rho}_1(\sigma_1\Delta + a_1 + f_1(u_0,0)) = 0$, and $u_0\frac{\partial f_1}{\partial u}(u_0,0) < 0$ by (3.7), we have $\hat{\rho}_1(\sigma_1\Delta + a_1 + f_1(u_0,0) + u_0\frac{\partial f_1}{\partial u}(u_0,0)) < 0$. Hence, all the eigenvalues ρ of the problem:

$$\sigma_1\Delta u + [a_1 + f_1(u_0,0) + u_0\frac{\partial f_1}{\partial u}(u_0,0)]u = \rho u \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega,$$

satisfy $\rho < 0$. However, u_1 satisfies this problem with $\rho = 0$. Thus $u_1 \equiv 0$. That is the operator $(I-L)$ is invertible on $\bar{K}_{(u_0,0)}$.

We next show that the operator L has property (α) on $\bar{W}_{(u_0,0)}$. Let $P>0$; observe that the eigenvalue $\hat{\rho}_1(\sigma_2\Delta - P + t[a_2+f_2(u_0,0)+P])$ is negative when $t=0$, and is positive when $t=1$. By continuity, there exists some $t^*\in(0,1)$, such that $\hat{\rho}_1(\sigma_2\Delta - P + t^*[a_2+f_2(u_0,0)+P])=0$. There exists $u_2^*>0$ in Ω, vanishing on $\partial\Omega$ such that $(-\sigma_2\Delta+P)u_2^* - t^*[a_2+f_2(u_0,0)+P]u_2^*=0$ in Ω. Since $S_{(u_0,0)}=C_0(\bar{\Omega})\times\{0\}$, we can readily verify that if we let $w=(0,u_2^*)$, then we have $w-t^*Lw\in S_{(u_0,0)}$ with $w\in\bar{K}_{(u_0,0)}\backslash S_{(u_0,0)}$. Consequently, by Lemma 2.7(i), we conclude that $index_K(A,(u_0,0))=0$.

We now consider the point $(0,v_0)$ and let $\tilde{L}=A'_+((0,v_0))$ be the Fréchet derivative of A at $(0,v_0)$ in K. Suppose that $(I-\tilde{L})(u_1,u_2)=0$, for some $u_1\geq 0, u_2\geq 0$ in $\bar{\Omega}$, and $u_1=u_2=0$ on $\partial\Omega$. Then

$$\begin{cases} \sigma_1\Delta u_1+[a_1+f_1(0,v_0)]u_1=0 & \\ & \text{in } \Omega, \\ \sigma_2\Delta u_2+v_0\frac{\partial f_2}{\partial u}(0,v_0)u_1+[a_2+f_2(0,v_0)+v_0\frac{\partial f_2}{\partial v}(0,v_0)]u_2=0 & \\ u_1=u_2=0 & \text{on } \partial\Omega. \end{cases}$$

From the first equation above and the first assumption in situation (i), we obtain $u_1\equiv 0$. Since $\hat{\rho}_1(\sigma_2\Delta+a_2+f_2(0.v_0))=0$ and $v_0\frac{\partial f_2}{\partial u_2}(0,v_0)\leq 0$ is not the trivial function, we have $\hat{\rho}_1(\sigma_2\Delta+a_2+f_2(0,v_0)+v_0\frac{\partial f_2}{\partial v}(0,v_0))<0$. We thus deduce from the second equation above that $u_2\equiv 0$. We thus conclude that the operator $I-\tilde{L}$ is invertible in $\bar{K}_{(0,v_0)}$.

From the first assumption in (i), we deduce that for $P>0$, the eigenvalue $\hat{\rho}_1(\sigma_1\Delta-P+t[a_1+f_1(0,v_0)+P])$ is negative if $t=0$ and is positive if $t=1$. Hence, there exist $t^*\in(0,1)$ and a nontrivial, non-negative function u_1^* vanishing on $\partial\Omega$, such that

$$-\sigma_1\Delta u_1^*+Pu_1^*-t^*(a_1+f_1(0,v_0)+P)u_1^*=0 \ \text{ in } \Omega.$$

Since $S_{(0,v_0)}=\{0\}\times C_0(\bar{\Omega})$, we can readily verify that if we let $\tilde{w}=(u_1^*,0)$, then we have $\tilde{w}-t^*\tilde{L}\tilde{w}\in S_{(0,v_0)}$ with $\tilde{w}\in\bar{K}_{(0,v_0)}\backslash S_{(0,v_0)}$. Consequently, by Lemma 2.7(i), we conclude that $index_K(A,(0,v_0))=0$.

We thus have

$$i_K(A,D)=1,\ index_K(A,(0,0))=index_K(A,(u_0,0))=index_K(A,(0,v_0))=0.$$

In order to avoid contradicting the additive property of the indices of the map on disjoint open subsets, there must be at least another fixed point of A in D. Hence for case (i), there must be more positive solution in D other than $(0,0),(u_0,0)$ or $(0,v_0)$.

We next consider the proof of case (ii). Let $L = A'_+((u_0, 0))$ be the Fréchet derivative of A at $(u_0, 0)$. Suppose that $(I - L)(u_1, u_2) = 0$, for some $u_1 \geq 0, u_2 \geq 0$ in $\bar{\Omega}$, and $u_1 = u_2 = 0$ on $\partial\Omega$. Then (u_1, u_2) satisfies (3.10) again. The second equation in (3.10) and the second assumption in (ii) implies that $u_2 \equiv 0$. We then obtain from the second equation in (3.10) that $u_1 \equiv 0$ in the same way as in case (i) above. Hence the operator $(I - L)$ is invertible on $\bar{K}_{(u_0,0)}$.

We next show that the operator L does not have property (α) on $\bar{K}_{(u_0,0)}$. We have $S_{(u_0,0)} = C_0(\bar{\Omega}) \times \{0\}$, and $\bar{W}_{(u_0,0)} \backslash S_{(u_0,0)} = C_0(\bar{\Omega}) \times \{C_0^+(\bar{\Omega}) \backslash \{0\}\}$. Suppose the operator L has property (α) on $\bar{K}_{(u_0,0)}$. Then there exists some $t^* \in (0, 1)$ and $(u_1^*, u_2^*) \in \bar{K}_{(u_0,0)} \backslash S_{(u_0,0)}$, such that

$$u_1^* - t^*(-\sigma_1\Delta + P)^{-1}([a_1 + f_1(u_0, 0) + u_0 \tfrac{\partial f_1}{\partial u}(u_0, 0) + P]u_1^* + u_0 \tfrac{\partial f_1}{\partial v}(u_0, 0)u_2^*) \in C_0(\bar{\Omega}),$$

$$u_2^* - t^*(-\sigma_2\Delta + P)^{-1}([a_2 + f_2(u_0, 0) + P]u_2^*) = 0.$$

The first equation above is always satisfied. The second equation above implies that

$$Tu_2^* = \frac{1}{t^*}u_2^* > u_2^* \text{ in } C_0(\bar{\Omega}),$$

where $T := (-\sigma_2\Delta + P)^{-1}[a_2 + f_2(u_0, 0) + P]$. By Theorem A2-6(i) in Chapter 6, we obtain $r(T) > 1$ for the spectral radius $r(T)$. On the other hand the second assumption in (ii) implies that there exists a positive eigenfunction ϕ for the negative eigenvalue $\beta := \hat{\rho}_1(\sigma_2\Delta + a_2 + f_2(u_0, 0))$ such that

$$(-\sigma_2\Delta + P)\phi - (a_2 + f_2(u_0, 0) + P)\phi = -\beta\phi > 0 \text{ in } \Omega.$$

We thus have $\phi - T\phi > 0$ in $C_0(\bar{\Omega})$. Thus by Theorem A2-6(ii) in Chapter 6, we obtain $r(T) < 1$. From this contradiction we conclude that L cannot have property α. Consequently, using Lemma 2.7, we have

$$index_K(A, (u_0, 0)) = index_{C_0(\bar{\Omega})\times\{0\}}(L, (0, 0)) = \pm 1.$$

In order to calculate $index_{C_0(\bar{\Omega})\times\{0\}}(L, (0, 0))$, we use Theorem A2-3 in Chapter 6 to find $index_{C_0(\bar{\Omega})\times\{0\}}(L, (0, 0)) = (-1)^m$, where m is the sum of multiplicities of eigenvalues of L greater than 1. Suppose $(\phi, \psi) \in C_0(\bar{\Omega}) \times \{0\}$ is an eigenvector of L with λ as eigenvalue. Then

$$(-\sigma_1\Delta + P)^{-1}[a_1 + f_1(u_0, 0) + u_0 \frac{\partial f_1}{\partial u}(u_0, 0) + P]\phi = \lambda\phi.$$

Let $\tilde{T} := (\sigma_1\Delta + P)^{-1}[a_1 + f_1(u_0,0) + u_0\frac{\partial f_1}{\partial u}(u_0,0) + P]$. We have $\tilde{\beta} := \hat{\rho}_1(\sigma_1\Delta + a_1 + f_1(u_0,0) + u_0\frac{\partial f_1}{\partial u}(u_0,0)) < 0$, and

$$(-\sigma_1\Delta + P)\tilde{\phi} - (a_1 + f_1(u_0,0) + u_0\frac{\partial f_1}{\partial u}(u_0,0) + P)\tilde{\phi} = -\tilde{\beta}\tilde{\phi} > 0 \text{ in } \Omega$$

for some positive eigenfunction $\tilde{\phi}$ for the eigenvalue $\tilde{\beta}$. We thus have $\tilde{\phi} - \tilde{T}\tilde{\phi} > 0$, and by Theorem A2-6(ii) in Chapter 6 again we obtain $r(\tilde{T}) < 1$. Consequently $\lambda < 1$ and $m = 0$. Thus we have $index_K(A,(u_0,0)) = (-1)^0 = 1$. Similarly, we can show that $index_K(A,(0,v_0)) = 1$.

We thus have

$$i_K(A,D) = 1, \; index_K(A,(0,0)) = 0,$$
$$index_K(A,(u_0,0)) = index_K(A,(0,v_0)) = 1.$$

In order to avoid contradicting the additive property of the indices (Theorem A2-1(ii) in Chapter 6), there must exist a positive solution of problem (1.1) in D other than $(0,0), (u_0,0)$ or $(0,v_0)$.

For each case (i) or (ii), the positive solution $(u(x),v(x))$ in D, other than $(0,0),(u_0,0)$ or $(0,v_0)$, has each component $\not\equiv 0, \geq 0$ in $\bar{\Omega}$. From the boundedness of the coefficients $[a_i + f_i(u(x),v(x))]$, we obtain from each equation in (1.1) and Lemma 1.1 that each component of $(u(x),v(x))$ is strictly positive in Ω.

For necessary conditions, we consider the special case when the intrinsic growth rates a_1 and a_2 of each species are the same

$$\sigma_1 = \sigma_2 = \sigma, \;\; a_1 + f_1(u,v) = p(u) - q(v), \;\; a_2 + f_2(u,v) = r(v) - s(u), \tag{3.11}$$

where $p(0) = a_1 = a_2 = r(0)$, $q(0) = s(0) = 0$, $p, r, -q, -s$ are $C^1([0,\infty))$ non-increasing functions, and $p' < 0, r' < 0$. Moreover, there exist constants c_1, c_2 such that

$$p(u) < 0 \text{ for } u > c_1; \;\; r(v) < 0 \text{ for } v > c_2. \tag{3.12}$$

Theorem 3.5. *Consider the boundary value problem (1.1), under the special conditions (3.11) and (3.12). Suppose further that both $p' + s', r' + q'$ have the same constant sign on $(0,c)$ where $c = max.\{c_1,c_2\}$. If the boundary value problem has a positive solution, then $p(0) > \lambda_1$, $r(0) > \lambda_1$, and one of the following three situations must hold:*

(i) $\hat{\rho}_1(\sigma\Delta + p(0) - q(v_0)) > 0$, and $\hat{\rho}_1(\sigma\Delta + r(0) - s(u_0)) > 0$;

(ii) $\hat{\rho}_1(\sigma\Delta + p(0) - q(v_0)) < 0$, and $\hat{\rho}_1(\sigma\Delta + r(0) - s(u_0)) < 0$;

(iii) $\hat{\rho}_1(\sigma\Delta + p(0) - q(v_0)) = 0$, and $\hat{\rho}_1(\sigma\Delta + r(0) - s(u_0)) = 0$.

Proof. First assume that

$$p' + s' > 0, \ \ r' + q' > 0. \tag{3.13}$$

Let $(\tilde{u}, \tilde{v})$ be a positive solution of the boundary value problem (1.1), (1.3) under the hypotheses of this theorem. The function $\tilde{u}$ is a positive lower solution to the scalar problem:

$$\sigma \Delta u + up(u) = 0 \ \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega; \tag{3.14}$$

and the constant function $u \equiv c_1$ is an upper solution for (3.14). Thus problem (3.14) has a positive solution, and Lemma 2.1 implies that $p(0) > \sigma\lambda_1$. Moreover, we have $r(0) = a_2 = a_1 = p(0) > \sigma\lambda_1$, and we have a positive solution u_0 for (3.14) and a positive solution v_0 for $\sigma\Delta v + vr(v) = 0$ in Ω, $v = 0$ on $\partial\Omega$. We have

$$p(0) - q(v_0) = r(0) - q(v_0) = r(0) + q(0) - q(v_0) < r(v_0),$$

where the last inequality is due to the second part of (3.13). Similarly, from the first inequality in (3.13), we obtain

$$r(0) - s(u_0) < p(u_0).$$

From the two inequalities above, we find

$$\hat{\rho}_1(\sigma\Delta + p(0) - q(v_0)) < \hat{\rho}_1(\sigma\Delta + r(v_0)) = 0,$$

$$\hat{\rho}_1(\sigma\Delta + r(0) - s(u_0)) < \hat{\rho}_1(\sigma\Delta + p(u_0)) = 0.$$

That is, we obtain the second conclusion (ii) of the statement of the theorem. If we reverse both inequalities in (3.13), the arguments above lead to conclusion (i) of the theorem. If both inequalities in (3.13) are changed to equality, then we obtain conclusion (iii).

Remark 3.3. When (3.13) holds, there are strong competitions between the two species, we obtain case (ii) in Theorem 3.5 when both eigenvalues involved are negative. If both inequalities in (3.13) are reversed, there are weaker competitions, and we obtain case (i) above when both eigenvalues involved are positive. The assumptions in Theorem 3.5 are very restrictive. In Theorem 3.11 to Theorem 3.13 below, we will consider cases when one species is much stronger than the other.

For fixed $f_i(u, v), i = 1, 2$, satisfying (3.7) and suppose a_1, a_2 satisfy (3.8), we now utilize Theorem 3.4 and comparison method to deduce a more detailed description of the set:

(3.15)

$$\Lambda := \{(a_1, a_2) |\ a_i > \rho_1(-\sigma_i\Delta), i = 1, 2;\ \text{the boundary value problem (1.1) has a strictly positive solution in } \Omega.\}$$

The analysis here is more general than that for the Volterra-Lotka prey-predator case given in the last section. By assumption (3.8), there exist positive solutions $u = u_0(a_1), v = v_0(a_2)$ respectively satisfying the following:

$$\sigma_1 \Delta u + u[a_1 + f_1(u,0)] = 0 \text{ in } \Omega, \ \ u = 0 \text{ on } \partial\Omega,$$

$$\sigma_2 \Delta v + v[a_2 + f_2(0,v)] = 0 \text{ in } \Omega, \ \ v = 0 \text{ on } \partial\Omega.$$

Define $\underline{v}, \bar{u}, \underline{u}, \bar{v}$ to be the maximal non-negative solutions of the following scalar boundary value problems:

$$(3.16) \qquad \begin{cases} \sigma_2 \Delta \underline{v} + \underline{v}(a_2 + f_2(u_0(a_1), \underline{v})) = 0 \text{ in } \Omega, \ \underline{v} = 0 \text{ on } \partial\Omega; \\ \sigma_1 \Delta \bar{u} + \bar{u}(a_1 + f_1(\bar{u}, \underline{v})) = 0 \text{ in } \Omega, \ \bar{u} = 0 \text{ on } \partial\Omega; \\ \sigma_1 \Delta \underline{u} + \underline{u}(a_1 + f_1(\underline{u}, v_0(a_2))) = 0 \text{ in } \Omega, \ \underline{u} = 0 \text{ on } \partial\Omega; \\ \sigma_2 \Delta \bar{v} + \bar{v}(a_2 + f_2(\underline{u}, \bar{v})) = 0 \text{ in } \Omega, \ \bar{v} = 0 \text{ on } \partial\Omega. \end{cases}$$

The four functions are not always nontrivial, and are completely determined by the two constants a_1 and a_2. For each fixed v, $\underline{v} \le v \le v_0(a_2)$, the functions $\bar{u}$ and 0 are respectively upper and lower solutions of

$$(3.17) \qquad \sigma_1 \Delta u + u(a_1 + f_1(u,v)) = 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega.$$

For each fixed u, $0 \le u \le \bar{u}$, the functions $v_0(a_2)$ and $\underline{v}$ are respectively upper and lower solutions of

$$(3.18) \qquad \sigma_2 \Delta v + v(a_2 + f_2(u,v)) = 0, \text{ in } \Omega, \ v = 0 \text{ on } \partial\Omega.$$

Similarly, for each fixed v, $0 \le v \le \bar{v}$, the functions $u_0(a_1)$ and $\underline{u}$ are respectively upper and lower solutions of problem (3.17). For each fixed u, $\underline{u} \le u \le u_0(a_1)$, the functions $\bar{v}$ and 0 are respectively upper and lower solutions of problem (3.18). Let $(u_1(x,t), v_1(x,t))$ and $(u_2(x,t), v_2(x,t))$ be respectively solutions of the initial boundary value problem (1.2) with initial conditions:

$$(3.19) \qquad (u_1(x,0), v_1(x,0)) = (\bar{u}, \underline{v}), \text{ and } (u_2(x,0), v_2(x,0)) = (\underline{u}, \bar{v}).$$

One can show by comparison that as $t \to +\infty$, $u_1(x,t)$ and $v_1(x,t)$ respectively tend from above and below to some $\bar{u}^s(x)$ and $\underline{v}^s(x)$ in $\bar{\Omega}$, since the initial conditions are upper and lower solutions for the steady state problem. Similarly as $t \to +\infty$, $u_2(x,t)$ and $v_2(x,t)$ respectively tend from below and above to some $\underline{u}^s(x)$ and $\bar{v}^s(x)$ in $\bar{\Omega}$, since the initial conditions are lower and upper solutions for the steady state problem. Note that $\bar{u}^s, \underline{v}^s, \underline{u}^s, \bar{v}^s$ are completely determined by the parameters a_1, a_2. Next, we define

$$
\begin{aligned}
G_1(a_1,a_2) &:= \rho_1(-\sigma_2\Delta - f_2(\underline{u}^s,0)) > \rho_1(-\sigma_2\Delta), \\
G_2(a_1,a_2) &:= \rho_1(-\sigma_1\Delta - f_1(0,\underline{v}^s)) > \rho_1(-\sigma_1\Delta).
\end{aligned}
\tag{3.20}
$$

For given (a_1, a_2) satisfying (3.8) and let (u, v) be a corresponding solution of the steady state problem (1.1), we can readily deduce by comparison that:

$$
\underline{u}^s \le u \le \bar{u}^s, \ \underline{v}^s \le v \le \bar{v}^s. \tag{3.21}
$$

If $(a_1, a_2) \in \Lambda$, then we have $\bar{u}^s \ge u > 0$ in Ω. Taking limit as $t \to +\infty$, we also find that $\bar{u}^s$ is a positive solution of:

$$
\sigma_1\Delta\bar{u}^s + \bar{u}^s(a_1 + f_1(\bar{u}^s, \underline{v}^s)) = 0 \text{ in } \Omega, \ \bar{u}^s = 0 \ \text{ on } \partial\Omega.
$$

Comparing with (3.20), we must have:

$$
a_1 > G_2(a_1, a_2).
$$

Similarly $\bar{v}^s$ is a positive solution of:

$$
\sigma_2\Delta\bar{v}^s + \bar{v}^s(a_2 + f_2(\underline{u}^s, \bar{v}^s)) = 0 \text{ in } \Omega, \ \bar{v}^s = 0 \ \text{ on } \partial\Omega.
$$

We conclude that

$$
a_2 > G_1(a_1, a_2).
$$

Define

$$
\begin{aligned}
H_1(a_1) &:= inf.\{\beta > \rho_1(-\sigma_2\Delta) : \beta > G_1(a_1, \beta)\}, \\
H_2(a_2) &:= inf.\{\alpha > \rho_1(-\sigma_1\Delta) : \alpha > G_2(\alpha, a_2)\}.
\end{aligned}
$$

Using comparison arguments as given in the last paragraph and more careful analysis by means of Theorem 3.4, one can obtain a more precise description of the set Λ as follows.

Theorem 3.6. *Consider problem (1.1) under assumptions (3.7) and (3.8). The set Λ defined in (3.15) is a connected region bounded by the two curves:*

$$
\Gamma_1 : a_1 = H_2(a_2), \quad \Gamma_2 : a_2 = H_1(a_1)
$$

in the following sense: for each $a_2 > \rho_1(-\sigma_2\Delta)$, the horizontal slice $\{a_1 : (a_1, a_2) \in \Lambda\}$ is a nonempty interval whose left endpoint is on Γ_1; and for each $a_1 > \rho_1(-\sigma_1\Delta)$, the vertical slice $\{a_2 : (a_1, a_2) \in \Lambda\}$ is a nonempty interval whose lower endpoint is Γ_2.

Details of the proof can be found in Ruan and Pao [195]. Moreover, the theorem is actually true for more general boundary conditions for the functions u, v respectively of the form $B_i = \alpha_i(x)\frac{\partial}{\partial\nu} + \beta_i(x), i = 1, 2$. Here, α_i and β_i are non-negative functions in $C^{1+\alpha}(\partial\Omega), 0 < \alpha < 1$, with either $\alpha_i = 0, \beta_i > 0$ or $\alpha_i > 0, \beta_i \geq 0$. The set Λ is illustrated in Fig. 1.3.1 below.

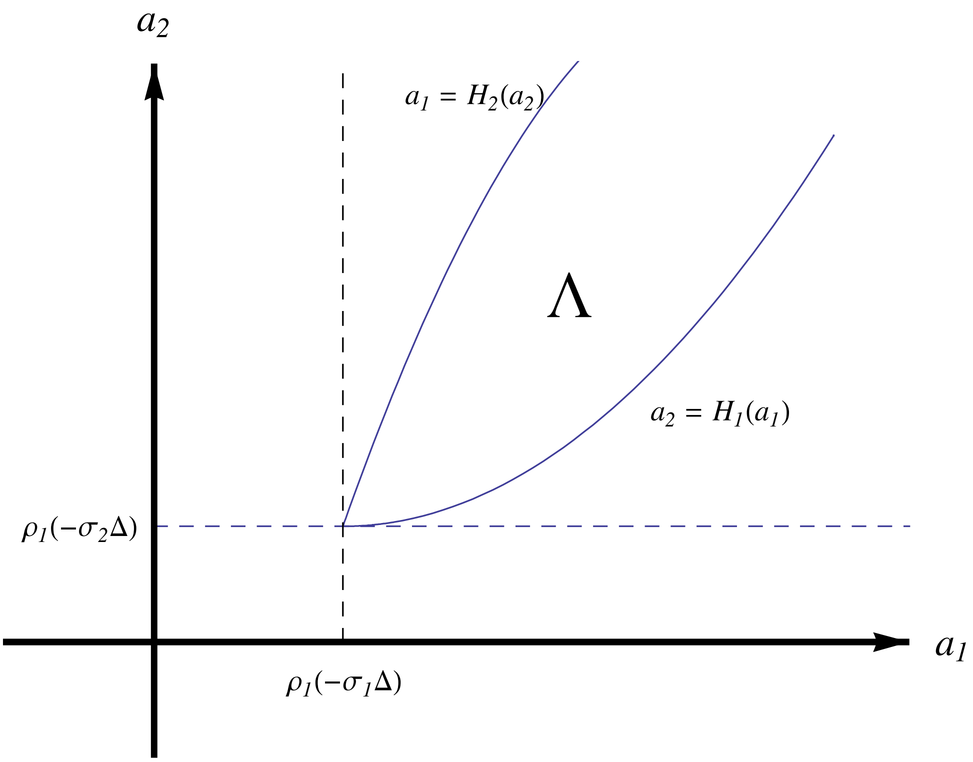

Figure 1.3.1: Coexistence Region in (a_1, a_2) Parameter Space.

Applying the results of Theorem 3.4 to system the Volterra-Lotka system (3.3), we see that when the interaction rates c and f are small, we will have case (i) when both of the related principal eigenvalues are positive. Thus Theorem 3.4 asserts the existence of positive coexistence solution. Actually, we can also obtain the existence of such solution when both c and f are small from the result in Corollary 3.2. On the other hand, when c and f are both large, we have strong competition between the species. In this case, we will have case (ii) in Theorem 3.4 when both related principal eigenvalues are negative. Thus we have coexistence positive solution again.

Part B: Extreme Strong Competition.

We now make a more careful study of the situation when both competition interaction parameters c and f are large. We shall see that the species may tend to segregate from each other as they coexist. For simplicity we restrict to $\sigma_1 = \sigma_2 = 1$, $a > \lambda_1, e > \lambda_1, b = g = 1$ and Dirichlet boundary condition. That is, we consider:

$$(3.22) \qquad \begin{cases} \Delta u + u(a - u - cv) = 0 & \\ & \text{in } \Omega, \\ \Delta v + v(e - v - fu) = 0 & \\ & \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Recall that by Lemma 2.1, there cannot be any non-negative solution of the above problem with $u \not\equiv 0$ if $a < \lambda_1$ (or $v \not\equiv 0$ if $e < \lambda_1$). When both c and f are large, there is one type of positive solution to the problem (3.22) closely related to the positive solutions of the reduced problem:

$$(3.23) \qquad \begin{cases} \Delta u + u(a - v) = 0 & \\ & \text{in } \Omega, \\ \Delta v + v(e - u) = 0 & \\ & \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

with $a > \lambda_1, e > \lambda_1$. Moreover, the existence of positive solution of the reduced problem (3.23) is related to the trivial solution of the problem:

$$(3.24) \qquad \Delta w + aw^+ + ew^- = 0 \quad \text{in } \Omega, \;\; w = 0 \quad \text{on } \partial\Omega.$$

We will find that if (3.24) has only the trivial solution which has nonzero index, then (3.23) has a positive solution. Moreover, for each isolated positive solution $(\hat{u}, \hat{v})$ of (3.23), there is a positive solution (u, v) of (3.22) with c, f large, fu close to $\hat{u}$ and cv close to $\hat{v}$.

There is another type of positive solution of (3.22) when $c, f \to \infty, c^{-1}f \to \alpha \in (0, \infty)$. Here, the corresponding positive solution (u, v) of (3.22) has the property that $c||u||_\infty, f||v||_\infty$ both tending to infinity. More precisely, if there is an isolated solution w_0 of the problem:

$$(3.25) \qquad \Delta w + w^+(a - \alpha^{-1}w^+) + w^-(e + w^-) = 0 \;\; \text{in } \Omega, \quad w = 0 \;\; \text{on } \partial\Omega$$

which changes sign in Ω and has non-zero index, then (3.22) has a positive solution near $(\alpha^{-1}w_0^+, -w_0^-)$ for c, f large. In this situation, one species is segregated to near where $w_0^+ \neq 0$ and the other species is segregated to near where $-w_0^- \neq 0$ in Ω. We now describe more carefully the first type of positive solution of (3.22).

Theorem 3.7 (Extreme Strong Competition for Both Species, I). *Suppose $(\hat{u}, \hat{v})$ is an isolated positive solution of (3.23) with non-zero index. Then for any $\epsilon > 0$, there exists a large $M > 0$, such that for any $c, f \geq M$, problem (3.22) has at least one positive solution (u, v) satisfying:*

$$||fu - \hat{u}||_\infty < \epsilon, \;\; ||cv - \hat{v}||_\infty < \epsilon. \tag{3.26}$$

By the index of $(\hat{u}, \hat{v})$, we mean the fixed point index denoted by $index_P(B, (\hat{u}, \hat{v}))$, where P represents the natural positive cone in $C(\bar{\Omega}) \times C(\bar{\Omega})$, and $B : C(\bar{\Omega}) \times C(\bar{\Omega}) \to C(\bar{\Omega}) \times C(\bar{\Omega})$ is defined by

$$B(u, v) = (-\Delta + k)^{-1}(ku + au - uv, kv + ev - uv),$$

with homogeneous Dirichlet boundary condition. Here, k is a large positive constant so that B maps some neighborhood N of $(\hat{u}, \hat{v})$ in P into P. (Recall the definition of fixed point index in Part B of Section 1.2.)

Proof. Let $\bar{u} = fu, \bar{v} = cv$. We readily verify that (u, v) is a positive solution of (3.22) if and only if $(\bar{u}, \bar{v})$ solves:

$$\begin{cases} \Delta u + u(a - v) - f^{-1}u^2 = 0 & \\ & \text{in } \Omega, \\ \Delta v + v(e - u) - c^{-1}v^2 = 0 & \\ & \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.27}$$

Consequently, in order to prove this theorem, it suffices to prove (3.27) has a positive solution near $(\hat{u}, \hat{v})$ when f and c are large. Comparing (3.27) with (3.23), we see that this can be achieved readily by homotopy invariance of degree argument.

Theorem 3.8. *Suppose problem (3.24) has only the trivial solution $w \equiv 0$ with $index_{C_0^1(\bar{D})}(\tilde{B}_1, 0) \neq 0$, where $\tilde{B}_1 w = (-\Delta)^{-1}(aw^+ + ew^-)$. Then the problem (3.23) has at least one positive solution. Moreover, there exists a constant $M > 0$ such that any solution (u, v) of problem (3.23) satisfies:*

$$||u||_\infty + ||v||_\infty \leq M;$$

and the sum of the indices of all the positive solutions of (3.23) is equal to $index_{C_0^1(\bar{\Omega})}(\hat{B}_1, 0)$. Here, $C_0^1(\bar{\Omega})$ denotes the functions in $C^1(\bar{\Omega})$ with zero boundary value on $\partial\Omega$.

Proof. The proof of this theorem can be divided into three steps.
Step 1. We first show that there exists $M > 0$ such that any positive solution (u, v) of the problem:

$$(3.28)\qquad \begin{cases} -\Delta u = tau + (1-t)a(u-v)^+ - uv & \\ & \text{in } \Omega, \\ -\Delta v = tev + (1-t)e(v-u)^+ - uv & \\ u = v = 0 & \text{on } \partial\Omega, \\ 0 \le t \le 1, & \end{cases}$$

must satisfy

$$(3.29)\qquad ||u||_\infty + ||v||_\infty < M.$$

Observe that neither component of a non-negative solution of (3.28) can vanish identically unless both components vanish identically.

Suppose, by contradiction, there exist $t_n \in [0,1]$ and positive solutions (u_n, v_n) of (3.28) with $t = t_n$ such that

$$||u_n||_\infty + ||v_n||_\infty \to \infty.$$

Then from the equation we find

$$(3.30)\qquad -\Delta \tilde{u}_n \le a\tilde{u}_n,\ -\Delta \tilde{v}_n \le e\tilde{v}_n,\ \tilde{u}_n|_{\partial\Omega} = \tilde{v}_n|_{\partial\Omega} = 0,$$

where

$$\tilde{u}_n = (||u_n||_\infty)^{-1}u_n,\ \tilde{v}_n = (||v_n||_\infty)^{-1}v_n.$$

From (3.30), we obtain

$$\int_\Omega |\nabla \tilde{u}_n|^2 dx \le a \int_\Omega \tilde{u}_n^2 dx \le a\ mes.(\Omega),$$

$$\int_\Omega |\nabla \tilde{v}_n|^2 dx \le e \int_\Omega \tilde{v}_n^2 dx \le e\ mes.(\Omega).$$

Here $mes.(\Omega)$ is the measure of the domain Ω. Thus $\{\tilde{u}_n\}, \{\tilde{v}_n\}$ are bounded in $W_0^{1,2}(\Omega)$, which is a Hilbert space. By compact embedding in $L^2(\Omega)$, we can choose a subsequence such that

$$\tilde{u}_n \to \tilde{u},\ \tilde{v}_n \to \tilde{v} \text{ weakly in } W_0^{1,2}(\Omega) \text{ and strongly in } L^2(\Omega).$$

Moreover, we can deduce by taking $(-\Delta)^{-1}$ with Dirichlet boundary conditions on both sides of (3.30) that if $\tilde{u} = 0$, then $\tilde{u}_n \to 0$ in $C(\bar{\Omega})$. This contradicts $||\tilde{u}_n||_\infty = 1$. Therefore we have $\tilde{u} \ne 0$. Similarly, $\tilde{v} \ne 0$.

