

The $n = 0$ Discovery

Thomas A. Witten

*James Franck Institute, University of Chicago
Chicago, Illinois 60637, USA
t-witten@uchicago.edu*

We describe Pierre-Gilles de Gennes' 1972 letter explaining polymer swelling as a form of critical phenomenon. We trace the impact of this " $n = 0$ " discovery on polymer theory and experiment. We discuss later developments in mainstream statistical physics that reflect the $n = 0$ insight of this paper. We collect the views of several leading statistical physicists on the significance of the discovery.

1. Introduction

During the midwinter holidays of 1971–1972, the statistical physics community was abuzz with rumors of a breakthrough. A professor named Kenneth Wilson had announced¹ a new way to understand the famous problem of critical fluctuations of certain dense gases. Under just the right conditions these critical gases have regions of high and low density. The regions range from molecular size to thousands of times larger, giving them a distinctive opalescent appearance. Wilson called his method the renormalization group. It embodied a symmetry relating the patterns of density seen at a large spatial scale to the sub-patterns within those patterns on smaller scales. This symmetry under spatial dilation had been qualitatively appreciated by others.² However, Wilson's method of expressing it gave unprecedented power. It allowed one to account for the peculiar measured power laws describing the growth of the fluctuations as the critical conditions of temperature and pressure were approached. Explaining these "critical exponents" was the nub of the problem of critical fluctuations.

The new theory got its power by connecting the intractable critical fluctuations to a simpler regime of weak fluctuations. The new approach had a generality that went beyond critically opalescent gases. Since the anomalous power-law behavior arose from a symmetry, any system with the same form of symmetry should show the same power laws. Moreover, fluctuating quantities more general than a scalar density field could be explained. For

example, the theory could also explain the vector fluctuations of the local magnetization in a metal on the verge of becoming a ferromagnet. Indeed, it could handle analogous fluctuations of a general vector field with an arbitrary number of components n . For the gas, $n = 1$; for the magnet, $n = 3$. The vector dimension n is often called a spin index, adopting the language of magnetic phase transitions.

As these rumors about the Wilson theory were circulating, Pierre-Gilles de Gennes was on a skiing vacation. But other things were on his mind. As reported by Françoise Brochard,³ he had been in the United States the previous summer and had discussed Wilson's ideas with physicists there. On his return to France in Autumn 1971, de Gennes had lectured on these ideas in his course at the College de France (lecture 15). He had seen a connection between Wilson's theory and a very different puzzle — the famous “excluded volume problem” of polymer physics (lecture 16). Polymers are long chain molecules whose links twist randomly under thermal fluctuations when they are dissolved in a suitable liquid. Such a chain is a type of random walk, whose spatial extent should vary as the square root of its length. However, the measured properties of real polymers were different from this prediction. The spatial extent grew too quickly with chain length, and there was no systematic explanation for the anomalous power law describing its growth. There were only *ad hoc* arguments that attempted to account for the excluded volume or self avoidance of the chain. These attempts tried to extrapolate from the simple cases of one contact between parts of the chain, then two contacts, and so forth.

De Gennes and others had noticed a resemblance between this counting of contacts and an analogous form of counting in a dense gas. The gas molecules — like the polymer sub-units — cannot intersect one another. This collective mutual repulsion must be treated by a painstaking calculational scheme in order to avoid double counting. It was this scheme which resembled the perturbations of a polymer owing to self avoidance. Both schemes kept track of the accounting by means of diagrams representing classes of self-intersecting configurations to be corrected for. As de Gennes puzzled over the two forms of diagrams, he had noticed closer and closer correspondences, especially when he used Wilson's style of accounting. In Wilson's method, the same set of diagrams described fluctuating fields with arbitrary numbers of vector components n . In this scheme there was a perfect correspondence between each polymer diagram and a corresponding vector field diagram. Moreover, the mathematical expression associated with a given polymer diagram was identical to that for the vector field, provided one chose the proper value of n . What was this magical value of n ? Zero!⁴

Now, these diagrams were the only input needed by the Wilson theory in order to determine its renormalization-group symmetry, and thence its critical exponents. The approximate formulas for these exponents were expressed as explicit functions of n . By setting $n = 0$ in these formulas, one was discovering information about self-avoiding polymers. Interpreting these power laws required another insight of interpretation. How could the critical exponents describing thermodynamic effects involving density and temperature apply to the purely geometric polymer properties of chain length and spatial extent? The field theory language of the Wilson theory allowed one to make the proper correspondence, by recalling the physical situations that each diagram represented. Now de Gennes could explain the

excluded volume exponent of polymers merely by transcribing from the Wilson theory of critical phenomena.

Armed with this insight, making this transcription was a simple matter. It could be explained in a 1 1/2-page paper. It was this paper that the journal *Physics Letters* received on January 10, 1972.⁵ It appeared, peppered with misprints, in the issue of February 28. With this paper two of the most important puzzles of statistical physics were recognized as aspects of a single phenomenon. Often when two very different phenomena are unified in this way, the implications are profound. De Gennes $n = 0$ discovery carried such implications. First, it cast important light on Wilson's renormalization symmetry. Up to that point, renormalization symmetry had been viewed as an exclusive feature of phase transitions. As such, the notion of renormalization was entangled with the separate notion of spontaneous symmetry breaking, a necessary feature of phase transitions. The new polymer realization of renormalization symmetry disentangled these two features. Polymers were not a phase transition and had no spontaneous symmetry breaking. Thus the polymer case provided insight into the essence of renormalization symmetry. As important as this insight was for renormalization symmetry, the implications for polymer physics were no less profound. Transcribing well-developed arguments from phase transitions to their polymeric counterparts led to clear cut predictions for a range of polymer properties where previous understanding had been murky and ambiguous.

