Chapter 1

Bernstein-Type Operators of One Complex Variable

In the Sections 1.1-1.4 first we obtain the exact degrees in approximation of analytic functions in compact disks by complex Bernstein polynomials and their Butzer's linear combination and in generalized Voronovskaja's results. These sections include approximation results on compact sets in $\mathbb{C}$ for the so-called Bernstein-Faber polynomials and their Butzer's linear combination in compact Faber sets. Convergence results on compact disks for the iterates of $B_n(f)(z)$ connected with the theory of the semigroups of operators and shape preserving properties of these iterates (in the sense that beginning with an index they preserve some properties of f in Geometric Function Theory, like the starlikeness, convexity and spirallikeness) also are proved.

Then in the next Sections 1.5-1.10 some similar properties for the complex q-Bernstein polynomials, Bernstein-Stancu polynomials, Bernstein-Kantorovich polynomials, Favard-Szász-Mirakjan operators, Baskakov operators and Balázs-Szabados operators are obtained.

For all kinds of Bernstein operators, the exact degrees of approximation mainly are obtained by three steps : 1) upper estimates ; 2) quantitative Voronovskaja-type formula ; 3) lower estimates by using step 2.

1.0 Auxiliary Results in Complex Analysis

In order to make the book more self-contained, in this section we briefly present the main known results and methods in Complex Analysis we use in our study.

The first one is called Vitali's theorem and can be stated as follows.

Theorem 1.0.1. (Vitali, see e.g. Kohr-Mocanu [118], p. 112, Theorem 3.2.10) *Let Ω be a domain in $\mathbb{C}$ and $F \subset \Omega$ a set having at least one accumulation point in Ω. If the sequence $(f_n)_{n\in\mathbb{N}}$ of analytic functions in Ω is bounded in each compact in Ω and $(f_n(z))_n$ is convergent for any $z \in F$, then $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent in any compact of Ω.*

In our applications, in general $\Omega = \mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ with $R > 1$, F is a segment included in $\mathbb{D}_R$ and the compact subsets considered will be the closed disks $\mathbb{D}_r = \{z \in \mathbb{C}; |z| \leq r\}$ with $1 \leq r < R$.

The second important result in Complex Analysis we use is the Cauchy's formula for disks.

Theorem 1.0.2. (Cauchy, see e.g. Kohr-Mocanu [118], p. 28, Theorem 1.2.20) *Let* $r > 0$ *and* $f : \overline{\mathbb{D}}_r \to \mathbb{C}$ *be analytic in* $\mathbb{D}_r$ *and continuous in* $\overline{\mathbb{D}}_r$. *Then, for any* $p \in \{0, 1, 2, ..., \}$ *and all* $|z| < r$ *we have*

$$f^{(p)}(z) = \frac{p!}{2\pi i} \int_{\Gamma} \frac{f(u)}{(u-z)^{p+1}} du,$$

where $\Gamma = \{z \in \mathbb{C}; |z| = r\}$ *and* $i^2 = -1$.

An immediate consequence of the Cauchy's formula is the so-called Weierstrass's theorem used in the proofs of shape preserving properties.

Theorem 1.0.3. (Weierstrass, see e.g. Kohr-Mocanu [118], p. 18, Theorem 1.1.6) *Let* $G \subset \mathbb{C}$ *be an open set. If the sequence* $(f_n)_{n\in\mathbb{N}}$ *of analytic functions on* G *converges to the analytic function* f*, uniformly in each compact in* G*, then for any* $p \in \mathbb{N}$*, the sequence of pth derivatives* $(f_n^{(p)})_{n\in\mathbb{N}}$ *converges to* $f^{(p)}$ *uniformly on compacts in* G.