Since $||u_n||_\infty + ||v_n||_\infty \to \infty$, without loss of generality we suppose $||v_n||_\infty \to \infty$. From (3.28), we find

$$-\Delta \tilde{u}_n = t_n a\tilde{u}_n + (1-t_n)a(\tilde{u}_n - \frac{v_n}{||u_n||_\infty})^+ - ||v_n||_\infty \tilde{u}_n \tilde{v}_n.$$

Multiplying both sides by $\phi \in C_0^\infty(\Omega)$ and integrating over Ω, we obtain

$$\int_\Omega \tilde{u}_n(\Delta\phi)dx + t_n \int_\Omega a\tilde{u}_n\phi dx + (1-t_n)\int_\Omega a\left(\tilde{u}_n - \frac{v_n}{||u_n||_\infty}\right)^+ \phi dx$$
$$= ||v_n||_\infty \int_\Omega \tilde{u}_n\tilde{v}_n\phi dx.$$

Since $\int_\Omega \tilde{u}_n\tilde{v}_n\phi dx \to \int_\Omega \tilde{u}\tilde{v}\phi dx, ||v_n||_\infty \to \infty$ and the left side of the above equation is uniformly bounded, we obtain $\int_\Omega \tilde{u}\tilde{v}\phi dx = 0$. Since ϕ is arbitrary, we conclude $\tilde{u}\tilde{v} = 0$ a.e. in Ω.

Let $\alpha_n = ||u_n||_\infty/||v_n||_\infty$. Without loss of generality we may assume $\alpha_n \to \alpha \in [0,\infty)$ (otherwise, we consider $||v_n||_\infty/||u_n||_\infty$). We also assume $t_n \to \bar{t} \in [0,1]$. From the equations in (3.28), we obtain

$$-\Delta(\alpha_n\tilde{u}_n - \tilde{v}_n) = t_n[a(\alpha_n\tilde{u}_n) - e\tilde{v}_n] + (1-t_n)[a(\alpha_n\tilde{u}_n - \tilde{v}_n)^+ - e(\tilde{v}_n - \alpha_n\tilde{u}_n)^+].$$

Multiplying the above equation by $\phi \in C_0^\infty(\Omega)$, integrating over Ω and passing to the limit, we obtain

$$\int_\Omega \nabla(\alpha\tilde{u} - \tilde{v})\nabla\phi dx = \bar{t}\int_\Omega [a(\alpha\tilde{u}) - e\tilde{v}]\phi dx + (1-\bar{t})\int_\Omega [a(\alpha\tilde{u}-\tilde{v})^+ - e(\tilde{v}-\alpha\tilde{u})^+]\phi dx.$$

Since $\tilde{u}, \tilde{v} \geq 0$ and $\tilde{u}\tilde{v} = 0$, we have

$$(\alpha\tilde{u} - \tilde{v})^+ = \alpha\tilde{u}, \ \ (\tilde{v} - \alpha\tilde{u})^+ = \tilde{v};$$

and hence

$$\int_\Omega \nabla(\alpha\tilde{u} - \tilde{v})\nabla\phi\, dx = \int_\Omega [a(\alpha\tilde{u}) - e\tilde{v}]\phi\, dx.$$

Let $w_0 = \alpha\tilde{u} - \tilde{v}$. We have $w_0^+ = \alpha\tilde{u}, w_0^- = -\tilde{v}$ and so

$$\int_\Omega \nabla w_0 \nabla\phi\, dx = \int_\Omega (aw_0^+ + ew_0^-)\phi\, dx, \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

This means $w_0 = \alpha\tilde{u} - \tilde{v}$ is a bounded weak solution of (3.24) and hence a classical solution. Since $w_0 \not\equiv 0$, we arrive at a contradiction. This completes the proof of step 1.

Step 2. Let B_M denotes the ball in $C(\bar{\Omega}) \times C(\bar{\Omega})$ centered at 0, with radius M as described in (3.29). Let P be the natural positive cone in $C(\bar{\Omega}) \times C(\bar{\Omega})$, and the operator B be defined as in Theorem 3.7. Here, we assume the positive constant k has been chosen sufficiently large for the definition of B so that B maps $P \cap B_M$ to P. We will show

$$deg_P(I - B, P \cap B_M, 0) = index_{C_0^1(\bar{\Omega})}(\tilde{B}_1, 0), \tag{3.31}$$

where $\tilde{B}_1$ is the mapping defined in the statement of this theorem.

First, by means of (3.29), the homotopy

$$H_t(u, v) = (-\Delta + k)^{-1}(tau + (1-t)a(u-v)^+ - uv + ku,$$
$$tev + (1-t)e(v-u)^+ - uv + kv)$$

with $k > 0$ leads to

$$deg_P(I - B, P \cap B_M, 0) = deg_P(I - H_0, P \cap B_M, 0). \tag{3.32}$$

Next, we consider another homotopy

$$\begin{cases} -\Delta u = a(u-v)^+ - tuv + (1-t)\epsilon_0 & \\ & \text{in } \Omega, \\ -\Delta v = e(v-u)^+ - tuv + (1-t)\epsilon_0 & \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.33}$$

where $t \in [0, 1]$. Here, ϵ_0 is a fixed positive number. If (u, v) is a non-negative solution of (3.33), then $u - v$ satisfies

$$-\Delta(u - v) = a(u - v)^+ + e(u - v)^- \ in\ \Omega,\ (u - v)|_{\partial\Omega} = 0.$$

Thus, by the assumption of the theorem we obtain $u = v$; and hence u satisfies

$$-\Delta u = (1-t)\epsilon_0 - tu^2 \ \ in\ \Omega,\ u|_{\partial\Omega} = 0. \tag{3.34}$$

Using an upper and lower solution argument and noting that the right hand side of (3.34) is concave with respect to u, we see that (3.34) has a unique non-negative solution ψ_t for $t \in [0, 1], 0 \le \psi_t \le (1-t)\epsilon_0(-\Delta)^{-1}(1)$ and $\psi_t > 0$ in Ω for $0 \le t < 1$. Here, $(-\Delta)^{-1}$ is taken with zero Dirichlet boundary condition. Thus (3.33) has a unique non-negative solution $(u, v) = (\psi_t, \psi_t)$. Define

$$\tilde{H}_t(u, v) = (-\Delta+k)^{-1}(a(u-v)^+ - tuv + (1-t)\epsilon_0 + ku, e(v-u)^+ - tuv + (1-t)\epsilon_0 + kv)$$

with $k > 0$ large and ϵ_0 chosen sufficiently small so that $||\psi_0||_\infty < M/2$. Then we obtain by homotopy invariance of degree that

$$deg_P(I - \tilde{H}_1, P \cap B_M, 0) = deg_P(I - \tilde{H}_0, P \cap B_M, 0) = index_P(\tilde{H}_0, (\psi_0, \psi_0)).$$

Since $\tilde{H}_1 = H_0$, we combine with (3.32) to find

$$deg_P(I - B, P \cap B_M, 0) = index_P(\tilde{H}_0, (\psi_0, \psi_0)). \tag{3.35}$$

Let $\tilde{P}$ denote the natural positive cone in $C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega})$ and $j : \tilde{P} \to P$ denote the inclusion. Since $\tilde{H}_0$ maps a neighborhood of (ψ_0, ψ_0) in P continuously into $\tilde{P}$, the commutativity property of the fixed point index (see Nussbaum [178]) leads to

$$index_P(\tilde{H}_0, (\psi_0, \psi_0)) = index_P(j\tilde{H}_0, (\psi_0, \psi_0)) = index_{\tilde{P}}(\tilde{H}_0 j, (\psi_0, \psi_0)). \tag{3.36}$$

Since $(\psi_0, \psi_0) \in int(\tilde{P})$, due to maximum principle, we further find that

$$index_{\tilde{P}}(\tilde{H}_0 j, (\psi_0, \psi_0)) = index_{\tilde{E}}(\tilde{H}_0 j, (\psi_0, \psi_0)), \tag{3.37}$$

where $\tilde{E}$ denotes $C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega})$.

Now consider the homeomorphism $h : \tilde{E} \to \tilde{E}$ defined by $h(u, v) = (u, u - v)$. Clearly $h^{-1} = h$ and h maps a neighborhood of (ψ_0, ψ_0) into a neighborhood of $(\psi_0, 0)$. Since (ψ_0, ψ_0) is an isolated fixed point of $\tilde{H}_0 j$, $(\psi_0, 0)$ is an isolated fixed point of $h^{-1}\tilde{H}_0 jh$. By the commutativity of the fixed point index, we have

$$index_{\tilde{E}}(\tilde{H}_0 j, (\psi_0, \psi_0)) = index_{\tilde{E}}(h^{-1}\tilde{H}_0 jh, (\psi_0, 0)), \tag{3.38}$$

where one readily verifies that

$$h^{-1}\tilde{H}_0 jh(u, w) = (-\Delta + k)^{-1}(aw^+ + \epsilon_0 + ku,\ aw^+ + ew^- + kw).$$

Since by assumption the problem $-\Delta w = aw^+ + ew^-\ in\ \Omega,\ w|_{\partial\Omega} = 0$ has only the trivial solution, we can use the homotopy $\hat{H}_t(u, w) = (-\Delta + tk)^{-1}(taw^+ + \epsilon_0 + tku,\ aw^+ + ew^- + tkw), 0 \le t \le 1$ to find

$$index_{\tilde{E}}(h^{-1}\tilde{H}_0 jh, (\psi_0, 0)) = index_{\tilde{E}}(\hat{H}_0, (\psi_0, 0)). \tag{3.39}$$

Now $\hat{H}_0(u, w) = ((-\Delta)^{-1}(\epsilon_0), (-\Delta)^{-1}(aw^+ + ew^-))$, and by the product theorem of degree, (cf. [75] or [178]), we find

(3.40)

$$\begin{aligned} &index_{\tilde{E}}(\hat{H}_0, (\psi_0, 0)) \\ &= index_{C_0^1(\bar{\Omega})}((-\Delta)^{-1}(\epsilon_0), \psi_0)) \cdot index_{C_0^1(\bar{\Omega})}((-\Delta)^{-1}(aw^+ + ew^-), 0) \\ &= index_{C_0^1(\bar{\Omega})}(\tilde{B}_1, 0). \end{aligned}$$

From (3.35) to (3.40), we obtain

$$deg_P(I - B, P \cap B_M, 0) = index_{C_0^1(\bar{\Omega})}(\tilde{B}_1, 0).$$

This proves the validity of (3.31).

Step 3. This last step will complete the proof of the theorem. By taking $t = 1$ in (3.28), the argument in step 1 above shows that any positive solution (u, v) of (3.23) satisfies $||u||_\infty + ||v||_\infty \le M$. Since $a, e > \lambda_1$, one can show as in Theorem 2.5 or 3.4 that (0,0) is a solution of (3.23) with

$$index_P(B, (0, 0)) = 0.$$

Choose a small ball B_r such that

$$deg_P(I - B, B_r \cap P, 0) = index_P(B, (0, 0)).$$

Then by the additivity of degree (cf. Theorem A2-1 in Chapter 6) and (3.31) we obtain

$$deg_P(I - B, (B_M \backslash \bar{B}_r) \cap P, 0) = index_{C_0^1(\bar{\Omega})}(\tilde{B}_1, 0) \neq 0.$$

Hence (3.23) has at least one non-negative solution in $(B_M \backslash \bar{B}_r) \cap P$, and the sum of the indices of all such solutions is equal to $index_{C_0^1(\bar{\Omega})}(\tilde{B}_1, 0)$. Since $a, e > \lambda_1$, (3.23) has no non-negative solution (u, v) with only one component identically zero. Since any non-negative solution of (3.23) must be in $B_M \cap P$ by step 1, and B_r is chosen so small that (3.23) has only the trivial solution in B_r, we see that all solutions with each component strictly positive in Ω of (3.23) are contained in $(B_M \backslash \bar{B}_r) \cap P$. Consequently, the sum of indices of such positive solutions of (3.23) is equal to $index_{C_0^1(\bar{\Omega})}(\tilde{B}_1, 0)$. This completes the proof of the theorem.

Corollary 3.9. *Suppose problem (3.24) has only the trivial solution $w \equiv 0$ with $index_{C_0^1(\bar{D})}(\tilde{B}_1, 0) \neq 0$, where $\tilde{B}_1 w = (-\Delta)^{-1}(aw^+ + ew^-)$. Then there exist large positive constants M and N such that for any $f, e \geq N$, problem (3.22) has at least one positive solution (u, v) satisfying:*

$$f||u||_\infty + c||v||_\infty \le M.$$

The next Theorem describes the second type of positive solution of (3.22) mentioned above.

Theorem 3.10 (Extreme Strong Competition for Both Species, II). *Let $\alpha \in (0, \infty)$. Suppose problem (3.25) has an isolated solution w_0 in $L^2(\Omega)$, which changes sign and has non-zero index. Then there exist respectively very large and small positive constants N and ϵ such that for any c, f satisfying*

$$c \geq N, \quad |c^{-1} f - \alpha| \le \epsilon,$$

the problem (3.22) has a positive solution (u, v) near $(\alpha^{-1} w_0^+, -w_0^-)$ in $L^2(\Omega) \times L^2(\Omega)$.

(The question of uniqueness and stability of the steady-state will be considered in Theorems 5.10 and 5.11 in Section 1.5 below).

Outline of Proof. The proof is similar to that of Theorem 3.8, thus we will only outline the main ideas. First, we consider the homotopy:

(3.41)

$$\begin{cases} -\Delta u = tau + (1-t)a(u-\alpha^{-1}v)^+ - tu^2 - (1-t)((u-\alpha^{-1}v)^+)^2 - cuv & \\ & \text{in } \Omega, \\ -\Delta v = tev + (1-t)e(v-\alpha u)^+ - tv^2 - (1-t)((v-\alpha u)^+)^2 & \\ \qquad -(t\beta + (1-t)\alpha)cuv & \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 \le t \le 1$. Here $\beta > 0$ is fixed. If (u, v) is any non-negative solution of (3.41), we can readily show that

$$-\Delta u \le \frac{a^2}{4}, \; -\Delta v \le \frac{d^2}{4} \text{ in } \Omega, \; u = v = 0 \text{ on } \partial\Omega,$$

since the function $g(s) := \lambda s - s^2$ is bounded by $\lambda^2/4$ for $s > 0$. We thus obtain that if (u, v) is any non-negative solution of (3.41), then

$$0 \le u \le M, \; 0 \le v \le M \tag{3.42}$$

for some $M > 0$. For simplicity, we next denote the right hand side of the first equation in (3.41) by $f_1(u, v, t)$, and that for the second equation by $f_2(u, v, t)$. Let

$$u_M = min.\{u, M\}, \; v_M = min.\{v, M\}.$$

Define

$$\tilde{f}_1(u, v, t) = f_1(u_M, v_M, t), \; \tilde{f}_2(u, v, t) = f_2(u_M, v_M, t).$$

Choose $\delta > 0$ so small such that in the neighborhood $N_\delta(w_0)$ in $L^2(\Omega)$, w_0 is the only solution of (3.25). Then choose $\delta_1 > 0$ so that $(u, v) \in \partial N_{\delta_1}(\alpha^{-1}w_0^+, -w_0^-)$ implies that $u \ne 0, v \ne 0$ and $\alpha u - v \in N_\delta(w_0)$. Here $\partial N_{\delta_1}(\alpha^{-1}w_0^+, -w_0^-)$ denotes the boundary of the δ_1-neighborhood $N_{\delta_1}(\alpha^{-1}w_0^+, -w_0^-)$ of $(\alpha^{-1}w_0^+, -w_0^-)$ in $L^2(\Omega) \times L^2(\Omega)$. We then show there exist positive N_1 large and ϵ small such that problem (3.41), with the first and second line on right hand side respectively replaced by $\tilde{f}_1(u, v, t)$ and $\tilde{f}_2(u, v, t)$, has no non-negative solution (u, v) with $(u, v) \in \partial N_{\delta_1}(\alpha^{-1}w_0^+, -w_0^-)$ whenever $c \ge N_1$, $|\beta - \alpha| \le \epsilon$ and $0 \le t \le 1$.

For any $c \ge N_1$, let $M_c > 0$ be large enough such that

$$\tilde{f}_1(u, v, t) + M_c u \ge 0, \; \tilde{f}_2(u, v, t) + M_c v \ge 0,$$

for any $u, v \geq 0$ and $t \in [0,1]$. Define the mapping

$$A_t(u,v) = (-\Delta + M_c)^{-1}(\tilde{f}_1(u,v,t) + M_c u, \tilde{f}_2(u,v,t) + M_c v)$$

which is completely continuous and maps the natural positive cone P in $L^2(\Omega) \times L^2(\Omega)$ into itself. We show that for large c, the problem with $t = 0$:

$$\text{(3.43)} \qquad \begin{cases} -\Delta u = a(u - \alpha^{-1}v)^+ - ((u - \alpha^{-1}v)^+)^2 - cuv & \\ & \text{in } \Omega, \\ -\Delta v = e(v - \alpha u)^+ - ((v - \alpha u)^+)^2 - \alpha cuv & \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique non-negative solution in $N_{\delta_1}(\alpha^{-1}w_0^+, -w_0^-)$, and the solution denoted by $(u_c, v_c) = (u_c, \alpha u_c - w_0)$ tends to $(\alpha^{-1}w_0^+, -w_0^-)$ in $L^2(\Omega) \times L^2(\Omega)$ as $c \to \infty$.

As in the proof of Theorem 3.8, we use the regularity of A_0, the homeomorphism $h(u,v) = (u, \alpha u - v)$ in $\tilde{E} = C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega})$, the commutativity of the fixed point index and the product formula (cf. [75] or [178]) to obtain:

$$\begin{aligned} &deg_P(I - A_1, P \cap N_{\delta_1}(\alpha^{-1}w_0^+, -w_0^-), 0) \\ &\qquad = deg_P(I - A_0, P \cap N_{\delta_1}(\alpha^{-1}w_0^+, -w_0^-), 0) \\ &\qquad = index_P(A_0, (u_c, v_c)) \\ &\qquad = index_{\tilde{E}}(A_0, (u_c, v_c)) \\ &\qquad = index_{C_0^1(\bar{\Omega})}(\tilde{B}_2, w_0) \cdot index_{C_0^1(\bar{\Omega})}(B, u_c) \\ &\qquad = index_{C_0^1(\bar{\Omega})}(\tilde{B}_2, w_0) \neq 0. \end{aligned}$$

Here, $\tilde{B}_2 w = (-\Delta)^{-1}[w^+(a - \alpha^{-1}w^+) + w^-(e + w^-)]$, $Bu = (-\Delta)^{-1}[a\alpha^{-1}w_0^+ - (\alpha^{-1}w_0^+)^2 - cu(\alpha u - w_0)]$, and we can obtain from uniqueness property that $index_{C_0^1(\bar{\Omega})}(B, u_c) = 1$.

This shows that $(u,v) = A_1(u,v)$ has at least one solution in the set $P \cap N_{\delta_1}(\alpha^{-1}w_0^+, -w_0^-)$ for $c \geq N > N_1$ and $|\beta - \alpha| \leq 1$. The solution of (3.22) will satisfy (3.41) for $t = 1$. For more details, see Dancer and Du [41].

Part C: One Much Stronger Competitor.

In the remaining part of this section, we finally consider the case of existence of positive solution for problem (3.22) when none of the conditions (i) or (ii) of

Theorem 3.4 is satisfied. (Here, in the notation of Theorem 3.4, $f_1(u,v) := -u - cv$, $f_2(u,v) := -fu - v, a_1 = a > \rho_1(-\Delta), a_2 = e > \rho_1(-\Delta), \sigma_1 = \sigma_2 = 1$. Also recall the definition of u_0 and v_0 in Remark 3.2.)

Define $\bar{c}, \bar{f}$ to be positive constants when

$$\hat{\rho}_1(\Delta + a - \bar{c}v_0) = 0, \;\; \hat{\rho}_1(\Delta + e - \bar{f}u_0) = 0. \tag{3.44}$$

For convenience, we define coexistence parameter sets as follows:

$$T^+ := \{(c,f) : c > \bar{c},\, 0 \le f < \bar{f}, \text{ and problem (3.22) has a strictly positive solution}\},$$

$$T^- := \{(c,f) : f > \bar{f},\, 0 \le c < \bar{c}, \text{ and problem (3.22) has a strictly positive solution}\}.$$

Let:

$$g_1(c) = \int_\Omega h^3\,dx - \bar{f}c\int_\Omega h^2(-\Delta - (a - 2u_0))^{-1}(u_0 h)\,dx, \tag{3.45}$$

where h is the positive eigenfunction which spans the kernel of $-\Delta - (e - \bar{f}u_0)$ and normalized so that $||h||_{L^2(\Omega)} = 1$. Similarly, define

$$g_2(f) = \int_\Omega k^3\,dx - \bar{c}f\int_\Omega k^2(-\Delta - (e - 2v_0))^{-1}(v_0 k)\,dx, \tag{3.46}$$

where k is the positive eigenfunction which spans the kernel of $-\Delta - (a - \bar{c}v_0)$ and normalized so that $||k||_{L^2(\Omega)} = 1$.

The coefficients c and f in (3.22) can be interpreted as coefficients of competition. The following theorem describes situations of coexistence when the competition coefficient of one species is relatively large compared with the other.

Theorem 3.11 (Positive Solution with One Competitor much Stronger). *Consider problem (3.22) with $a > \rho_1(-\Delta), e > \rho_1(-\Delta)$ and $\bar{c}, \bar{f}$ as defined in (3.44). The coexistence parameter set described above has the following properties.*
(i) The set T^+ is nonempty if either $g_1(\bar{c}) > 0$ or $g_2(\bar{f}) < 0$.
(ii) For almost all (a,e) in $(\lambda_1,\infty)\times(\lambda_1,\infty)$, either T^+ is nonempty or T^- is nonempty.
(Here, g_1, g_2 are defined in (3.45) and (3.46).)

Proof. Linearizing equations (3.22) at $(u_0, 0)$ leads to the system:

$$\begin{cases} -\Delta y + (2u_0 - a)y = -cu_0 z & \\ & \text{in } \Omega, \\ -\Delta z + fu_0 z = ez & \\ y = z = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.47}$$

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Since $u_0 > 0$ in Ω, by comparison we have $\rho_1(-\Delta+(2u_0-a)) > \rho_1(-\Delta+(u_0-a)) = 0$. Thus by [3], the operator $[-\Delta+(2u_0-a)]^{-1}$ exists and is a compact positive operator on $C_0^{1,\alpha}(\bar{\Omega}), 0<\alpha<1$. Equation (3.47) is thus equivalent to the system:

$$
\begin{aligned}
y &= [-\Delta+(2u_0-a)]^{-1}(-cu_0 z),\\
z &= f[\Delta+e]^{-1}(u_0 z).
\end{aligned}
\tag{3.48}
$$

For convenience, let $A_1 = [\Delta+e]^{-1}, M_a z = u_0 z, A_2 = [-\Delta+(2u_0-a)]^{-1}$, $M_{ca}z = -cu_0 z$, and $q=(y,z)^T$ then (3.48) can be written as

$$q = fB_1 q + B_2 q,$$

where

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & A_1 M_a \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & A_2 M_{ca} \\ 0 & 0 \end{bmatrix}$$

are compact operators on the Banach space $[C_0^{1,\alpha}(\Omega)]^2$. Thus $I - fB_1 - B_2$ is a Fredholm operator on $[C_0^{1,\alpha}(\Omega)]^2$, with index 0. Furthermore, since $ker(I - \bar{f}B_1 - B_2)$ has dimension 1, $\bar{f}$ will be a simple eigenvalue of the pair $(I-B_2, B_1)$ provided

$$B_1\phi \notin Range(I - B_2 - \bar{f}B_1), \tag{3.49}$$

where ϕ is any element in the $ker(I-\bar{f}B_1-B_2)$ (cf. Chow and Hale [28]). To verify (3.49), let $\phi=(y,z)^T$ and

$$
\begin{aligned}
y &= [-\Delta+(2u_0-a)]^{-1}(-cu_0 z),\\
z &= \bar{f}[\Delta+e]^{-1}(u_0 z),
\end{aligned}
\tag{3.50}
$$

with $z \neq 0$. Then the second component of $B_1\phi$ is $[\Delta+e]^{-1}u_0 z = \bar{f}^{-1}z$. For any $\phi^* = (y^*, z^*) \in [C_0^{1,\alpha}(\Omega)]^2$, the second component of $(I-B_2-\bar{f}B_1)\phi^*$ is $I - \bar{f}[\Delta+e]^{-1}(u_0 z^*)$. Hence if $B_1\phi \in Range(I-B_2-\bar{f}B_1)$, then $z \in Range(I-\bar{f}[\Delta+e]^{-1}M_a)$. However, by (3.50), the kernel $ker(I-\bar{f}[\Delta+e]^{-1}M_a)$ is spanned by z, leading to a contradiction.

The analysis above justifies the application of the bifurcation theorem of Crandall and Rabinowitz [33], with fixed $a>\lambda_1, e>\lambda_1, c>0$, while the parameter f varies across $\bar{f}$. For all f near $\bar{f}$, $(u_0, 0)$ is a solution of problem (3.22). There exists $\delta_0 > 0$ and smooth functions $f:(-\delta_0,\delta_0)\to R$, $u:(-\delta_0,\delta_0)\to C_0^{1,\alpha}(\bar{\Omega})$, $v:(-\delta_0,\delta_0)\to C_0^{1,\alpha}(\bar{\Omega})$ such that:

$$f(0)=\bar{f}, \quad u(s) = u_0 + sy_0 + \tilde{y}(s), \quad v(s) = sz_0 + \tilde{z}(s),$$

where

$$z_0 \text{ spans the } ker(I - \bar{f}[\Delta + e]^{-1} M_a),$$
$$z_0(x) > 0, \text{ for } x \in \Omega, \int_\Omega z_0^2 \, dx = 1,$$
$$y_0 = [-\Delta + (2u_0 - a)]^{-1}(-cu_0 z_0);$$
$$||\tilde{y}(s)||_{C_0^{1,\alpha}(\Omega)} = o(|s|), \quad ||\tilde{z}(s)||_{C_0^{1,\alpha}(\Omega)} = o(|s|), \text{ as } s \to 0.$$

Moreover, in a sufficiently small neighborhood of $(\bar{f}, u_0, 0)$ in $R \times C_0^{1,\alpha}(\bar{\Omega}) \times C_0^{1,\alpha}(\bar{\Omega})$, the triples $(f(s), u(s), v(s)), |s| \leq \delta_0$ are the only solutions to (3.22) other than $(f, u_0, 0)$. In particular, we have positive solutions to (3.22) when $s > 0$.

We now let $\lambda(s) = f(s) - \bar{f}$ and calculate $\lambda'(0)$. For this purpose, we consider the equation

$$-\Delta v(s) = ev(s) - v^2(s) - f(s)u(s)v(s),$$

which is equivalent to:

(3.51)

$$(-\Delta + \bar{f}u_0 - e)(sz_0 + \tilde{z}(s)) = -\lambda(s)(u_0 + sy_0 + \tilde{y}(s))(sz_0 + \tilde{z}(s)) \\ - (sz_0 + \tilde{z}(s))^2 - \bar{f}(sy_0 + \tilde{y}(s))(sz_0 + \tilde{z}(s)).$$

Differentiating (3.51) with respect to s yields

(3.52)

$$(-\Delta + \bar{f}u_0 - e)(\tilde{z}'(s)) \\ = \lambda'(s)(u_0 + sy_0 + \tilde{y}(s))(sz_0 + \tilde{z}(s)) - \lambda(s)(y_0 + \tilde{y}'(s))(sz_0 + \tilde{z}(s)) \\ - \lambda(s)(u_0 + sy_0 + \tilde{y}(s))(z_0 + \tilde{z}'(s)) - 2(sz_0 + \tilde{z}(s))(z_0 + \tilde{z}'(s)) \\ - \bar{f}(y_0 + \tilde{y}'(s))(sz_0 + \tilde{z}(s)) - \bar{f}(sy_0 + \tilde{y}(s))(z_0 + \tilde{z}'(s)).$$

If we differentiate with respect to s once more and evaluate at $s = 0$, we obtain

$$(3.53) \qquad (-\Delta + \bar{f}u_0 - e)(\tilde{z}''(s)) = -2\lambda'(0)u_0 z_0 - 2(z_0 + \tilde{z}'(0))^2 \\ - 2\bar{f}(y_0 + \tilde{y}'(0))(z_0 + \tilde{z}'(0)).$$

We deduce $\tilde{z}'(0) = \tilde{y}'(0) = 0$ and obtain from (3.53)

$$(3.54) \quad \int_\Omega z_0(-\Delta + \bar{f}u_0 - e)(\tilde{z}''(s)) \, dx = \int_\Omega [-2\lambda'(0)u_0 z_0^2 - 2(z_0)^3 - 2\bar{f}y_0 z_0^2] \, dx.$$

Integrating by parts, we obtain

$$(3.55) \qquad \lambda'(0) = -[\int_\Omega u_0 z_0^2 \, dx]^{-1}[\int_\Omega (z_0^3 + \bar{f}y_0 z_0^2) \, dx].$$

Identifying z_0 with h in (3.45), and noting that $y_0 = [-\Delta + (2u_0 - a)]^{-1}(-cu_0 z_0)$, we find

$$(3.56) \qquad \lambda'(0) = -[\int_\Omega u_0 z_0^2 \, dx]^{-1} g_1(c),$$

where $g_1(c)$ is given in (3.45). Suppose $g_1(\bar{c}) > 0$, then $g_1(c)$ changes sign at some $\tilde{c} > \bar{c}$, and for $c = c_1 \in (\bar{c}, \tilde{c})$, we have $g_1(c_1) > 0$, and $\lambda'(0) < 0$. Consequently for $\delta > 0$ sufficiently small, (3.22) has a positive solution if $(c, f) = (c_1, \bar{f} - \delta)$. That is T^+ is non-empty.

If $g_1(\bar{c}) > 0$ and c_0 is slightly less than $\bar{c}$, positive solution bifurcates to the left of $\bar{f}$ at $(c_0, \bar{f})$. If $g_1(\bar{c}) < 0$, then positive solution bifurcates to the right of $\bar{f}$ at $(c_0, \bar{f})$. By symmetry, if $g_2(\bar{f}) < 0$ and f_0 is slightly less than $\bar{f}$, positive solution bifurcates to the right of $\bar{c}$ at $(\bar{c}, f_0)$. Thus T^+ is also nonempty. This proves part (i).

We shall not show the proof of part (ii), which can be found in Dancer [40].

The following lemma can be proved readily, and can be used for applying part (i) of Theorem 3.11 to find positive coexistence states.