In this chapter I explore the implications of the $n = 0$ paper. I first describe the paper itself, providing a gloss on each paragraph in turn. Then I review the impact on polymer physics. Next I review the impact on mainstream renormalization theory and statistical physics. Finally I report the reflections of several prominent physicists whose work was influenced. Since our task is to judge the importance of an idea, the outcome is necessarily subjective. The account below comes from an avowed devotee of the $n = 0$ paper. Other observers might relegate the $n = 0$ paper to a much smaller historical importance. A thoughtful weighing of the evidence might prove them right. With these caveats, I give my own subjective and tentative impressions below.

2. Gloss

The note is titled "Exponents for the excluded volume problem as derived by the Wilson method." Clearly this terse account is addressed to experts. The excluded volume problem is the polymer physicist's name for explaining how the size R of a self-repelling polymer grows with its length N . This amounts to finding the exponent ν in the formula $R \sim N^\nu$.

The Wilson method refers specifically to a technique of calculating the diagrams mentioned above. Since the numerical value corresponding to a given diagram depends only on distances between selected interaction points, each diagram has a simple, explicit dependence on the dimension d of space. The critical point where the interesting behavior appears is just the point where the numerical values diverge. The technical achievement of the renormalization group is that it provides a way to make sense of the diagrams despite this divergence. Still, concrete predictions using the diagrams require definite numbers, not divergent integrals. Here a final observation by Wilson was crucial. The divergences

went away above a special dimension: $d = 4$. If one pretended that the dimension of space were very close to four, one could obtain finite quantities by the technique of performing a Taylor expansion in the small parameter $\epsilon \equiv 4 - d$. (Thus Wilson's famous renormalization group paper — still unpublished at the time of de Gennes' writing — was titled "Critical exponents in 3.99 dimensions.")¹ The Wilson paper had calculated critical exponents by expanding to second order in this ϵ , and then obtaining the prediction for three-dimensional space by evaluating the expansion at $\epsilon = 1$.

De Gennes' two-sentence abstract reports the result of this procedure using his $n = 0$ realization to calculate ν .

The body of the paper begins by defining the partition function needed to determine the average end-to-end separation \bar{R} of a self-avoiding polymer. It is the number of distinct self-avoiding walks of length N with ends separated by R , denoted^a $\Gamma_N(R)$. If $\bar{R} \sim N^\nu$, this Γ quantity must show it. In order to make contact with Wilson's quantities, de Gennes immediately introduces the quantity G , a transform of Γ . The spatial variable is replaced by its Fourier transform variable k . The chain length variable is likewise replaced by its Laplace transform P , nowadays known as the monomer chemical potential. At this point, G is claimed to have a divergence as this potential approaches a certain value P_c : $G(0, P) \sim (P - P_c)^{-\gamma}$. This amounts to saying $Z_N \equiv \sum_R \Gamma_N(R) \sim N^{\gamma-1} e^{NP_c}$. This kind of scaling behavior had been suspected on numerical grounds^{12, 13} but not established. Thus this claim is justified by the arguments to follow.

A second claim follows, regarding the dependence of $G(k, P)$ on k when P takes its critical value P_c . G is said to vary as a power of k involving a second exponent denoted η . The justification for this claim is also found in the arguments to follow. Now the desired exponent ν governing the size \bar{R} of the chain is claimed to be related to the γ and η just defined. The "usual scaling arguments" are invoked to justify it.

These arguments amount to saying that $G(k, P)$ has a fixed functional form as $\Delta P \equiv P - P_c \rightarrow 0$. The fixed functional form means that the only difference between $G(k, \Delta P)$ and the same quantity evaluated λ times farther from the critical point is a difference of scale factors in the k and G dependence:

$$G(k, \Delta P) = \mu_G G(\mu_k k, \lambda(\Delta P)),$$

where $\mu_G(\lambda)$ and $\mu_k(\lambda)$ are the scale factors needed to reflect the expansion of ΔP by a factor λ . This homogeneity or scaling law connects the claimed behavior of G for $k = 0$ to that for $\Delta P = 0$. From the $k = 0$ behavior we infer $\mu_G \propto \lambda^\gamma$, (by taking $\lambda = 1/\Delta P$). From the $\Delta P = 0$ behavior, we infer $\mu_G \propto \mu_k^{2-\eta}$. Thus $\mu_k \propto \lambda^{\gamma/(2-\eta)}$. That is, a change in Δp (or inverse chain length) by a factor λ is equivalent to a change in k (or inverse spatial distance) by a factor $\lambda^{\gamma/(2-\eta)}$.

This is the content of the statement^b $\nu = \gamma/(2 - \eta)$ in de Gennes' note. Such "usual scaling arguments" were common and accepted in the phase transition literature,¹¹ but not among polymer physicists.¹² The reason for this difference may be experimental. For phase transitions, experimental support for these scaling arguments in the vicinity of the critical

^aHere an apparent misprint was corrected.