Indeed, note that by the above Cauchy's formula we can write

$$f_n^{(p)}(z) - f^{(p)}(z) = \frac{p!}{2\pi i} \int_{\Gamma} \frac{f_n(u) - f(u)}{(u-z)^{p+1}} du,$$

from which by passing to modulus the theorem easily follows. In our applications, $G = \mathbb{D}_R$ with $R > 1$ and the compact subsets in G are $\mathbb{D}_r$ with $1 \leq r < R$.

Another well-known result used in the proof of shape preserving properties is the following.

Theorem 1.0.4. (see e.g. Graham-Kohr [105], Theorem 6.1.18) *If* $f_n, f : \Omega \to \mathbb{C}$, $n \in \mathbb{N}$ *are analytic in the domain* Ω, f *is univalent in* Ω *and* $f_n \to f$ *uniformly in the compact* $K \subset \Omega$, *then there exists* $n_0(K)$ *such that for all* $n \geq n_0$, f_n *is univalent in* K.

The classical so called Maximum Principle (or Maximum Modulus Theorem) will be frequently used in the proofs of error estimates.

Theorem 1.0.5. (see e.g. Kohr-Mocanu [118], p. 2, Corollary 1.1.20) *If* $\Omega \subset \mathbb{C}$ *is a bounded domain and* $f : \overline{\Omega} \to \mathbb{C}$ *is analytic in* Ω *and continuous in* $\overline{\Omega}$, *then denoting by* Γ *the boundary of* Ω *we have*

$$\max\{|f(z)|; z \in \overline{\Omega}\} = \max\{|f(z)|; z \in \Gamma\}.$$

For our applications again Ω will be an open disk centered at origin.

Useful in some of our proofs will be the well-known so called theorem on the zeroes of analytic functions, which in essence says that the zeroes of an analytic function (non-identical null) necessarily are isolated points. More exactly we can state the following.

Theorem 1.0.6. (see e.g. Kohr-Mocanu [118], p. 20, Theorem 1.1.12) *Suppose that f is analytic in the domain Ω and that f is not identical null in Ω. If a is a zero for f then there exists $r = r(a) > 0$ such that $\mathbb{D}(a, r) = \{z \in \mathbb{C}; |z - a| < r\} \subset \Omega$ and $f(z) \neq 0$, for all $z \in \mathbb{D}(a, r) \setminus \{a\}$.*

Also, we will use the classical so called theorem on the identity of analytic functions.

Theorem 1.0.7. (see e.g. Kohr-Mocanu [118], p. 21, Theorem 1.1.14) *Let $\Omega \subset \mathbb{C}$ be a domain. If $f, g : \Omega \to \mathbb{C}$ are analytic in Ω then $f \equiv g$ on Ω is equivalent with the fact that the set $\{z \in \Omega; f(z) = g(z)\}$ has at least one accumulation point in Ω.*

Finally, we state a basic result very useful in the proofs of the approximation results and called Bernstein's inequality for complex polynomials in compact disks.

Theorem 1.0.8. (Bernstein [43], p. 45, relation (80) for general $r > 0$, see also e.g. Lorentz [126], p. 40, Theorem 4, for $r = 1$) *Let $P(z) = \sum_{k=0}^{n} a_k z^k$ be with $a_k \in \mathbb{C}$, for all $k \in \{0, 1, 2, ..., \}$ and for $r > 0$ denote $\|P_n\|_r = \max\{|P_n(z)|; |z| \leq r\}$.*

(i) For all $|z| \leq 1$ we have $|P_n'(z)| \leq n\|P_n\|_1$;

(ii) If $r > 0$ then for all $|z| \leq r$ we have $|P_n'(z)| \leq \frac{n}{r}\|P_n\|_r$.

One observes that (ii) immediately follows from (i). Indeed, denoting $Q_n(z) = P_n(rz)$, $|z| \leq 1$, by (i) applied to $Q_n(z)$ it easily follows $r|P_n'(rz)| \leq n\|P_n\|_r$, for all $|z| \leq 1$, which proves (ii).