Lemma 3.1. *Consider problem (3.22) with $a > \lambda_1, e > \lambda_1$ and $\bar{c}, \bar{f}, g_1(\bar{c}), g_2(\bar{f})$ be as described in Theorem 3.11. If e is sufficiently large, then $g_1(\bar{c}) > 0$ and $g_2(\bar{f}) < 0$.*

Proof. We first deduce a more convenient expression for $g_1(\bar{c})$ and $g_2(\bar{f})$. By definition in (3.45)

$$-\Delta h = eh - \bar{f} u_0 h \ \text{ in } \ \Omega, \ \ h = 0 \ \text{ on } \partial\Omega.$$

We have

$$-\Delta h - (a - 2u_0)h = (e - a)h + (2 - \bar{f})u_0 h \quad \text{in } \Omega.$$

That is

$$h = (e - a)Lh + (2 - \bar{f})L(u_0 h) \tag{3.57}$$

where L is the operator $[-\Delta - (a - 2u_0)]^{-1}$, under zero Dirichlet boundary conditions. Using (3.57), we can rewrite $g_1(\bar{c})$ by means of (3.45) as:

$$g_1(\bar{c}) = (2 - \bar{f})^{-1}(2 - \bar{f} - \bar{f}\bar{c}) \int_\Omega h^3 \, dx - (2 - \bar{f})^{-1}(a - e)\bar{f}\bar{c} \int_\Omega h^2 Lh \, dx, \tag{3.58}$$

if $\bar{f} \neq 2$. Note that h is positive and L is a positive operator. Similarly, we obtain:

$$k = (a - e)L_2 k + (2 - \bar{c})L_2(v_0 k),$$

where L_2 is the operator $[-\Delta - (e - 2v_0)]^{-1}$, under zero Dirichlet boundary conditions. Moreover from (3.46), we have

$$g_2(\bar{f}) = (2 - \bar{c})^{-1}(2 - \bar{c} - \bar{c}\bar{f}) \int_\Omega k^3 \, dx - (2 - \bar{c})^{-1}(e - a)\bar{c}\bar{f} \int_\Omega k^2 Lk \, dx. \tag{3.59}$$

It is easy to see that $g_2(\bar{f}) < 0$ if $\bar{c} < 2, 2 - \bar{c} - \bar{f}\bar{c} < 0$ and $e > a$. For fixed $a > \lambda_1$, equations (3.44) indicate that if e is large, then both $\bar{f}$ and v_0 are increased. This in turn leads to smaller $\bar{c}$. We thus have to carefully estimate $\bar{f}\bar{c}$ as e becomes large.

Let $\tilde{v} = e^{-1}v_0$. Then $\tilde{v}$ is a solution of

$$-e^{-1}\Delta\tilde{v} = \tilde{v}(1-\tilde{v}) \text{ in } \Omega, \quad \tilde{v} = 0 \text{ on } \partial\Omega.$$

It can be shown that as $e \to \infty$, we have $\tilde{v} \to 1$ in $L^p(\Omega)$ for each $p \in (1,\infty)$ (see e.g. Dancer [38] or similar proof in Theorem 5.2 in Section 1.5). Let $\hat{c} = e\bar{c}$. Note that $\bar{c}$ depends on e, and the spectral radius satisfies $r((-\Delta)^{-1}(a-\hat{c}\tilde{v})I) = 1$. Now, if $c > 0$, $r((-\Delta)^{-1}(a-c\tilde{v})I) \to r((-\Delta)^{-1}(a-c)I) = \lambda_1^{-1}(a-c)$ as $e \to \infty$, since $\tilde{v} \to 1$ in $L^p(\Omega)$. Suppose $\hat{c} = e\bar{c}$ is unbounded as $e \to \infty$. Let $e_i \to \infty$ and $\hat{c}(e_i) := e_i\bar{c} > c^*$ for some $c^* > a - \lambda_1$. Then

$$1 = r((-\Delta)^{-1}(a-\hat{c}\tilde{v})I) < r((-\Delta)^{-1}(a-c^*\tilde{v})I) \to \lambda_1^{-1}(a-c^*)$$

as $e_i \to \infty$. This implies $c^* < a - \lambda_1$, contradicting the assumption on c^*. We may thus assume $\hat{c} = e\bar{c} \to \alpha$ for some α as $e \to \infty$. Since $1 = r((-\Delta)^{-1}(a-\hat{c}\tilde{v})I) \to \lambda_1^{-1}(a-\alpha)$ as $e \to \infty$. We must have $e\bar{c} \to a - \lambda_1$ as $e \to \infty$.

We next consider the change of $\bar{f}$ for large e. Let $\tilde{r} > 0$ be an arbitrary number. We will show that if e is large, then

$$r((-\Delta)^{-1}(e-\tilde{r}eu_0)I) > 1, \text{ that is } \hat{\rho}_1(\Delta + (e-\tilde{r}eu_0)I) > 0. \tag{3.60}$$

Thus, from the definition of $\bar{f}$ in (3.44) and comparison, we must have $\bar{f} > \tilde{r}e$. Since $\tilde{r}$ is arbitrary, it follows that $e^{-1}\bar{f} \to \infty$ as $e \to \infty$. To prove (3.60), it suffices to find a $\mu_e > 1$ and a non-negative nontrivial function $w = s_e \in W_0^{1,2}(\Omega)$ which is a weak lower solution (as described in Section 9.3 in Chapter 9 of Evans [57]) for the problem:

$$-\Delta w = \mu_e^{-1}(e - reu_0)w \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega. \tag{3.61}$$

This follows because a simple calculation shows that this implies that $(-\Delta + KI)^{-1}(e + K - \tilde{r}eu_0)s_e \geq \beta s_e$ for some $\beta > 1$. Here, $K > \tilde{r}eu_0$ is chosen to ensure the operator acting on s_e is positive. Thus by p. 265, in Schaefer [205] or a theorem similar to Theorem A2-6 in Chapter 6, we find the spectral radius satisfies

$$r((-\Delta + KI)^{-1}(e + K - \tilde{r}eu_0)I) \geq \beta > 1.$$

In order to construct the weak lower solution as described above, we choose a neighborhood N of $\partial\Omega$ in Ω such that $u_0(x) \leq (2\tilde{r})^{-1}$ in N. Let z denote the principal non-negative eigenfunction of $(-\Delta)$ on N, with zero Dirichlet boundary conditions on ∂N. Define $s_e(x)$ to be $z(x)$ on N and to be zero otherwise. Then $s_e \in W_0^{1.2}(\Omega)$. Since $e - \tilde{r}eu_0(x) \geq (1/2)e$ on N, we have

$$-\Delta s_e \leq \mu_e^{-1}(e - \tilde{r}eu_0)s_e \tag{3.62}$$

pointwise in N if e is large. Moreover (3.62) is trivially valid in $\Omega\backslash\bar{N}$ pointwise. We can then deduce as in Section 4.4 of Chapter 4 or Lemma 1.1 in Berestyki and Lions [7] that s_e is a weak lower solution for (3.61). This completes the proof that $e^{-1}\bar{f} \to \infty$ as $e \to \infty$. It follows that $\bar{f}\bar{c} \to \infty$ as $e \to \infty$. Since $\bar{c} \to 0$ as $e \to \infty$, the comments after (3.59) imply that $g_2(\bar{f}) < 0$ for large e.

By means of further analysis of the asymptotics for $\bar{f}, \bar{c}$ as $e \to \infty$ using formula (3.58), we can deduce as above that $g_1(\bar{c}) > 0$ as $e \to \infty$. For more details see Dancer [40].

The following theorem provides more information concerning the coexistence states as the parameters (c, f) changes near $(\bar{c}, \bar{f})$ as described in (3.44).

Theorem 3.12. *Consider problem (3.22) with hypotheses as described in Theorem 3.11. Assume that $(c_1, f_1) \in T^+$. Suppose further that $0 \le c \le c_1, f_1 \le f < \bar{f}$ and either $c < c_1$ or $f_1 < f$, then problem (3.22) has a strictly positive solution which is an "asymptotically stable" solution of the corresponding parabolic problem.*

Remark 3.4. Here, by an "asymptotically stable" solution, we mean a solution (u, v) such that the spectral radius satisfies $r(A'(u, v)) \le 1$, (u, v) is an isolated solution and has index 1 in

$$D = \{(u, v) : C_0(\bar{\Omega}) \times C_0(\bar{\Omega}), 0 \le u \le a, 0 \le v \le e \text{ } in \text{ } \Omega\},$$

where A is the map whose fixed points are solutions of (3.22), described in (3.65) below. $A'(u, v)$ denotes the Fréchet derivative of A at (u, v).

Proof. We first prove the existence of a strictly positive solution. Since $(c_1, f_1) \in T^+$, there exists a strictly positive solution (u_1, v_1) satisfying

$$\begin{cases} -\Delta u_1 = u_1(a - u_1 - c_1 v_1) & \\ & \text{in } \Omega, \\ -\Delta v_1 = v_1(e - v_1 - f_1 u_1) & \\ u_1 = v_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

By comparison, we have $0 \le u_1 \le u_0 \le a, 0 \le v_1 \le v_0 \le e$ in $\bar{\Omega}$. Since $c \le c_1$, we have

$$-\Delta u_1 \le u_1(a - u_1 - cv_1) \text{ in } \Omega.$$

Moreover, strict inequality holds in Ω if $c < c_1$, since $u_1 > 0$ and $v_1 > 0$ in Ω. Thus

$$u_1 \le (-\Delta + \hat{k}I)^{-1}(u_1(a + \hat{k} - u_1 - cv_1)), \tag{3.63}$$

and equality does not hold if $c < c_1$. Similarly

$$v_1 \ge (-\Delta + \hat{k}I)^{-1}(v_1(e + \hat{k} - v_1 - fu_1)), \tag{3.64}$$

and equality does not hold if $f > f_1$. Here $\hat{k}$ is a positive constant satisfying $\hat{k} \geq max.\{a + ce, e + fa\}$ such that the mapping:

$$A(u,v) = (-\Delta + \hat{k})^{-1}(u(a + \hat{k} - u - cv), v(e + \hat{k} - v - fu)) \tag{3.65}$$

is monotone on the set:

$$D = \{(u,v) \in C_0(\bar{\Omega}) \times C_0(\bar{\Omega}) : 0 \leq u \leq a, 0 \leq v \leq e \ \ in \ \bar{\Omega}\}.$$

If we define $(u_2, v_2) = A(u_1, v_1)$, we see from (3.63) and (3.64) that

$$u_0 \geq u_2 \geq u_1, \ \ 0 \leq v_2 \leq v_1.$$

Defining $(u_{n+1}, v_{n+1}) = A(u_n, v_n), n = 2, 3, 4 \ldots$, we have

$$u_0 \geq u_{n+1} \geq u_n, \ \ 0 \leq v_{n+1} \leq v_n, \ n = 2, 3, 4 \ldots$$

By theory explained in Leung [125], the function $(\tilde{u}, \tilde{v}) = lim_{n\to\infty}(u_n, v_n)$ is a strictly positive solution of (3.22), unless $\tilde{v} = 0$. In this case $(\tilde{u}, \tilde{v}) = (u_0, 0)$. It is also shown in Dancer [40] that

$$\begin{matrix} \tilde{u} \leq \hat{u} \\ \tilde{v} \geq \hat{v} \end{matrix} \qquad \text{in } \ \bar{\Omega},$$

if $(\hat{u}, \hat{v})$ is any solution of (3.22) satisfying:

$$\begin{matrix} u_1 \leq \hat{u} \leq u_0 \\ 0 \leq \hat{v} \leq v_1 \end{matrix} \qquad \text{in } \ \bar{\Omega}.$$

Let $C = \{(u,v) \in D : u_1 \leq u \leq u_0, 0 \leq v \leq v_1\}$. If the limit is such that $(\tilde{u}, \tilde{v}) = (u_0, 0)$, then $(u_0, 0)$ is the only fixed point of the mapping A in C. The set C is closed and convex (thus contractible); and $AC \subseteq C$ by monotonicity. Hence by basic properties of fixed point index, the sum of the indices of fixed points of A in C (counted relative to C) is 1, see Amann [3]. If $(u_0, 0)$ is the only fixed point in C, then we must have $index_C(A, (u_0, 0)) = 1$. On the other hand, by using Theorem 1 and Lemma 2 in Dancer [37] and part of Proposition 1 in Dancer [39], we can show that $index_C(A, (u_0, 0)) = 0$, provided we have

$$r(A'(u_0, 0)) > 1, \text{and } A'(u_0, 0)(h, k) \neq (h, k) \text{ if } (h, k) \in (C_0^+(\bar{\Omega}) \times C_0^+(\bar{\Omega})) \backslash \{0, 0\}. \tag{3.66}$$

Note that the first property above is related to property (α) of Definition 2.1, as indicated in the proof of case (ii) in Theorem 3.4. In order to analyze the spectral radius indicated in (3.66), we note that

$$A'(u_0, 0)(h, k) = (-\Delta + \hat{k}I)^{-1}(a + \hat{k} - 2u_0)h - cu_0k, (e + \hat{k} - fu_0)k).$$

We thus have the following relationship for the various spectrum

$$\sigma(A'(u_0,0)) = \sigma((-\Delta+\hat{k}I)^{-1}(a+\hat{k}-2u_0)I) \cup \sigma((-\Delta+\hat{k}I)^{-1}(e+\hat{k}-fu_0)I).$$

It follows that

$$r(A'(u_0,0)) \geq r((-\Delta+\hat{k}I)^{-1}(e+\hat{k}-fu_0)I) > 1.$$

The last inequality above is due to the fact that $f < \bar{f}$. The second property of (3.66) can be proved by procedures as in the proof of Theorem 3.4. We can thus conclude that we also have $index_C(A,(u_0,0)) = 0$, contradicting the fact that $index_C(A,(u_0,0)) = 1$ deduced above. Consequently, we must have $(\tilde{u},\tilde{v})$ is a strictly positive solution of (3.22).

The details of the proof of the "asymptotic stability" of the solution as described in the above remark will be given later in Section 1.5 of this chapter. By applying Remark 4 on p. 58 of Dancer [39], with $E = L^p(\Omega) \times L^p(\Omega)$ for large p, we can deduce that the solution is actually asymptotically stable with respect to the corresponding parabolic problem in the space $X^\alpha \times X^\alpha$, where X^α is a fractional power space in the sense of p. 29 in Henry [84] (cf. Section 6.4 in Chapter 6). For more explanations, see Dancer [40].

By means of Theorem 3.12, we can obtain the following more detailed information.

Theorem 3.13. *Consider problem (3.22) with hypotheses described in Theorem 3.11. Assume T^+ is nonempty. Then there exist $\mu > \bar{c}$, $\nu \in (0,\bar{f})$ and a continuous strictly increasing function $g^+ : [\bar{c},\mu] \to (0,\bar{f}]$ such that $g^+(\bar{c}) = \nu, g^+(\mu) = \bar{f}$, and $T^+ = \{(c,f) : c > \bar{c}, 0 < f < \bar{f}, f \geq g^+(c)\}$. Moreover, if $(c,f) \in int\ T^+$, then problem (3.22) has at least two solutions, at least one of which is "asymptotically stable". Furthermore, problem (3.22) has a strictly positive asymptotically stable solution if $c = \bar{c}$, $\nu < f < \bar{f}$, and a strictly positive solution if $f = \bar{f}$, $\bar{c} < c < \mu$.*

Notes.

Theorem 3.1 and Corollary 3.2 are found in Leung [121] and Pao [182]. Theorem 3.3 is due to Blat and Brown [11]. Theorems 3.4 and 3.5 are obtained from Li and Logan [151]. Theorem 3.6 is due to Ruan and Pao [195]. Theorems 3.7, 3.8, Corollary 3.9 and Theorem 3.10 are results in Dancer and Du [41]. Theorem 3.11 is given in Dancer [40] and the proof of part (i) follows an argument in Cantrell and Cosner [18]. Lemma 3.1, Theorem 3.12 and Theorem 3.13 are obtained from Dancer [40].

1.4 Strictly Positive Coexistence for Diffusive Cooperating Systems

In this section we study problem (1.1) when the functions $f_1(u, v)$ and $f_2(u, v)$ simulate cooperation or mutualism between the two species populations $u(x)$ and $v(x)$ in a bounded domain Ω, with conditions as described in Section 1.1. For simplicity, we first consider the Volterra-Lotka type of interaction when f_1 and f_2 are linear. More precisely, we consider

$$(4.1) \qquad \begin{cases} \sigma_1 \Delta u + u(a - bu + cv) = 0 & \\ & \text{in } \Omega, \\ \sigma_2 \Delta v + v(e + fu - gv) = 0 & \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where a, b, c, e, f and g are all positive constants. The signs of the interaction coefficients $+c$ and $+f$ indicate mutualism. The first theorem shows that when each species can survive by itself (i.e. they satisfy (4.2)), and the cooperation coefficients are not very large (i.e. they satisfy (4.3)), then there will be coexistence equilibrium state. The main idea in the proof is that condition (4.3) will impose a bound on both populations.

Theorem 4.1. *Suppose*

$$(4.2) \qquad a > \sigma_1 \lambda_1 \ \ \textit{and} \ \ e > \sigma_2 \lambda_1,$$

then the boundary value problem (4.1) has a solution with each component strictly positive in Ω if and only if

$$(4.3) \qquad cf < bg.$$

Proof. By hypothesis (4.3), there exists (x_0, y_0) in the first open quadrant where

$$a - bx_0 + cy_0 \leq 0, \quad e + fx_0 - gy_0 \leq 0.$$

Define $(\bar{u}(x), \bar{v}(x)) \equiv (x_0, y_0)$ for $x \in \Omega$. Let $\omega(x) > 0$ be the principal eigenfunction for the operator $(-\Delta)$ on Ω with principal eigenvalue $\lambda_1 > 0$ and zero

Dirichlet boundary conditions. We readily verify that for $\delta > 0$ sufficiently small,
(4.4)

$$\begin{cases} \sigma_1 \Delta \bar{u} + \bar{u}(a - b\bar{u} + cv) \le 0 \ \text{ in } \ \Omega, \ \text{ for } \ \delta\omega(x) \le v \le \bar{v}, \\[2ex] \sigma_2 \Delta \bar{v} + \bar{v}(e + fu - g\bar{v}) \le 0 \ \text{ in } \ \Omega, \ \text{ for } \ \delta\omega(x) \le u \le \bar{u}, \\[2ex] \sigma_1 \Delta(\delta\omega) + \delta\omega(a - b\delta\omega + cv) \\ \qquad\qquad = \delta\omega(-\sigma_1\lambda_1 + a - b\delta\omega + cv) \ge 0 \ \text{ in } \ \Omega, \ \text{ for } \ \delta\omega(x) \le v \le \bar{v}, \\[2ex] \sigma_2 \Delta(\delta\omega) + \delta\omega(e + fu - g\delta\omega) \\ \qquad\qquad = \delta\omega(-\sigma_1\lambda_1 + e + fu - g\delta\omega) \ge 0 \ \text{ in } \ \Omega, \ \text{ for } \ \delta\omega(x) \le u \le \bar{u}. \end{cases}$$

Thus the functions $(\bar{u}(x), \bar{v}(x))$ and $(\delta\omega(x), \delta\omega(x))$ form a pair of coupled ordered upper-lower solutions for the boundary value problem (4.1). By Theorem 1.4-2 in Leung [125], the problem (4.1) has a solution $(u(x), v(x))$ satisfying

$$\delta\omega(x) \le u(x) \le x_0, \quad \delta\omega(x) \le v(x) \le y_0 \ \text{ for } \ x \in \bar{\Omega}.$$

To prove the converse, suppose (4.1) has a nontrivial positive solution $(\tilde{u}, \tilde{v})$ and $cf \ge bg$. Choose k such that

$$\frac{b}{c} \le k \le \frac{f}{g}.$$

Define $(u_\alpha(x), v_\alpha(x)) = (\alpha\omega(x), \alpha k\omega(x))$ for $x \in \bar{\Omega}$. Considering the equations (4.1) in a neighborhood of the boundary, we readily obtain by maximum principle that the outward normal derivative of $\tilde{u}$ and $\tilde{v}$ are strictly negative at the boundary. Thus we have

$$\tilde{u}(x) \ge u_{\alpha_0}(x), \ \ \tilde{v}(x) \ge v_{\alpha_0}(x), \quad x \in \bar{\Omega} \ \text{ for some } \ \alpha_0 > 0. \tag{4.5}$$

By the choice of k, we readily verify that

$$\begin{cases} \sigma_1 \Delta u_\alpha + u_\alpha(a - bu_\alpha + cv_\alpha) \ge 0 \ \text{ in } \ \Omega, \\[2ex] \sigma_2 \Delta v_\alpha + v_\alpha(e + fu_\alpha - gv_\alpha) \ge 0 \ \text{ in } \ \Omega \end{cases} \tag{4.6}$$

for all $\alpha \ge \alpha_0$. Using (4.5), (4.6) and the sweeping principle for quasimonotone nondecreasing system by means of a family of lower solutions, we assert that

$$\tilde{u}(x) \ge u_\alpha(x), \ \ \tilde{v}(x) \ge v_\alpha(x), \quad x \in \bar{\Omega} \ \text{ for all } \ \alpha > \alpha_0.$$

(The sweeping principle is an extension of Theorem 1.4-2 for the scalar case described in Leung [125]. The extension to quasimonotone nondecreasing system is described in Theorem A3-9 in Chapter 6.) We thus obtain a contradiction by letting $\alpha \to \infty$. Consequently, we must have (4.3).

We next consider the more general cooperating system:

$$\begin{cases} \Delta u + uM(u,v) = 0 & \\ & \text{in } \Omega, \\ \Delta v + vN(u,v) = 0 & \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.7}$$

where $M, N \in C^1(R \times R)$,

$$M_v(u,v) > 0,\ N_u(u,v) > 0 \text{ for } u,v \geq 0. \tag{4.8}$$

(4.9) For $u,v \geq 0, -D \leq M_u \leq 0,\ -D \leq N_v \leq 0,$ for some $D > 0$; moreover, either M_u or N_v is not identically zero.

Let Γ_M and Γ_N be points on the open uv-plane defined respectively by the equations $M(u,v) = 0$ and $N(u,v) = 0$. For convenience, define the functions $M_1(u,v) = M(u,v) - M(0,0), N_1(u,v) = N(u,v) - N(0,0)$; and let Γ_{M_1} and Γ_{N_1} be points on the open uv-plane defined respectively by the equations $M_1(u,v) = 0$ and $N_1(u,v) = 0$. We will assume that

(4.10)
Γ_M and Γ_N are two distinct curves; and the set Γ_{M_1} and Γ_{N_1} are represented by two distinct positive functions $u = \phi_1(v), u = \psi_1(v)$ respectively for $v \geq 0$.

Theorem 4.2. *Under hypotheses (4.8) to (4.10), suppose $M(0,0) > \rho_1(-\Delta) = \lambda_1$, and $N(0,0) > \rho_1(-\Delta) = \lambda_1$.*

(i) If Γ_M and Γ_N intersect at a point (x_0, y_0) in the first open quadrant, then problem (4.7) has a solution with each component strictly positive in Ω.

(ii) If the problem (4.7) has a positive solution (with each component strictly positive in Ω), then

$$sup_{x>0} \frac{\psi_1(x)}{x} > inf_{x>0} \frac{\phi_1(x)}{x}. \tag{4.11}$$

Proof. We have $M(x_0, y_0) = N(x_0, y_0) = 0$. Define $(\bar{u}(x), \bar{v}(x)) \equiv (x_0, y_0)$ for $x \in \bar{\Omega}$. Let $\omega(x), \lambda_1$ be as defined in the proof of Theorem 4.1. For $\delta > 0$ sufficiently small, we have:

(4.12)
$$\begin{cases} \Delta\bar{u} + \bar{u}M(\bar{u}, v) = x_0 M(x_0, v) \le x_0 M(x_0, y_0) = 0 \text{ in } \Omega, \text{ for } \delta\omega(x) \le v \le \bar{v}, \\ \Delta\bar{v} + \bar{v}N(u, \bar{v}) = y_0 N(u, y_0) \le y_0 N(x_0, y_0) = 0 \text{ in } \Omega, \text{ for } \delta\omega(x) \le u \le \bar{u}, \\ \Delta(\delta\omega) + \delta\omega M(\delta\omega, v) \ge \Delta(\delta\omega) + \delta\omega M(\delta\omega, 0) \text{ in } \Omega, \text{ for } \delta\omega(x) \le v \le \bar{v}, \\ \qquad = -\lambda_1\delta\omega + \delta\omega M(\delta\omega, 0) \\ \qquad > -\lambda_1\delta\omega + \delta\omega\lambda_1 = 0 \text{ in } \Omega, \text{ for } \delta > 0 \text{ sufficiently small}, \\ \Delta(\delta\omega) + \delta\omega N(u, \delta\omega) \ge \Delta(\delta\omega) + \delta\omega N(0, \delta\omega) \text{ in } \Omega, \text{ for } \delta\omega(x) \le u \le \bar{u}, \\ \qquad = -\lambda_1\delta\omega + \delta\omega N(0, \delta\omega) \\ \qquad > -\lambda_1\delta\omega + \delta\omega\lambda_1 = 0 \text{ in } \Omega, \text{ for } \delta > 0 \text{ sufficiently small}. \end{cases}$$

Thus the functions $(\bar{u}(x), \bar{v}(x))$ and $(\delta\omega(x), \delta\omega(x))$ form a pair of coupled ordered upper-lower solutions for the boundary value problem (4.7). By Theorem 1.4-2 in [125], the problem (4.7) has a solution $(u(x), v(x))$ satisfying

$$\delta\omega(x) \le u(x) \le x_0, \quad \delta\omega(x) \le v(x) \le y_0 \text{ for } x \in \bar{\Omega}.$$

This proves (i). For part (ii), suppose (4.7) has a positive solution $(\tilde{u}(x), \tilde{v}(x))$ in $\bar{\Omega}$ and (4.11) is false. That is, assume

$$inf_{x>0}\frac{\phi_1(x)}{x} \ge sup_{x>0}\frac{\psi_1(x)}{x}.$$

Then there exists a constant $\tau > 0$ such that $\phi_1(\sigma\omega(x)) \ge \tau\sigma\omega(x) \ge \psi_1(\sigma\omega(x))$ for all $x \in \Omega$, and all $\sigma > 0$. Note that by definition and (4.9), we have $0 = M(\phi_1(\sigma\omega(x)), \sigma\omega(x)) - M(0,0) \le M(\tau\sigma\omega(x), \sigma\omega(x)) - M(0,0)$. Thus, $M(0,0) \le M(\tau\sigma\omega(x), \sigma\omega(x))$ for $x \in \Omega$. Similarly, we obtain $N(0,0) \le N(\tau\sigma\omega(x), \sigma\omega(x))$ for $x \in \Omega$. We thus arrive at the following inequalities for $x \in \Omega$, all $\sigma > 0$:

(4.13)
$$\begin{cases} -\Delta(\tau\sigma\omega) = \tau\lambda_1\sigma\omega < \tau\sigma\omega M(0,0) \le \tau\sigma\omega M(\tau\sigma\omega, \sigma\omega), \\ -\Delta(\sigma\omega) = \lambda_1\sigma\omega < \sigma\omega N(0,0) \le N(\tau\sigma\omega, \sigma\omega). \end{cases}$$

Moreover, by means of maximum principle at the boundary, we can verify that $\tilde{u} \ge \sigma_0\tau\omega, \tilde{v} \ge \sigma_0\omega$, for $x \in \bar{\Omega}$, $\sigma_0 > 0$ sufficiently small. Thus using the family of lower solutions $(\sigma\tau\omega, \sigma\omega), \sigma \ge \sigma_0$ for the quasimonotone nondecreasing system (4.7), we obtain a contradiction, $\tilde{u} \ge \sigma\tau\omega, \tilde{v} \ge \sigma\omega$, as $\sigma \to \infty$.

For a simple special case for (4.7), we consider

$$(4.14)\qquad \begin{cases} \Delta u + u(m_1(v) - m_2(u)) = 0 & \\ & \text{in } \Omega, \\ \Delta v + v(n_1(u) - n_2(v)) = 0 & \\ & \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $m_1, m_2, n_1, n_2 \in C^2(R); m_2(0) = n_2(0) = 0$. As in (4.8) and (4.9), we assume

$$(4.15)\qquad m_i' > 0,\ n_i' > 0,\ \ for\ i = 1, 2.$$

$$(4.16)\qquad |m_2'| \leq D, |n_2'| \leq D,\ \ for\, some\, constant\, D > 0.$$

Theorem 4.3. *Assume the hypotheses on m_i, n_i above and (4.15) and (4.16). Suppose $m_1(0) > \rho_1(-\Delta)$, and $n_1(0) > \rho_1(-\Delta)$, and further*

$$(4.17)\qquad m_1'', n_1'' \leq 0,\ \ m_2'', n_2'' \geq 0.$$

Then the problem (4.14) has a solution with each component strictly positive in Ω if and only if the two simultaneous equations: $m_1(v) = m_2(u), n_1(u) = n_2(v)$ has a solution in the first open quadrant in the uv-plane.

Proof. Hypotheses (4.15) to (4.17) ensures that conditions (4.8) to (4.10) are satisfied for problem (4.7) with $M(u,v) = m_1(v) - m_2(u), N(u,v) = n_1(u) - n_2(v)$,(except that the function $\psi_1(v)$ may possibly be only defined in a bounded subinterval of $v \geq 0$). If the simultaneous equations: $m_1(v) = m_2(u), n_1(u) = n_2(v)$ have a solution in the first open quadrant, then we can apply the same proof as the first part of Theorem 4.2 to assert that problem (4.14) has a positive solution.

Next, assume that (4.14) has a positive solution. Recall that $M_1(u,v) = m_1(v) - m_1(0) - m_2(u), N_1(u,v) = n_1(u) - n_1(0) - n_2(v)$. We verify $\phi_1(v) = m_2^{-1}(m_1(v) - m_1(0)), \psi_1(v) = n_1^{-1}(n_2(v) + n_1(0))$. The functions ϕ_1 is concave down and ψ_1 is concave up. Therefore, $\frac{\phi_1(x)}{x}$ is nonincreasing and $\frac{\psi_1(x)}{x}$ is nondecreasing. Note also that both ϕ_1 and ψ_1 are increasing functions. For $v \geq 0$, let $\phi(v) = m_2^{-1}(m_1(v)), \psi(v) = n_1^{-1}(n_2(v))$, we have $\phi(0) = m_2^{-1}(m_1(0)) > m_2^{-1}(0) = 0 > n_1^{-1}(0) = \psi(0)$. If the $lim_{u\to\infty} n_1(u)$ is finite, then $\psi(v)$ tends to ∞ as v tends to a finite number. Then there must be a number x^* where $\psi(x^*) > \phi(x^*)$. Hence ψ and ϕ must be equal at some positive number, and the simultaneous equations must have a solution in the first open quadrant.

If $lim_{u\to\infty} n_1(u) = \infty$, then $\psi_1(x)$ is defined for all $x > 0$, and by Theorem 4.2, we have (4.11):

$$sup_{x>0} \frac{\psi_1(x)}{x} \ > \ inf_{x>0} \frac{\phi_1(x)}{x}.$$

The last inequality implies that there exist $x_0 > 0$ such that $\psi_1(x_0) > \phi_1(x_0)$. Since $\psi_1(0) = 0 = \phi_1(0)$, there must be a $x_1 \in (0, x_0)$ at which $\psi_1'(x_1) > \phi_1'(x_1)$. This can be written as

$$\frac{n_2'(x_1)}{n_1'(n_1^{-1}(n_2(x_1) + n_1(0)))} > \frac{m_1'(x_1)}{m_2'(m_2^{-1}(m_1(x_1) - m_1(0)))}. \tag{4.18}$$

Since $n_2'' \geq 0, n_2' \geq n_2'(0) > 0$, we must have $n_2(x) \to \infty$ as $x \to \infty$. Consequently, there must exists $x_2 \in (x_1, \infty)$ such that $n_2(x) > n_2(x_1) + n_1(0)$ and $m_1(x) > m_1(x_1) - m_1(0)$ for $x > x_2$. It then follows readily from (4.18) that

$$\begin{aligned} \frac{n_2'(x)}{n_1'(n_1^{-1}(n_2(x)))} &\geq \frac{n_2'(x_1)}{n_1'(n_1^{-1}(n_2(x_1)+n_1(0)))} \\ &> \frac{m_1'(x_1)}{m_2'(m_2^{-1}(m_1(x_1)-m_1(0)))} \geq \frac{m_1'(x)}{(m_2'(m_2^{-1}(m_1(x)))}. \end{aligned} \tag{4.19}$$

for $x \geq x_2$. This means

$$\frac{d}{dx} n_1^{-1}(n_2(x)) > \frac{d}{dx} m_2^{-1}(m_1(x))$$

for $x \geq x_2$. Note that $\frac{d}{dx} n_1^{-1}(n_2(x))$ is nondecreasing in x while $\frac{d}{dx} m_2^{-1}(m_1(x))$ is nonincreasing. Consequently, we must have $n_1^{-1}(n_2(\tilde{x})) \geq m_2^{-1}(m_1(\tilde{x}))$ for some $\tilde{x} > x_2$; i.e. $\psi(\tilde{x}) \geq \phi(\tilde{x})$. We can then conclude the proof as in the last paragraph.