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temperature T_c spanned 4 to 5 decades in $T - T_c$ in 1971.¹¹ The corresponding range of N in polymer systems was scarcely two decades, and was subject to technical uncertainties in determining the chain length N . Thus, de Gennes' invocation of the "usual scaling arguments" rested chiefly on the correspondence to phase transitions that was to be argued in the sequel.

Shortly de Gennes will describe the diagrammatic correspondence between the polymer problem and the phase transition problem. The structure of the polymer diagrams, which accounted for a self-repulsion between monomers with a strength v_0 had been articulated e.g. by Fixman.¹⁴ The $n = 0$ correspondence with phase-transition diagrams is to be asserted below. But first de Gennes makes a digression to comment on the "renormalized repulsion" denoted v_R . The digression is aimed at explaining a widely believed property of polymers in Wilson's renormalized language. The strength of the repulsion v_0 was believed not to affect the value of the exponent ν . This fact is naturally explained if the diagrams can be cast into a form in which v_0 is replaced by a fixed quantity. The renormalization procedure aims to perform this recasting.

Renormalization aims to describe the calculation of a quantity such as $G(k, P)$ by replacing the original lattice degrees of freedom such as monomer density by local averages. One may account for the overall effects of self-avoidance using only these local averages. Moreover, the locally averaged calculation has the same form as the original one, except that the input parameters such as v_0 have different values. The quantity playing the role of v_0 in the locally averaged calculation is denoted v_R . One may determine v_R from v_0 for a given type of averaging. On the other hand, one may use the anticipated scaling dependence of the calculated quantity e.g. $G \sim k^{-2+\eta}$ to determine how v_R must vary with the averaging length scale. Equation (3) is an expression for v_R of this form. It does not refer to an averaging length scale directly; rather, it uses $P - P_c$ the monomer chemical potential that would generate polymers of the desired scale. The aim of this equation is to show that v_R reaches a fixed value.

In the following paragraph de Gennes sketches a way to reach the same conclusion using a variant of the polymer problem in which the polymer is confined to a finite volume with a fixed, small, average monomer density. Both of these arguments about v_R are sketchy and obscure. In any case they address a side issue. They aim to justify a point that will be better justified by the simple observation in the following paragraph.

The next paragraph gives the central observation of the paper: the G function for polymers is a special case of a corresponding G function for phase transitions. The argument is based on the perturbation diagrams described above. Figure 1 shows a typical diagram for the counterpart of G for an n -vector field. Each line in the diagram accounts for the correlations of the field at the two end points of the line. The lines terminate at interaction points shown as heavy rectangles. The field interacts with itself at every point; the statistical weight of the field at any point differs from what it would be without the interaction. This altered statistical weight influences the field elsewhere because of its correlations, in particular, the field at other interaction points is altered. By allowing the interaction points to be at arbitrary positions one accounts for part of the effects of the interaction on G . This diagram represent a mathematical expression that contributes to the Taylor expansion for

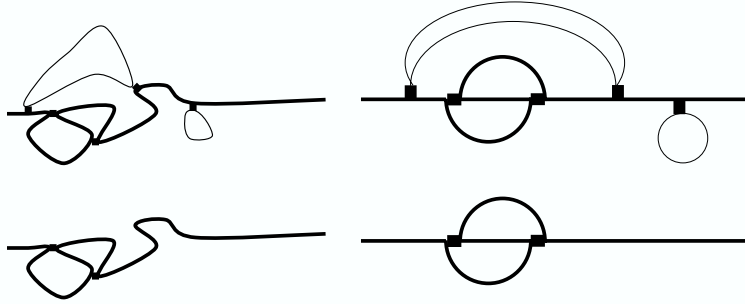


Fig. 1. Top: Perturbation diagram for the n -vector field shown in two equivalent drawings. Meaning of the lines and rectangles is explained in the text. Light lines denote loops. Bottom: Perturbation diagram for a self-repelling polymer, also shown by two equivalent drawings. The lower pair is obtained from the upper pair by removing the loops.

G in powers of v_0 . Since there are five interaction points, this diagram is part of the v_0^5 term in this expansion. By connecting the interaction points in all possible ways and adding the corresponding mathematical expressions together, one obtains the full v_0^5 contribution.

One feature of the illustrated diagram is the appearance of the two loop lines shown as narrow lines. These lines indicate field correlations that pass from one interaction point to another and eventually return to the original point. The light and dark lines have the same mathematical meaning. Such a loop disturbance can occur for any of the n degrees of freedom of the vector field. Thus the overall disturbance carried by such a loop is proportional to n . A factor n for this loop must appear in the mathematical expression for the diagram. The same is true for both loops in this diagram and for all other loops in other diagrams.

Now we consider a polymer diagram for the G of Eq. (1). Its interpretation is simpler. The thick line represents the polymer, following some arbitrary path through space. As pictured in the diagram this polymer intersects itself at two points indicated by black rectangles. The statistical weight of such paths is altered by the self repulsion of strength v_0 . This diagram, with its two interaction points, thus contributes to the power series expansion in v_0 in order v_0^2 .