Concerning the approximation of analytic functions by sequences of complex polynomials, as it will be seen in the next sections of this chapter and in the next chapters, the main results one refer to approximation in compact disks centered at origin (in particular in the compact unit disk). The advantage consists in the fact that in these kinds of disks constructive methods can be indicated. But of course that it is very important to obtain approximation results in more general domains in the complex plane. In what follows we briefly present the standard method based on the so-called Faber polynomials introduced by Faber [70], which allows to extend all the constructive methods from the closed unit disk to more general domains. The method is less constructive because a generally unknown mapping function (generated from the Riemann's mapping theorem) enters into considerations. For all the details below on this method see e.g. the book of Gaier [76], pp. 42-54. Also, for other important contributions to the topic of constructive complex approximation see the book of Dzjadyk [69].

Definition 1.0.9. (i) $\gamma : [a, b] \to \mathbb{C}$ is called Jordan curve if it is closed (i.e. $\gamma(a) = \gamma(b)$) and simple (i.e. injective). The length of the curve γ is defined by

$$L(\gamma) = \sup\{\sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})|; n \in \mathbb{N}, a = t_0 < ... < t_n = b\}.$$

γ is called rectifiable if $L < +\infty$.

The interior of a Jordan curve is called Jordan domain and the curve is called boundary curve of that domain.

(ii) (Radon [158]) Suppose that $\gamma : [a, b] \to \mathbb{C}$ is a rectifiable Jordan curve. Because $L < +\infty$, it is known that γ has a tangent γ' almost everywhere. Then γ is called of bounded rotation if γ' can be extended to a function of bounded variation on the whole curve.

Remark. Simple examples of Jordan curve of bounded rotation can be made up of finitely many convex arcs (where corners are permitted).

Now, if G is a Jordan domain, then (by the Riemann's mapping theorem) let us denote by Ψ the conformal mapping of $\mathbb{C} \setminus \overline{\mathbb{D}}_1$ onto $\mathbb{C} \setminus \overline{G}$, normalized at ∞, that is $0 < \lim_{w\to\infty} \frac{\Psi(w)}{w} < \infty$. Also, denote by Φ the inverse function of Ψ. Obviously that Ψ and Φ depend on $\overline{G}$, but for the simplicity of notation we will not write them as $\Psi_{\overline{G}}$ and $\Phi_{\overline{G}}$, considering in our presentation that $\overline{G}$ is arbitrary but fixed.

For a Jordan domain G, denote by $A(\overline{G})$ the class of all functions continuous in $\overline{G}$ and analytic in G. In what follows we sketch a method by which any $f \in A(\overline{G})$ can be approximated by polynomials. For our considerations, it is sufficient to suppose that the boundary curve of G is rectifiable and of bounded rotation.

First, one considers the Laurent expansion of $[\Phi(z)]^n$, $n \in \mathbb{N} \cup \{0\}$, valid for large z

$$[\Phi(z)]^n = a_0^{(n)} + ... + a_n^{(n)} z^n + \sum_{k=1}^{\infty} a_{-k}^{(n)}/z^k.$$

Definition 1.0.10. (Faber [70]) (i) The polynomial $F_n(z) = a_0^{(n)} + ... + a_n^{(n)} z^n$, $n \in \mathbb{N} \cup \{0\}$ is called the Faber polynomial of degree n attached to the domain G. (Note that for $z \in \mathbb{D}_R$, $R > 1$ we can write

$$F_n(z) = \frac{1}{2\pi i} \int_{|u|=R} \frac{[\Phi(u)]^n}{u - z} du.)$$

(ii) If $f \in A(\overline{G})$ then

$$a_n(f) = \frac{1}{2\pi i} \int_{|u|=1} \frac{f[\Psi(u)]}{u^{n+1}} du = \frac{1}{2\pi i} \int_{-\pi}^{\pi} f[\Psi(e^{it})] e^{-int} dt, n \in \mathbb{N} \cup \{0\}$$

are called the Faber coefficients of f and $\sum_{n=0}^{\infty} a_n(f) F_n(z)$ is called the Faber series attached to f on G. (Here $i^2 = 1$.) The Faber series represent a natural generalization of Taylor series when the unit disk is replaced by an arbitrary simply connected domain bounded by a "nice" curve.