We next consider a generalization of Theorem 4.1, when the interaction coefficients b, c, f and g may change with position, and the Laplacian is replaced by two second order uniformly elliptic operators as follow:

$$L_k = \sum_{i,j=1}^{N} a_{ijk}(x) \partial_i \partial_j + \sum_{j=1}^{N} b_{jk}(x) \partial_j - c_k(x), \quad k = 1, 2 \tag{4.20}$$

with

$$a_{ijk} \in C(\bar{\Omega}), \quad b_{jk}, c_k \in L^\infty(\Omega), \quad i, j = 1, ..., N, \; k = 1, 2, \tag{4.21}$$

We will consider the problem:

$$\begin{cases} L_1 u + u[a - b(x)u + c(x)v] = 0 & \\ & \text{in } \Omega, \\ L_2 v + v[e + f(x)u - g(x)v] = 0 & \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.22}$$

where $b, c, f, g \in C(\bar{\Omega})$ satisfy $b(x) > 0, g(x) > 0$ for each $x \in \bar{\Omega}$, and $c \geq 0, f \geq 0$ in Ω, $c \not\equiv 0, f \not\equiv 0$; the parameters $a, e \in R$ are constants. For any function $h \in L^\infty(\Omega)$, we denote

$$h_L := ess\,inf_\Omega\ h, \quad h_M := ess\,sup_\Omega\ h.$$

We will consider solutions of (4.22) with u, v in $W^{2,p}(\Omega), p > N$, and the equations are satisfied almost everywhere. By the Sobolev embedding, we have $W^{2,p}(\Omega) \subset C^{2-N/p-\epsilon}$ for any small $\epsilon > 0$. Moreover, the functions $u, v \in W^{2,p}(\Omega)$ is twice classically differentiable almost everywhere in Ω. Actually, many of the theorems in the last two sections can be extended in analogous fashions as below. For convenience, we will let $w = \theta_{[-L_k, p(x), q(x)]}$ denote the positive solution of

$$-L_k w = w[p(x) - q(x)w] \ \text{ in } \Omega, \ \ w = 0 \ \text{ on } \partial\Omega,$$

if $\hat{\rho}_1(L_k + p(x)) > 0$. Otherwise, let $\theta_{[-L_k, p(x), q(x)]} \equiv 0$ in $\bar{\Omega}$. (Here, we assume $q(x) > 0$ in $\bar{\Omega}$.) Recall the definition of the principal eigenvalues, $\rho_1(-\sigma\Delta + p(x))$ and $\hat{\rho}_1(\sigma\Delta + \tilde{p}(x))$, given in (1.4) and (1.7) in Section 1.1. We now extend the definitions naturally when $\sigma\Delta$ is replaced by a second order uniformly elliptic operator. Moreover, for the corresponding Dirichlet problem in a different domain G, the principal eigenvalues are denoted by $\rho_1^G(-L_k + p(x))$ and $\hat{\rho}_1^G(L_k + \tilde{p}(x))$. For more detailed description of the properties of such solutions, the maximum and comparison theorems in $W^{2,p}(\Omega)$ theory, the reader is referred to Theorems A3.1 to A3.5 in Section 6.3 in Chapter 6. The following theorem is an extension of Theorem 4.1 of this section.

Theorem 4.4 (Positive Solution under Weak Cooperation). *Suppose*

$$\text{(4.23)} \quad \begin{cases} c_M f_M < b_L g_L; \ \textit{and} \\ \\ \hat{\rho}_1(L_1 + a + c(x)\theta_{[-L_2, e, g(x)]}) > 0, \ \hat{\rho}_1(L_2 + e + f(x)\theta_{[-L_1, a, b(x)]}) > 0, \end{cases}$$

then the boundary value problem (4.22) has a solution with each component strictly positive in Ω.

Proof. From (4.22), we see that if (u, v) is a positive solution of problem (4.22), then

$$u = \theta_{[-L_1, a+cv, b(x)]}, \quad v = \theta_{[-L_2, e+fu, g(x)]}.$$

By comparison, we readily deduce

$$\theta_{[-L_1, a+cv, b(x)]} \leq \theta_{[-L_1, a+c_M v_M, b_L]} \leq \frac{a + c_M v_M - (c_1)_L}{b_L}.$$

Thus,

$$\text{(4.24)} \qquad u_M \leq \frac{a + c_M v_M - (c_1)_L}{b_L}.$$

Similarly, we deduce

$$v_M \le \frac{e + f_M u_M - (c_2)_L}{g_L}. \tag{4.25}$$

From (4.24) and (4.25) we obtain a bound of any positive solution of problem (4.22) in terms of a and e as follows:

$$\begin{cases} u_M \le \frac{(a-(c_1)_L)g_L + (e-(c_2)_L)c_M}{b_L g_L - c_M f_M}, \\ v_M \le \frac{(e-(c_2)_L)b_L + (a-(c_1)_L)f_M}{b_L g_L - c_M f_M}. \end{cases} \tag{4.26}$$

As in (2.14) we consider the problem:

$$-L_2 v - f(x)uv = v(e - g(x)v) \text{ in } \Omega, \; v = 0 \text{ on } \partial\Omega. \tag{4.27}$$

Define the map $v(u)$ from $C^1(\bar\Omega)$ to $C^1(\bar\Omega)$ as in (2.19) with $\rho_1(-\sigma_2\Delta - fu)$ replaced by $\rho_1(-L_2 - f(x)u)$.

Note that if both $a \le \rho_1(-L_1)$ and $e \le \rho_1(-L_2)$, then the second and third inequality of assumptions (4.23) cannot be satisfied. Suppose $e > \rho_1(-L_2)$, we write the first equation of (4.22) as

$$-L_1 u - c(x)v(0)u = au - b(x)u^2 + c(x)[v(u) - v(0)]u \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega \tag{4.28}$$

and bifurcate with increasing parameter a at $a = \rho_1(-L_1 - c(x)\theta_{[-L_2,e,g(x)]})$ when $(u,v) = (0, \theta_{[-L_2,e,g(x)]})$. From the bound (4.26) of positive solutions in terms of a, we can show as in Lemma 2.3 that there exists a continuum of solutions S^+ of (4.28) contained in $R \times P$, i.e. $u \ge 0$ whenever $(a,u) \in S^+$ and $\{a \in R : (a,u) \in S^+\} = (\rho_1(-L_1 - c(x)\theta_{[-L_2,e,g(x)]}, +\infty)$.

If $(a,u) \in S^+$, then $u \ge 0$ and so $v(u) \ge v(0)$, i.e. $v(u)$ is not the trivial solution. Consequently, the continuum of solutions $\{(a,u,v(u)) : (a,u) \in S^+\}$ for system (4.22) cannot be connected with the continuum of solutions $\{(a,u(0),0) : a > \rho_1(-L_1)\}$. Thus both components of the solutions of (4.22) on the continuum $\{(a,u,v(u)) : (a,u) \in S^+\}$ are positive in Ω; and by comparison, both the second and third inequality in (4.23) are satisfied.

Similarly, suppose $a > \rho_1(-L_1)$, we bifurcate with e to obtain a solution of problem (4.22) with both components positive in Ω for each $e > \rho_1(-L_2 - g(x)\theta_{[-L_1,a,b(x)]})$. This completes the proof

Remark 4.1. From the methods in the previous two sections, we can deduce that the last two inequalities in (4.23) imply that the related indices of both solutions $(0, \theta_{[-L_2,e,g(x)]})$ and $(\theta_{[-L_1,a,b(x)]}, 0)$ of (4.22) are zero.

The previous theorems in this section essentially concentrate on finding steady states when cooperative interaction coefficients between the different species are relatively small. These are reflected for instance in assumptions (4.3) and the first part of (4.23). We next concentrate on situations when the cooperative coefficients are relatively large. For simplicity, we will assume

$$L_1 = L_2 = L. \tag{4.29}$$

The conditions on large cooperative coefficients will be imposed in the form:

$$c_L f_L - b_M g_M > b_M c_M - b_L c_L, \tag{4.30}$$

or

$$c_L f_L - b_M g_M > g_M f_M - g_L f_L. \tag{4.31}$$

In the next theorem, we will see for instance that under condition (4.30), for any given fixed $a < \rho_1(-L)$, there exists a constant $e(a)$ such that for $e > e(a)$, the problem (4.22), (4.29) cannot have positive equilibrium. Roughly speaking, the cooperation rates and growth rate of one species is too large for any possible coexistence equilibrium.

Theorem 4.5 (Nonexistence under Strong Cooperation). *Assume (4.29) for problem (4.22).*
(i) Suppose (4.30) holds, then for any fixed $a < \rho_1(-L)$, there exists a number $e = e(a)$ such that $a > \rho_1(-L - c(x)\theta_{[-L,e(a),g(x)]})$, and problem (4.22) does not have any coexistence positive solution if $e > e(a)$.
(ii) Suppose (4.31) holds, then for any fixed $e < \rho_1(-L)$, there exists a number $a = a(e)$ such that $e > \rho_1(-L - f(x)\theta_{[-L,a(e),b(x)]})$, and problem (4.22) does not have any coexistence positive solution if $a > a(e)$.

Before proving this theorem, we first consider the following two lemmas which estimate the sizes of the solutions, and will be used to prove the theorem.

Lemma 4.1. *Assume (4.29), and let (u, v) be any positive coexistence solution of (4.22). Then,*
(i) If $e \geq a$, then

$$u \leq \frac{c_M + g_M}{f_L + b_L} v. \tag{4.32}$$

(ii) If $a \geq e$, then

$$v \leq \frac{f_M + b_M}{c_L + g_L} u. \tag{4.33}$$

Proof. Assume (4.29), $e \geq a$, and let (u, v) be any coexistence positive solution of (4.22). Define

$$w = (c_M + g_M)v - (f_L + b_L)u. \tag{4.34}$$

We can deduce from (4.22) that we have in Ω,

$$(-L - a + b_L u + g_M v)w \geq 0. \tag{4.35}$$

Moreover, from the second equation of (4.22), we find

$$e = \rho_1(-L - fu + gv). \tag{4.36}$$

Thus the monotonic dependence of the principal eigenvalue on the potential implies that

$$a \leq e \leq \rho_1(-L - f_L u + g_M v).$$

This gives $\rho_1(-L - a - f_L u + g_M v) \geq 0$, and

$$\rho_1(-L - a + b_L u + g_M v) > \rho_1(-L - a - f_L u + g_M v) \geq 0. \tag{4.37}$$

Consequently, (4.35), (4.37), the strong maximum principle (cf. Theorem A3-1 in Chapter 6), and the argument for strong maximum principle for Theorem 3.5 in p. 35 of [71] imply that $w \geq 0$. This completes the proof of part (i) of this Lemma. Part (ii) is proved similarly.

Lemma 4.2. *(i) For a fixed $a < \rho_1(-L_1)$, let $e_0(a) > \rho_1(-L_2)$ be such that*

$$a > \rho_1(-L_1 - c(x)\theta_{[-L_2,e,g]}) \quad \textit{for each} \quad e > e_0(a). \tag{4.38}$$

Assume that there exists a sequence of positive coexistence solutions (e_n, u_n, v_n) of (4.22), $n \geq 1$, such that $e_n > max\{e_0(a), 0\}$ for each $n \geq 1$ and $lim_{n\to\infty} e_n = \infty$. Then, for any compact subset $K \subset \Omega$ there exists a positive constant $\alpha = \alpha(K) > 0$ such that for each $n \geq 1$

$$\frac{v_n}{e_n} \geq \alpha \quad in \quad K. \tag{4.39}$$

(ii) Similarly, for a fixed $e < \rho_1(-L_2)$, let $a_0(e) > \rho_1(-L_1)$ be such that

$$e > \rho_1(-L_2 - f(x)\theta_{[-L_1,a,b]}) \quad \textit{for each} \quad a > a_0(e). \tag{4.40}$$

Assume that there exists a sequence of positive coexistence solutions (a_n, u_n, v_n) of (4.22), $n \geq 1$, such that $a_n > max\{a_0(e), 0\}$ for each $n \geq 1$ and $lim_{n\to\infty} a_n = \infty$. Then, for any compact subset $K \subset \Omega$ there exists a positive constant $\beta = \beta(K) > 0$ such that for each $n \geq 1$

$$\frac{u_n}{a_n} \geq \beta \quad \text{in} \quad K. \tag{4.41}$$

Proof. We first prove the existence of $e_0(a)$ with property as stated in inequality (4.38). Since $c \in C(\bar{\Omega}), c \geq 0, c \not\equiv 0$, there exists a ball B with $\bar{B} \subset \Omega$ such that

$$\tilde{c}_L := min_{\bar{B}} c > 0.$$

On the other hand, by Theorem 3.4 in [45] or Theorem A3-4 in Chapter 6

$$lim_{e\to\infty}\frac{\theta_{[-L_2,e,g]}}{e} = g^{-1} \text{ uniformly in } \bar{B};$$

and hence, there exists $\hat{e}$ such that for $e > \hat{e}$, we have

$$\theta_{[-L_2,e,g]} > \frac{e}{2\,max_{\bar{B}}g} \quad \text{in } \bar{B}.$$

Consequently, by comparison of principal eigenvalues (Theorem 2.3 in [45] or Theorem A3-5 in Chapter 6), we obtain

$$\rho_1(-L_1 - c\theta_{[-L_2,e,g]}) < \rho_1^B(-L_1 - c\theta_{[-L_1,e,g]}) < \rho_1^B(-L_1) - \frac{c_L}{2\,max_{\bar{B}}g}e$$

for each $e > \hat{e}$. Thus, for a fixed $a < \rho_1(-L_1)$, there must exists $e_0(a) > \rho_1(-L_2)$ such that inequality (4.38) is satisfied.

Let $(e_n, u_n, v_n), n \geq 1$, be a sequence of positive solutions of (4.22) with $e_n > max\{e_0(a), 0\}$ and $lim_{n\to\infty}e_n = \infty$. Then, from the second equation of (4.22), we find

$$-L_2 v_n = e_n v_n - g v_n^2 + f u_n v_n \geq e_n v_n - g v_n^2 \text{ in } \Omega,$$

with $f \not\equiv 0$; thus v_n is a strict positive upper solution of

$$-L_2 w = e_n w - g w^2 \text{ in } \Omega, \;\; w = 0 \text{ on } \partial\Omega.$$

By Lemma 3.2 in Delgado, López-Gómez and Suarez [45] (cf. Theorem A3-3 in Chapter 6), we find

$$v_n \geq \theta_{[-L_2,e_n,g]}. \tag{4.42}$$

Substituting (4.42) into the first equation of (4.22) and repeating the previous arguments, we obtain

$$u_n \geq \theta_{[-L_1 - c(x)\theta_{[-L_2,e_n,g]},a,b(x)]}. \tag{4.43}$$

Note that the function on the right of the above inequality is well defined and positive because of (4.38). From (4.43) we find

$$lim\,inf_{n\to\infty}\frac{u_n}{e_n} \geq lim\,inf_{n\to\infty}\frac{\theta_{[-L_1-c(x)\theta_{[-L_2,e_n,g]},a,b(x)]}}{e_n}. \tag{4.44}$$

We now show that

$$lim\,inf_{n\to\infty}\frac{\theta_{[-L_1-c(x)\theta_{[-L_2,e_n,g]},a,b(x)]}}{e_n} \geq \frac{c_L}{b_M g_M} \tag{4.45}$$

uniformly in compact subsets of Ω. Let Ω_1, Ω_2 be two arbitrary subdomains of Ω such that

$$\bar{\Omega}_1 \subset \Omega_2, \;\; \bar{\Omega}_2 \subset \Omega.$$

Define

$$\Theta_n := \frac{\theta_{[-L_1 - c(x)\theta_{[-L_2, e_n, g]}, a, b(x)]}}{e_n},$$

which is the unique positive solution of

$$(4.46) \qquad -\frac{1}{e_n} L_1 w = (\frac{a}{e_n} + c\frac{\theta_{[-L_2, e_n, g]}}{e_n})w - bw^2 \quad \text{in } \Omega, \quad w = 0 \text{ on } \partial\Omega.$$

By Theorem 3.4 in [45] or Theorem A3-4 in Chapter 6,

$$lim_{n\to\infty} \frac{\theta_{[-L_2, e_n, g]}}{e_n} = g^{-1} \quad \text{uniformly in } \bar{\Omega}_2.$$

Thus, for any $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ such that for each $n \geq n_0$ we have

$$(4.47) \qquad \frac{a}{e_n} + c\frac{\theta_{[-L_2, e_n, g]}}{e_n} \geq \frac{c_L}{g_M} - \epsilon \;\text{ in } \Omega_2.$$

Since Θ_n is the unique positive solution of (4.46), it follows from (4.47) that for each $n \geq n_0$, the function Θ_n is a strict positive upper solution of the problem:

$$(4.48) \qquad -\frac{1}{e_n} L_1 w = (\frac{c_L}{g_M} - \epsilon)w - bw^2 \quad \text{in } \Omega_2, \quad w = 0 \text{ on } \partial\Omega_2.$$

Suppose that $\epsilon > 0$ is sufficiently small such that $\frac{c_L}{g_M} - \epsilon > 0$, then for n sufficiently large, we have

$$\frac{c_L}{g_M} - \epsilon > \rho_1^{\Omega_2}(\frac{-1}{e_n} L_1) = \frac{\rho_1^{\Omega_2}(-L_1)}{e_n} \to 0 \;\text{ as }\; n \to \infty.$$

Consequently (4.48) has a unique positive solution, say $\Theta_n^{\Omega_2}$; and by comparison we have

$$\Theta_n \geq \Theta_n^{\Omega_2} \quad \text{in } \;\Omega_2$$

for all n sufficiently large. Moreover, from (4.48) we obtain from Theorem 3.4 in [45] or Theorem A3-4 in Chapter 6 that

$$lim_{n\to\infty} \Theta_n^{\Omega_2} = \frac{c_L}{b g_M} - \frac{\epsilon}{b} \;\text{ uniformly in } \;\Omega_1.$$

Thus,

$$lim\, inf_{n\to\infty} \Theta_n \geq \frac{c_L}{b_M g_M} - \frac{\epsilon}{b_L} \;\text{ uniformly in } \;\Omega_1.$$

Since the above is valid for any $\epsilon > 0$, we obtain (4.45) uniformly in any compact subset of Ω. We then obtain from (4.44) that

$$lim\,inf_{n\to\infty}\frac{u_n}{e_n} \geq \frac{c_L}{b_M g_M} \tag{4.49}$$

uniformly in any compact subset of Ω, and in particular in $\bar{\Omega}_1$. We next define

$$\hat{u}_n := \frac{u_n}{e_n}, \quad \hat{v}_n := \frac{v_n}{e_n},$$

and obtain from the second equation of (4.22) that

$$\frac{-1}{e_n}L_2\hat{v}_n = \hat{v}_n - g\hat{v}_n^2 + f\hat{u}_n\hat{v}_n.$$

Consequently, from (4.49) we see that for any $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ such that $\hat{v}_n$ is a strict positive upper solution of the problem

$$\frac{-1}{e_n}L_2 w = (1 + \frac{f_L c_L}{b_M g_M} - \epsilon)w - gw^2 \text{ in } \Omega_1, \quad w = 0 \text{ on } \partial\Omega_1 \tag{4.50}$$

for each $n \geq n_0$. Choose $\epsilon > 0$ sufficiently small so that $1 + \frac{f_L c_L}{b_M g_M} - \epsilon > 0$. then we see that for n sufficiently large, problem (4.50) has a unique positive solution, which we denote by $\hat{\Theta}_n^{\Omega_1}$. Moreover, by Lemma 3.2 in [45] or Theorem A3-3 in Chapter 6, we find

$$\hat{v}_n = \frac{v_n}{e_n} \geq \hat{\Theta}_n^{\Omega_1}, \tag{4.51}$$

for sufficiently large n.

Let K be an arbitrary compact subset of Ω, we choose subdomains $\Omega_1 \subset \Omega_2$ as described above, and $K \subset \Omega_1$. Then by Theorem 3.4 in [45] or Theorem A3-4 in Chapter 6,

$$\lim_{n\to\infty}\hat{\Theta}_n^{\Omega_1} = (1 + \frac{f_L c_L}{b_M g_M} - \epsilon)g^{-1} \text{ uniformly in } K.$$

Since the limit above is bounded away from zero in K, we obtain (4.39) as described in part (i). Part (ii) is proved similarly.

Proof of Theorem 4.5.

Assume (4.29), (4.30) and fix $a < \rho_1(-L)$. Suppose there exists a sequence of positive coexistence solutions of (4.22), $(e_n, u_n, v_n), n \geq 1$, such that $e_n > max\{e_0(a), 0\}$ and $lim_{n\to\infty}e_n = \infty$. Let $\Omega_1 \subset \Omega$ be an arbitrary subdomain of Ω with $\bar{\Omega}_1 \subset \Omega$. By Lemma 4.2, there exists $\alpha = \alpha(\Omega_1) > 0$ such that for each $n \geq 1$

$$\frac{v_n}{e_n} \geq \alpha \text{ in } \Omega_1.$$

Moreover, by Lemma 4.1, we have for each $n \geq 1$

$$\frac{u_n}{e_n} \leq \frac{c_M + g_M}{f_L + b_L} \frac{v_n}{e_n}.$$

Thus by (4.30), there exists $\epsilon > 0$ such that for each $n \geq 1$

$$\frac{u_n}{e_n} \leq \frac{c_L v_n}{b_M e_n} - \epsilon \text{ in } \Omega_1.$$

That is, for all $n \geq 1$, we have

$$b_M u_n - c_L v_n \leq -\epsilon b_M e_n \text{ in } \Omega_1. \tag{4.52}$$

On the other hand, we find from the first equation in (4.22) that

$$a = \rho_1(-L + bu_n - cv_n) \leq \rho_1^{\Omega_1}(-L + b_M u_n - c_L v_n).$$

Consequently, we find from (4.52) that

$$a \leq \rho_1^{\Omega_1}(-L) - \epsilon b_M e_n \to -\infty \text{ as } n \to \infty.$$

This contradiction shows that problem (4.22) cannot have any positive coexistence state for e large enough. This completes the proof of Theorem 4.5.

The following theorem concerning a-priori uniform bound for positive solutions of (4.22) will lead to sufficient conditions for coexistence state in the case of large cooperative coefficients.

Theorem 4.6. *Assume (4.29) for problem (4.22) and $N \leq 5$. Suppose that*

$$c_L f_L > b_M g_M,$$

and for some $\alpha > 0$

$$max\{|a|, |e|\} \leq \alpha;$$

then there exists a constant $C = C(\alpha, \Omega, b, c, f, g)$ such that

$$||u||_{L^\infty(\Omega)} \leq C, \quad ||v||_{L^\infty(\Omega)} \leq C,$$

for any positive coexistence solution (u, v) of problem (4.22).

Proof. We shall prove this theorem under the condition $a \geq e$. By symmetry, the result can be proved similarly if $e \geq a$. Suppose that the conclusion of the theorem is false, and there exists a sequence of positive coexistence solutions $(a_k, e_k, u_k, v_k), k \geq 1$ with $-\alpha \leq e_k \leq a_k \leq \alpha$, such that

$$lim\, sup_{k\to\infty}(||u_k||_{L^\infty(\Omega)} + ||v_k||_{L^\infty(\Omega)}) = \infty. \tag{4.53}$$

We claim that

(4.54) $$lim\, sup_{k\to\infty}||u_k||_{L^\infty(\Omega)} = \lim\, sup_{k\to\infty}||v_k||_{L^\infty(\Omega)} = \infty.$$

Otherwise, if $\{||v_k||_{L^\infty(\Omega)}\}_{k\geq 1}$ is bounded by a positive constant β, then the first equation in (4.22) leads to

$$-Lu_k \leq (\alpha + c_M\beta)u_k - bu_k^2 \ \text{ in } \ \Omega;$$

and by comparison, we deduce that $\{||u_k||_{L^\infty(\Omega)}\}_{k\geq 1}$ is also bounded. However, this contradicts (4.53). Similarly, if $\{||u_k||_{L^\infty(\Omega)}\}_{k\geq 1}$ is bounded, then $\{||v_k||_{L^\infty(\Omega)}\}_{k\geq 1}$ is also bounded. Consequently, (4.54) must hold. By choosing a subsequence, if necessary, we may assume that

(4.55) $$lim_{k\to\infty}||u_k||_{L^\infty(\Omega)} = \infty, \quad lim_{k\to\infty}(a_k, e_k) = (a_\infty, e_\infty),$$

for some $(a_\infty, e_\infty) \in R^2$ satisfying $-\alpha \leq e_\infty \leq a_\infty \leq \alpha$. From Lemma 4.1(ii), we obtain

(4.56) $$v_k \leq \frac{f_M + b_M}{c_L + g_L} u_k, \quad \text{for all } \ k \geq 1.$$

For each $k \geq 1$, let $x_k \in \Omega$ be such that

(4.57) $$u(x_k) = M_k := ||u_k||_{L^\infty(\Omega)}.$$

Since Ω is bounded, we may assume without loss of generality that

(4.58) $$lim_{k\to\infty}x_k = x_\infty \in \bar{\Omega}.$$

We now consider the two different cases where (i) $x_\infty \in \Omega$, or (ii) $x_\infty \in \partial\Omega$.

For case (i), denote

$$\delta := d(x_\infty, \partial\Omega)/2 > 0, \ \ \epsilon_k := M_k^{-1/2}, \ k \geq 1.$$

Since $lim_{k\to\infty}M_k = \infty$, we have $lim_{k\to\infty}\epsilon_k = 0$. The change of variables

(4.59) $$y := \frac{x - x_k}{\epsilon_k}, \ \ (z_k, w_k) := \epsilon_k^2(u_k, v_k), \ \ k \geq 1,$$

transforms the system (4.22) into

(4.60) $$\begin{cases} \mathcal{A}_k z_k = \epsilon_k^2 a_k z_k - b(x_k + \epsilon_k y)z_k^2 + c(x_k + \epsilon_k y)z_k w_k, \\ \mathcal{A}_k w_k = \epsilon_k^2 e_k w_k - g(x_k + \epsilon_k y)z_k^2 + f(x_k + \epsilon_k y)z_k w_k, \end{cases}$$

where

(4.61) $$\mathcal{A}_k = -\Sigma_{i,j=1}^N a_{ij1}(x_k + \epsilon_k y)\partial_i\partial_j - \epsilon_k\Sigma_{j=1}^N b_{j1}(x_k + \epsilon_k y)\partial_j + \epsilon_k^2 c_1(x_k + \epsilon_k y),$$

provided $x_k + \epsilon_k y \in \Omega$. If $y \in R^N$ satisfies $|y| \leq \frac{\delta}{\epsilon_k}$, then $x = x_k + \epsilon_k y \in \Omega$, and thus (4.60) holds. For any $\rho > 0$, let B_ρ be the ball of radius ρ centered at the origin, we have $B_\rho \subset B_{\delta/\epsilon_k}$ for k sufficiently large, since $lim_{k\to\infty}\epsilon_k = 0$. From definition (4.59) we have $z_k = u_k/M_k$, thus

(4.62) $$||z_k||_{L^\infty(B_\rho)} = 1, \;\; z_k(0) = 1, \;\; \text{for all} \;\; k \geq 1.$$

Moreover, from (4.56) and (4.62), we have

(4.63) $$||w_k||_{L^\infty(B_\rho)} \leq \frac{f_M + b_M}{c_L + g_L}, \;\; \text{for all} \;\; k \geq 1.$$

Using compactness argument as in Section 5.1 or Section A.3 in [125], we can choose subsequence, again labeled by k, such that there exists non-negative functions $z, w \in W^{2,p}(B_\rho) \cap C^{1,\nu}(B_\rho), 0 < \nu < 1$, $p > N$ sufficiently large, with

$$lim_{k\to}(z_k, w_k) = (z, w) \;\; \text{in} \;\; (W^{2,p}(B_\rho) \cap C^{1,\nu}(B_\rho))^2.$$

We thus have $z(0) = 1$, and passing to limit as $k \to \infty$ in (4.60), we find (z, w) satisfies:

(4.64) $$\begin{cases} -\Sigma_{i,j=1}^N a_{ij1}(x_\infty)\partial_i\partial_j z = -b(x_\infty)z^2 + c(x_\infty)zw, \\ \\ -\Sigma_{i,j=1}^N a_{ij1}(x_\infty)\partial_i\partial_j w = -g(x_\infty)z^2 + f(x_\infty)zw. \end{cases} \quad \text{in} \;\; B_\rho.$$

Since ρ is arbitrary, by a standard diagonal sequence argument we can assert that $z, w \in W^{2,p}_{loc}(R^N)$ and (4.64) holds in the whole R^N. Moreover, standard elliptic regularity theory implies that $z, w \in C^2(R^N)$. Furthermore, by a linear change of coordinates, (4.64) can be reduced to

(4.65) $$\begin{cases} -\Delta z = -b(x_\infty)z^2 + c(x_\infty)zw \\ \\ -\Delta w = -g(x_\infty)w^2 + f(x_\infty)zw \end{cases} \quad \text{in} \;\; R^N.$$

From (4.65), we obtain

(4.66) $$(-\Delta + b(x_\infty)z + g(x_\infty)w)(w - \frac{f(x_\infty) + b(x_\infty)}{c(x_\infty) + g(x_\infty)}z) = 0 \;\; \text{in} \;\; R^N.$$

Since the functions z, w are non-negative and $z(0) = 1$, the potential coefficient $V := b(x_\infty)z + g(x_\infty)w$ of the above equation has the property:

$$V \geq 0, \;\; V \not\equiv 0 \;\; \text{in} \;\; R^N.$$

By a Liouville type Theorem (see Lemma 7.5 in [45] or Theorem A3-6 in Chapter 6), the bounded solution in R^N of (4.66) must satisfy:

(4.67) $$w - \frac{f(x_\infty) + b(x_\infty)}{c(x_\infty) + g(x_\infty)}z = 0 \quad \text{in} \;\; R^N.$$

Using the relation (4.67), the first equation in (4.65) becomes

$$-\Delta z = \frac{c(x_\infty)f(x_\infty) - b(x_\infty)g(x_\infty)}{c(x_\infty) + g(x_\infty)} z^2 \quad \text{in } R^N. \tag{4.68}$$

Since $c_L f_L > b_M g_M$, we have $c(x_\infty)f(x_\infty) > b(x_\infty)g(x_\infty)$. By Theorem 1.1 in Gidas and Spruck [70] or Theorem A3-7 in Chapter 6, the non-negative solution of the above equation must satisfy $z = 0$ in R^N, because $N \leq 5$. This contradicts the fact that $z(0) = 1$; therefore we must have case (ii) with $x_\infty \in \partial\Omega$.

For case (ii), we use the same argument as in the second part of the proof of Theorem 1.1 in [70] or Theorem A3-7 in Chapter 6 to show that the problem:

$$\begin{cases} -\Delta z = -b(x_\infty)z^2 + c(x_\infty)zw \\ \\ -\Delta w = -g(x_\infty)w^2 + f(x_\infty)zw \end{cases} \quad \text{in } R^N_+, \tag{4.69}$$

where $R^N_+ = \{x \in R^N : x_N \geq 0\}$, has a non-negative solution (z, w) with $z(0) = 1$. Then, using the same argument as above with R^N replaced with R^N_+, we arrive again at a contradiction. We thus conclude that there must exist a-prior bound for the positive coexistence solution of (4.22) as described in the statement of this Theorem.

Theorem 4.7 (Positive Solution under Strong Cooperation). *Consider problem (4.22) with $L_1 = L_2$ and $N \leq 5$.*
(i) Suppose

$$\begin{cases} c_L f_L - b_M g_M > b_M c_M - b_L c_L, \ \textit{and} \\ \\ \hat{\rho}_1(L_1 + a + c(x)\theta_{[-L_2,e,g(x)]}) < 0, \ \textit{i.e. } a < \rho_1(-L_1 - c(x)\theta_{[-L_2,e,g(x)]}); \end{cases} \tag{4.70}$$

then the boundary value problem (4.22) has a solution with each component strictly positive in Ω.
(ii) Suppose

$$\begin{cases} c_L f_L - b_M g_M > g_M f_M - g_L f_L; \ \textit{and} \\ \\ \hat{\rho}_1(L_2 + e + f(x)\theta_{[-L_1,a,b(x)]}) < 0, \ \textit{i.e. } e < \rho_1(-L_2 - f(x)\theta_{[-L_1,a,b(x)]}); \end{cases} \tag{4.71}$$

then the boundary value problem (4.22) has a solution with each component strictly positive in Ω.