We now observe that the polymer diagram may be obtained from the n -vector field diagram by simply removing the two loops. This fact, illustrated with this diagram, is true in general. If one considers an arbitrary n -vector diagram and removes the loops, one obtains a polymer diagram. Conversely, every polymer diagram is contained (exactly once) within the set of loop-removed n -vector diagrams. As noted above, the mathematical expression corresponding to the polymer diagram is identical to that for the n -vector diagram. This equivalence arises partly from a fundamental similarity between the two systems. In the polymer diagrams, the lines between interaction points represent random walks (segments of noninteracting polymer) between the points. These random walks are equivalent to a diffusive motion, as Einstein noted in his famous paper relating Brownian motion and diffusion.⁶ In the n -vector diagrams the corresponding line represents spatial correlations between the fluctuating vector field at two different points. In effect, these correlations

also arise by diffusion: a statistical weight tending to make the field at a point equal to the average of the field at neighboring points. The equivalence between the polymer and the n -vector diagrammatic expressions also required some mathematical jiggering. The quantity G constructed in de Gennes' Eq. (1) has been devised to make the mathematical correspondence exact.

Once one knows the mathematical expression for the n -vector diagrams, it is absurdly simple to extract the expression for the polymer diagrams. This simplicity arises from our observation above that each loop in an n -vector diagram contributes a factor n to the mathematical expression for that diagram. We have noted that the sum of contributions from all the (infinite number of) diagrams completely accounts for the effects of the interaction v_0 on G . Once one has the expression encompassing all n -vector diagrams, the corresponding expression encompassing all loopless diagrams is thus obtained by simply setting $n = 0$ in the n -vector expression. Thus the G function for polymers is a mere special case of the G for the n -vector field. Naturally this equivalence extends to the critical exponents of the n -vector model.

De Gennes' paragraph points out this diagrammatic correspondence, without pictures, in a way that an expert in these diagrams can follow. The next step is to make use of the correspondence. He quotes Wilson's calculation for the exponents γ and η expanded to next leading order in the the difference of spatial dimensionality d from 4. He evaluates these expressions for $n = 0$ and $d = 3$, then uses the exponent relations cited above to get the desired exponent ν . This ν is then compared with the best available value and observed to be in good agreement. He also compares with two theoretical hypotheses which, like the Wilson values, can be evaluated for general dimension d . He notes that their expansions in $d - 4$ do not agree with the Wilson counterparts.

Though de Gennes cites a precise value of ν and compares it closely with other values, one should note that the $d - 4$ expansion provides only ambiguous information about the three-dimensional world. For example, if one expands $2\nu = \gamma/(2 - \eta)$ to order $(4 - d)^2$, and evaluates it for $d = 3$, one obtains $2\nu = 1.1835\dots$, rather than de Gennes' value of 1.195. By using such schemes, one could obtain a wide range of estimates for ν . The Wilson prescription does not provide a way to choose among such alternative estimates.

The claims of this brief paper seem to have been immediately accepted by the statistical physics community. The $n = 0$ finding is reported as established in the Cargese lectures on renormalization group⁷ in 1973 and in the review article by Wilson and Kogut⁸ in 1974.

3. Polymer Impact

3.1. *Antecedents*

The excluded volume problem addressed in Ref. 5 is at the heart of much of polymer physics. Any property related to a concentration or molecular weight dependence in a good solvent necessarily involves the exponent ν cited above. In order to deal with these properties decisively, one must thus have a framework for understanding this ν and hence the excluded volume problem.

The importance of the excluded volume problem had been recognized since early 20th century, when it was first established that polymers were flexible chains⁹ of many independent statistical segments. By mid-century this common feature of polymers had given rise to a powerful body of theory led by Paul Flory.¹⁰ Flory devised a simple and appealing explanation of why self-repulsion or excluded volume causes a polymer's size R to grow as its molecular weight M to a power ν greater than $1/2$. The Flory theory predicted $\nu = 3/(d+2)$ in d dimensions, and this value was completely consistent with experiments. The explanation was deemed adequate by those who worked with polymers daily. Yet efforts^{12,14} to justify its assumptions and develop the idea into a systematic theory remained at the level of *ad hoc* attempts.

Meanwhile, the analogous statistical problem of the self-avoiding walk on a lattice was recognized as fundamental. Statistical theorists like Fisher¹⁵ and Domb¹⁶ were devising systematic ways to infer the asymptotic functional form of $R(N)$ and the number of walks Z_N of length N . One approach was to enumerate all self-avoiding walks up to a given size, and then examine the consistency of these exact results with a given asymptotic form. A second approach was to view the self-avoidance constraint as a perturbation on the ensemble of unrestricted random walks. Since the large- M behavior of unrestricted walks was completely understood, the limitations of the small- M enumerations were removed. One could then calculate the effects of this perturbation systematically by assigning a statistical penalty v_0 for each intersection and then performing an expansion in powers of v_0 .¹⁴ This approach leads to the diagrams discussed in the last section. However, the limit of interest is one where the unrestricted walks have many self-intersections and hence many complicated diagrams. Thus the potential of this method seemed very limited. Only for dense and strongly interpenetrating polymer solutions could these diagrams be reduced to a tractable form,¹⁷ in this concentrated regime all signs of excluded volume swelling had vanished. Yamakawa's¹⁸ widely-read 1971 monograph reviewed these different approaches. It was clear that understanding of the excluded-volume effect was fragmentary and unsatisfactory.