(iii) The mapping T defined by $T[P_n](z) = \sum_{k=0}^{n} c_k F_k(z)$, where $P_n(w) = \sum_{k=0}^{n} c_k w^k$ is called the Faber mapping.

Remark. By Definition 1.0.10, (iii), the Faber mapping T is linear and defined on the set of all polynomials $\mathcal{P}$ defined on $\overline{\mathbb{D}}_1$ and with values in the set of polynomials $\mathcal{P}$ defined on $\overline{G}$. In some cases it can be extended as a linear and bounded mapping between the Banach spaces $A(\overline{\mathbb{D}}_1)$ and $A(\overline{G})$ (both endowed with the corresponding uniform norms). Below we briefly point out this extension (for full details see e.g.

APPROXIMATION BY COMPLEX BERNSTEIN AND CONVOLUTION TYPE OPERATORS

the book of Gaier [76], pp. 48-49) under the hypothesis that the boundary of G is a rectifiable Jordan curve of bounded rotation. In this case for G, first it follows that $\|T(P)\| \leq C\|P\|$ for each $P \in \mathcal{P}$, where $C > 0$ depends only on G. Then T can be extended to the closure of $\mathcal{P}$ and since $\overline{\mathcal{P}} = A(\overline{\mathbb{D}}_1)$, T can be extended as a linear and bounded operator from $A(\overline{\mathbb{D}}_1)$ into $A(\overline{G})$, with the property that $\|T(f)\| \leq C\|f\|$ for each $f \in A(\overline{\mathbb{D}}_1)$.

Now, since the Faber mapping has the integral representation

$$T[P_n](z) = \frac{1}{2\pi i}\int_C \frac{P_n[\Phi(u)]}{u-z}du,$$

valid for each polynomial P_n, by passing to limits we obtain the formula

$$T[F](z) = \frac{1}{2\pi i}\int_C \frac{F[\Phi(u)]}{u-z}du,\ z \in G,\ F \in A(\overline{D}_1).$$

Also, the converse formula

$$F(w) = \frac{1}{2\pi i}\int_{|u|=1} \frac{T[F](\Psi(u))}{u-w}du,\ w \in \mathbb{D}_1$$

holds.

The following two known results are of great importance for approximation.

Theorem 1.0.11. (more precisely see e.g. Gaier [76], p. 50, Theorem 3) *If* $F \in A(\overline{\mathbb{D}}_1)$, $F(w) = \sum_{n=0}^{\infty} c_n w^n$ *then the Faber coefficients of* $T[F]$ *are* c_n.

Theorem 1.0.12. (more precisely see e.g. Theorem 4 in Gaier [76], p. 51) *Suppose that the boundary of G is a rectifiable Jordan curve of bounded rotation and let* $f \in A(\overline{G})$. *There exists* $F \in A(\overline{\mathbb{D}}_1)$ *with* $f = T[F]$ *if and only if as function of* $w \in \overline{\mathbb{D}}_1$, *the Cauchy integral* $\int_{|u|=1} \frac{f[\Psi(u)]}{u-w}du$ *belongs to* $A(\overline{\mathbb{D}}_1)$ *and in this case we have*

$$F(w) = \frac{1}{2\pi i}\int_{|u|=1} \frac{f[\Psi(u)]}{u-w}du,\ w \in \overline{\mathbb{D}}_1,$$

(F is extended by continuity on $\partial\mathbb{D}_1$).