(Outline of Proof.) Let $G(e) := \rho_1(-L_1 - c\theta_{[-L,e,g]})$, $G(e)$ is a decreasing function of e. For a fixed $a < \rho_1(-L_1)$, we find from Theorem 4.5 above that there exists a number $e(a)$ such that if $a > \rho_1(-L_1 - c(x)\theta_{[-L_2,e(a),g(x)]}) = G(e(a))$, then there is no coexistence state, and if $e > e(a)$ there is no coexistence. Since

$\theta_{[-L_2,e,g(x)]} = 0$ if $e = 0$ and $a < \rho_1(-L_1 - c(x)\theta_{[-L_2,e,g(x)]})$ when $e = 0$, there is a number e_a such that $a = G(e_a)$. Theorem 4.6 above gives uniform bound for all solutions under first inequality in (4.70). Hence with e as the bifurcation parameter, the branch of unbounded curve of solutions has to connect e to minus infinity. That is bifurcating (e, u, v) at $(e_a, 0, \theta_{[-L_2,e_a,g(x)]})$, the branch of positive solutions connects e from e_a to minus infinity. However, if $e < e_a$, then $G(e) > G(e_a) = a$. This means the second inequality in (4.70) is satisfied. This proves part (i). The second part is proved in the same way by symmetry. (See Fig. 1.4.1 and Theorem 4.9 below for clarification, we allow both e and a to be $\leq \rho_1(-L_1)$ simultaneously.)

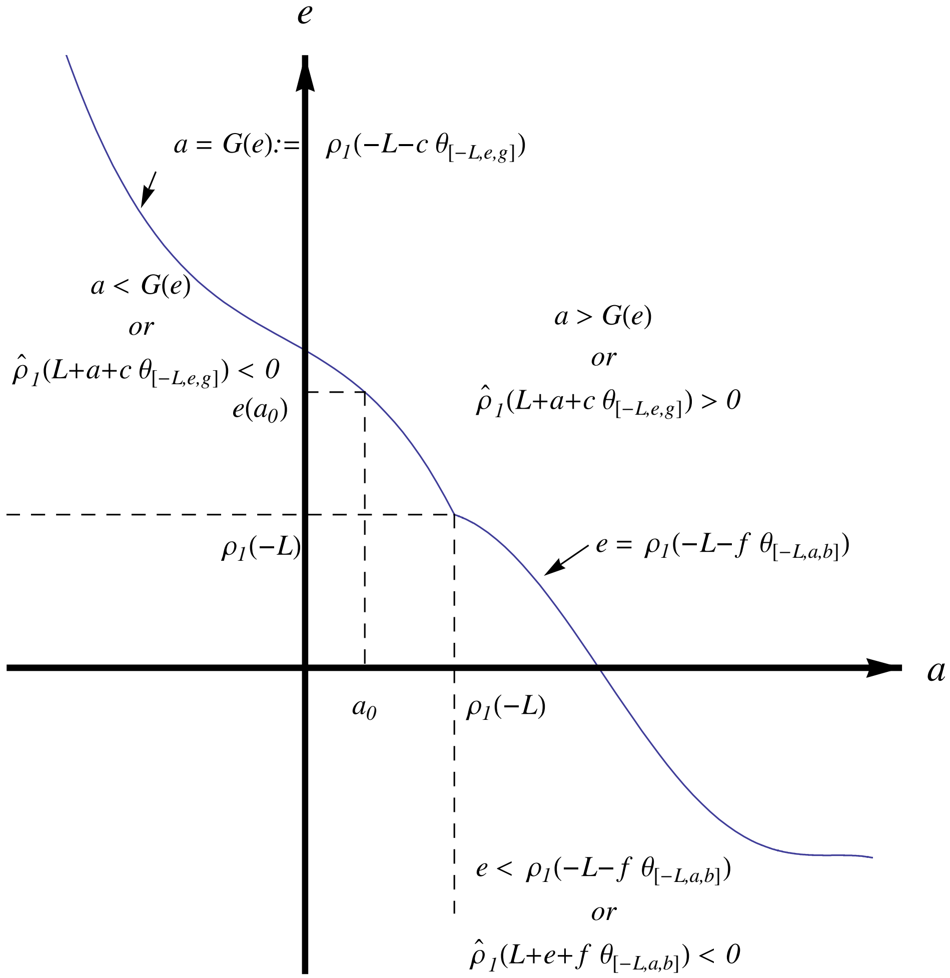

Figure 1.4.1: (For large c, f) Curves bounding regions of coexistence on (a, e) plane when b, c, f, g are fixed, and $L = L_1 = L_2$.

Remark 4.2. Roughly speaking, suppose

$$c_L f_L - b_M g_M > max.\{b_M c_M - b_L c_L, g_M f_M - g_L f_L\} \tag{4.72}$$

and some of the semitrivial positive solution is "linearly stable" (with index 1), then there exists positive coexistence state to (4.22).

The following theorems describe more carefully the set of parameters a, e when one or more coexistence state may occur. For fixed a, we define interval for the parameter e so that there exist coexistence solution(s) by I_2^a. For fixed e, define interval for parameter a so that there exist coexistence solution(s) by I_1^e. The following theorem first considers the case when the cooperative coefficients are relatively small.

Theorem 4.8. *Assume the first inequality in (4.23) for problem (4.22), and let* I_1^e, I_2^a *be defined as above.*

(i) Suppose $e > \rho_1(-L_2)$*, then either* $I_1^e = (\rho_1(-L_1 - c(x)\theta_{[-L_2,e,g(x)]}, \infty)$ *or there exists* $a_* \leq \rho_1(-L_1 - c(x)\theta_{[-L_2,e,g(x)]})$ *such that* $I_1^e = [a_*, \infty)$*. If* $a_* < \rho_1(-L_1 - c(x)\theta_{[-L_2,e,g(x)]})$*, then there exists at least two positive coexistence states for* $a \in (a_*, \rho_1(-L_1 - c(x)\theta_{[-L_2,e,g(x)]}))$*.*

(ii) Suppose $a > \rho_1(-L_1)$*, then either* $I_2^a = (\rho_1(-L_2 - f(x)\theta_{[-L_1,a,b(x)]}, \infty)$ *or there exists* $e_* \leq \rho_1(-L_2 - f(x)\theta_{[-L_1,a,b(x)]})$ *such that* $I_2^a = [e_*, \infty)$*. If* $e_* < \rho_1(-L_2 - f(x)\theta_{[-L_1,a,b(x)]})$*, then there exists at least two positive coexistence states for* $e \in (e_*, \rho_1(-L_2 - f(x)\theta_{[-L_1,a,b(x)]}))$*.*

Remark 4.3. The details of the proof of the above theorem can be found in Theorems 8.8 and 8.14 in Delgado, López-Gómez and Suarez [45]. The idea of the proof of part (i) is as follows. In case I_1^e is larger than $(\rho_1(-L_1 - c(x)\theta_{[-L_2,e,g(x)]}, \infty)$, then there exists a coexistence state (u_*, v_*) when $a = a_*$. Moreover for such e there will be a maximal coexistence state (u^e, v^e) satisfying $u_* \leq u^e \leq K_1, v_* \leq v^e \leq K_2$ for some large constants K_1, K_2. Using degree theory method as in the last chapter, it can be shown that the index of this maximal coexistence solution is 1. In order to satisfy the homotopy invariance of degree in an appropriate set of positive functions, there must be at least one more positive coexistence solution.

Similar multiplicity results can also be obtained in the case for large cooperative coefficients. In this case, we use homotopy invariance and show that the index of an appropriate minimal coexistence state is 1 to conclude that there must be another positive coexistence solution.

Theorem 4.9. *Consider problem (4.22) with* $L_1 = L_2 = L$ *and* $N \leq 5$*.*

(i) Assume the first inequality in (4.70) and $a < \rho_1(-L)$*. Then either* $I_2^a = (-\infty, e_a)$ *or* $I_2^a = (-\infty, e^*]$ *for some* $e^* \geq e_a$ *where* e_a *is the unique value of*

e satisfying $a = \rho_1(-L - c(x)\theta_{[-L,e_a,g(x)]})$. If $I_2^a = (-\infty, e^*]$ with $e^* > e_a$, then problem (4.22) has at least two coexistence state for each $e \in (e_a, e^*)$.
(ii) Assume the first inequality in (4.71) and $e < \rho_1(-L)$. Then either $I_1^e = (-\infty, a_e)$ or $I_1^e = (-\infty, a^*]$ for some $a^* \geq a_e$ where a_e is the unique value of a satisfying $e = \rho_1(-L - f(x)\theta_{[-L,a_e,b(x)]})$. If $I_1^e = (-\infty, a^*]$ with $a^* > a_e$, then problem (4.22) has at least two coexistence state for each $a \in (a_e, a^*)$.

The details of the proof of the theorem above can be found in Theorems 8.9 and 8.10 in [45].

Notes.

Theorem 4.1 is due to Korman and Leung [107]. Theorems 4.2 and 4.3 are found in Li and Ghoreishi [149]. Theorems 4.4 to 4.9 are obtained from Delgado, López-Gómez and Suarez [45].

1.5 Stability of Steady-States as Time Changes

In this section, we discuss the stabilities of the steady states found in the previous sections. Here, stability can be interpreted slightly differently in various cases. We might prove directly that certain smooth solutions of the corresponding parabolic problem stay close and tend to the steady state. Sometimes, only linearized stabilities are considered, and the steady states are stable or unstable with respect to solutions of the corresponding parabolic problem in appropriate functions spaces by means of applying standard stability theorems. In case that the linearized problem has zero as its eigenvalue, more sophisticated theorem will be applied. We will call nontrivial non-negative steady-state solutions with one component identically zero semi-trivial solutions.

Part A: Prey-Predator Case.

We first consider the prey-predator case discussed in Section 1.2. Before discussing the stability of the coexistence states, we note a very remarkble necessary and sufficient condition relating the existence of positive coexistence state and the linearized stability of the trivial and semi-trivial non-negative solutions.

Theorem 5.1. *Consider problem (2.23) under hypotheses (2.24) to (2.27) and additionally:*

$$h(0) - m(0) \neq \lambda_1 d, \text{ and } M_u(u, v) \leq 0 \text{ if } u, v \geq 0.$$

Then problem (2.23) has a positive solution iff the point spectrum of the linearized system at each of its trivial and semi-trivial non-negative solutions contains a positive number.

Proof. We first prove necessity, and assume (2.23) has a positive solution $(\bar{u}, \bar{v})$. The possible trivial or semi-trivial non-negative solutions are $(0,0), (u_0, 0), (0, v_0)$. We thus have to consider the linearization of the operator:

$$F : \begin{bmatrix} u \\ v \end{bmatrix} \to \begin{bmatrix} \Delta u + uM(u,v) \\ d\Delta v + v(h(u) - m(v)) \end{bmatrix} \tag{5.1}$$

at these three solutions. By comparison, we have $u_0 \geq \bar{u}$; thus by Lemma 2.1, we have $M(0,0) > \lambda_1$. That is, $\hat{\rho}_1(\Delta + M(0,0)I) > 0$. The Fréchet derivative $F'(0,0)$ is given by:

$$F'(0,0) = \begin{bmatrix} \Delta + M(0,0)I & 0 \\ 0 & d\Delta + (h(0) - m(0))I \end{bmatrix}. \tag{5.2}$$

Hence, the spectrum of $F'(0,0)$ contains a positive real number. We next consider the Fréchet derivative $F'(u_0, 0)$:

$$F'(u_0,0)\begin{pmatrix} w \\ z \end{pmatrix} = \begin{bmatrix} \Delta w + (M(u_0,0) + u_0 M_u(u_0,0))w + u_0 M_v(u_0,0)z \\ d\Delta z + (h(u_0) - m(0))z \end{bmatrix}. \tag{5.3}$$

We see that $F'(u_0, 0)$ has only pure point spectrum σ_p given by $\sigma_p = \{\xi_1, \xi_2, \dots\} \cup \{\theta_1, \theta_2, \dots\}$ where $\{\xi_1, \xi_2, \dots\}$ is the point spectrum of the operator $\Delta + (M(u_0, 0) + u_0 M_u(u_0, 0))$, while $\{\theta_1, \theta_2, \dots\}$ is the point spectrum of the operator $d\Delta + (h(u_0) - m(0))$. By Theorem 2.5 (ii), (iii) we have $\theta_1 = \hat{\rho}_1(d\Delta + (h(u_0) - m(0)) > 0$. This means σ_p contains a positive number.

In case the solution $(0, v_0)$ of (2.23) exists with v_0 nontrivial, then Lemma 2.1 implies $h(0) > \lambda_1 d + m(0)$. We apply Theorem 2.5(iii) to assert $\hat{\rho}_1(\Delta + M(0, v_0)) > 0$. We then consider the Fréchet derivative: $F'(0, v_0)$

$$F'(0,v_0)\begin{pmatrix} w \\ z \end{pmatrix} = \begin{bmatrix} \Delta w + M(0,v_0)w \\ v_0 h'(0)w + d\Delta z + (h(0) - m(v_0) - v_0 m'(v_0))z \end{bmatrix}. \tag{5.4}$$

As in the above case, we deduce that the spectrum of $F'(0, v_0)$ contains a positive number.

We next prove the sufficiency part of this Theorem, and assume the point spectrum of the linearized system at each of the trivial and semi-trivial solutions contains a positive number. First, consider the point $(0,0)$. From the representation (5.2) for $F'(0,0)$, we must have either $\hat{\rho}_1(\Delta + M(0,0)I) > 0$ or $\hat{\rho}_1(d\Delta + (h(0) - m(0))I) > 0$. There are thus three possible cases (a) $M(0,0) > \lambda_1$, $h(0) < \lambda_1 d + m(0)$; (b) $M(0,0) > \lambda_1$, $h(0) > \lambda_1 d + m(0)$; or (c) $M(0,0) \leq \lambda_1$, $h(0) > \lambda_1 d + m(0)$.

We first consider case (a). Since $M(0,0) > 0$, we have a solution $(u_0, 0)$ with u_0 nontrivial. Consider the linearization of the operator $u :\to \Delta u + uM(u,0)$ at u_0. The principal eigenvalue for the operator $\Delta + M(u_0, 0)$ is zero. By comparison, the principal eigenvalue ξ_1 of the corresponding linear operator $\Delta + [M(u_0, 0) + u_0 M_u(u_0, 0)]$ must have $\xi_1 < 0$. From (5.3), the spectrum σ_p of $F'(u_0, 0)$ satisfies $\sigma_p = \{\xi_1, \xi_2, \dots\} \cup \{\theta_1, \theta_2, \dots\}$ where $\{\xi_1, \xi_2, \dots\}$ is the point spectrum of the operator $\Delta + [M(u_0, 0) + u_0 M_u(u_0, 0)]$, while $\{\theta_1, \theta_2, \dots\}$ is the point spectrum of the operator $d\Delta + (h(u_0) - m(0))$. Thus $\xi_1 < 0$ implies that the principal eigenvalue θ_1 of the operator $d\Delta + (h(u_0) - m(0))$ must be > 0. From Theorem 2.5(ii), we conclude that problem (2.23) has a positive solution.

We next consider case (b). Since $h(0) > \lambda_1 d + m(0)$, we have a solution $(0, v_0)$ with v_0 nontrivial. Consider the linearization of the operator $v :\rightarrow d\Delta v + v(h(0) - m(v))$ at v_0. By comparison, the principal eigenvalue $\tilde{\xi}_1$ of the corresponding linear operator $d\Delta + [h(0) - m(v_0) - v_0 m'(v_0)]$ must have $\tilde{\theta}_1 \leq 0$. From (5.4), the spectrum σ_p of $F'(0, v_0)$ satisfies $\sigma_p = \{\tilde{\xi}_1, \tilde{\xi}_2, \dots\} \cup \{\tilde{\theta}_1, \tilde{\theta}_2, \dots\}$ where $\{\tilde{\xi}_1, \tilde{\xi}_2, \dots\}$ is the point spectrum of the operator $\Delta + M(0, v_0)I$, while $\{\tilde{\theta}_1, \tilde{\theta}_2, \dots\}$ is the point spectrum of the operator $d\Delta + [h(0) - m(v_0) - v_0 m'(v_0)]$. Thus $\tilde{\theta}_1 \leq 0$ implies that the principal eigenvalue $\tilde{\xi}_1$ of the operator $\Delta + M(0, v_0)I$ must be > 0. From Theorem 2.5(iii), we conclude that problem (2.23) has a positive solution.

Finally, we now show that case (c) cannot occur. Since $h(0) > \lambda_1 d + m(0)$, we have a solution $(0, v_0)$ with v_0 nontrivial. As in the last paragraph, the principal eigenvalue $\tilde{\xi}_1$ of the linear operator $\Delta + [h(0) - m(v_0) - v_0 m'(v_0)]$ must have $\tilde{\theta}_1 \leq 0$, and the principal eigenvalue $\tilde{\xi}_1$ of the operator $\Delta + M(0, v_0)I$ must be > 0. However, by assumption $M(0, 0) \geq M(0, v_0)$; thus $\hat{\rho}_1(\Delta + M(0, 0)) > 0$. This contradicts the condition $M(0, 0) \leq \lambda_1$ of case (c).

This completes the proof of Theorem 5.1

The problem of uniqueness and stability of positive solutions of (2.23) is usually quite difficult. We now consider the uniqueness and stability of positive solution for a special case of (2.23) when the diffusion parameters are small. This is a singular perturbation problem. The result can also be used to study the situation when the space domain is large. We will see that the effect of the boundary condition will become less significant. More precisely, consider

(5.5)
$$\begin{cases} \epsilon\Delta u + u(a - bu - cv) = 0 & \\ & \text{in } \Omega, \\ \epsilon\Delta v + d^{-1} v(h(u) - m(v)) = 0 & \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

(5.6) The functions h and m belong to $C^1(R)$, with $h' > 0$ and $m' \geq 0$; a, b, c, d and ϵ are positive constants.

For convenience, we denote

(5.7)
$$F(u, v) = (u(a - bu - cv), d^{-1} v(h(u) - m(v)),$$

$X = L^p(\Omega) \times L^p(\Omega),\ p > 1,\ A = diag(\Delta, \Delta)$ is an operator on X.

Theorem 5.2 (Uniqueness near Constant Equilibrium). *Assume that the equation $F(u, v) = 0$ has an isolated root $w_0 = (\hat{u}, \hat{v})$ in the first open quadrant in R^2, and there exists a constant B_0 such that $m(B_0) > h(a/b)$. Then the problem (5.5) has a unique solution w_ϵ in a neighborhood $N(w_0)$ of the constant function w_0 in X for sufficiently small $\epsilon > 0$. Moreover, $||w_\epsilon - w_0||_X \to 0$ as $\epsilon \to 0^+$.*

Proof. From the proof of Theorem 2.5. we have an a-priori bound on the values of all positive solutions of (5.5), independent of $\epsilon > 0$. We can modify the function $F(u, v)$ for large $|u| + |v|$, and for $u < 0$ or $v < 0$, without affecting the equilibrium positive solutions we are seeking. We may thus assume without loss of generality that $F(u, v)$ and all its first partial derivatives are bounded for all $(u, v) \in R^2$, and the first or second component of F is zero when $u \leq 0$ or $v \leq 0$ respectively. By comparison and sweeping principle argument for scalar equations, we can readily justify that the solutions we found will be positive solutions of the original problem.

Let p be a large positive number greater that $max\{2, dim\,\Omega\}$, and consider the operator A on $L^p(\Omega) \times L^p(\Omega)$:

$$A = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \tag{5.8}$$

with domain $D(A) = (W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \times (W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$. We may consider the functions F to be a mapping from $L^p(\Omega) \times L^p(\Omega)$ into $L^\infty(\Omega) \times L^\infty(\Omega)$, and thus into $L^r(\Omega) \times L^r(\Omega)$ for any $r \in [1, \infty)$. Due to the structure of F, we can assert that the operator is continuous from $L^p(\Omega) \times L^P(\Omega)$ into $L^r(\Omega) \times L^r(\Omega)$. (See Theorem 19.2 in Vainberg [222] or Theorem A4-1 in Chapter 6). Let F' be the Jacobian matrix of F, we can similarly obtain the mapping from $L^p(\Omega) \times L^p(\Omega)$ into the entries of F' in $L^r(\Omega)$ is continuous. Moreover, using Hölder's inequality $i.e. ||fg||_q \leq ||f||_p ||g||_r$ for $1/q = 1/p + 1/r$, and the argument in Section 20 in [222], we can show that F maps $L^p \times L^p$ into $L^q \times L^q$ with continuous Gateaux derivatives expressible by means of $F' \in M_r^2$ where M_r^2 denotes 2×2 matrices with entries in $L^r(\Omega)$. Since its Gateaux derivative F' is continuous, the map F is Fréchet differentiable as a mapping from $L^p(\Omega) \times L^p(\Omega)$ into $L^q(\Omega) \times L^q(\Omega)$ (cf. Theorem 3.3 in Vainberg [222] or Theorem A4-2 in Chapter 6). More precisely, we obtain

$$\begin{aligned} &F(w) - F(w_1) = F'(w_1)(w - w_1) + \tilde{\theta}(w), \\ &\text{with } F'(w_1) \in M_r^2,\ ||\tilde{\theta}(w)||_q = o(||w - w_1||_p), \end{aligned} \tag{5.9}$$

where w, w_1 are elements of $L^p(\Omega) \times L^p(\Omega)$. Note that r can be chosen arbitrarily large so that q can be made large and satisfies $q > max\{2,\ dim\,\Omega\}$. Moreover we have $q < p$ and $L^q \supset L^p$. The operator A defined in (5.8) can be extended from $L^p(\Omega) \times L^p(\Omega)$ into $L^q(\Omega) \times L^q(\Omega)$, with domain $D(A) = (W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \times (W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega))$. We will denote by A as operator in both $L^p(\Omega) \times L^p(\Omega)$ and $L^q(\Omega) \times L^q(\Omega)$ without causing confusion.

The function w_0 is a constant function in Ω, and thus $F'(w_0)$ is a constant matrix which commutes with the operator A. We have:

$$F'(w_0) = \begin{bmatrix} -b\hat{u} & -c\hat{u} \\ d^{-1}\hat{v}h'(\hat{u}) & -d^{-1}\hat{v}m'(\hat{v}) \end{bmatrix}. \tag{5.10}$$

Let μ_1, μ_2 be eigenvalues of the matrix $F'(w_0)$. Then we have

$$\mu_1 + \mu_2 = -b\hat{u} - d^{-1}\hat{v}m'(\hat{v}) < 0,$$

$$\mu_1 \cdot \mu_2 = bd^{-1}\hat{u}\hat{v}m'(\hat{v}) + cd^{-1}\hat{u}\hat{v}h'(\hat{u}) > 0.$$

This implies that $Re\, \mu_1 < 0$ and $Re\, \mu_2 < 0$. Thus we have the spectrum $\sigma(F'(w_0)) \subset \{z : z \in C,\ Re\, z \leq -\bar{c} < 0\}$ for some constant $\bar{c}$.

The C_0 semigroup $U(t)$ generated by the bounded operator $F'(w_0)$ satisfies $||U(t)|| \leq Me^{-\bar{c}t}$ for some constant $M > 0$. For each $\epsilon > 0$, the operator ϵA generate a C_0 semigroup T_ϵ with $||T_\epsilon|| \leq M_0$, where $M_0 \geq 1$. Since ϵA commutes with $F'(w_0)$, we have $T_\epsilon(t)U(t) = U(t)T_\epsilon(t)$, and $||(T_\epsilon(t)U(t))^n|| = ||(T_\epsilon(nt)U(nt))|| \leq M_0 M e^{-\bar{c}nt}$. Let $S_\epsilon(t)$ be the C_0 semigroup generated by $\epsilon A + F'(w_0)$, then the Trotter product formula (Corollary 5.5, in Chapter 3 of Pazy [184] or Theorem A4-6 in Chapter 6) yields for all $x \in L^q(\Omega) \times L^q(\Omega)$:

$$S_\epsilon(t)x = lim_{n\to\infty}(T_\epsilon(t/n)U(t/n))^n x = lim_{n\to\infty}(T_\epsilon(t)U(t))x. \tag{5.11}$$

We thus have

$$||S_\epsilon(t)|| \leq M_0 M e^{-\bar{c}t}, \tag{5.12}$$

which is independent of $\epsilon > 0$. We can thus assert that 0 is not an element of $\sigma(\epsilon A + F'(w_0))$, and the resolvent operator satisfies:

$$||(\epsilon A + F'(w_0))^{-1}|| \leq \frac{MM_0}{\bar{c}} \tag{5.13}$$

by using the general version of Hille-Yosida Theorem or Theorem A4-3 in Chapter 6.

For a small $\delta > 0$, let $N_\delta(w_0) = \{w \in L^p(\Omega) \times L^p(\Omega) : ||w - w_0||_p < \delta\}$ be the δ-neighborhood of w_0 in $L^p(\Omega) \times L^p(\Omega)$. We can consider problem (5.5) as finding a solution of

$$-\epsilon Aw = F(w) \tag{5.14}$$

in a neighborhood $N_\delta(w_0)$ of $X = L^p(\Omega) \times L^p(\Omega)$. Since $F(w_0) = 0$, we can set w_1 to be w_0 in (5.9) to rewrite (5.14) as

$$(5.15) \qquad (\epsilon A + F'(w_0))w = F'(w_0)w_0 + \theta(w),$$

where $||\theta(w)||_q = o(||w - w_0||_p)$ (Here, the function θ is determined by w_0). For $w \in X$, let

$$(5.16) \qquad Q_\epsilon(w) := (\epsilon A + F'(w_0)^{-1}[F'(w_0)w_0 + \theta(w)].$$

We now show that for $\epsilon > 0$ sufficiently small, Q_ϵ maps $N_\delta(w_0)$ into itself. Note that $\theta(w) \in L^q(\Omega) \times L^q(\Omega)$, and thus $\theta_1(w) := (\epsilon A + F'(w_0))^{-1}\theta(w) \in W^{2,q}(\Omega) \times W^{2,q}(\Omega)$, by the regularity theory of elliptic equations. Since $q > dim\,\Omega$, we obtain by Sobolev embedding that $||\theta_1(w)||_p \le c_1||\theta_1(w)||_{W^{2,q}} \le c_2||\theta(w)||_q = o(||w - w_0||_p$, for some constants c_1, c_2. Hence, for $\delta > 0$ sufficiently small, we have

$$(5.17) \qquad ||\theta_1(w)||_p < \delta/2 \quad \text{for} \quad w \in N_\delta(w_0).$$

We next consider the term $(\epsilon A + F'(w_0)^{-1}F'(w_0)w_0$ in (5.16). The C_0 semigroup generated by $\epsilon A + F'(w_0)$ satisfies (5.12), and $(\epsilon A + F'(w_0))x \to F'(w_0)x$ as $\epsilon \to 0$ for any $x \in D(A)$, where $D(A)$ is the domain of A, which is dense in X. By the The Trotter-Neveu-Kato Semigroup Convergence Theorem (see Theorem 7.2, on p. 44 in Goldstein [74] or Theorem A4-7 in Chapter 6), we find $S_\epsilon(t)x \to U(t)x$ for all $t \ge 0$, $x \in X$. Moreover, the resolvent satisfies $R(\lambda, \epsilon A + F'(w_0))x \to R(\lambda, F'(w_0))x$ for any $x \in X$ as $\epsilon \to 0$ with $\lambda > -\bar{c}$, where $R(\lambda, A)$ denotes the operator $(\lambda - A)^{-1}$. Putting $\lambda = 0$, we find $(\epsilon A + F'(w_0))^{-1}x \to (F'(w_0))^{-1}x$. In particular

$$(5.18) \qquad (\epsilon A + F'(w_0))^{-1}F'(w_0)z \to z \;\text{ for any }\; z \in X,$$

as $\epsilon \to 0$. Thus for sufficiently small ϵ, we have $||(\epsilon A + F'(w_0))^{-1}F'(w_0)w_0 - w_0||_p < \delta/2$. Since

$$||Q_\epsilon(w) - w_0||_p \le ||(\epsilon A + F'(w_0))^{-1}F'(w_0)w_0 - w_0||_p + ||\theta_1(w)||_p,$$

we find from (5.17) and the last inequality that Q_ϵ maps $N_\delta(w_0)$ into itself. Note that for $w_1, w_2 \in N_\epsilon(w_0)$, we have

$$Q_\epsilon(w_1) - Q_\epsilon(w_2) = (\epsilon A + F'(w_0))^{-1}(\theta(w_1) - \theta(w_2)).$$

By means of Sobolev embedding and elliptic $W^{2,q}$ estimates, we find

$$||Q_\epsilon(w_1) - Q_\epsilon(w_2)||_p \le K||\theta(w_1) - \theta(w_2)||_q,$$

for some constant K. Using the continuity of Gateaux derivative F', Lagrange' formula, and Hölder's inequality as mentioned above, we deduce:

$$||\theta(w_1) - \theta(w_2)||_q \leq \beta ||w_1 - w_2||_p$$

where β is arbitrarily small if w_1, w_2 are close enough to w_0 in X (cf. [222]). From the last two inequalities we find that Q_ϵ is a contraction in $N_\delta(w_0)$ for sufficiently small δ. We thus obtain in the neighborhood a unique fixed point w_ϵ, which is the solution of the problem.

Remark 5.1. The solution in $[W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]^2$ found in the above theorem is actually a classical solution. However, it converges to the constant solution as $\epsilon \to 0$ only in $L^p(\Omega) \times L^p(\Omega)$. The fact that the product of the eigenvalues of $F'(w_0)$ is positive is a consequence of the prey-predator interaction. It leads to the fact that all eigenvalues have negative real parts.

In order to study the stability of the steady-state found for (5.5), we now consider the parabolic problem:

$$\begin{cases} \bar{u}_t(\bar{x}, t) = R^{-2}\Delta\bar{u} + \bar{u}(a - b\bar{u} - c\bar{v}) & \\ & (\bar{x}, t) \in \Omega \times (0, t), \\ \bar{v}_t(\bar{x}, t) = R^{-2}\Delta\bar{v} + d^{-1}\bar{v}(h(\bar{u}) - m(\bar{v})) & \\ \bar{u} = \bar{v} = 0 & (\bar{x}, t) \in \partial\Omega \times \{0\}, \end{cases} \tag{5.19}$$

$$(\bar{u}(\bar{x}, 0), \bar{v}(\bar{x}, 0)) = (\bar{u}_0(\bar{x}), \bar{v}_0(\bar{x})) \qquad \bar{x} \in \Omega. \tag{5.20}$$

For $R > 0$ sufficiently large, we have an equilibrium solution $(\bar{u}_R(\bar{x}), \bar{v}_R(\bar{x}))$ of (5.19), which is in an arbitrary small neighborhood of w_0 in X. Here, we may define $(u_R(x), v_R(x)) := (\bar{u}_R(\bar{x}), \bar{v}_R(\bar{x}))$ for $x \in R\Omega$ where $\bar{x} = x/R \in \Omega$.

Let

$$B := \{(\bar{u}, \bar{v}) : (\bar{u}, \bar{v}) \in C(\bar{\Omega}) \times C(\bar{\Omega}),\ \bar{u} = \bar{v} = 0 \ on\ \partial\Omega\}. \tag{5.21}$$

The operator $A_1 := diag.(R^{-2}\Delta, R^{-2}\Delta)$ is an infinitesimal generator of an analytic semigroup on B for $t \geq 0$, with domain $D(A_1) = \{(\bar{u}_1, \bar{u}_2) : \bar{u}_i \in W^{2,p}(\Omega)$ for all p, $\bar{u}_i = 0$ and $\Delta\bar{u}_i = 0$ on $\partial\Omega$, $i = 1, 2\}$. If $(\bar{u}_0, \bar{v}_0) \in B$, we can consider the solution of the initial boundary value problem (5.19), (5.20) as a function

$$(\bar{u}(\cdot, t), \bar{v}(\cdot, t)) \in C([0, T], B) \cap C^1((0, T], B), \tag{5.22}$$

with $(\bar{u}(\cdot, 0), \bar{v}(\cdot, 0)) \in B$, $(\bar{u}(\cdot, t), \bar{v}(\cdot, t)) \in D(A_1)$ for all $t \in (0, T]$. We have the following stability theorem.