3.2. *Early impact*

Almost immediately after the $n = 0$ discovery, de Gennes' colleague Des Cloizeaux proved an important generalization.¹⁹ The $n = 0$ paper concerns the critical state of a gas or magnet at a temperature T_c at which a symmetry begins to be spontaneously broken. Here the spatial correlations grow to infinity in their range. For systems with $n > 1$, a form of long-range correlation persists in the symmetry-broken, magnetized state below T_c . Spontaneously broken symmetry means that all directions of magnetization are equal in energy. Thus the magnetization is free to rotate throughout the sample with no energy cost. This freedom of rotation is known as a Goldstone mode. It amounts to an infinite range correlation. This range is only made finite by applying a symmetry-breaking magnetic field.

Remarkably, an analogous phenomenon occurs in the polymer domain. To find the analogy, one has only to ask what the perturbation diagrams for $T < T_c$ look like when $n = 0$. The diagrams consist of many polymer lines interacting with each other. The density of ends is proportional to the applied magnetic field. Evidently this is equivalent to the concentration of polymer chains. When this end concentration is low for a given

monomer concentration, the polymers must be long. As the magnetic field goes to zero, the length and size of these polymers evidently diverge. This is the polymer analog of the divergent correlation length below T_c for a magnet. Again, a complicated system of fluctuating spins has been replaced by a prosaic liquid containing flexible chain molecules.

A concrete interpretation of these so-called semi-dilute solutions soon emerged in the neutron scattering group connected with de Gennes. The solution could be viewed as a dense packing of self-avoiding “blobs” linked end-to-end to form chains. These chains should show no self-avoidance beyond the scale of a blob. This picture led to several direct predictions for how the scattered neutron intensity should depend on wave-vector, chain length and concentration. Soon after the $n = 0$ paper, these predictions were satisfyingly confirmed.²⁰ A year before the $n = 0$ paper de Gennes had conceived the notion of reptation to describe the kinetics of relaxation of an entangled polymer solution.²¹ The notion of blobs complemented the reptation idea and extended its predictive power.

3.3. *Further implications*

Theoretical reflection turned up further striking analogies. One was an extension of the critical amplitude ratios seen in phase transitions. For example one may observe the divergence of the correlation length ξ by approaching the critical temperature T_c from either above or below. The rate of divergence is governed by the same critical exponent ν in both cases. Thus the ratio of the two lengths $\xi(T_c + \Delta T)/\xi(T_c - \Delta T)$ must remain finite as $\Delta T \rightarrow 0$. It was observed that this ratio took the same value for all critical points of the same type. The Wilson theory justified this similarity. Like the critical exponents, these amplitude ratios were calculable by the renormalized theory, which was independent of the specifics of the critical system being studied. Similar universal ratios should exist in polymer solutions. The most accessible ones were not comparisons between the quantities above and below T_c . But they did preserve the general feature of being ratios of diverging quantities known to diverge with the same power.²²

A variant of critical phase transitions proved to have a striking polymer analog. It can happen that magnetism can set in at the surface of a metal slightly before magnetism appears in the bulk. The correlations (the lines in the perturbation diagrams) have altered mathematical expressions in the vicinity of the surface, owing to the boundary conditions it imposes on the field. The result is a critical point with a new renormalization symmetry, known as the “special critical point.” By examining the corresponding diagrams for $n = 0$ one can divine the polymer analog. Polymers are weakly attracted to the surface of the solution — so weakly that the adsorbed polymers extend arbitrarily far into the solution. The new exponents appear in the form of the concentration profile.²³ Again an esoteric phenomenon in the domain of phase transitions proves to have a prosaic and accessible polymer analog. Soon polymer theorists found further variants. Further new exponents arose when the polymers were adsorbed in a wedge or a corner.²⁴

Without motivation from phase transitions, polymer theorists sought to find how exponents such as ν would be influenced by changing the architecture of the polymers. The chemists could make star-shaped polymers with a desired number f of arms. They could also make loops. These each showed instructive behavior when one examined the quantity

Z_N giving the number of configurations. As noted above, this Z_N for an ordinary polymer is governed by the exponent γ . However, for loops, one need only know the exponent ν governing the size. For the stars, the exponent γ_f depended on the number of arms f in a nontrivial way. These exponents γ_f are experimentally relevant. They dictate the thermodynamic work required to assemble the star from separated arms. Analogous quantities in phase transitions were soon found. By comparing the diagrams, one could identify the γ_f exponents as “anomalous dimensions of composite operators.”⁷ visible only by evaluating multi-point correlation functions in special limits. In the polymer case, all this subtlety was sidestepped simply by synthesizing a polymer with the proper architecture.

Like polymers, diffusing particles are random walks. Thus it is easy to represent the interaction between diffusing particles and polymers by simple modifications of the $n = 0$ diagrams. One had only to remove the self-interaction lines from the diagram lines representing the diffusing particles.²⁵ If f of these particles meet the polymer at a point, one may again ask how the number of configurations varies with f . The corresponding exponents $\tilde{\gamma}_f$ describe the interaction of a self-avoiding polymer that absorbs a diffusing substance. These $\tilde{\gamma}_f$ exponents again prove to have a nontrivial dependence on f . This fact had conceptual importance. It showed that the probability distribution of absorption on a dilation-symmetric structure like a polymer is a multifractal or fractal measure.²⁶ These fractal measures had been postulated in other phenomena, such as the distribution of vorticity in a highly turbulent fluid. A fractal measure has a complicated form of dilation symmetry that requires a one-index set of critical exponents to characterize it. The polymer absorber example showed in a straightforward way how fractal measures were a natural consequence of diffusion onto a dilation-symmetric object.

