On abnormal maximal subgroups of finite groups

A. BALLESTER-BOLINCHES*

Departament d'Àlgebra, Universitat de València
Dr. Moliner, 50; 46100, Burjassot, València, Spain
E-mail: Adolfo.Ballester@uv.es

JOHN COSSEY

Mathematics Department, Mathematical Sciences Institute, Australian National University
Canberra, ACT 0200, Australia
E-mail: John.Cossey@anu.edu.au

R. ESTEBAN-ROMERO

Institut Universitari de Matemàtica Pura i Aplicada, Universitat Politècnica de València
Camí de Vera, s/n; 46022 València, Spain
E-mail: resteban@mat.upv.es

In this survey we show the influence of the abnormal maximal subgroups of finite groups in their structure.

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1. Introduction

This survey is about finite groups. Hence the unspoken rule is that all groups considered are finite.

> "What is the role of the abnormal subgroups in the structure of a group?"

is the motivating question in this survey. In fact, the results we present here are contributions to the long-running investigation of the influence on a group of its proper abnormal structure.

*Corresponding author

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Consequently, abnormality, a subgroup embedding property introduced by P. Hall in his Cambridge lectures, is one of the central concepts here. Recall that a subgroup U of a group G is said to be *abnormal* in G if for all $g \in G$, $g \in \langle U, U^g \rangle$.

Theorem 1.1 (Hall, Cambridge lectures). *The following conditions are together both necessary and sufficient for U to be abnormal in G:*

(1) Every subgroup of G containing U is self-normalising.
(2) U is not contained in two distinct conjugate subgroups of G.

Condition 1 is already sufficient if G is soluble. However A. Feldman[1] showed that it is not sufficient in the general case. The unitary group $\mathrm{U}_3(3)$ has a non-abnormal subgroup U isomorphic to Σ_4 such that every subgroup containing it is self-normalising.

Recall the obvious but convenient fact that a maximal subgroup is either normal or abnormal in G. Hence the abnormal maximal subgroups of a group are precisely its non-normal maximal subgroups. Moreover, every abnormal subgroup is contained in an abnormal maximal subgroup.

The history of our results probably begins with Dedekind groups: groups with all subgroups normal. They were first investigated by R. Dedekind.[2] The motivation was algebraic number theory. Dedekind wanted to determine the algebraic number fields with the property that every subfield is normal.

Dedekind groups form a proper subclass of the one composed of all groups with all maximal subgroups normal: the class of all nilpotent groups.

Typical examples of groups in which the set of the abnormal maximal subgroups is non-empty but nonetheless restricted are critical groups with respect to classes of groups containing the class of all nilpotent groups. Recall that a group G is said to be $\mathcal{X}$-critical, for a class of groups $\mathcal{X}$, if $G \notin \mathcal{X}$, but all proper subgroups of G are in $\mathcal{X}$. It seems clear that detailed knowledge of the structure of $\mathcal{X}$-critical groups can give some insight into what makes a group to belong to $\mathcal{X}$. For example, when $\mathcal{X}$ is closed under taking subgroups, a group of least order which does not belong to $\mathcal{X}$ is $\mathcal{X}$-critical. In this case, it is enough to check the condition on proper subgroups only for the maximal subgroups.

Many authors have studied $\mathcal{X}$-critical groups for a number of classes of groups. The focus here is on the classes of nilpotent and supersoluble groups.

The critical groups for the class of all nilpotent groups were studied by O. J. Schmidt.[3] These groups are called nowadays Schmidt's groups. He

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proves that if a group G has all its proper subgroups nilpotent, then G is soluble. In fact, this hypothesis has much stronger implications for the structure of G than solubility.

Theorem 1.2 ([3]). *(1) If every proper subgroup of a group G is nilpotent, then G is soluble.*

(2) Assume that every proper subgroup of G is nilpotent, but G is not nilpotent. Then G satisfies:

(a)
- $|G| = p^a q^b$ *for prime numbers* $p \neq q$,
- *the Sylow p-subgroup is normal in G,*
- *the Sylow q-subgroups are cyclic, and*
- *for every Sylow q-subgroup Q of G,* $\Phi(Q) \leq \mathrm{Z}(G)$.

(b) The nilpotency class of the Sylow p-subgroup P of G is at most two. Moreover, $\Phi(P) \leq \mathrm{Z}(G)$.

(c)
- *For $p > 2$, P has exponent p;*
- *for $p = 2$, the exponent of P is at most 4.*

The complete classification of Schmidt groups was given by L. Rédei.[4] An alternative proof of this result can be found in.[5]

J. S. Rose[6] considered the effects of replacing *proper* by *proper abnormal* in Schmidt's result, and proved the following:

Theorem 1.3. *If every proper abnormal subgroup of a group G is nilpotent, then G is soluble. Moreover, G has a normal Sylow subgroup P such that G/P is nilpotent.*

The hypothesis of the above theorem holds in every epimorphic image of G. Hence, using induction, the solubility of G is a consequence of the following theorem:

Theorem 1.4 (R. Baer[7]). *Let G be a primitive group. If every core-free maximal subgroup of G is nilpotent, then G is soluble.*

The complete characterisation of the groups in the above class is the following:

Theorem 1.5. *Every abnormal maximal subgroup of a group G is nilpotent if and only if either the nilpotent residual A of G is trivial or A/A' is a chief factor of G such that $A' \leq \mathrm{Z}(G)$.*

Proof. Assume that $A \neq 1$ and A/A' is a chief factor of G such that $A' \leq \mathrm{Z}(G)$. Then G is soluble. Let M be an abnormal maximal subgroup

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of G. Then $G = AM$. Since A/A' is nilpotent and $A' \leq \mathrm{Z}(G)$, we have that $A' \leq \Phi(A) < A$. Hence $A' = \Phi(A) \leq \Phi(G)$ and so $A' \leq M$. Consequently $G/A' = (A/A')(M/A')$ and $A \cap M = A'$. Since G/A is nilpotent, it follows that M/A' is nilpotent. Therefore M is nilpotent.