Theorem 5.3 (Asymptotic Stability under Small Diffusion). *Assume all the hypotheses concerning $F(u,v)$ in Theorem 5.2. For $R>0$ sufficiently large, the equilibrium solution $(\bar{u}_R(\bar{x}),\bar{v}_R(\bar{x}))$ of (5.19) is asymptotically stable in the sense that any solution of the initial boundary value problem (5.19), (5.20), with initial condition in B, considered as a function described in (5.22) will satisfy:*

$$||\bar{u}(\cdot,t),\bar{v}(\cdot,t)-(\bar{u}_R(\bar{x}),\bar{v}_R(\bar{x}))||_B \to 0, \text{ as } t\to+\infty, \tag{5.23}$$

provided that $||(\bar{u}_0,\bar{v}_0)-(\bar{u}_R(\bar{x}),\bar{v}_R(\bar{x}))||_B$ is sufficiently small, where

$$(\bar{u}(\bar{x},0),\bar{v}(\bar{x},0)=(\bar{u}_0(\bar{x}),\bar{v}_0(\bar{x})) \qquad \bar{x}\in\Omega. \tag{5.24}$$

Proof. (Outline) To prove the asymptotic stability of $(\bar{u}_R,\bar{v}_R)$, we apply a stability result of Mora [176] or Theorem A4-9 in Chapter 6. We see that it suffices to show that the spectrum of the linearization A_R+B_R of the elliptic system corresponding to (5.19) at $(\bar{u}_R,\bar{v}_R)$ is in a subset of $\{z: Re\, z\le -c_1\}$ for some $c_1>0$ where

$$A_R=\begin{bmatrix} R^{-2}\Delta & 0 \\ 0 & R^{-2}\Delta \end{bmatrix},$$

$$B_R=\begin{bmatrix} a\bar{u}_R-2b\bar{u}_R-c\bar{v}_R & -c\bar{u}_R \\ d^{-1}\bar{v}_R h'(\bar{u}_R) & d^{-1}(h(\bar{u}_R)-\bar{v}_R m'(\bar{v}_R)-m(\bar{v}_R)) \end{bmatrix}$$

are operators on B given in (5.21). Let $S_M:=\{z: Re\, z\le -M\}$. For any large $M>0$, the spectrum of the operator A_R is contained in S_M for all large enough $R>0$. Since the functions $\bar{u}_R$ and $\bar{v}_R$ are uniformly bounded for all R, the norms of the operators B_R are uniformly bounded as operators on the Banach space B. Moreover, the operators A_R and B_R commute. We thus obtain from the semicontinuity of the spectrum of closed operator (see Sections 3.1-3.2 of Chapter 4 in Kato [102] or Theorem A4-10 in Chapter 6.) that the spectrum of (A_R+B_R) is contained in a closed subset of the left open complex plane of the form $\{z:\ Re\, z\le -c_1\}$, for some $c_1>0$, provided R is sufficiently large.

The last theorem gives the stability of a steady-state for only a very special situation. More general theorem will be more elaborate. In view of Theorems 5.2 and 5.3, one is interested in conditions which insure the uniqueness of positive solutions for certain prey-predator systems. In the case of Volterra-Lotka type of interaction, there are some simple conditions. Without loss of generality, we consider problem (2.1) with $\sigma_1=\sigma_2=1$ as follows:

$$\begin{cases} \Delta u+u(a-bu-cv)=0 & \\ & \text{in } \Omega, \\ \Delta v+v(e+fu-gv)=0 & \\ u=v=0 & \text{on } \partial\Omega, \end{cases} \tag{5.25}$$

where a, b, c, e, f and g are positive constants.

Theorem 5.4 (Uniqueness under Weak Interaction). *Consider the boundary value problem (5.25) under hypotheses:*

$$\begin{cases} a > \lambda_1, \;\; e > \lambda_1, \\ cf < gb, \\ a > gb(gb-cf)^{-1}[\lambda_1 + ce/g]. \end{cases} \quad \text{and} \tag{5.26}$$

There exists a positive constant $k < 1$ such that if

$$cf < k(bg), \tag{5.27}$$

then (5.25) has a unique coexistence solution with each component strictly positive in Ω, and in $C^{2+\alpha}(\bar{\Omega})$.

Proof. Let C, F be positive constants such that $c \leq C, f \leq F$ and

$$CF < gb, \quad a > gb(gb - CF)^{-1}[\lambda_1 + Ce/g]. \tag{5.28}$$

Let $\hat{U}, \tilde{U}, \hat{V}, \tilde{V} \in C^{2+\alpha}(\Omega)$ be strictly positive functions in Ω satisfying the following scalar problems:

$$\begin{aligned} &\Delta\hat{U} + \hat{U}(a - b\hat{U}) = 0 \text{ in } \Omega, \;\; \hat{U} = 0 \text{ on } \partial\Omega, \\ &\Delta\hat{V} + \hat{V}(e + \tfrac{Fa}{b} - g\hat{V}) = 0 \text{ in } \Omega, \;\; \hat{V} = 0 \text{ on } \partial\Omega, \\ &\Delta\tilde{U} + \tilde{U}(a - b\tilde{U} - C\hat{V}) = 0 \text{ in } \Omega, \;\; \tilde{U} = 0 \text{ on } \partial\Omega, \\ &\Delta\tilde{V} + \tilde{V}(e - g\tilde{V}) = 0 \text{ in } \Omega, \;\; \tilde{V} = 0 \text{ on } \partial\Omega. \end{aligned} \tag{5.29}$$

Note that $\hat{U}, \hat{V}, \tilde{V}$ exist because $a, e, e+Fa/b$ are $> \lambda_1$; and $\hat{U}, \hat{V}, \tilde{V}$ are $\geq \delta\phi > 0$ in Ω for sufficiently small $\delta > 0$. One can readily deduce by upper lower solutions method that $\hat{V}(x) \leq \frac{1}{g}(e + \frac{Fa}{b})$, hence $a - C\hat{V} \geq a - \frac{C}{g}(e + \frac{Fa}{b}) > \lambda_1$ for all $x \in \bar{\Omega}$. Consequently, we obtain

$$0 < \delta\phi \leq \tilde{U} \leq \hat{U}, \;\; 0 < \delta\phi \leq \tilde{V} \leq \hat{V}$$

for $x \in \Omega, \delta > 0$ sufficiently small. Since the outward normal derivatives of ϕ are negative on the boundary, there must exist a constant $\bar{K} > 0$ such that

$$\hat{U} \leq \bar{K}\tilde{U}, \;\; \hat{V} \leq \bar{K}\tilde{V}, \;\; \hat{U} \leq \bar{K}\tilde{V}, \;\; \hat{V} \leq \bar{K}\tilde{U} \tag{5.30}$$

for all $x \in \bar{\Omega}$.

Define $u_1 := \hat{U}$. Let v_1 be the positive solution of

$$\Delta v_1 + v_1(e + fu_1 - gv_1) = 0 \text{ in } \Omega, \ \ v_1 = 0 \text{ on } \partial\Omega,$$

and $u_i, v_i, i = 2, 3, \dots$ be defined inductively by:

$$(5.31) \qquad \begin{cases} \Delta u_i + u_i(a - bu_i - cv_{i-1}) = 0 \\ \Delta v_i + v_i(e + fu_i - gv_i) = 0 \end{cases} \text{ in } \Omega, \qquad u_i = v_i = 0 \text{ on } \partial\Omega.$$

As described in Leung [123] or Section 5.3 in [125], the sequence satisfies:

$$(5.32) \qquad \begin{aligned} &\tilde{U} \le u_2 \le u_4 \le u_6 \le \cdots \le u_5 \le u_3 \le u_1 \le \hat{U}, \\ &\tilde{V} \le v_2 \le v_4 \le v_6 \le \cdots \le v_5 \le v_3 \le v_1 \le \hat{V} \end{aligned}$$

for all $x \in \bar{\Omega}$. From (5.31), we find for $i \ge 1$:

$$\begin{aligned} 0 &= \int_\Omega (u_{2i+2}\Delta u_{2i+1} - u_{2i+1}\Delta u_{2i+2})dx \\ &= -\int_\Omega u_{2i+1}u_{2i+2}[b(u_{2i+2} - u_{2i+1}) + c(v_{2i+1} - v_{2i})]dx, \end{aligned}$$

which implies

$$(5.33) \qquad b\int_\Omega (u_{2i+1} - u_{2i+2})u_{2i+1}u_{2i+2}dx = c\int_\Omega (v_{2i+1} - v_{2i})u_{2i+1}u_{2i+2}dx.$$

Also for $i \ge 1$, we have

$$\begin{aligned} 0 &= \int_\Omega (v_{2i+1}\Delta v_{2i} - v_{2i}\Delta v_{2i+1})dx \\ &= -\int_\Omega v_{2i}v_{2i+1}[f(u_{2i} - u_{2i+1}) + g(v_{2i+1} - v_{2i})]dx, \end{aligned}$$

which implies

$$(5.34) \qquad g\int_\Omega (v_{2i+1} - v_{2i})v_{2i}v_{2i+1}dx = f\int_\Omega (u_{2i+1} - u_{2i})v_{2i}v_{2i+1}dx.$$

Moreover, for $i \ge 1$

$$0 = \int_\Omega (u_{2i}\Delta u_{2i+1} - u_{2i+1}\Delta u_{2i})dx$$

$$= -\int_\Omega u_{2i}u_{2i+1}[b(u_{2i} - u_{2i+1}) + c(v_{2i-1} - v_{2i})]dx,$$

$$0 = \int_\Omega (v_{2i-1}\Delta v_{2i} - v_{2i}\Delta v_{2i-1})dx$$

$$= -\int_\Omega v_{2i-1}v_{2i}[f(u_{2i} - u_{2i-1}) + g(v_{2i-1} - v_{2i})]dx,$$

respectively gives

$$b\int_\Omega (u_{2i+1} - u_{2i})u_{2i}u_{2i+1}dx = c\int_\Omega (v_{2i-1} - v_{2i})u_{2i}u_{2i+1}dx, \tag{5.35}$$

$$g\int_\Omega (v_{2i-1} - v_{2i})v_{2i-1}v_{2i}dx = f\int_\Omega (u_{2i-1} - u_{2i})v_{2i-1}v_{2i}dx. \tag{5.36}$$

Using (5.33), (5.34) and (5.30), (5.32) we deduce that:

$$\begin{aligned} &\textstyle\int_\Omega (u_{2i+1} - u_{2i+2})u_{2i+1}u_{2i+2}dx = \frac{c}{b}\int_\Omega (v_{2i+1} - v_{2i})u_{2i+1}u_{2i+2}dx \\ &\le \textstyle\frac{c}{b}\int_\Omega \bar{K}^2(v_{2i+1} - v_{2i})v_{2i}v_{2i+1}dx = \bar{K}^2\frac{cf}{bg}\int_\Omega (u_{2i+1} - u_{2i})v_{2i}v_{2i+1}dx. \end{aligned} \tag{5.37}$$

Then, we use (5.35), (5.36) and (5.30), (5.32) again to obtain:

$$\begin{aligned} &\textstyle\int_\Omega (u_{2i+1} - u_{2i})u_{2i}u_{2i+1}dx = \frac{c}{b}\int_\Omega (v_{2i-1} - v_{2i})u_{2i}u_{2i+1}dx \\ &\le \textstyle\frac{c}{b}\int_\Omega \bar{K}^2(v_{2i-1} - v_{2i})v_{2i-1}v_{2i}dx = \bar{K}^2\frac{cf}{bg}\int_\Omega (u_{2i-1} - u_{2i})v_{2i-1}v_{2i}dx. \end{aligned} \tag{5.38}$$

Combining (5.37), (5.38) and (5.30), (5.32) once more, we obtain:

$$\int_\Omega (u_{2i+1} - u_{2i+2})u_{2i+1}u_{2i+2}dx = \bar{K}^8(\frac{cf}{bg})^2\int_\Omega (u_{2i-1} - u_{2i})u_{2i-1}u_{2i}dx \tag{5.39}$$

for each integer $i \ge 1$. From (5.39), we conclude that if

$$cf < (\bar{K})^{-4}(bg),$$

then $\lim_{i\to\infty}\int_\Omega (u_{2i+1} - u_{2i+2})u_{2i+1}u_{2i+2}dx = 0$. By dominated convergence, and $lim_{i\to\infty}u_{2i+1} = u^* > 0$ in Ω, $lim_{i\to\infty}u_{i+2} = u_* > 0$ in Ω, $lim_{i\to\infty}(u_{2i+1} - u_{2i+2}) = u^* - u_* \ge 0$ in Ω, we conclude that $u^* = u_*$ for all $x \in \Omega$. Similarly, from $lim_{i\to\infty}(v_{2i+1} - v_{2i+2}) = v^* - v_* \ge 0$ in Ω, we deduce $v^* = v_*$. By [125], the solution of (u, v) of (5.25) satisfies $u_* \le u \le u^*, v_* \le v \le v^*$ in Ω. We thus obtain (5.27) by choosing $k = \bar{K}^{-4}$.

Part B: Competing Species Case.

We next discuss some stability results for the competing species case, and consider system (1.2) with initial conditions:

$$u(x,0) = u^0(x),\ v(x,0) = v^0(x) \qquad \text{for } x \in \Omega. \tag{5.40}$$

We assume that

(5.41)

The functions f_i have Hölder continuous partial derivatives up to second order in compact sets, $i = 1, 2$; a_1, a_2, σ_1 and σ_2 are positive constants.

Moreover

$$\begin{cases} f_i(0,0) = 0,\ i = 1,2; \\ \frac{\partial f_i}{\partial u}, \frac{\partial f_i}{\partial v} < 0,\ i = 1,2 \ \text{ for } (u,v) \text{ in the first open quadrant.} \end{cases} \tag{5.42}$$

Under appropriate conditions, we will prove the local asymptotic stability of steady states by the method of upper lower solutions for the corresponding parabolic problem. The main assumption essentially means the competitions between the species are relatively small. The method of proof here avoids the problem of locating the spectrum of the linearized equation. It may not be readily justified that the spectrum is on the right half plane as in proof of Theorem 5.3 above.

Theorem 5.5 (Asymptotic Stability under Weak Competition). *Consider the initial-boundary value problem (1.2), (5.40), under hypotheses (5.41), (5.42) and*

$$a_i > \sigma_i \lambda_1,\ i = 1, 2. \tag{5.43}$$

Suppose $(u, v) = (\bar{u}_1(x), \bar{u}_2(x))$ *is an equilibrium solution of (1.2) with each* $\bar{u}_i$ *in* $C^{2+\alpha}(\bar{\Omega})$, $\bar{u}_i(x) > 0$ *in* Ω, $\partial \bar{u}_i/\partial \nu < 0$ *on* $\partial\Omega$, *for* $i = 1, 2$, *and*

$$\begin{aligned} &sup_{x\in\Omega} \left| \frac{\bar{u}_i(x)\cdot(\partial f_j/\partial u_i)(\bar{u}_1(x),\bar{u}_2(x))}{\bar{u}_j(x)\cdot(\partial f_j/\partial u_j)(\bar{u}_1(x),\bar{u}_2(x))} \right| \\ &\qquad < inf_{x\in\Omega} \left| \frac{\bar{u}_i(x)\cdot(\partial f_i/\partial u_i)(\bar{u}_1(x),\bar{u}_2(x))}{\bar{u}_j(x)\cdot(\partial f_i/\partial u_j)(\bar{u}_1(x),\bar{u}_2(x))} \right| < \infty \end{aligned} \tag{5.44}$$

for each $1 \le i, j \le 2, i \ne j$, *then* $(\bar{u}_1(x), \bar{u}_2(x))$ *is asymptotically stable. Here asymptotically stable means that if* $(u, v) = (u_1(x,t), u_2(x,t))$ *is a solution of (1.2), (5.40) with* $u_i \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega} \times [0, T])$, *each* $T > 0, i = 1, 2$, *then* $u_i(x,t) \to \bar{u}_i(x)$ *uniformly as* $t \to +\infty, i = 1, 2$ *in* $\bar{\Omega}$, *provided that* $(u_1(x, 0),$

$u_2(x,0)) = (u^0(x), v^0(x))$ and its first partial derivatives are close enough to that of $\bar{u}_i(x)$ respectively for all $x \in \bar{\Omega}, i = 1, 2$.

Remark 5.2. Recall that in Section 1.3, there are many theorems giving sufficient conditions for the existence of positive equilibrium. For (3.3) with sufficiently small c and f, one can show that there exists equilibrium with property as described in (5.44).

Proof. Hypothesis (5.44) implies that there are constants ρ_1, ρ_2 close enough to 1, with $\rho_1 < 1 < \rho_2$ such that for each $x \in \Omega$,

$$(5.45)\qquad \begin{aligned} &0 < \frac{\bar{u}_i(x)\cdot max_{\rho_1 \le s,\tau \le \rho_2}|(\partial f_j/\partial u_i)(s\bar{u}_1(x), \tau\bar{u}_2(x))|}{\bar{u}_j(x)\cdot min_{\rho_1 \le s \le 1}|(\partial f_j/\partial u_j)(s\bar{u}_1(x), \bar{u}_2(x))|} \\ &< inf_{x\in\Omega}\frac{\bar{u}_i(x)}{\bar{u}_j(x)}\{\frac{min_{\rho_1 \le s,\tau \le \rho_2}|(\partial f_i/\partial u_i)(s\bar{u}_1(x), \tau\bar{u}_2(x))|}{max_{\rho_1 \le s \le 1}|(\partial f_i/\partial u_j)(s\bar{u}_1(x), \bar{u}_2(x))|}\} - \epsilon_1 < \infty \end{aligned}$$

for each $1 \le i, j \le 2, i \ne j$, where ϵ_1 is a small positive number. We will construct appropriate lower and upper solutions v_i, w_i, and apply a comparison theorem to obtain the results here. Let

$$G(x) = \frac{\bar{u}_2(x)\, min_{\rho_1 \le s \le \tau \le \rho_2}|\frac{\partial f_2}{\partial u_2}(s\bar{u}_1(x), \tau\bar{u}_2))|}{\bar{u}_1(x)\, max_{\rho_1 \le s \le 1}|\frac{\partial f_2}{\partial u_1}(s\bar{u}_1(x), \bar{u}_2(x))|},$$

for $x \in \Omega$; and let α be a number, $1 < \alpha < \rho_2$ such that $(1-\rho_1) > (\alpha - 1)inf_{x\in\Omega}G(x)$. Define $w_2(x,t) = p(x,t)\bar{u}_2(x), p(x,t) = 1+(\alpha-1-\epsilon_4\bar{u}_2(x))e^{-mt}$, where ϵ_4 and m are positive constants to be determined later(one condition on ϵ_4 is $\epsilon_4 max_{x\in\bar{\Omega}}\bar{u}_2(x) < \alpha - 1$). On the other hand, define $v_1(x,t) = q(x,t)\bar{u}_1(x)$, $q(x,t) = 1 - (1-\beta(x))e^{-mt}$, where $\beta(x) = 1 - (\alpha-1)inf_{x\in\Omega}G(x) + \epsilon_2(\alpha - 1) + \epsilon_3(\alpha-1)\bar{u}_1(x)$, ϵ_2 and ϵ_3 are small positive constants satisfying $\epsilon_2 + \epsilon_3 max_{x\in\bar{\Omega}}\bar{u}_1(x) < \epsilon_1 < inf_{x\in\Omega}G(x)$. (Observe that $\rho_1 < \beta(x) < 1$). We have

$$(5.46)\qquad \begin{aligned} &\sigma_2\Delta w_2[a_2 + f_2(v_1, w_2)] - \frac{\partial w_2}{\partial t} \\ &= p(x,t)\bar{u}_2[f_2(v_1,w_2) - f_2(v_1,\bar{u}_2) + f_2(v_1,\bar{u}_2) - f_2(\bar{u}_1,\bar{u}_2)] \\ &\quad + e^{-mt}[m(\alpha - 1 - \epsilon_4\bar{u}_2(x))\bar{u}_2 - \bar{u}_2\sigma_2\epsilon_4\Delta\bar{u}_2 - 2\sigma_2\epsilon_4\textstyle\sum_{i=1}^n \bar{u}_{2x_i}^2] \\ &\le p(x,t)\bar{u}_2[max_{1\le\tau\le\rho_2}\{\frac{\partial f_2}{\partial u_2}(v_1,\tau\bar{u}_2)\}\{(\alpha-1)\bar{u}_2e^{-mt} - \epsilon_4\bar{u}_2^2e^{-mt}\} \\ &\quad - min_{\rho_1\le s\le 1}\{\frac{\partial f_2}{\partial u_1}(s\bar{u}_1,\bar{u}_2)\}\cdot\{(1-\hat{\beta})\bar{u}_1e^{-mt} - \epsilon_3(\alpha-1)\bar{u}_1^2e^{-mt}\}] + e^{-mt}[\cdots] \end{aligned}$$

where $[\cdots]$ represents the terms inside the brackets immediately before the inequality sign $\le$, and $\hat{\beta} = 1 - (\alpha-1)inf_{x\in\Omega}G(x) + \epsilon_2(\alpha-1)$. Set $\epsilon_4 = m = \epsilon_3$;

thus

$$|p(x,t)\bar{u}_2[max_{1\le\tau\le\rho_2}\{\tfrac{\partial f_2}{\partial u_2}(v_1,\tau\bar{u}_2)\}\{(-\epsilon_4\bar{u}_2^2e^{-mt})$$

$$+min_{\rho_1\le s\le 1}\{\tfrac{\partial f_2}{\partial u_1}(s\bar{u}_1,\bar{u}_2)\}\epsilon_3(\alpha-1)\bar{u}_1^2e^{-mt}]$$

$$+e^{-mt}[m(\alpha-1-\epsilon_4\bar{u}_2(x))\bar{u}_2-\bar{u}_2\sigma_2\epsilon_4\Delta\bar{u}_2]|\le\epsilon_4e^{-mt}\bar{u}_2(x)K_1$$

for all $x\in\Omega$, where K_1 is some positive constant. In a neighborhood $\emptyset$ of $\partial\Omega$ in Ω, we have $-2\sigma_2\epsilon_4\sum_{i=1}^n\bar{u}_{2x_i}^2e^{-mt}+\epsilon_4e^{-mt}\bar{u}_2(x)K_1<0$, for all $t\ge 0$, since $\bar{u}_2=0$ on $\partial\Omega$. Further,

$$max_{1\le\tau\le\rho_2}\{\tfrac{\partial f_2}{\partial u_2}(v_1(x,t),\tau\bar{u}_2(x))\}(\alpha-1)\bar{u}_2(x)$$

$$-min_{\rho_1\le s\le 1}\{\tfrac{\partial f_2}{\partial u_1}(s\bar{u}_1(x),\bar{u}_2(x))\}(1-\hat{\beta})\bar{u}_1(x)$$

$$\le max_{1\le s\le\tau\le\rho_2}\{\tfrac{\partial f_2}{\partial u_2}(s\bar{u}_1(x),\tau\bar{u}_2(x))\}(\alpha-1)\bar{u}_2(x)$$

$$+max_{\rho_1\le s\le 1}|\tfrac{\partial f_2}{\partial u_1}(s\bar{u}_1(x),\bar{u}_2(x))|((\alpha-1)G(x)-\epsilon_2(\alpha-1))\bar{u}_1(x)$$

$$=-min_{1\le s\le\tau\le\rho_2}|\tfrac{\partial f_2}{\partial u_2}(s\bar{u}_1(x),\tau\bar{u}_2(x))|(\alpha-1)\bar{u}_2(x)$$

$$+\bar{u}_2(x)min_{\rho_1\le s\le\tau\le\rho_2}|\tfrac{\partial f_2}{\partial u_2}(s\bar{u}_1(x),\tau\bar{u}_2(x))|(\alpha-1)$$

$$-\epsilon_2(\alpha-1)\bar{u}_1(x)max_{\rho_1\le s\le 1}|\tfrac{\partial f_2}{\partial u_1}(s\bar{u}_1(x),\bar{u}_2(x))|<0,$$

for all $x\in\Omega, t\ge 0$. Consequently, we have $\sigma_2\Delta w_2+w_2[a_2+f_2(v_1,w_2)]-\partial w_2/\partial t<0$, for $x\in\emptyset,,t\ge 0$. For $x\in\Omega\backslash\emptyset$, two terms in (5.46) satisfy the inequality:

$$p(x,t)\bar{u}_2[max_{1\le\tau\le\rho_2}\{\tfrac{\partial f_2}{\partial u_2}(v_1,\tau\bar{u}_2)\}(\alpha-1)\bar{u}_2e^{-mt}$$

$$-min_{\rho_1\le s\le 1}\{\tfrac{\partial f_2}{\partial\bar{u}_1}(s\bar{u}_1,\bar{u}_2)\}\cdot(1-\hat{\beta})\bar{u}_1e^{-mt}]<-\epsilon_2K_2e^{-mt},$$

for some $K_2>0$, all $t\ge 0$; and for such (x,t), the sum of all the other remaining terms after the inequality sign $\le$ in (5.46) can be reduced to less than $(1/2)\epsilon_2K_2e^{-mt}$ in absolute value, by choosing $\epsilon_4=m=\epsilon_3$ sufficiently small. We therefore have $\sigma_2\Delta w_2+w_2[a_2+f_2(v_1,w_2)]-\partial w_2/\partial t<0$, for $(x,t)\in\Omega\times[0,\infty)$, and $w_2(x,t)$ is an upper solution.

For v_1, we have the inequality:

(5.47)

$$\sigma_1 \Delta v_1 [a_1 + f_1(v_1, w_2)] - \frac{\partial v_1}{\partial t}$$

$$= q(x,t)\bar{u}_1[f_1(v_1, w_2) - f_1(v_1, \bar{u}_2) + f_1(v_1, \bar{u}_2) - f_1(\bar{u}_1, \bar{u}_2)]$$

$$+ e^{-mt}[-m(1-\beta(x))\bar{u}_1 + \bar{u}_1\sigma_1\epsilon_3(\alpha-1)\Delta\bar{u}_1 + 2\sigma_1\epsilon_3(\alpha-1)\sum_{i=1}^n \bar{u}_{1x_i}^2]$$

$$\geq q(x,t)\bar{u}_1[min_{1\leq\tau\leq\rho_2}\{\frac{\partial f_1}{\partial u_2}(v_1, \tau\bar{u}_2)\}\{(\alpha-1)\bar{u}_2 e^{-mt} - \epsilon_4\bar{u}_2^2 e^{-mt}\}$$

$$- max_{\rho_1\leq s\leq 1}\{\frac{\partial f_1}{\partial u_1}(s\bar{u}_1, \bar{u}_2)\} \cdot \{(1-\hat{\beta})\bar{u}_1 e^{-mt} - \epsilon_3(\alpha-1)\bar{u}_1^2 e^{-mt}\}] + e^{-mt}[\cdots]$$

where $[\cdots]$ represents the terms inside the brackets immediately before the inequality sign $\geq$. Due to the choice of $\epsilon_4 = m = \epsilon_3$ made previously, one has the inequality

$$| q(x,t)\bar{u}_1[min_{1\leq\tau\leq\rho_2}\{\frac{\partial f_1}{\partial u_2}(v_1, \tau\bar{u}_2)\}(-\epsilon_4\bar{u}_2^2 e^{-mt})$$

$$- max_{\rho_1\leq s\leq 1}\{\frac{\partial f_1}{\partial u_1}(s\bar{u}_1, \bar{u}_2)\}(-\epsilon_3(\alpha-1)\bar{u}_1^2 e^{-mt})]$$

$$+ e^{-mt}[-m(1-\beta(x))\bar{u}_1 + \bar{u}_1\sigma_1\epsilon_3(\alpha-1)\Delta\bar{u}_1]| \leq \epsilon_4 e^{-mt}\bar{u}_1 K_3$$

for all $x \in \Omega$, where K_3 is some positive constant. In a neighborhood $\tilde{\emptyset}$ of $\partial\Omega$ in Ω, we have $2\sigma_1\epsilon_3(\alpha-1)\sum_{i=1}^n \bar{u}_{1x_i}^2 e^{-mt} - \epsilon_4 e^{-mt}\bar{u}_1(x)K_3 > 0$, for all $t \geq 0$, since $\bar{u}_1 = 0$ on $\partial\Omega$. Further,

$$min_{1\leq\tau\leq\rho_2}\{\frac{\partial f_1}{\partial u_2}(v_1, \tau\bar{u}_2)\}\{(\alpha-1)\bar{u}_2 - max_{\rho_1\leq s\leq 1}\{\frac{\partial f_1}{\partial u_1}(s\bar{u}_1, \bar{u}_2)\}(1-\hat{\beta})\bar{u}_1$$

$$\geq -max_{\rho_1\leq s,\tau\leq\rho_2}|\frac{\partial f_1}{\partial u_2}(s\bar{u}_1, \tau\bar{u}_2)|(\alpha-1)\bar{u}_2 + min_{\rho_1\leq s\leq 1}|\frac{\partial f_1}{\partial u_1}(s\bar{u}_1, \bar{u}_2)|(1-\hat{\beta})\bar{u}_1$$

$$\geq -(\alpha-1)\bar{u}_1\, min_{\rho_1\leq s\leq 1}|\frac{\partial f_1}{\partial u_1}(s\bar{u}_1, \bar{u}_2)|(inf_{x\in\Omega}G(x) - \epsilon_1)$$

$$+ min_{\rho_1\leq s\leq 1}|\frac{\partial f_1}{\partial u_1}(s\bar{u}_1, \bar{u}_2)|(1-\hat{\beta})\bar{u}_1$$

$$= -\bar{u}_1(x) min_{\rho_1\leq s\leq 1}|\frac{\partial f_1}{\partial u_1}(s\bar{u}_1, \bar{u}_2)| \cdot (\epsilon_2 - \epsilon_1)(\alpha-1) > 0,$$

for all $x \in \Omega, t \geq 0$. The second $\geq$ sign in the last sentence is due to hypothesis (5.45). Consequently, we have $\sigma_1\Delta v_1 + [a_1 + f_1(v_1, w_2)] - \partial v_1/\partial t > 0$, for $x \in \tilde{\emptyset}, t \geq 0$. For $x \in \Omega\backslash\tilde{\emptyset}$, two terms in (5.47) satisfies the inequality:

$$q(x,t)\bar{u}_1[min_{1\leq\tau\leq\rho_2}\{\frac{\partial f_1}{\partial u_2}(v_1, \tau\bar{u}_2)\} \cdot (\alpha-1)\bar{u}_2 e^{-mt}$$

$$- max_{\rho_1\leq s\leq 1}\{\frac{\partial f_1}{\partial u_1}(s\bar{u}_1, \bar{u}_2)\}(1-\hat{\beta})\bar{u}_1 e^{-mt}] > (\epsilon_1 - \epsilon_2)K_4 e^{-mt}$$

for some $K_4 > 0$, all $t \geq 0$; and for such (x,t), the sum of all the other remaining terms after the inequality sign $\geq$ in (5.47) can be reduced to less than $(1/2)(\epsilon_1 - \epsilon_2)K_4 e^{-mt}$ in absolute value, by reducing the size of $\epsilon_4 = m = \epsilon_3$. We therefore have $\sigma_1 \Delta v_1 + [a_1 + f_1(v_1, w_2)] - \partial v_1/\partial t > 0$, for $(x,t) \in \Omega \times [0,\infty)$, and $v_1(x,t)$ is a lower solution.

Since all the first partial derivatives of f_1 and f_2 have the same sign, we can interchange the role of $\bar{u}_1, f_1$ with $\bar{u}_2, f_2$ respectively and construct lower and upper solutions v_2, w_1 in exactly the same manner as before. Here v_2, w_1 are of the form $v_2 = \tilde{q}(x,t)\bar{u}_2, w_1 = \tilde{p}(x,t)\bar{u}_1(x)$ with $\tilde{p}(x,t), \tilde{q}(x,t)$ analogous to $p(x,t), q(x,t)$ respectively. ($\tilde{p}(x,t) \to 1^+, \tilde{q}(x,t) \to 1^-$, as $t \to \infty$, all $x \in \bar{\Omega}$).