By the early nineties, the methodology of renormalization was fully accepted by polymer theorists as the way to treat excluded volume effects. It had grown beyond its origins in phase transition theory as an independent realization of the basic renormalization ideas.²⁷ The methodology was explained in textbooks and monographs by polymer theorists.^{28–30} Polymer theorists could readily extract the polymer implications from discoveries in the phase-transition domain. They could also generalize the formalism to discover new forms of dilation symmetry in polymer phenomena with no obvious counterpart in phase transitions. An example is the new exponent governing the demixing of immiscible polymers in a common solvent.³¹ The $n = 0$ paper opened the door to this culmination.

4. Field Theory Implications: A Paradoxical Limit

The aim of the $n = 0$ paper was to address polymer phenomena. Still, the $n = 0$ discovery inevitably had an influence on the statistical field theory that made the discovery possible. Some of this influence has been indicated above. The polymeric realization of renormalization symmetry sharpened understanding of this symmetry by decoupling it from the domain of phase transitions and spontaneously broken symmetry. It added a new level of concreteness to our idea of what it means for a system to embody renormalization symmetry. The polymer aspect of phenomena like composite operators enabled insight into field theory that would otherwise have been more difficult.³² The polymer example was the first instance of

self-organized criticality³³ where dilation symmetry is attained without the need to tune a parameter like temperature to its critical value.

In this section we deal with a deeper implication of $n = 0$. The idea of $n = 0$ forces one to embrace a paradox. How could a vector field with zero components represent anything but empty space? How could one imagine a broken-symmetry field with minus one Goldstone modes? The perturbation diagrams give a perfectly unambiguous answer to these questions. But since the diagrammatic correspondence is complete, $n = 0$ represents a polymer beyond the confines of any calculational technique. The passage to the paradoxical state reveals a completely new kind of object. The $n = 0$ paper does not dwell on this paradox. Still, it establishes a clear meaning for a state where no meaning would have been thought possible beforehand.

Since the $n = 0$ paper, several other $n = 0$ states have been discovered. Like the polymer case, these states, when viewed naively, represent nothing but empty space. However, when they are treated as de Gennes treated the polymer case, they too reveal surprising states. The simplest case was the generalization from de Gennes' self-avoiding curves to arbitrary self-avoiding lattice clusters, the so-called lattice animals. In 1975 Lubensky³⁴ found that these animals emerge from another field theory in just the way that polymers emerge from the conventional critical field theory. His discovery established the equivalence between lattice animals and the randomly-branched polymers long considered by the polymer community.³⁵ The broken symmetry phase of this theory represented another object important in stochastic geometrical objects: the spanning clusters of a randomly-filled lattice at its percolation threshold. Following the polymeric counterpart of this regime led to striking experimental discoveries about the how branched polymers interpenetrate and compress one another.³⁶ Striking differences from linear polymers were thus revealed.

A second and more pervasive instance of $n = 0$ is Edwards' replica trick.³⁷ It was invented to represent fields such as the vector fields above in the presence of a frozen, disordered environment. One must account for the frozen randomness of the environment differently from the fluctuating randomness of the fields. The replica trick is a way of accounting for the frozen randomness. One imagines n copies or replicas of the vector field, all having the same state of the random environment. Next one allows the random variables representing the environment to fluctuate as the field variables do. With the frozen variables removed, one may calculate the properties of the n -replica state using ordinary methods of statistical physics. Finally, to represent the frozen randomness, one takes the limit in which the number of replicas goes to zero. Edwards had certainly thought about de Gennes' $n = 0$ paper before he formulated the replica trick.⁴ However, there is no evidence that the two discoveries were causally related.

The replica trick reveals an $n = 0$ state, like the cases above, with no apparent relation to the $n \neq 0$ states that gave rise to it. Here the limit reveals not a new geometry but a new state of randomness. As with the cases above, the ordered phase yields additional insight. By construction all the replicas are equivalent, yet at low temperature the replicas may become spontaneously different. This spontaneously broken replica symmetry³⁸ persists to $n = 0$ and represents an important transition in the state of frozen disorder.

5. Assessment

The ultimate measure of the importance of an idea is its impact on people's thinking. How would physicists have thought differently about phase transitions and polymers if the $n = 0$ paper had not been written? Potential answers to this question range widely. At one extreme, one might view the discovery as a simple bookkeeping trick, like the Dyson formula for summing repeated insertions into a diagram.³⁹ Such a trick would likely have been quickly discovered elsewhere if the original discovery had not been reported. At the other extreme are profound notions such as the duality linking high temperature and low temperature behavior of a phase transition system, discovered by Kramers and Wannier in 1941.⁴⁰ No student first learning of duality can avoid a sense of wonder that interacting fields can embody such a unifying symmetry.

In order to gauge the effect of the $n = 0$ paper on people's thinking, the author sought the opinion of several experts within the polymer field and outside it. These scientists gave their permission for their remarks to be used in this chapter. Still, they should not be held responsible for the author's misinterpretations of their statements.