Remark. Theorem 1.0.12 allows to reduce the approximation of $f \in A(\overline{G})$ to the approximation of $F \in A(\overline{\mathbb{D}}_1)$. Indeed, let $(S_n(F)(w))_{n\in\mathbb{N}}$, $S_n(F)(w) = \sum_{k=0}^{m_n} a_k(F)w^k$, be an approximation sequence for F. Then $T[S_n(F)](z) = \sum_{k=0}^{m_n} a_k(F)F_k(z)$, $n \in \mathbb{N}$ will represent an approximation sequence for f in the set $\overline{G}$ (here $F_k(z), k \in \mathbb{N}$ denote the Faber polynomials attached to G). Indeed, denoting the uniform norms by $\|\cdot\|$, this follows from the relation (6.19), p. 51 in Gaier [76],

$$\|f - \sum_{k=0}^{m_n} a_k(F)F_k\|_{\overline{G}} = \|T(F - \sum_{k=0}^{m_n} a_k(F)e_k)\|_{\overline{G}} \leq |||T||| \cdot \|F - \sum_{k=0}^{m_n} a_k(F)e_k\|_{\overline{\mathbb{D}}_1},$$

where $e_k(w) = w^k$ and $|||T||| \leq M < \infty$, because of the hypothesis on the boundary of G.

1.1 Bernstein Polynomials

In this section, we find the exact orders in simultaneous uniform approximation of analytic functions by complex Bernstein polynomials in closed disks, an upper estimate in Voronovskaja's result and we prove that the complex Bernstein polynomials attached to an analytic function, preserve the univalence, starlikeness, convexity and spirallikeness. Also, to Jordan domains Bernstein-type polynomials are attached and approximation results on connected compact sets with estimates are obtained.

1.1.1 *Bernstein Polynomials on Compact Disks*

Concerning the approximation properties (uniform convergence), the results in Wright [199], Kantorovich [113], Bernstein [39; 40; 41], Lorentz [125] and Tonne [190] are well-known. It is worth nothing that an entire Chapter 4 of 38 pages is dedicated to this topic in the book of Lorentz [125]. In that book interesting convergence properties of $B_n(f)(z)$ and of its so-called degenerate form, in various domains in $\mathbb{C}$, like compact disks, ellipses, loops, autonomous sets are presented.

For example, the following three approximation results due to Bernstein, Tonne and Kantorovich concerning the uniform approximation of Bernstein polynomials in the unit disk and in an ellipse hold.

Theorem 1.1.1. *(i) (Bernstein, see e.g. Lorentz [125], p. 88) For the open* $G \subset \mathbb{C}$, *such that* $\overline{\mathbb{D}}_1 \subset G$ *and* $f : G \to \mathbb{C}$ *is analytic in* G, *the complex Bernstein polynomials* $B_n(f)(z) = \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} f(k/n)$, *uniformly converge to* f *in* $\overline{\mathbb{D}}_1$. *Here* $\mathbb{D}_1$ *denotes the open unit disk.*

(ii) (Tonne [190]) If $f(z) = \sum_{k=0}^{\infty} c_k z^k$ *is analytic in the open unit disk* $\mathbb{D}_1$, $f(1)$ *is a complex number and there exist* $M > 0$ *and* $m \in \mathbb{N}$ *such that* $|c_k| \le M(k+1)^m$, *for all* $k = 0, 1, 2, ...$, *then* $B_n(f)(z)$ *converges uniformly (as* $n \to \infty$*) to* f *on each closed subset of* $\mathbb{D}_1$.

(iii) (Kantorovich, see e.g. Lorentz [125], p. 90) If f *is analytic in the interior of an ellipse of foci* 0 *and* 1, *then* $B_n(f)(z)$ *converges uniformly to* $f(z)$ *in any closed set contained in the interior of ellipse.*

But in all the previous mentioned work no quantitative estimates of these convergence results were obtained. In what follows, first we obtain upper quantitative estimates on compact disks. For this purpose, denote $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$.