Conversely assume that every abnormal maximal subgroup of G is nilpotent. Then G is soluble by Theorem 1.3. Assume that G is not nilpotent. Then $A \neq 1$. By [8, V, 3.6], G has an abnormal maximal subgroup M such that $G = M\mathrm{F}(G)$. Then M is nilpotent and so M is a Carter subgroup of G. Applying [8, V, 4.2], M is a system normaliser of G. On the other hand, A is nilpotent. Hence $A' \leq \Phi(A) \leq \Phi(G) \leq M$. Thus $G/A' = (A/A')(M/A')$ and, by [8, IV, 5.18], we conclude that $M \cap A = A'$. Therefore A/A' is a chief factor of G. Let p be the prime dividing the order of A/A'. Since $A' \leq \Phi(A)$, it follows that A is a p-group. Since $\Phi(A) \leq \Phi(G) \leq M$, we have that $A' \leq \mathrm{Core}_G(M)$. By [8, I, 5.9], $\mathrm{Core}_G(M) = \mathrm{Z}_\infty(G)$, the hypercentre of G. Applying [8, IV, 6.14], $\mathrm{Z}_\infty(G)$ centralises A. This implies that $A' \leq \mathrm{Z}(A)$. Since M is nilpotent and $\mathrm{O}_p\big(M/\mathrm{Core}_G(M)\big) = 1$, the Sylow p-subgroup of M is contained in $\mathrm{Core}_G(M)$ and so it centralises A. In particular, the p-subgroup A' is centralised by the Sylow p-subgroup of M. Since M is nilpotent, $A' \leq \mathrm{Z}(G)$, as desired. □

Among other published extensions of Schmidt's result, one due to B. Huppert and K. Doerk is of particular interest. B. Huppert[9] proved that if every maximal subgroup of G is supersoluble, then G is soluble. This is of course equivalent to assume that all proper subgroups are supersoluble. K. Doerk[10] studied with more detail the critical groups with respect to the class of all supersoluble groups. We summarise here Huppert and Doerk's results.

Theorem 1.6. *Let G be a critical group for the class of all supersoluble groups. Then:*

(1) G is soluble.
(2) G has a unique non-trivial normal Sylow subgroup P.
(3) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.
(4) The Frattini subgroup $\Phi(P)$ of P is contained in the supersoluble hypercentre of G, i.e., there exists a series $1 = N_0 \leq N_1 \leq \cdots \leq N_m = \Phi(P)$ such that N_i is a normal subgroup of G and $|N_i/N_{i-1}|$ is prime for $1 \leq i \leq m$.
(5) $\Phi(P) \leq \mathrm{Z}(P)$; in particular, P has class at most 2.
(6) The derived subgroup P' of P has at most exponent p, where p is the prime dividing $|P|$.

(7) *For* $p > 2$, *P has exponent p; for* $p = 2$, *P has exponent at most* 4.
(8) *Let Q be a complement to P in G. Then* $Q \cap C_G\big(P/\Phi(P)\big) = \Phi(G) \cap \Phi(Q) = \Phi(G) \cap Q$.
(9) *If* $\overline{Q} = Q/\big(Q \cap \Phi(G)\big)$, *then $\overline{Q}$ is a critical group for the class of abelian groups or a cyclic group of prime power order.*

The critical groups with respect to the class of supersoluble groups were completely classified by V. T. Nagrebeckiĭ.[11] Rose[6] observed that imposing supersolubility only on abnormal maximal subgroups is not sufficient to get solubility. He shows that in $\mathrm{PGL}_2(7)$, every maximal subgroup except $\mathrm{PSL}_2(7)$ is supersoluble. Hence before the classification of finite simple groups, there was no hope of describing those groups. In fact, this classification is used by S. Li and W. Shi[12] to prove the following:

Theorem 1.7. *If every abnormal maximal subgroup of a group G is supersoluble, then the composition factors of G are isomorphic to* $\mathrm{PSL}_2(p)$ *or* C_q, *where p and q are primes and* $p^2 - 1 \equiv 0 \pmod{16}$.

However, a structural description of these groups remains open. In this context, M. Asaad[13] asked for a supersoluble version of Theorem 1.4:

Question 1.1. What can be said about the structure of a primitive group in which all core-free maximal subgroups are supersoluble?

The following result contains the answer to the above questions:

Theorem 1.8 ([14]). *Let G be a group. Then every abnormal maximal subgroup is supersoluble if and only if G satisfies the following conditions:*

(1) *If G is insoluble, then the following conditions hold:*
   (a) $G/F(G) \cong \mathrm{PGL}_2(p)$, *where p is a prime such that* $p^2 - 1 \equiv 0 \pmod{16}$,
   (b) *the soluble residual and the nilpotent residual of G coincide and it is isomorphic to* $\mathrm{PSL}_2(p)$ *or* $\mathrm{SL}_2(p)$, *and*
(2) *If G is soluble, then either G is supersoluble or G satisfies the following conditions, where A is the supersoluble residual of G and M is a supersoluble projector of G:*
   (a) A/A' *is a complemented non-cyclic chief factor of G.*
   (b) $C_M(A/A')$ *contains no non-central complemented chief factors.*

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(c) *Either M is nilpotent or B/B' is a non-central complemented cyclic chief factor of M, where B the nilpotent residual of M. If C is a complement of B/B' in G, then C is abelian of exponent dividing $p-1$, where p is the prime dividing $|A/A'|$.*

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