Sir Sam Edwards' impact spans both polymer physics and statistical field theory. His development of the diagrammatic description set the stage for de Gennes' insight. He is one of the fathers of the paradigmatic state of frozen disorder: the spin glass.³⁷ In his reply to the author's query, he focussed on the central problem of computing the exponent ν . He summarized the various *ad hoc* theories of ν such as Flory's and noted that he had found a more systematic argument to justify the Flory result. He recalls the series expansion methods noted above. Then he comments on the $n = 0$ paper:

"The paper of dG left me puzzled. Clearly the mean field approach did not give correct indices even though it was pretty accurate so for complex problems one had to work with methods like mean field but there seemed no way of getting that last bit of accuracy. RG [the Renormalization Group] is hopelessly too complicated for real life problems even though, as in Karl [Freed]'s book,²⁸ it offers a rigorous basis for polymer theory. My own view is [that] this whole area is overtaken by the work of Baumgartner and Muthukumar,⁴¹ who showed that one can write a program that allows the series expansion to be computerized and the (divergent) series can be summed by Pade summation easily and quickly. It is enormously easier than RG."

Edwards addresses the $n = 0$ paper on its own terms, as a means to determine the numerical value of ν . He notes that if this is the goal, the most efficient means need not be based on the renormalization symmetry that ν represents. Edwards views the paper as a clever idea to address an academic issue about the means to achieve arbitrary accuracy for a fundamental number that was well-enough known for practical purposes. The implications of deeper understanding argued above were not the point of the $n = 0$ paper.

Giorgio Parisi is another giant of statistical physics. He was an early participant in developing the renormalization group for phase transitions.⁷ He co-discovered the notion of replica symmetry breaking in the spin glass.³⁸ Parisi writes:

"I do not remember exactly when I learned the de Gennes paper. Quite soon, I guess, because it was used in the framework of the epsilon $[4 - d]$ expansion

to confirm the prediction for the critical exponents. I think that this paper has played a very important role and opened the way to others people to make analytic continuation in n . I think that also the formalism for localization originates for that paper. I do not know how much Edwards and Anderson³⁷ have been influenced by that paper. There was also a paper by Brout in the 50's⁴² [titled] "Statistical Mechanical Theory of a Random Ferromagnetic System," ... but I do not know if it has influenced the developing of the replica method. Certainly I arrived to spin glasses because I wanted to understand better the $n = 0$ limit for branched polymers that was developed by Lubensky (I guess)."

Parisi goes farther than Edwards in acknowledging a conceptual advance in the $n = 0$ paper. Yet it does not seem to have exerted a pivotal influence on his own thinking. Instead, he cites the more direct influence of Lubensky's use of $n = 0$ in his treatment of branched polymers, summarized above. This impact was evidently strong. It led from a polymeric form of $n = 0$ to its replica realization to implement frozen randomness. Parisi also raises the possibility of previous awareness of the replica trick in Brout's early paper. (The author has not investigated this possibility.)

Tom Lubensky, the inspiration for Parisi's work, offered his own views about the importance of the $n = 0$ paper. As noted above, Lubensky showed that branched polymers, random lattice animals and percolation clusters are realizations of renormalization symmetry, using an $n = 0$ representation. He is the coauthor of the leading textbook about soft matter statistical physics.⁴³ Lubensky writes:

"I don't remember the exact moment I heard about the $n = 0$ paper. I believe I saw it in preprint form before it was published. I found it to be a major discovery — so simple yet so powerful. It opened up the statistics of polymers to the entire arsenal of renormalization group techniques. In the end, other RG techniques not relying on the $n = 0$ trick produced equivalent results in a more physical context, but the $n = 0$ idea showed that all of this was possible.

This paper and the subsequent ones by Des Cloizeaux and by de Gennes on adding an external field to treat semi-dilute systems did have a great impact on my thinking though not immediately. In the late 70's I extended the $n = 0$ ideas to branched polymers, where the number of components of the field was merely the fugacity for polymer number, so $n \rightarrow 0$ allowed me to develop field theory techniques to study the statistics of individual branched polymers and lattice animals. The de Gennes work also had an impact on my thinking about percolation and the statistics of clusters near the percolation threshold.

I think that the $n = 0$ approach has fallen from favor. The modern polymer focus more on scaling and the Direct RG approach (developed by Des Cloizeaux if memory serves me) and discussed in de Gennes' book.⁴⁴ The original concept, however, stands as a very important idea that brought modern RG ideas to polymers."

Lubensky accords the $n = 0$ paper a fundamental role. He views it as leading the way to understand renormalization symmetry on a deeper and more general basis than was previously seen via phase transitions. He calls this "direct renormalization," and notes that it is no longer dependent on the phase-transition context that gave birth to it. Thus the $n = 0$ connection is no longer of great relevance for calculations of polymers. In this sense it has "fallen out of favor." The cross-fertilization between polymer and phase transition

phenomena is no longer primary in his view. Still its fundamental significance of unifying two dissimilar forms of dilation symmetry is acknowledged.

The author queried Elihu Abrahams, a father of the theory of waves in disordered media.⁴⁵ The purpose was to judge the impact of the $n = 0$ paper on this domain. Abrahams writes:

... “I would not say that the paper (or rather, note) affected my own work, since I was interested in an entirely different sector of critical phenomena. On the other hand, I think it does represent a substantial contribution, when one recognizes how influential it was on the work of lots of other people.”