Theorem 1.1.2. (Gal [77], p. 264, Theorem 3.4.1, (iii)-(v)) *Suppose that* $R > 1$ *and* $f : \mathbb{D}_R \to \mathbb{C}$ *is analytic in* $\mathbb{D}_R$, *that is* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{D}_R$.

(i) Let $1 \le r < R$ *be arbitrary fixed. For all* $|z| \le r$ *and* $n \in \mathbb{N}$, *we have*

$$|B_n(f)(z) - f(z)| \le \frac{C_r(f)}{n},$$

where $0 < C_r(f) = \frac{3r(1+r)}{2} \sum_{j=2}^{\infty} j(j-1)|c_j| r^{j-2} < \infty$.

(ii) For the simultaneous approximation by complex Bernstein polynomials, we have : if $1 \le r < r_1 < R$ *are arbitrary fixed, then for all* $|z| \le r$ *and* $n, p \in \mathbb{N}$*,*

$$|B_n^{(p)}(f)(z) - f^{(p)}(z)| \le \frac{C_{r_1}(f)p!r_1}{n(r_1 - r)^{p+1}},$$

where $C_{r_1}(f)$ *is given as at the above point (i).*

Proof. (i) Denoting $e_k(z) = z^k$ and $\pi_{k,n}(z) = B_n(e_k)(z)$, we evidently have $B_n(f)(z) = \sum_{k=0}^{\infty} c_k \pi_{k,n}(z)$ and we get

$$|B_n(f)(z) - f(z)| \le \sum_{k=0}^{\infty} |c_k| \cdot |\pi_{k,n}(z) - e_k(z)|,$$

so that we need an estimate for $|\pi_{k,n}(z) - e_k(z)|$.

For this purpose we use the recurrence proved for the real variable case in Andrica [24]

$$\pi_{k+1,n}(z) = \frac{z(1-z)}{n} \pi'_{k,n}(z) + z\pi_{k,n}(z),$$

for all $n \in \mathbb{N}$, $z \in \mathbb{C}$ and $k = 0, 1, ...$ Since the relationship in Andrica [24] proved for the real case is a simple algebraic manipulation, it is valid for complex variable as well. Taking into account that the paper Andrica [24] is less accessible, let us reproduce here the idea of proof. It consists of the simple algebraic relationship

$$S'_{k,n}(z) = \frac{S_{k+1,n}(z)}{z(1-z)} - n\frac{S_{k,n}(z)}{1-z},$$

which is divided by n^k, where

$$S_{k,n}(z) = \sum_{j=0}^{n} j^k \binom{n}{j} z^j (1-z)^{n-j}.$$

(Note that the cases $z = 0$ and $z = 1$ are trivial in the recurrence for $\pi_{k,n}(z)$.)

From this recurrence, we easily obtain that degree $(\pi_{k,n}(z)) = \min\{n, k\} \le k$. Also, it easily implies the next recurrence

$$\pi_{k,n}(z) - z^k$$

$$= \frac{z(1-z)}{n}[\pi_{k-1,n}(z) - z^{k-1}]' + \frac{(k-1)z^{k-1}(1-z)}{n} + z[\pi_{k-1,n}(z) - z^{k-1}].$$

Denoting with $\|\cdot\|_r$ the norm in $C(\overline{\mathbb{D}}_r)$, where $\overline{\mathbb{D}}_r = \{z \in \mathbb{C}; |z| \le r\}$, one observes that by a linear transformation the Bernstein's inequality in the closed unit disk becomes $|P'_k(z)| \le \frac{k}{r}\|P_k\|_r$, for all $|z| \le r$, $r \ge 1$, where P_k represents an algebraic polynomial of degree $\le k$. Therefore, from the above recurrence we get