Finally, we have $v_i(x,t) \to \bar{u}_i(x)$ from below, and $w_i(x,t) \to \bar{u}_i(x)$ from above, as $t \to \infty$, uniformly for $x \in \bar{\Omega}, i = 1,2$. When the initial conditions $u_i(x,0)$ and their partial derivatives are close to that of $\bar{u}_i(x)$ in the sense described in the theorem, we have $v_i(x,0) \leq u_i(x,0) \leq w_i(x,0), x \in \bar{\Omega}$. (Note that we have $\partial \bar{u}_i/\partial \nu < 0$ on $\partial\Omega$). Applying appropriate comparison or differential inequalities as in Section 1.2 in [125], we obtain

$$v_i(x,t) \leq u_i(x,t) \leq w_i(x,t) \qquad \text{for } (x,t) \in \bar{\Omega} \times [0,\infty),$$

and thus we have $(\bar{u}_1(x), \bar{u}_2(x))$ as an asymptotically stable equilibrium solution.

In the situation when the intrinsic growth rate of one species is small, we can prove the following theorem in a similar fashion.

Theorem 5.6. *Consider the initial-boundary value problem (1.2), (5.40), under hypotheses (5.41), (5.42) and*

$$a_1 < \sigma_1 \lambda_1, \;\; a_2 > \sigma_2 \lambda_1; \tag{5.48}$$

and $u_2^*(x) \in C^{2+\alpha}(\bar{\Omega})$ *is a solution of*

$$\sigma_2 \Delta v + v[a_2 + f_2(0,v)] = 0 \text{ in } \Omega, \;\; v = 0 \text{ on } \partial\Omega, \tag{5.49}$$

with $u_2^*(x) > 0$ *for* $x \in \Omega$. *Let* $(u,v) = (u_1(x,t), u_2(x,t))$ *be a solution of (1.2), (5.40) with* $u_i \in C^{2+\alpha, 1+\alpha/2}(\bar{\Omega} \times [0,T])$, *each* $T > 0, i = 1,2$, *where* $u^0(x), v^0(x)$ *are both non-negative functions in* $C^{2+\alpha}(\bar{\Omega})$ *satisfying compatibility conditions as described in Ladyzhenskaya, Solonnikov and Ural'ceva [113] or Section 1.3 in [125]. Then* $(u_1(x,t), u_2(x,t)) \to (0, u_2^*(x))$ *as* $t \to \infty$, *uniformly for* $x \in \bar{\Omega}$, *provided that* u^0, v^0 *and all their first partial derivatives are close enough to* $0, u_2^*$ *respectively and their corresponding first partial derivatives.*

Under the stronger assumption of uniqueness of positive steady state, we can obtain a global stability result as follows. (We shall discuss the problem of such uniqueness in later theorems in this section.)

Theorem 5.7 (Global Asymptotic Stability in case of Uniqueness). *Consider problem (3.3) restricted to $\sigma_1 = \sigma_2$. Assume condition (3.4) is satisfied and that problem (3.3) only has a unique solution $(u^*(x), v^*(x))$ with both components strictly positive in Ω. Let $(u(x,t), v(x,t)$ be a solution of the initial boundary value problem:*

$$(5.50) \qquad \begin{cases} u_t = \Delta u + u(a - bu - cv) & \\ & \text{in } \Omega \times [0,\infty), \\ v_t = \Delta v + v(e - fu - gv) & \\ u = v = 0 & \text{on } \partial\Omega \times [0,\infty), \\ u(x,0) = u^0(x), v_(x,0) = v^0(x) & \text{in } \Omega \end{cases}$$

with both $u^0, v^0 \geq 0, \not\equiv 0$ in $C^\alpha(\bar{\Omega}), 0 < \alpha < 1$, and vanishing on $\partial\Omega$, then

$$(u(x,t), v(x,t)) \to (u^*(x), v^*(x)), \quad \text{as } t \to \infty$$

uniformly in $\bar{\Omega}$.

Proof. We first choose numbers a_1 and e_1 such that

$$(5.51) \qquad a_1 > a, \quad e_1 > e$$

and

$$(5.52) \qquad a > \lambda_1 + \frac{ce_1}{g}, \quad e > \lambda_1 + \frac{fa_1}{b}.$$

By hypothesis (3.4), such a_1, e_1 must exist.

For convenience, we introduce the following notation: If $w \in C^1(\bar{\Omega}), w(x) > 0$ for all $x \in \Omega$, and $\partial w/\partial \nu < 0$ everywhere on $\partial\Omega$, we write $w >> 0$. If $w, z \in C^1(\bar{\Omega})$,we write $w << z$ if $z - w >> 0$. We first prove the theorem under the additional conditions $u^0, v^0 \in C^1(\bar{\Omega})$,

$$(5.53) \qquad u^0 >> 0, \quad v^0 >> 0,$$

and for all $x \in \bar{\Omega}$,

$$(5.54) \qquad u^0(x) \leq \frac{\theta_{a_1}}{b}, \quad v^0(x) \leq \frac{\theta_{e_1}}{g}.$$

Here, for any $A > \lambda_1$, the symbol θ_A denotes the unique positive solution of

$$\Delta Z + Z[A - Z] = 0 \text{ in } \Omega, \quad Z|_{\partial\Omega} = 0.$$

Let ϕ_1 be the positive principal eigenfunction for the Laplacian on Ω with Dirichlet boundary condition. Choose $\epsilon > 0$ so small such that

$$\epsilon\phi_1(x) \le u^0(x), \ \ \epsilon\phi_1(x) \le v^0(x) \tag{5.55}$$

and

$$\begin{aligned} a &> \lambda_1 + \tfrac{ce_1}{g} + b\epsilon\phi_1(x), \\ e &> \lambda_1 + \tfrac{fa_1}{b} + g\epsilon\phi_1(x), \end{aligned} \tag{5.56}$$

for all $x \in \bar{\Omega}$. If we set $\bar{u} = \theta_{a_1}/b, \bar{v} = \theta_{e_1}/g$ and $\underline{u} = \underline{v} = \epsilon\phi_1$, then

$$\Delta\bar{u} + \bar{u}[a - b\bar{u} - c\underline{v}] = (a - a_1)\bar{u} - c\bar{u}\underline{v} < 0$$

for $x \in \Omega$; and since $\bar{u} \le a_1/b$,

$$\begin{aligned} \Delta\underline{v} + \underline{v}[e - f\bar{u} - g\underline{v}] &= \underline{v}[e - \lambda_1 - f\bar{u} - g\underline{v}] \\ &\ge \underline{v}[e - \lambda_1 - fa_1/b - g\underline{v}] > 0 \end{aligned}$$

on Ω. Similarly, we have

$$\begin{aligned} &\Delta\underline{u} + \underline{u}[a - b\underline{u} - c\bar{v}] > 0, \\ &\Delta\bar{v} + \bar{v}[e - f\underline{u} - g\bar{v}] < 0. \end{aligned}$$

The conclusion of the theorem follows from the uniqueness assumption, the inequalities $\underline{u}(x) \le u^0(x) \le \bar{u}(x), \underline{v}(x) \le v^0(x) \le \bar{v}(x), x \in \bar{\Omega}$, and comparison with solutions of the differential system (5.50) with initial conditions replaced at the steady-state upper lower solutions $(\bar{u}(x), \underline{v}(x))$. Solutions with such initial conditions converges monotonically to a maximum-minimum pair of steady - state. (See Pao [183] or Theorem 1.3 in [32].)

We next remove condition (5.54) on the initial functions $u^0(x), v^0(x)$. First, observe that there exists large $K > 1$, such that

$$u^0(x) < \frac{K\theta_a(x)}{b}, \quad v^0(x) < \frac{K\theta_e(x)}{g}$$

on Ω. Define $(\bar{U}(x,t), \underline{V}(x,t))$ to be the solution of problem (5.50) with initial conditions replaced with

$$(\bar{U}(x,0), \underline{V}(x,0)) = (\frac{K\theta_a(x)}{b}, 0).$$

It is clear that $\underline{V} \equiv 0$, $\bar{U}$ is non-negative in $\Omega \times [0, \infty)$ and

$$lim_{t\to\infty}\bar{U}(x,t) = \frac{\theta_a(x)}{b}. \tag{5.57}$$

Moreover, the convergence above is monotone, because $\bar{U}(x,0), \underline{V}(x,0)$ satisfies

$$\Delta\bar{U}(x,0) + \bar{U}(x,0)[a - b\bar{U}(x,0) - c\underline{V}(x,0)] = b(\bar{U}(x,0))^2[K^{-1} - 1] < 0,$$

$$\Delta\underline{V}(x,0) + \underline{V}[e - f\bar{U} - g\underline{V}] = 0.$$

The convergence in (5.57) is also in $C^1(\bar{\Omega})$ norm by using $W^{2,p}$ estimates, compact embedding and equations (5.50). (See e.g. pp. 87–89 in Fife [59]). Similarly, define $(\underline{U}(x,t), \bar{V}(x,t))$ to be the solution of problem (5.50) with initial conditions replaced with

$$(\underline{U}(x,0), \bar{V}(x,0)) = (0, \frac{K\theta_e(x)}{g}).$$

We have $\underline{U} \equiv 0$, $\bar{V}$ is non-negative in $\Omega \times [0,\infty)$ and the monotone $C^1(\bar{\Omega})$ convergence

$$lim_{t\to\infty}\bar{V}(x,t) = \frac{\theta_e(x)}{g}. \tag{5.58}$$

On the other hand, one readily verifies that the functions $\underline{U}(x,t), \bar{U}(x,t), \underline{V}(x,t), \bar{V}(x,t)$ satisfies:

$$\begin{aligned}
&\Delta\bar{U} + \bar{U}[a - b\bar{U} - c\underline{V}] - \partial\bar{U}/\partial t < 0\\
&\Delta\underline{V} + \underline{V}[e - f\bar{U} - g\underline{V}] - \partial\underline{V}/\partial t \geq 0\\
&\Delta\bar{V} + \bar{V}[e - f\underline{U} - g\bar{V}] - \partial\bar{V}/\partial t < 0\\
&\Delta\underline{U} + \underline{U}[a - b\underline{U} - c\bar{V}] - \partial\bar{U}/\partial t \geq 0
\end{aligned} \tag{5.59}$$

for $(x,t) \in \Omega \times (0,\infty)$, and

$$\begin{aligned}
&0 = \underline{U}(x,0) \leq u^0(x) \leq \bar{U}(x,0) = \tfrac{K\theta_a(x)}{b}\\
&0 = \underline{V}(x,0) \leq v^0(x) \leq \bar{V}(x,0) = \tfrac{K\theta_e(x)}{g}
\end{aligned} \tag{5.60}$$

for $x \in \bar{\Omega}$. From comparison theorems (cf. pp. 24–26 in [125]), we assert that

$$0 = \underline{U}(x,t) \leq u(x,t) \leq \bar{U}(x,t), \quad 0 = \underline{V}(x,t) \leq v(x,t) \leq \bar{V}(x,t) \tag{5.61}$$

for $(x,t) \in \Omega \times [0,\infty)$. We next observe that $\Delta\bar{u} + \bar{u}[a - b\bar{u}] = (a - a_1)\bar{u} < 0$ in Ω, $\bar{u}|_{\partial\Omega} = 0$, thus $\bar{u} = \theta_{a_1}/b$ is a strict upper solution of the problem

$$\Delta z + z[a - bz] = 0 \text{ in } \Omega, \quad z|_{\partial\Omega} = 0.$$

Similarly, θ_{e_1}/g is a strict upper solution of the problem

$$\Delta z + z[e - gz] = 0 \text{ in } \Omega, \quad z|_{\partial\Omega} = 0.$$

By monotone iteration and comparison, we obtain

$$\frac{\theta_a}{b} << \frac{\theta_{a_1}}{b}, \quad \frac{\theta_e}{g} << \frac{\theta_{e_1}}{g}. \tag{5.62}$$

For $s > 0$, let $u^s(x) = u(x,s), v^s(x) = v(x,s)$ for $x \in \bar{\Omega}$. We obtain from (5.57), (5.58), (5.61) and (5.62) that for $s > 0$ sufficiently large

$$u^s(x) \leq \frac{\theta_{a_1}(x)}{b}, \quad v^s(x) \leq \frac{\theta_{e_1(x)}}{g} \tag{5.63}$$

for $x \in \bar{\Omega}$. On the other hand for $s > 0$, we obtain from the theory of parabolic equations and strong maximum principle that u^s, v^s are in $C^1(\bar{\Omega})$ and

$$u^s >> 0, \quad v^s >> 0. \tag{5.64}$$

Comparing (5.63) and (5.64) respectively with (5.54) and (5.53), we obtain the conclusion of this theorem by using the beginning part of the proof.

When the intrinsic growth rates of both species are the same, the following theorem gives sufficient conditions for uniqueness of coexistence solution. It reflects the situation when the crowding effect of each species on itself is greater than its competing effecting on the growth rate on the other species.

Theorem 5.8 (Uniqueness under Weak Competition). *Consider problem (3.3) with $\sigma_1 = \sigma_2 = 1$. Suppose that*

$$a = e > \lambda_1, \ b > f, \text{ and } c < g, \tag{5.65}$$

then (3.3) has a unique coexistence solution with each component in $C^{2+\alpha}(\bar{\Omega})$ and strictly positive in Ω.

Proof. Suppose $\sigma_1 = \sigma_2 = 1$, (5.65) holds and (u, v) is a solution of (3.3) with each component in $C^{2+\alpha}(\bar{\Omega})$ and strictly positive in Ω. We claim that if z is a function in $C^1(\bar{\Omega})$ satisfying

$$\Delta z + z[a - bu - gv] = 0 \text{ in } \Omega, \ \ z = 0 \text{ on } \partial\Omega, \tag{5.66}$$

then $z \equiv 0$ in Ω. Note that the eigenvalue problem

$$\Delta w + w[a - bu - cv] + \lambda w = 0 \text{ in } \Omega, \ w = 0 \text{ on } \partial\Omega, \tag{5.67}$$

had eigenvalue $\lambda = 0$ with eigenfunction $w = u$ which is strictly positive in Ω. It follows that $\lambda = 0$ is the smallest eigenvalue of the problem (5.67). Thus from

Rayleigh's quotient, we find that for any nontrivial function $\psi \in C^1(\bar{\Omega})$ which vanishes at $\partial\Omega$, we have

$$0 \leq \frac{\int_\Omega |\nabla\psi|^2 - [a - bu - cv]\psi^2)dx}{\int_\Omega \psi^2 ds}. \tag{5.68}$$

Suppose z satisfies (5.66), we integrate by parts and obtain from (5.68) that

$$0 = \int_\Omega(-z\Delta z - z^2[a - bu - gv])dx$$

$$= \int_\Omega(|\nabla z|^2 - z^2[a - bu - cv])dx + \int_\Omega v(g - c)z^2 dx \geq \int_\Omega v(g - c)z^2 dx.$$

Since $g > c$ and $v > 0$ in Ω, we justify the claim that $z \equiv 0$ in $\bar{\Omega}$.

The differential equations in (3.3) can be written as:

$$\begin{cases} \Delta u + u[a - bu - gv] + (g - c)uv = 0 \\ \\ \Delta v + v[a - bu - gv] + (b - f)uv = 0 \end{cases} \quad \text{in } \Omega.$$

Multiplying the first and second equation above respectively by $(b - f)$ and $(g - c)$ and subtracting, we obtain $\Delta\psi + \psi[a - bu - gv] = 0$ in Ω, where $\psi = (b-f)u-(g-c)v$. We thus have $\psi \equiv 0$; that is $v \equiv ru$, where $r = (b-f)/(g-c)$. From the first equation in (3.3), we obtain

$$\Delta u + u[a - \frac{bg - cf}{g - c}u] = 0 \text{ in } \Omega, \;\; u = 0 \text{ on } \partial\Omega.$$

Hence the function $\theta = \frac{bg-cf}{g-c}u$ satisfies

$$\Delta\theta + \theta[a - \theta] = 0 \text{ in } \Omega, \;\; \theta = 0 \text{ on } \partial\Omega. \tag{5.69}$$

Consequently (u, v) must satisfy

$$(u, v) = (\frac{g - c}{bg - cf}\theta, \frac{b - f}{bg - cf}\theta) \text{ in } \bar{\Omega}.$$

where θ is uniquely defined as the solution of problem (5.69).

Other sufficient conditions for unique positive coexistence state even for $a \neq e$ can also be found.

Theorem 5.9 (Uniqueness under Weak Competition). *Consider problem (3.3) with $\sigma_1 = \sigma_2 = 1$ and assume (3.4) is satisfied. Suppose that*

$$4bg > \frac{gc^2\theta_a}{b\theta_{(e-af/b)}} + 2cf + \frac{bf^2\theta_e}{g\theta_{(a-ec/g)}}, \tag{5.70}$$

then (3.3) has a unique coexistence solution with each component in $C^{2+\alpha}(\bar{\Omega})$ *and strictly positive in* Ω.

Remark 5.3. Here, for any $A > \lambda_1$, θ_A denotes the unique positive solution of (5.69) where a is replaced by A. Thus, by (3.4), $\theta_{(e-af/b)}$ and $\theta_{(a-ec/g)}$ are positive functions in Ω. For fixed a, b, e and g, hypothesis (5.70) will be satisfied for c, f sufficiently small. This is true because $\theta_{(e-af/b)}$(or $\theta_{(a-ec/g)}$) increases as f (or c) decreases for $x \in \Omega$. Thus $\frac{gc^2\theta_a}{b\theta_{(e-af/b)}}$ (or $\frac{bf^2\theta_e}{g\theta_{(a-ec/g)}}$) decreases as f (or c) decreases.

Proof. Assume all the hypotheses of this theorem, and $(u_1, v_1), (u_2, v_2)$ are two strictly positive solutions of (3.3) in Ω. Let $p = u_1 - u_2, q = v_1 - v_2$, then

$$(5.71)\qquad \begin{cases} \Delta p + [a - bu_1 - cv_1]p - bu_2 p - cu_2 q = 0 & \\ & \text{in } \Omega, \\ \Delta q + [e - fu_2 - gv_2]q - fv_1 p - gv_1 q = 0 & \\ & \\ p = q = 0 & \text{on } \partial\Omega. \end{cases}$$

Since u_1 is a strictly positive solution of

$$(5.72)\qquad \begin{cases} \Delta\psi + [a - bu_1 - cv_1]\psi + \alpha\psi = 0 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\alpha = 0$, the number $\alpha = 0$ must be the smallest eigenvalue of the above problem. Moreover, by variational properties, we have

$$(5.73)\qquad \int_\Omega z(-\Delta z - [a - bu_1 - cv_1]z)dx \geq 0,$$

for any $z \in C^2(\bar{\Omega})$ which vanishes on $\partial\Omega$. Similarly, since v_2 is a strictly positive solution of

$$(5.74)\qquad \begin{cases} \Delta\psi + [e - fu_2 - gv_2]\psi + \alpha\psi = 0 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\alpha = 0$, the number $\alpha = 0$ must be the smallest eigenvalue of the above problem. Moreover,

$$(5.75)\qquad \int_\Omega z(-\Delta z - [e - fu_2 - fv_2]z)dx \geq 0,$$

for any $z \in C^2(\bar{\Omega})$ which vanishes on $\partial\Omega$. Multiplying the first equation of (5.71) by $-p$, the second by $-q$, integrating over Ω and adding, we deduce from (5.73) and (5.75) that

$$(5.76)\qquad \int_\Omega (bu_2 p^2 + (cu_2 + fv_1)pq + gv_1 q^2)dx \leq 0.$$

By comparison of scalar equations using upper and lower solutions we can readily obtain for $i = 1, 2$, $x \in \Omega$,

$$(5.77) \qquad \begin{aligned} (\tfrac{1}{b})\theta_{(a-ec/g)} &\leq u_i \leq (\tfrac{1}{b})\theta_a, \\ (\tfrac{1}{g})\theta_{(e-af/b)} &\leq v_i \leq (\tfrac{1}{g})\theta_e. \end{aligned}$$

From (5.77), we have

$$(5.78) \qquad \frac{c^2 g\theta_a}{b\theta_{(e-af/b)}} + 2cf + \frac{f^2 b\theta_e}{g\theta_{(a-ec/g)}} > c^2\frac{u_2}{v_1} + 2cf + f^2\frac{v_1}{u_2}$$

in Ω. Thus by hypothesis (5.70), the quadratic expression in the integrand of (5.76) is positive definite for each $x \in \Omega$. Consequently, we must have p and q identically equal to zero in Ω. That is $(u_1, v_1) \equiv (u_2, v_2)$ in $\bar{\Omega}$.

Theorem 5.9 is relevant for weak competition (i.e. small c, f). In case of strong competition, we consider problem (3.22) in Section 1.3 with large c, f. With a modification of the proof of Theorem 3.10 and more carefully analysis of the indices we can extend Theorem 3.10 to obtain the following "uniqueness" result.

Theorem 5.10 (Local Uniqueness of Segregated Coexistence under Strong Competition). *Suppose $w_0 \in C_0^1(\bar{\Omega})$ is a non-degenerate solution of (3.25) which changes sign. Let $max\{2, N/2\} < p < \infty$, then there exist respectively very large and small positive constants $\tilde{N}$ and ϵ such that for any c, f satisfying*

$$(5.79) \qquad c \geq \tilde{N}, \quad |cf^{-1} - \alpha| \leq \epsilon,$$

the problem (3.22) has a "unique" positive solution (u, v) near $(\alpha^{-1}w_0^+, -w_0^-)$ in $L^p(\Omega) \times L^p(\Omega)$. (Recall that α is any fixed number satisfying $\alpha \in (0, \infty)$.)

We now discuss the stability of the "unique" positive solution described in Theorem 5.10. Consider the problem:

$$(5.80) \qquad \begin{cases} u_t = \Delta u + u(a - u - cv) & \\ & \text{in } \Omega \times [0, \infty), \\ \tau u_t = \Delta v + v(e - v - fu) & \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

where $a > \lambda_1, e > \lambda_1$ and $\tau > 0$. The stability problem of the positive steady-state solution of the system is reduced by the following theorem to the study of the stability of the steady-state w_0 with non-zero index of the scalar problem (3.25) described in Theorem 3.10 of Section 1.3, Part B.

Theorem 5.11 (Stability of the Segregated Coexistence Solution). *Assume that the hypotheses of Theorem 5.10 are valid. Then the "unique" positive solution (u, v) for problem (3.22) found in Theorem 5.10 is a stable steady state solution for the parabolic problem (5.80) if w_0 described in Theorem 5.10 is a stable solution of (3.25) (for the corresponding parabolic problem); and it is unstable if w_0 is unstable. (Here, stable or unstable is interpreted as in Theorem A4-11 or Theorem A4-12 in Chapter 6 for solutions in the fractional power space $X^{\tilde{\alpha}} \times X^{\tilde{\alpha}}, 0 \leq \tilde{\alpha} < 1, X = L^p(\Omega)$).*

Proof. Suppose w_0 is a stable solution, and there exist $i \to \infty$ such that the unique positive solution (u_i, v_i) of problem (3.22) with $(c, f) = (c_i, f_i)$ satisfying (5.79) and $u_i \to \alpha^{-1} w_0^+, v_i \to (-w_0^-)$ in $L^p(\Omega)$ for $p > max\{2, N/2\}$ is unstable. (Recall in Theorem 3.10, we consider the solution (u, v) for (3.22) near $(\alpha^{-1} w_0^+, -w_0^-)$ in L^2; however, by Lemma 1.4 in Dancer and Guo [42], convergence in L^2 together with $||\cdot||_\infty$ bound imply convergence in $L^p, p > 2$). Consider the eigenvalues for linearized problem of (3.22) at (u_i, v_i) with the second equation multiplied by τ^{-1}:

$$(5.81) \qquad \begin{cases} \Delta h_i + (a - 2u_i - c_i v_i) h_i - c_i u_i k_i = \lambda h_i & \\ & \text{in } \Omega, \\ \Delta k_i - f_i v_i h_i + (e - 2v_i - f_i u_i) k_i = \lambda \tau k_i & \\ h_i = k_i = 0 \ \text{ on } \partial\Omega. & \end{cases}$$

If we let $w_i = -k_i$, (5.81) becomes:

$$(5.82) \qquad \begin{cases} \Delta h_i + (a - 2u_i - c_i v_i) h_i + c_i u_i w_i = \lambda h_i & \\ & \text{in } \Omega, \\ \Delta w_i + f_i v_i h_i + (e - 2v_i - f_i u_i) w_i = \lambda \tau w_i & \\ h_i = k_i = 0 \ \text{ on } \partial\Omega. & \end{cases}$$

By the stability theorems in Henry [84] (cf. Theorem A4-11 in Chapter 6), and the assumption that (u_i, v_i) is unstable, we deduce that the principal eigenvalue $\tilde{\lambda}_i$ of (5.82) must satisfy

$$\tilde{\lambda}_i \geq 0.$$

Let the eigenfunctions corresponding to $\lambda = \tilde{\lambda}_i$ in (5.82) be $(\tilde{h}_i, \tilde{w}_i) \in K \backslash \{0, 0)\}$, where K is the cone of non-negative functions in $L^p(\Omega) \times L^p(\Omega)$, $||\tilde{h}_i||_p + ||\tilde{w}_i||_p = 1$. We first show that $\{\tilde{\lambda}_i\}$ is uniformly bounded. Suppose $\tilde{\lambda}_i \to \infty$ as $i \to \infty$, we obtain from (5.82)

$$(5.83) \quad -\Delta(\beta_i \tilde{h}_i + \tilde{w}_i) = (a - 2u_i)\beta_i \tilde{h}_i + (e - 2v_i)\tilde{w}_i - \tilde{\lambda}_i(\beta_i \tilde{h}_i + \tau \tilde{w}_i) \ \text{ in } \Omega,$$

where $\beta_i = f_i/c_i$. Hence,

$$
\begin{aligned}
(5.84)\qquad \textstyle\int_\Omega |\nabla(\beta_i\tilde{h}_i + \tilde{w}_i|^2 dx &= \textstyle\int_\Omega[\beta_i(a-\tilde{\lambda}_i)\tilde{h}_i + (e-\tilde{\lambda}_i\tau)\tilde{w}_i](\beta_i\tilde{h}_i + \tilde{w}_i)dx \\
&\quad - 2\textstyle\int_\Omega(\beta_i u_i\tilde{h}_i + v_i\tilde{w}_i)(\beta_i\tilde{h}_i + \tilde{w}_i)dx < 0.
\end{aligned}
$$

Here, we use $a - \tilde{\lambda}_i < 0$, $e - \tilde{\lambda}_i\tau < 0$ for large i and $\tilde{h}_i, \tilde{w}_i \geq 0, \not\equiv 0$. This is a contradiction, and thus $\{\tilde{\lambda}_i\}$ is uniformly bounded. Consequently, we may assume without loss of generality that $lim_{i\to\infty}\tilde{\lambda}_i = \tilde{\lambda}$ with $\tilde{\lambda} \geq 0$. Note that the $L^p(\Omega)$ norm of the expression on the right of (5.83) is uniformly bounded, we thus assert by regularity theory that $\{||\beta_i\tilde{h}_i + \tilde{w}_i||_{2,p}\}$ is uniformly bounded. By compact embedding, there exists a subsequence (still denoted as $\{\beta_i\tilde{h}_i + \tilde{w}_i\}$) such that $\beta_i\tilde{h}_i + \tilde{w}_i \to y$ in $L^p(\Omega)$ as $i \to \infty$, and $y \geq 0$. We must have $y \not\equiv 0$; otherwise it follows readily from $0 \leq \beta_i\tilde{h}_i \leq \beta_i\tilde{h}_i + \tilde{w}_i, 0 \leq \tilde{w}_i \leq \beta_i\tilde{h}_i + \tilde{w}_i$ that $||\tilde{h}_i||_p + ||\tilde{w}_i||_p \to 0$ as $i \to \infty$, contradicting $||\tilde{h}_i||_p + ||\tilde{w}_i||_p = 1$. We also know that there exist $\tilde{h}, \tilde{w} \in L^p(\Omega)$ such that $\tilde{h}_i \to \tilde{h}, \tilde{w}_i \to \tilde{w}$ weakly in $L^p(\Omega)$. Hence, we have $y = \alpha\tilde{h} + \tilde{w}$. Note that by Sobolev embedding, $||\tilde{h}_i||_\infty$ and $||\tilde{w}_i||_\infty$ are also uniformly bounded. Let ϕ be a C^2 function with compact support in Ω, and multiply the first equation in (5.81) by ϕ when $(h_i, k_i, \lambda) = (\tilde{h}_i, \tilde{k}_i, \tilde{\lambda}_i), \tilde{k}_i = -\tilde{w}_i$, and integrate by parts, we find

$$(5.85)\qquad (\tilde{h}_i, -\Delta\phi) = (a - 2u_i - c_i v_i - \tilde{\lambda}_i, \tilde{h}_i\phi) - (c_i u_i, \tilde{k}_i\phi),$$

where $(\cdot,\cdot)$ denotes the integral of the product over Ω. Dividing both sides above by c_i, we find

$$(5.86)\qquad \frac{1}{c_i}(\tilde{h}_i, -\Delta\phi) = (\frac{a}{c_i} - \frac{2}{c_i}u_i - v_i - \frac{\tilde{\lambda}_i}{c_i}, \tilde{h}_i\phi) - (u_i, \tilde{k}_i\phi).$$

Passing to the limit as $i \to \infty$ and noting that $c_i \to \infty$, $u_i \to \alpha^{-1}w_0^+, v_i \to -w_0^-$ in $L^r(\Omega)$ for any $r > 2$, we obtain

$$(5.87)\qquad (w_0^-\tilde{h} - \alpha^{-1}w_0^+\tilde{k}, \phi) = 0,$$

where $\tilde{k} = \tilde{w}$. Since the C^2 functions ϕ satisfying (5.87) are dense in $L^q(\Omega)$ for $1/q + 1/p = 1$, we obtain

$$(5.88)\qquad w_0^-\tilde{h} = \alpha^{-1}w_0^+\tilde{k} \quad \text{in } \Omega.$$

Let

$$D_1 = \{x : w_0(x) < 0\}, \;\; D_2 = \{x : w_0(x) > 0\}.$$

Since $w_0 \in C_0^1(\bar{\Omega})$ and w_0 changes sign on Ω, both D_1 and D_2 are not empty we must have the property

$$(5.89)\qquad \tilde{h} \equiv 0 \text{ in } D_1, \;\; \tilde{w} \equiv 0 \text{ in } D_2.$$

Let ϕ be a C^2 function with compact support in Ω, if we multiply (5.83) by ϕ and integrate by parts, we obtain

$$(\beta_i\tilde{h}_i + \tilde{w}_i, -\Delta\phi) = ((a\beta_i\tilde{h}_i + e\tilde{w}_i), \phi) \tag{5.90}$$
$$-2((\beta_i u_i\tilde{h}_i + v_i\tilde{w}_i), \phi) - \tilde{\lambda}_i(\beta_i\tilde{h}_i + \tau\tilde{w}_i), \phi),$$

where $(\cdot, \cdot)$ denotes the integral of the product over Ω. Since $u_i \to \alpha^{-1}w_0^+, v_i \to (-w_0^-)$ in $L^\gamma(\Omega)$ for any $\gamma \in (2, \infty)$, we can pass to the limit above as $i \to \infty$ to obtain

$$(\beta_i\tilde{h} + \tilde{w}, -\Delta\phi) = ((a\alpha\tilde{h} + e\tilde{w}), \phi) \tag{5.91}$$
$$-2((w_0^+\tilde{h} + (-w_0^-)\tilde{w}), \phi) - \tilde{\lambda}(\alpha\tilde{h} + \tau\tilde{w}), \phi).$$

Note that $y = \alpha\tilde{h} + \tilde{w}$, and since C^2 functions ϕ in (5.91) above are dense in $L^q(\Omega)$, where $1/q + 1/p = 1$, we find by means of property (5.89) that

$$\begin{aligned} -\Delta y &= (a\alpha\tilde{h} + e\tilde{w}) - 2(w_0^+\tilde{h} + (-w_0^-)\tilde{w}) - \tilde{\lambda}(\alpha\tilde{h} + \tau\tilde{w}) \\ &= [(a - 2\alpha^{-1}w_0^+)sgn^+w_0 + (e + 2w_0^-)sgn^-w_0 \\ &\quad - \tilde{\lambda}(sgn^+w_0 + \tau sgn^-w_0)]y \equiv B(\tilde{\lambda})y, \end{aligned} \tag{5.92}$$

and $y = 0$ on $\partial\Omega$. Here, sgn^+w_0 (or sgn^-w_0) is the function with value 1 (or 0) and 0 (or 1) respectively at points where w_0 is positive or negative. The expression $(B(\lambda)y, y)$ defined above is decreasing in λ for λ real. Hence by (5.91) and the fact that $\tilde{\lambda} \geq 0$, we deduce $(\Delta y + B(0)y, y) \geq 0$. It follows from the characterization of eigenvalues that $\Delta + B(0)I$ has a non-negative real eigenvalue. By our non-degeneracy assumption, this eigenvalue must be positive. Thus by Theorem A4-12 in Chapter 6, we find w_0 is not stable as a solution of the corresponding parabolic equation. This contradicts the assumption in the beginning that w_0 is stable, unless the unique positive solutions (u_i, v_i) for large i are all stable.