Bernard Nienhuis is another father of modern statistical physics. The power of dilation symmetry has proven much greater in two dimensions than in three, and much of this power was brought to light by Nienhuis.⁴⁶ Nienhuis writes:

“I remember precisely how I learned of this result. It was quoted and used by H. Hilhorst in a lecture, in 1976. I was very pleased to learn it. Not many years later I heard de Gennes speak about the subject in Vlieland, a small island of the Netherlands, at national theoretical physics meeting. Indeed it influenced my thinking. Especially the usefulness of generalizing models and parameters beyond their original definition. Later I worked with the $O(n)$ [n -vector] model including the limit $n \rightarrow 0$, in a paper with Domany Mukamel and Schwimmer (Nucl Phys B190). Soon it enabled me to calculate critical exponents of the $O(n)$ model in two dimensions (PRL 49, 1982). I think de Gennes indeed had an essential influence on my thinking, and his work attracted me to the subject of polymers. Concerning the question if this was a unexpected breakthrough or a follow up on earlier work, I tend to lean to the latter opinion. I think de Gennes’ major contribution was that he understood and used the full consequences of the connection, and developed it into a complete scaling theory.”

Nienhuis acknowledges de Gennes’ influence and the power of the $n = 0$ thinking to draw new implications from field theory. He views the $n = 0$ finding as one element of an extended process rather than a pivotal discovery.

At the time of the $n = 0$ paper Russia was an active center for polymer field theory, as it is today. The author consulted Alexander Grosberg to learn of its impact there. Grosberg went on to co-author a highly respected monograph on polymer statistical physics.²⁹ Grosberg writes:

“I was a student in 1972, and I could understand nothing in this $n = 0$ paper until after I learned a little bit of Wilson concurrently with de Gennes. Ilya M. Lifshitz, my teacher, was busy at the time working on the collapsed state of chains, on globules. When the strength of de Gennes approach became evident, there was a huge excitement about the fact that de Gennes approach to what happens above theta-point and Lifshitz approach to what happens below are complementary and together cover the whole range in a way that was seen as a uniformly solid physics (not solid state physics, but solidly built physics). In the subsequent years, there was a huge effort in Moscow to learn the polymer papers by de Gennes and his group. Special seminar was organized, chaired by Lifshitz and working every Tuesday for several years. A significant fraction of the seminar was about reviewing and discussing the French group’s papers. De Gennes’ book⁴⁴

was translated and published in Russian very soon after the original. In this sense, $n = 0$ and subsequent work very much shaped our thinking exactly along the lines of complementarity between this approach to good solvent and Lifshitz approach to the bad solvent. This is how our book with Khokhlov²⁹ is written.”

Grosberg and his colleagues clearly found the $n = 0$ notion and its implications exciting. For them it was an integral part of the “scaling concepts”⁴⁴ that gave de Gennes work on polymers their major impact.

The author himself was greatly influenced by the paper:

“Like several of the others quoted above, I learned of the $n = 0$ discovery second-hand. In this period I was learning about the renormalization group as a postdoc. I viewed the $n = 0$ discovery as a profound realization. On the one hand, it meant that all the powerful results from phase transition renormalization were immediately applicable to the polymer domain, where understanding was much more primitive. On the other hand, the polymer interpretation of the diagrams revealed the renormalization symmetry as a phenomenon of striking and concrete simplicity. It was much more accessible than its phase transition counterpart. These implications were clear from the statement of the $n = 0$ correspondence, even though none of these implications were stated in the paper. The implications that this paper brought to light were the seed leading to the advances much celebrated above.”

6. Conclusion

Almost forty years have passed since the $n = 0$ paper of Ref. 5 was published. The strong activity of exploring the implications of anomalous scaling properties like the exponent ν have largely subsided. The main activity in polymers concerns dense fluids and solids where these good-solvent scaling properties are of little importance. Likewise the search for analogous “anomalous scaling” phenomena in the domain of phase transitions has subsided. How should we interpret this decline in activity? Were the $n = 0$ paper and those that followed it a distraction from the problems of ongoing importance? Another interpretation seems more likely. These papers about excluded volume effects, both theoretical and experimental, transformed our understanding of this issue. What had been a central puzzle of polymers is now a settled problem. We now have the means to understand why exponents such as ν appear and to calculate these exponents as accurately as needed. We know how to devise well-founded scaling laws that show the consequences of these exponents for measured properties. Finally, we know how to recognize combinations of measured quantities that must approach universal, system-independent values for sufficiently long chains.

For polymer physics the $n = 0$ paper revealed the key idea that enabled this framework of understanding to be built for polymers. But the benefits of the paper do not stop there. The recognition that polymers are a realization of the renormalization group gave profound insight into the nature of renormalization symmetry. Using the polymer example, the process of establishing the dilation symmetry of a system via renormalization was shown to be simpler and more direct than in the phase transition phenomena where renormalization was invented. One could infer the symmetry merely by examining a single long polymer

configuration. As noted above, the scope of dilation symmetric phenomena has expanded greatly since that time. Stochastic geometric objects like random animals, disordered quantum states, stochastic growth processes and chaotic dynamics have all shown dilation symmetry. For all such phenomena, the polymer example served as an early inspiration and continues to serve as one of the simplest paradigms of nontrivial dilation symmetry.

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