$$\begin{aligned}|\pi_{k,n}(z)-e_k(z)| &\le (k-1)\frac{r(1+r)}{rn}\|\pi_{k-1,n}-e_{k-1}(z)\|_r\\ &\quad+\frac{r^{k-1}(1+r)(k-1)}{n}+r|\pi_{k-1,n}(z)-e_{k-1}(z)|\\ &\le (k-1)\frac{(1+r)}{n}\cdot[\|\pi_{k-1,n}\|_r+\|e_{k-1}\|_r]\\ &\quad+\frac{r^{k-1}(1+r)(k-1)}{n}+r|\pi_{k-1,n}(z)-e_{k-1}(z)|\\ &\le r|\pi_{k-1,n}(z)-e_{k-1}(z)|\\ &\quad+[2(1+r)r^{k-1}+(1+r)r^{k-1}]\frac{k-1}{n}.\end{aligned}$$

Above we used that for all $k,n\in\mathbb{N}$ and $|z|\le r$, $r\ge 1$, we have $|\pi_{k,n}(z)|\le r^k$ (see relation (4) in the proof of Theorem 4.1.1 in Lorentz[125], p. 88) and $|e_k(z)|\le r^k$.

Now, by taking $k=1,2,...,$ in the inequality

$$|\pi_{k,n}(z)-e_k(z)|\le r|\pi_{k-1,n}(z)-e_{k-1}(z)|+3(1+r)r^{k-1}\frac{k-1}{n},$$

by recurrence we easily obtain the required inequality

$$\begin{aligned}|\pi_{k,n}(z)-e_k(z)| &\le \frac{3(1+r)}{n}[r^{k-1}+2r^{k-1}+...+(k-1)r^{k-1}]\\ &=\frac{3(1+r)}{n}\cdot\frac{k(k-1)}{2}r^{k-1}\le\frac{3r(1+r)}{2n}\cdot k(k-1)r^{k-2}.\end{aligned}$$

This immediately implies the estimate in (i).

Note that since by hypothesis, $f(z)=\sum_k^{\infty}c_kz^k$ is absolutely and uniformly convergent in $|z|\le r$, for any $1\le r<R$, it follows that the power series obtained by differentiating twice, i.e. $f''(z)=\sum_{k=2}^{\infty}k(k-1)c_kz^{k-2}$, also is absolutely convergent for $|z|\le r$, which implies $\sum_{k=2}^{\infty}k(k-1)|c_k|r^{k-2}<+\infty$.

(ii) Denoting by γ the circle of radius $r_1>1$ and center 0, since for any $|z|\le r$ and $v\in\gamma$, we have $|v-z|\ge r_1-r$, by the Cauchy's formulas it follows that for all $|z|\le r$ and $n\in\mathbb{N}$, we have

$$\begin{aligned}|B_n^{(p)}(f)(z)-f^{(p)}(z)| &= \frac{p!}{2\pi}\left|\int_\gamma\frac{B_n(f)(v)-f(v)}{(v-z)^{p+1}}dv\right|\\ &\le\frac{C_{r_1}(f)}{n}\frac{p!}{2\pi}\frac{2\pi r_1}{(r_1-r)^{p+1}}=\frac{C_{r_1}(f)}{n}\frac{p!r_1}{(r_1-r)^{p+1}},\end{aligned}$$

which proves the theorem. □

Remarks. 1) An analogue to Theorem 1.1.2, (i), case $r=1$, has been obtained by a different method in Ostrovska [146].

