Preface

Substantial progress had been made in the last two decades in the theory of nonlinear systems of partial differential equations. Much of the developments are motivated by applications to the natural sciences of biology, physics and chemistry. There is a considerable amount of results concerning positive solutions for the study of ecological and medical sciences. Other applications involve reactor dynamics, fluid, plasma, display technology etc. There are several excellent books published in such topics in the last decade; however, due to numerous recent developments of new methods and results there is a need for a book to collect them for convenient reference and study. The gathering of many existing theorems enhances the understanding of the subject and leads to directions for further research or applications. In the mean time, the demand for reliable applications encourages deeper understanding of the underlying mathematical methods of nonlinear partial differential equations. Many of the problems were introduced in my first book in 1989. In the last twenty years, there is tremendous progress in the mathematical formulation for studies in cancer, cardiology, epidemiology and cell development etc., leading to larger systems of nonlinear partial differential equations. More thorough understanding of the interaction between a few components is crucial for building to large systems. Serious efforts are made to make this book self-contained. Although many theorems used in the book are presented in other books or papers, we include their explanations in the Appendices so that this book is more readable to many graduate students and researchers who are not specialists in these topics.

For the study of positive solutions, several methods are used extensively in this book. Topological degree theory method is extremely fruitful in proving the existence of positive equilibrium for several coupled elliptic systems. One of the most important tools in nonlinear analysis is the Leray-Schauder degree. Due to the fact that the positive cone is a retract of a Banach space, it is possible to define a fixed point index for compact maps as introduced by H. Amann. The fixed point index is equivalent to Leray-Schauder degree. Many existence theorems follow from the property of homotopic invariance of degree. Another powerful method in nonlinear analysis is the use of bifurcation theory as developed by Crandall and Rabinowitz. Bifurcation of solutions may occur at points where the implicit function theorem does not apply. Estimation of solutions by means of maximum principle combined with global bifurcation theory provides convenient analysis of the behavior of positive equilibria as various parameters changes. Many diagrams are included describing the range of parameters so that coexistence can occur. Through the use of maximum principle for $W^{2,p}$ solutions, the book considers the theory of non-classical solutions for interacting species. By means of weak upper-lower solutions, it studies solutions with discontinuous and highly spatially varying growth rates.

For the study of parabolic time-dependent problems, we use both semigroup methods and the classical Schauder's theory. The semigroup method provides existence of solution of initial value problems in various function spaces. Combined with the spectral analysis of the related linearized parabolic system, we obtain many time-stability results for positive equilibria. Comparison theorems for parabolic systems under various boundary conditions also provide estimates of solutions by means of upper and lower solutions.

A significant part of the book is devoted to the study of optimal control of systems of nonlinear partial differential equations as developed by J. L. Lions. They are systems motivated by applications involving equilibrium or time dependent problems. The object is to control the coefficients of the systems so that certain properties of the subsequent solutions are maximized. Both the theories of weak and classical solutions are used. A larger optimality system of equation is deduced for the optimal control. Combined with the method of upper-lower solutions, we construct monotone sequences converging to estimates of the optimality system.

The book also describes results concerning systems of nonlinear wave equations and traveling wave solutions for parabolic systems. The system of wave equations is analyzed by semigroup method. In contrast to the popular method of finding traveling wave solutions by means of dynamical system theory, we carefully explain the method of finding traveling solutions for parabolic systems by using upper-lower solution in an unbounded domain. Other topics studied include invariant manifolds for coupled parabolic-hyperbolic systems, cross-diffusion for elliptic systems, persistence, blow-up due to boundary inflow, coupled elliptic-parabolic system related to display technology, degenerate diffusive systems and other related topics. Although the systems are motivated by applications, the techniques of analyzing such types of problems are carefully explained. They involve extensions of methods described in the above paragraphs.

Chapter 1 considers systems of two coupled nonlinear elliptic or parabolic equations. The nonlinear terms incorporate the interactions between two life species occupying a common domain. The cases of competition, cooperation or prey-predator relationship are covered in detail in separate sections. The boundary values are given in the Dirichlet type. The trivial vector function is always a solution of the systems. The major concern is the additional possibility of coexistence solutions when both species survive together. We use the methods described in the last few paragraphs to find coexistence states under various configuration of interaction parameters, diffusion rates and size of the environment. Many results are related to the principal eigenvalues of various scalar problems induced by the original larger system. The time-stability of the coexistence states for all the three cases are discussed in the last section of the chapter. Some other long-time behavior of the corresponding reaction-diffusion systems are also studied. Most of the results are published by many researchers in the nineties or afterwards, and cannot be found in other books. Chapter 2 extends our discussion of problems in the first chapter to larger systems of equations. The species components may now be classified into groups inside which they interact in competition, cooperative or food-chain manner, while the different groups interact in various ways. The conditions for coexistence becomes more complex. However, we see the methods developed in the first chapter can be extended to cover many different cases. For practical applications, we consider analysis of epidemics, fission reactor engineering and other problems.

Chapter 3 studies the optimal control of nonlinear systems analyzed in the first two chapters. We control the interaction parameters or boundary conditions in order to optimize an expression involving the solutions of the systems. Using the understanding of the uncontrolled systems in the last chapters, we deduce conditions when optimal control is possible. We consider the control of elliptic, parabolic and time-periodic systems. For biological systems, we maximize the economic return of species-harvesting; and for reactor problems, we optimize the target temperature profile. From the original systems together with the optimization criteria we deduce larger optimality systems which describe the optimal controls. We further analyze the solution of the optimality systems by means of monotone convergence schemes. So far, results for such systems have not been gathered coherently in a book form.

Chapter 4 emphasizes on other aspects of the solutions of the reaction-diffusion systems. We consider conditions on the equations when certain components can persist indefinitely in time. We study the effect of diffusion rates which may depend on the concentrations of other species. Such self and cross-diffusion property can have significant effect on the coexistence problem. Questions concerning blow-up, extinction, degenerate diffusion rates and others are also investigated by various methods described above. Chapter 5 first considers traveling wave solutions for competitive parabolic systems. There had been numerous results on such topics over two decades ago found by means of dynamical systems technique. Here, we present some very new recent results found by means of upper-lower solution method in an unbounded domain. We also study a system of hyperbolic equations and the stability of their equilibrium. We further discuss the problem of invariant manifold for solutions of coupled Navier-Stokes and wave equations. Roughly speaking, we find a relationship between the fluid velocity field and the magnetic field so that it is invariant as time changes. Finally, we consider a coupled elliptic-parabolic problem motivated by research on plasma display technology. We estimate whether the sizes of the ion concentrations can reach a high enough level for light emission.

We painstakingly itemize in the Appendices those theories and theorems used in deducing the results in Chapters 1 to 5. They include many standard theorems in scalar and systems of partial differential equations, methods in linear and nonlinear functional analysis and topology. In real world applications, models are very complicated and they have to be continuously improved with deeper investigation. It is therefore important to understand how the applicable results in Chapters 1 to 5 are deduced from the more fundamental theories in the Appendices. With the standard tools conveniently displayed, one can then readily modify the theorems in the first five chapters to forms more suitable for proper utilization. The stress of this book is consequently different from others with similar titles existing in the literature. On the other hand, the presentation of the topics in the first five chapters are motivated by practical applications. Consequently, the results can be applied to real world problems by non-specialists, even if the rigorous proofs presented are not completely understood. Finally, this book can only cover those topics which have interest me, my friends and colleagues. Many other subjects concerning systems of nonlinear partial differential equations are beyond our present scope. I hope that this book is helpful for researchers who will continue to explore on the subject.

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