We next prove the converse part of the theorem, and assume w_0 is unstable. Suppose the conclusion is false; then there exists a sequence of stable solutions (u_i, v_i) with corresponding $c_i \to \infty, c_i^{-1}f_i \to \alpha$, and principal eigenvalues $\tilde{\lambda}_i$ for corresponding linearized problem (5.82) satisfying $\tilde{\lambda}_i \leq 0$. Hence, there exists $(\tilde{h}_i, \tilde{w}_i) \in K\backslash\{(0,0)\}$ with $||\tilde{h}_i||_p + ||\tilde{w}_i||_p = 1$ such that $(\tilde{h}_i, \tilde{w}_i, \tilde{\lambda}_i)$ satisfies (5.82). We first show $\{\tilde{\lambda}_i\}$ is uniformly bounded. Suppose, not, there exists a subsequence, denoted again by $\{\tilde{\lambda}_i\}$ such that $\tilde{\lambda}_i \to -\infty$ as $i \to \infty$. Let $\beta_i = f_i/c_i$, then

$$-\Delta(\beta_i\tilde{h}_i + \tilde{w}_i) = (a - 2u_i)\beta_i\tilde{h}_i + (e - 2v_i)\tilde{w}_i - \tilde{\lambda}_i(\beta_i\tilde{h}_i + \tau\tilde{w}_i) \text{ in } \Omega. \tag{5.93}$$

Therefore, from the non-negativity of $\tilde{h}_i, \tilde{w}_i$, we find

$$-\Delta(\beta_i\tilde{h}_i + \tilde{w}_i) \geq (-\theta_i - \tilde{\lambda}_i\alpha_1)(\beta_i\tilde{h}_i + \tilde{w}_i) \text{ in } \Omega, \tag{5.94}$$

where $\theta_i = max\{||(a-2u_i)||_\infty, ||(e-2v_i)||_\infty\}$, $\alpha_1 = min\{1, \tau\}$. From the proof of Theorem 3.10, we have uniform bound for $\{||u_i||_\infty\}$, $\{||v_i||_\infty\}$, and thus $\{\theta_i\}$ is uniformly bounded. Thus $-\theta_i - \tilde{\lambda}_i \to +\infty$ as $i \to \infty$. Let ϕ_1 be a positive principal eigenfunction of the Δ, we obtain from (5.94)

$$\lambda_1 \int_\Omega (\beta_i\tilde{h}_i + \tilde{w}_i)\phi_1 dx \geq (-\theta_i - \tilde{\lambda}_i\alpha_1) \int_\Omega (\beta_i\tilde{h}_i + \tilde{w}_i)\phi_1 dx. \tag{5.95}$$

This is impossible as $i \to \infty$. Thus $\{\tilde{\lambda}_i\}$ is uniformly bounded; then, we use the same argument as in the proof of the stable case to obtain:

$$\beta_i\tilde{h}_i + \tilde{w}_i \to y \text{ in } L^p(\Omega),\ y \not\equiv 0 \text{ in } L^p(\Omega) \text{ and } y \geq 0,$$

since $\tilde{h}_i \geq 0, \tilde{w}_i \geq 0$. Moreover, we have $\tilde{\lambda}_i \to \tilde{\lambda}, \tilde{\lambda} \leq 0$, and $y = \alpha\tilde{h} + \tilde{w}$ satisfies

$$\begin{cases} -\Delta y = [(a - 2\alpha^{-1}w_0^+)sgn^+w_0 + (e + 2w_0^-)sgn^-w_0 \\ \qquad\qquad -\tilde{\lambda}(sgn^+w_0 + \tau sgn^-w_0)]y & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.96}$$

Since $y \geq 0, \not\equiv 0$, the characterization of eigenvalues implies that the eigenvalue problem for λ in

$$\begin{cases} -\Delta h = [(a - 2\alpha^{-1}w_0^+)sgn^+w_0 + (e + 2w_0^-)sgn^-w_0 \\ \qquad\qquad -\tilde{\lambda}(sgn^+w_0 + \tau sgn^-w_0) + \lambda]h & \text{in } \Omega, \\ h = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.97}$$

has principal eigenvalue equal to zero. Moreover, since $-\tilde{\lambda}(sgn^+w_0 + \tau sgn^-w_0) \geq 0$, we obtain by comparison that the principal eigenvalue $\lambda = \bar{\lambda}$ of the problem

$$\begin{cases} -\Delta h = [(a - 2\alpha^{-1}w_0^+)sgn^+w_0 + (e + 2w_0^-)sgn^-w_0 + \lambda]h & \text{in } \Omega, \\ h = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.98}$$

must have $\bar{\lambda} \geq 0$. However, from the fact that w_0 is non-degenerate, we have $\bar{\lambda} \neq 0$. Consequently we have $\bar{\lambda} > 0$. This implies that w_0 is stable, contradicting the assumption of the second half of the proof. This completes the proof of this theorem.

We now come to the discussion of the case when the competition coefficients c, f for system (3.22) are not both small or both large. Recall that in Theorem 3.12 in Section 1.3, we find the existence of positive solutions when c and f are such that $\hat{\rho}_1(\Delta + a - cv_0)$ and $\hat{\rho}_1(\Delta + e - fu_0)$ are of different signs, where $(u_0, 0)$ and $(0, v_0)$ are the semi-trivial solutions. We shall now prove the "asymptotic stability" of the positive solution asserted by Theorem 3.11.

Recall that for Theorem 3.12, we set

$$D = \{(u, v) : C_0(\bar{\Omega}) \times C_0(\bar{\Omega}), 0 \leq u \leq a, 0 \leq v \leq e \text{ in } \Omega\},$$

where A is the map given by (3.65) whose fixed points are solutions of (3.22). By an "asymptotically stable" solution (u, v) in Theorem 5.12 below, we mean the spectral radius satisfies

(5.99)

$$r(A'(u, v)) \leq 1, \ (u, v) \text{ is an isolated solution, and } index_D(A, (u, v)) = 1.$$

Note that the case when $r(A'(u, v)) = 1$ is usually undetermined. However, if we also find that the index is 1 and the solution is isolated, then we can use a relevant theorem involving stability on the "center manifold" to obtain the solution is asymptotically stable with respect to flows in an appropriate function subspace $X^\alpha \times X^\alpha$ of $X \times X, X = L^p(\Omega)$. This will be explained in the proof of the following theorem.

Theorem 5.12. *Under the hypotheses of Theorem 3.12, one of the positive solution for (3.22) found in Theorem 3.12 is asymptotically stable in the sense described in (5.99) if* $0 \leq c \leq c_1, f_1 \leq f < \bar{f}$ *and either* $c < c_1$ *or* $f_1 < f$. *Here* $(c_1, f_1) \in T^+$ *is defined for Theorem 3.11, so that positive solution exist.*

Proof. (Outline) For convenience, we define the cone $\tilde{K} = \{(u, v) \in C_0(\bar{\Omega}) \times C_0(\bar{\Omega}) : u \geq 0 \text{ in } \Omega, \ v \leq 0 \text{ in } \Omega\}$ and denote the corresponding induced order by $\geq_S$. Recall that in the proof of Theorem 3.12, we choose k to be a positive constant satisfying $k \geq max.\{a + ce, e + fa\}$, and define that the mapping:

$$A(u, v) := (-\Delta + k)^{-1}(u(a + k - u - cv), v(e + k - v - fu))$$

on the set:

$$D := \{(u, v) \in C_0(\bar{\Omega}) \times C_0(\bar{\Omega}) : 0 \leq u \leq a, 0 \leq v \leq e \ \ in \ \bar{\Omega}\}.$$

For $(c_1, f_1) \in T^+$, we have a strictly positive solution (u_1, v_1) satisfying

$$\begin{cases} -\Delta u_1 = u_1(a - u_1 - c_1 v_1) & \text{in } \Omega, \\ -\Delta v_1 = v_1(e - v_1 - f_1 u_1) & \\ u_1 = v_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

We can write $C := \{(u,v) \in D : u_1 \le u \le u_0, 0 \le v \le v_1\}$ as an order interval $C = [(u_1,v_1),(u_0,0)]$ in D with order $\ge_S$ induced by $\tilde{K}$. The mapping A is increasing on the order interval C.

We have shown in the proof of Theorem 3.12 that $A(u_1,v_1) >_S (u_1,v_1)$(where $>_S$ means $\ge_S$ and equality does not hold). Let $w = A(u_1,v_1) - (u_1,v_1) >_S 0$. Since A is increasing, the map A_t defined by $A_t(u,v) = A(u,v) - tw$, for $0 < t < 1$, is an increasing C^1 map of C into itself. Let x_t denote its minimal fixed point in C, which can be obtained by iterating from (u_1,v_1). Moreover, by iteration, x_t increases as t decreases. Since $\{x_t : t \in (0,1)\}$ lies in a compact set (by the boundedness of C and the compactness of A), we readily see that $x_0 = \lim_{t\to 0^+} x_t$ exists, is in C, and is a fixed point of A.

We will prove the solution x_0 is an "asymptotically stable" solution. Since $x_t \ge_S (u_1,v_1)$, the first component of x_t is positive in Ω. Since, $x_t = Ax_t - tw \le_S A(u_0,0) - tw = (u_0,0) - tw$, and both components of w are positive in Ω, we find that the second component of x_t must be positive in Ω. By argument as in the proof of the last theorem $A'(x_t)$ is a demi-interior operator to $\tilde{K}$. That is, for any $y \in \tilde{K}\backslash\{(0,0)\}$, we have $f(A'(x_t)y) > 0$ for all $f \in \tilde{K}^*\backslash\{0\}$, where $\tilde{K}^* = \{g \in (C_0(\bar{\Omega}) \times C_0(\bar{\Omega}))^* : g(z) \ge 0 \text{ for all } z \in \tilde{K}\}$. (Note that this is true by using the Riesz representation of linear functional, and the fact that such $A'(x_t)y$ is positive in Ω by the maximum principle applied to the linearized system of the form (5.82)). However, as described in p. 50 of Dancer [39], if $A'(x_t)$ is a demi-interior operator, then $(\lambda I - A'(x_t))^{-1}$ is a demi-interior operator for some $\lambda > r(A'(x_t))$. Then, using the geometric expansion for $(\lambda I - A'(x_t))^{-1}$ as described in the appendix of Schaefer [205], we can obtain $f(y) > 0$ if $y \in \tilde{K}\backslash\{(0,0)\}$ for any $f \in \tilde{K}^*\backslash\{0\}$, which is an eigenfunction corresponding to the eigenvalue $r(A'(x_t))$. Moreover, we have $r(A'(x_t))$ is a simple eigenvalue of $A'(x_t)$ and is the only non-zero eigenvalue to which there corresponds a positive eigenfunction (cf. Lemma 2.4 in Dancer and Guo [42] or Theorem 3.2 on p.632 of Amann [3], or equivalently Theorem A3-8 in Chapter 6). Then, applying a variant of the remark on p. 143 of Dancer [37] to the increasing mapping $A_t : C \to C$ with minimal solution x_t in C and the fact that $f(x_t) > 0$ as deduced above, we find $r(A'(x_t)) \le 1$. Further, from the continuity of spectral radius, we obtain $r(A'(x_0)) \le 1$ as $t \to 0^+$.

From Theorem 3.12, we have $index_C(A,(u_0,0)) = 0$ and $r(A'(u_0,0)) > 1$. Thus, from the conclusion of the above paragraph, $x_0 \ne (u_0,0)$. Hence, x_0 is a strictly positive solution. By the arguments given in the last paragraph, we find $A'(x_0)$ is a demi-interior operator. If $r(A'(x_0)) < 1$, then we can use Theorem A2-3 in Chapter 6 to obtain $index_C(A,x_0) = 1$. Moreover, this implies that the principal eigenvalue of the linearized equation at x_0 is negative. By the first theorem in Chapter 5 in Henry [84] or Theorem A4-11 in Chapter 6, we obtain asymptotic stability with respect to solutions of parabolic problems corresponding to (3.22) with initial conditions in the subspace $X^\alpha \times X^\alpha, 0 \le$

$\alpha < 1$, with $X = L^p(\Omega)$, for large p.

On the other hand, suppose $r(A'(x_0)) = 1$. Then all fixed points x_t of A_t in a neighborhood of $(x,t) = (x_0, 0)$ are represented by $(x_t, t) = (x_0 + \alpha h + z(\alpha), \phi(\alpha))$, where z and ϕ are C^1 functions, with $\phi : (-\epsilon, \epsilon) \to R, \phi(0) = 0, z(0) = 0, h$ spans $N(I - A'(x_0)), f$ spans $N(I - A'(x_0)^*)$ and $f(w(\alpha)) = 0$, for all small α (cf Dancer [37]). Moreover, we have $\phi(\alpha) > 0$ when $\alpha \in (-\epsilon, 0)$. We choose a number $\alpha_0 < 0$, with $\tau := \phi(\alpha_0) > 0$ where $\phi'(\alpha_0) \neq 0$, $I - A'(x_\tau)$ is invertible. Since we also have $r(A'(x_\tau)) \leq 1$ and by Krein-Rutman theorem $r(A'(x_\tau))$ is in the spectrum of $A'(x_\tau)$, thus $r(A'(x_\tau)) < 1$ (cf. Theorem A2-5 in Chapter 6), and we obtain $index_C(A_\tau, x_\tau) = 1$. We next deduce that x_0 is isolated. Suppose not, then we obtain from the analyticity of A that $\phi(\alpha) = 0$ for all small α. Thus, any solution of $x = A(x) - tw$ near $(x_0, 0)$ must has $t = 0$, contradicting (x_t, t) are solutions. To calculate the $index_C(A, x_0)$, we can construct a neighborhood V containing $x_t, 0 \leq t \leq \tau$ so that by homotopy invariance:

$$deg_C(A_\tau, V) = deg_C(A, V) = index_C(A, x_0). \tag{5.100}$$

Then, by means of the functional f and the isolated property of x_0, we can construct appropriate neighborhood to show:

$$deg_C(A_\tau, V) = index_C(A_\tau, x_\tau) = 1. \tag{5.101}$$

By means of (5.100) and (5.101), we obtain $index_C(A, x_0) = 1$. For more details, see the arguments for proving Proposition 3 in p. 144 and Remark 4 in p. 146 of Dancer [37].

We next observe that the $index_C(A, x_0)$ is the same as $index_D(A, x_0)$, when we assume $\partial\Omega$ is smooth. To see this, we use the space $V \times V$ where V denotes the space of functions u in $C_0(\bar{\Omega})$ for which $\phi_1^{-1}u$ extends to a continuous function on $\bar{\Omega}$ with the norm $||u|| := sup_{x\in\Omega}|\phi_1^{-1}(x)u(x)|$, where ϕ_1 denotes the positive eigenfunction corresponding to the principal eigenvalue for $-\Delta$ on Ω with Dirichlet boundary condition. The set V is a Banach space under the norm $|| \cdot ||$. The functions u for which $inf_{x\in\Omega}\phi_1^{-1}u(x) > 0$ are interior elements of the cone $K \cap V$, where K is the usual cone in $C_0(\bar{\Omega})$. In particular, this holds for $u \in C^1(\bar{\Omega})$ with $u(x) > 0$ in Ω and $\partial u/\partial\nu < 0$ on $\partial\Omega$. The mapping A is completely continuous from $E := C_0(\bar{\Omega}) \times C_0(\bar{\Omega})$ into $V \times V$. Moreover, if x_0 is an isolated fixed point of A in E, the commutativity theorem for degree (see Granas [75] or Nussbaum [178]) ensures that $index_E(A, x_0) = index_{V\times V}(A, x_0)$. Similarly, if $x_0 \in C$, the index of x_0 in C is the same as that in $C \cap (V \times V)$. That is, we only need to prove our results for indices in the space $V \times V$. In this case, we simply have to prove that the fixed point is interior to $C \cap (V \times V)$ or $D \cap (V \times V)$, and the result then follows. This can be readily justified by using maximum principle for the corresponding system as explained before.

Next, suppose (i) $r(A'(x_0)) = 1$, (ii) the fixed point x_0 is isolated and (iii) the index $index_D(A, x_0) = 1$. From (i) we have the principal eigenvalue of the corresponding linearized system at x_0 is $= 0$; then we justify as in above that the eigenspace (center manifold) is one dimensional. From Theorem 6.2.1 in Chapter 6 of Henry [84], we assert that if x_0 is asymptotically stable in the center manifold, then it is asymptotically stable in $X^\alpha \times X^\alpha$. We can use the argument in Theorem 9.3.2 of Chow and Hale [28] to assert the stability of x_0 on the manifold is determined by Liapunov Schmidt reduction since the related function F there is C^2. In particular, x_0 is stable if 0 has index 1 for the Liapunov Schmidt reduction. The Liapunov Schmidt reduction for $I - A$ is equivalent to that for $L - F$. Moreover, by Theorem 24.2 in Krasnosel'skii and Zabrieko [109], we can relate the $index_{L^p \times L^p}(A, x_0)$ or $index_D(A, x_0)$ with the index of the 0 of the bifurcation equation, and find they are equal in this simple case. Thus by property (iii), the index of 0 of the bifurcation equation is 1, and x_0 is asymptotically stable on the manifold and $X^\alpha \times X^\alpha$ in $L^p \times L^p$. The details are too technical to be included here (cf. Dancer [40]).

Part C: Cooperative Species Case.

We now come to the discussion of the stability of some of the positive steady states (i.e. coexistence states) for cooperating species found in Section 1.4. We will use the operators L_1 and L_2 as defined in (4.20) and (4.21). Recall the general cooperative system (4.22), and the part concerning weak cooperation in assumptions (4.23). Also, recall the symbol $\theta_{[-L,a,b]}$ defined immediately before Theorem 4.4, denoting the solution for the related scalar problem. Under some further restrictive conditions, we have the following uniqueness and stability theorem.

Theorem 5.13 (Uniqueness under Weak Cooperation and Asymptotic Stability). *Assume hypotheses (4.23),*

$$c(x) > 0, f(x) > 0 \textit{ for } x \in \Omega,$$

and that any coexistence state (u^*, v^*) of problem (4.22) satisfies

$$sup_\Omega(u^*/v^*) \cdot sup_\Omega(v^*/u^*) < inf_\Omega(b/c) \cdot inf_\Omega(g/f). \tag{5.102}$$

Then the boundary value problem (4.22) possesses a unique coexistence solution. Moreover, it is asymptotically stable.

Proof. (Outline) Under the assumptions (4.23), we can obtain uniform a-priori bound for all non-negative solutions of (4.22). Thus we can use the fixed point cone index method to study non-negative solutions as in Sections 2 and 3. As in Theorem 2.5 and part (i) of Theorem 3.4, we can show that both semi-trivial positive solutions $(\theta_{[-L_1,a,b]}, 0)$ and $(0, \theta_{[-L_2,e,g]})$ have zero local index. Moreover, the solution $(0, 0)$ has index zero and the global index of the related mapping equals one. Consequently, it suffices to show that under condition (5.102), all the eigenvalues of the corresponding linearized problem at (u^*, v^*) have real parts less than some negative constant. By Theorem A2-3 in Chapter 6, we can then infer that the solution (u^*, v^*) has local index one; and by the additivity of indices, such positive solution must be unique. The linearized problem at (u^*, v^*) is given by:

$$(5.103) \qquad \begin{cases} L_1 h + (a - 2bu^* + cv^*)h + cu^*k = \lambda h & \\ & \text{in } \Omega, \\ L_2 k + fv^*h + (e + fu^* - 2gv^*)k = \lambda k & \\ h = k = 0 & \text{on } \partial\Omega. \end{cases}$$

By Theorem A3-8 in Chapter 6 (or Theorem 8.3 and 8.4 in [45]), the principal eigenvalue of problem (5.103) is negative if we can find functions $h^* > 0, k^* > 0$ in Ω satisfying

(5.104)

$$L_1 h^* + (a - 2bu^* + cv^*)h^* + cu^*k^* < 0, \quad L_2 k^* + fv^*h^* + (e + fu^* - 2gv^*)k^* < 0$$

in Ω. Then, by Theorem A3-8 in Chapter 6 again, the real parts of all other eigenvalues of problem (5.103) are less than some negative number. In order to construct h^*, k^* satisfying (5.104), we first find positive constants α, β such that

$$(5.105) \qquad \frac{inf_\Omega(b/c)}{sup_\Omega(v^*/u^*)} > \frac{\beta}{\alpha} > \frac{sup_\Omega(u^*/v^*)}{inf_\Omega(g/f)}.$$

Such constants exist due to assumption (5.102). Then, define

$$h^* := \alpha u^*, \ \ k^* := \beta v^*.$$

We have

$$(5.106) \qquad L_1 h^* + (a - 2bu^* + cv^*)h^* + cu^*k^* = u^*(\beta cv^* - \alpha bu^*).$$

Moreover, for each $x \in \Omega$, we obtain from (5.105)

$$(5.107) \qquad \begin{aligned} &(\beta cv^* - \alpha bu^*)\tfrac{1}{cu^*} = \beta\tfrac{v^*}{u^*} - \alpha\tfrac{b}{c} \\ &\qquad\qquad \le \beta sup_\Omega(\tfrac{v^*}{u^*}) - \alpha\, inf_\Omega(\tfrac{b}{c}) < 0. \end{aligned}$$

Thus we obtain the first inequality in (5.104) from (5.106) and (5.107). Similarly, we can verify the second inequality in (5.104). The assertion on asymptotic stability of the solution (u^*, v^*) can be deduced from Theorem A4-11 in Chapter 6. This completes the proof of Theorem 5.13.

The following corollary is more readily applicable, using only information from the coefficients of the system (4.22). Recall the definitions of c_M, g_M, b_L and g_L etc. for Theorem 4.4.

Corollary 5.14. *Assume $L_1 = L_2$, $c(x) > 0, f(x) > 0$ for each $x \in \Omega$,*

$$c_M f_M < b_L g_L,\ \rho_1(-L_1) > 0,\ a > \rho_1(-L_1),\ e > \rho_1(-L_2), \tag{5.108}$$

and

$$\frac{b_M g_M}{16 b_L g_L (b_L g_L - c_M f_M)^2} \cdot \frac{(g_L a^2 + c_M e^2)(b_L e^2 + f_M a^2)}{(a - \rho_1(-L_1))(e - \rho_1(-L_1))} \cdot (sup_\Omega \frac{\phi}{\psi})^2 < \frac{1}{c_M f_M}, \tag{5.109}$$

where $\phi > 0$ is the principal eigenfunction associated with $\rho_1(-L_1)$, normalized so that $||\phi||_\infty = 1$ and $\psi > 0$ is the unique solution of

$$-L_1 \psi = 1, \quad in\ \Omega, \quad \psi|_{\partial\Omega} = 0.$$

Then the boundary value problem (4.22) has exactly one coexistence solution. Furthermore, it is asymptotically stable.

Proof. We first show that the family of functions $(\bar{u}_t, \bar{v}_t), t \geq 1$, defined by

$$\bar{u}_t := \frac{t(g_L a^2 + c_M b^2)}{4(b_L g_L - c_M f_M)} \psi, \ \ \bar{v}_t := \frac{t(b_L e^2 + f_M a^2)}{4(b_L g_L - c_M f_M)} \psi, \tag{5.110}$$

are upper solutions to problem (4.22). To verify this, it suffices to show for $x \in \Omega$,

$$\begin{aligned} 1 &\geq \psi \cdot [a - t(b(x)K_1 - c(x)K_2)\psi], \\ 1 &\geq \psi \cdot [e - t(g(x)K_2 - f(x)K_1)\psi], \end{aligned} \tag{5.111}$$

where

$$K_1 := \frac{(g_L a^2 + c_M e^2)}{4(b_L g_L - c_M f_M)}, \ \ K_2 := \frac{(b_L e^2 + f_M a^2)}{4(b_L g_L - c_M f_M)}.$$

For positive A and B, we have $sup_{\xi \geq 0}(A - B\xi)\xi = A^2/(4B)$. Thus we find that for each $t \geq 1$,

$$\psi \cdot [a - t(b(x)K_1 - c(x)K_2)\psi] \leq \frac{a^2}{4t(b(x)K_1 - c(x)K_2)} \leq \frac{a^2}{4(b_L K_1 - c_M K_2)}. \tag{5.112}$$

Similarly, we find

$$\psi \cdot [e - t(g(x)K_2 - f(x)K_1)\psi] \leq \frac{e^2}{4(b_L K_2 - c_M K_1)}. \tag{5.113}$$

From the definition of K_1 and K_2, we have:

$$a^2 = 4(b_L K_1 - c_M K_2), \quad e^2 = 4(b_L K_2 - c_M K_1). \tag{5.114}$$

We thus obtain (5.111) from (5.112), (5.113) and (5.114). From the generalized sweeping principle or Theorem A3-9 in Chapter 6, substituting at $t = 1$ in (5.110) we obtain the estimates:

$$u^* \leq \frac{(g_L a^2 + c_M b^2)}{4(b_L g_L - c_M f_M)}\psi, \quad v^* \leq \frac{(b_L e^2 + f_M a^2)}{4(b_L g_L - c_M f_M)}\psi, \tag{5.115}$$

for any positive solution (u^*, v^*) of (4.22). By comparison with the scalar equations, we readily obtain

$$u^* \geq \theta_{[-L_1,a,b]} \geq \frac{a - \rho_1(-L_1)}{b_M}\psi, \quad v^* \geq \theta_{[-L_2,e,g]} \geq \frac{e - \rho_1(-L_1)}{g_M}\psi. \tag{5.116}$$

We can readily verify that (5.102) is satisfied by using (5.109), (5.115) and (5.116). Consequently, we can apply Theorem 5.13 to complete the proof of this corollary.

As in the above corollary, we can apply the generalized sweeping principle and Theorem 5.13 to deduce other uniqueness and asymptotic stability results as follows. (Since the technique is similar, we will omit the details which is given in Corollary 9.5 in [45].)

Corollary 5.15. *Assume* $L_1 = L_2$, $c(x) > 0$, $f(x) > 0$ *for each* $x \in \Omega$,

$$c_M f_M < b_L g_L, \quad a \geq e > \rho_1(-L_1),$$

and

$$\frac{N_1}{M_2} \cdot \frac{N_2}{M_1} \cdot (sup_{\bar{\Omega}} \frac{\theta_{[-L_1,a,b(x)]}}{\theta_{[-L_2,e,g(x)]}})^2 < \frac{b_L g_L}{c_M f_M}. \tag{5.117}$$

Then problem (4.22) has a unique coexistence state, which is asymptotically stable. Here

$$N_1 = \frac{b_M(g_L + c_M)}{b_L g_L - c_M f_M}, \quad N_2 = \frac{b_M(b_L + f_M)}{b_L g_L - c_M f_M},$$

$$M_1 = max\{\frac{g_L(c_L + g_M)}{b_M g_M - c_L f_L}, \frac{(c_L + g_L)[g_M(b_M + f_M) - f_L g_L]}{b_M[g_M(b_M + f_M) - f_L(c_L + g_L)]}\},$$

$$M_2 = max\{\frac{g_L(f_L + b_M)}{b_M g_M - c_L f_L}, \frac{g_M(b_M + f_M)}{g_M(b_M + f_M) - f_L(c_L + g_L)}\}.$$

In the Volterra-Lotka model (4.1) for cooperating species with constant coefficients, we may perform stretching of variables in u and v to attain without loss of generality $\sigma_1 = \sigma_2 = b = g = 1$. In this case, Corollary 5.15 simplifies into the following result as in Theorem 3.3 and 3.4 in Korman and Leung [107].

Corollary 5.16. *Consider problem (4.1) with*

$$\sigma_1 = \sigma_2 = b = g = 1. \tag{5.118}$$

Suppose

$$cf < 1, \;\; a \geq e > \rho_1(-\Delta),$$

and

$$(sup_{\bar{\Omega}} \frac{\theta_{[-\Delta,a,1]}}{\theta_{[-\Delta,e,1]}})^2 < \frac{1}{cf}. \tag{5.119}$$

Then problem (4.1) has a unique coexistence state, which is asymptotically stable.

Proof. Under the assumptions of this corollary, the constants M_1, M_2, N_1 and N_2 of Corollary 5.15 satisfies

$$N_1 = M_1 = \frac{1+c}{1-cf}, \quad N_2 = M_2 = \frac{1+f}{1-cf}.$$

Consequently hypothesis (5.117) becomes (5.119). The result follows from Corollary 5.15.

There are some results for global attractivity of positive solution. In the situation when we have uniform a-priori bound as in Theorem 5.13, we can apply the following topological result in Hirsch [86] to the parabolic system associated with (4.22).

Theorem 5.17. *Assume that T is a strongly positive monotone continuous dynamical system on X where the cone K has non-empty interior and X is separable. Moreover, assume that the closure of the positive semi-orbit $O(x)$ of x is compact for each $x \in X$. Then, there exists a dense subset A of X such that if $x \in A$, then $\omega(x)$ (the ω-limit of x), is contained in the set of stationary points.*

Due to excessive technicalities, we will omit the details of the above theorem. As a consequence of the theorem, we have the following result.

Theorem 5.18 (Global Attractivity). *Assume that $c_M f_M < b_L g_L$, $a > \rho_1(-L_1)$, $e > \rho_1(-L_2)$, $c(x) > 0$, $f(x) > 0$ for each $x \in \Omega$, and that problem (4.22) has a unique coexistence state, say (u^*, v^*). Consider the following* *corresponding parabolic problem:*

(5.120)
$$\begin{cases} u_t = L_1 u + u[a - b(x)u + c(x)v] & \\ & \text{in } \Omega \times (0,\infty), \\ v_t = L_2 v + v[e + f(x)u - g(x)v] & \\ u = v = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = \hat{u}_0(x),\ v(x,0) = \hat{v}_0(x) & x \in \Omega, \end{cases}$$

where $\hat{u}_0, \hat{v}_0 \in C(\bar{\Omega})$. Then the solution of (5.120) is defined for all $t > 0$ and there exists a dense subset A of $(C(\bar{\Omega}))^2$ such that if $\hat{u}_0 > 0, \hat{v}_0 > 0$ and $(\hat{u}_0, \hat{v}_0) \in A$, then

(5.121)
$$lim_{t\to\infty}||u(x,t) - u^*||_\infty = lim_{t\to\infty}||v(x,t) - v^*||_\infty = 0.$$

The idea of the proof is as follows. If $(\hat{u}_0, \hat{v}_0) \in A$, we first show by means of Theorem 5.17 that the solution of the parabolic problem (5.120) converges to some steady state. Then we use comparison method to show that if $\hat{u}_0 > 0$ and $\hat{v}_0 > 0$, the solution converges to a positive coexistence state. Finally, (5.121) follows from the assumption on uniqueness of positive coexistence state. More details can be found in the proof of Theorem 9.8 in [45]. They are omitted in order to condense the length of this section.

Notes.

Theorem 5.1 and Theorem 5.2 can be found respectively in Li [148] and Li and Ghoreishi [149]. Theorem 5.3 is obtained from Li and Ramm [152], and Theorem 5.4 is adopted from methods in Leung [123]. Theorem 5.5 is found in Leung [122]; Theorem 5.7 to Theorem 5.9 are due to Cosner and Lazer [32]. Theorems 5.10 and 5.11 are found in Dancer and Guo [42]. Theorem 5.12 is obtained from Dancer [40]. Theorem 5.13, Corollaries 5.14 to 5.16 and Theorem 5.18 are due to Delgado, López-Gómez and Suarez [45]. Theorem 5.17 is obtained by Hirsch [86].