2) Let us give a proof of the relationship $B_n(f)(z)=\sum_{k=0}^{\infty}c_kB_n(e_k)(z)$ used at the beginning of the proof of Theorem 1.1.2, (i). Denoting $f_m(z)=\sum_{j=0}^{m}c_jz^j$, $|z|\le r$, $m\in\mathbb{N}$, since from the linearity of B_n we obviously have $B_n(f_m)(z)=$

$\sum_{k=0}^{m} c_k B_n(e_k)(z)$, it suffices to prove that for any fixed $n \in \mathbb{N}$ and $|z| \le r$ with $r \ge 1$, we have $\lim_{m\to\infty} B_n(f_m)(z) = B_n(f)(z)$. But this is immediate from $\lim_{m\to\infty} \|f_m - f\|_r = 0$ and from the inequality

$$|B_n(f_m)(z) - B_n(f)(z)| \le \sum_{k=0}^{n} \binom{n}{k} |z^k(1-z)^{n-k}| \cdot \|f_m - f\|_r \le M_{r,n}\|f_m - f\|_r,$$

valid for all $|z| \le r$.

In what follows we present the Voronovskaja-type formula with a quantitative upper estimate.

Theorem 1.1.3. (Gal [78]) *Let $R > 1$ and suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$, that is we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$.*

(i) The following Voronovskaja-type result in the closed unit disk holds

$$\left|B_n(f)(z) - f(z) - \frac{z(1-z)}{2n} f''(z)\right| \le \frac{|z(1-z)|}{2n} \cdot \frac{10M(f)}{n},$$

for all $n \in \mathbb{N}, z \in \overline{\mathbb{D}}_1$, where $0 < M(f) = \sum_{k=3}^{\infty} k(k-1)(k-2)^2 |c_k| < \infty$.

(ii) Let $r \in [1, R)$. Then for all $n \in \mathbb{N}, |z| \le r$, we have

$$\left|B_n(f)(z) - f(z) - \frac{z(1-z)}{2n} f''(z)\right| \le \frac{5(1+r)^2}{2n} \cdot \frac{M_r(f)}{n},$$

where $M_r(f) = \sum_{k=3}^{\infty} |c_k| k(k-1)(k-2)^2 r^{k-2} < \infty$.

Proof. (i) Denoting $e_k(z) = z^k$, $k = 0, 1, ...,$ and $\pi_{k,n}(z) = B_n(e_k)(z)$, we can write $B_n(f)(z) = \sum_{k=0}^{\infty} c_k \pi_{k,n}(z)$, which immediately implies

$$\left|B_n(f)(z) - f(z) - \frac{z(1-z)}{2n} f''(z)\right| \le \sum_{k=3}^{\infty} |c_k| \cdot \left|\pi_{k,n}(z) - e_k(z) - \frac{z^{k-1}(1-z)k(k-1)}{2n}\right|,$$

for all $z \in \overline{\mathbb{D}}_1$, $n \in \mathbb{N}$.

In what follows, we will use the recurrence obtained in the proof of Theorem 1.1.2, (i)

$$\pi_{k+1,n}(z) = \frac{z(1-z)}{n} \pi'_{k,n}(z) + z\pi_{k,n}(z),$$

for all $n \in \mathbb{N}$, $z \in \mathbb{C}$ and $k = 0, 1, ...$

If we denote

$$E_{k,n}(z) = \pi_{k,n}(z) - e_k(z) - \frac{z^{k-1}(1-z)k(k-1)}{2n},$$

then it is clear that $E_{k,n}(z)$ is a polynomial of degree $\le k$ and by a simple calculation and the use of the above recurrence we obtain the following relationship

$$\begin{aligned} E_{k,n}(z) &= \frac{z(1-z)}{n} E'_{k-1,n}(z) + zE_{k-1,n}(z) \\ &\quad + \frac{z^{k-2}(1-z)(k-1)(k-2)}{2n^2}[(k-2) - z(k-1)], \end{aligned}$$

APPROXIMATION BY COMPLEX BERNSTEIN AND CONVOLUTION TYPE OPERATORS

for all $k \geq 2$, $n \in \mathbb{N}$ and $z \in \overline{\mathbb{D}}_1$.

According to the Bernstein's inequality $\|E'_{k-1,n}\| \leq (k-1)\|E_{k-1,n}\|$, the above relationship implies for all $|z| \leq 1$, $k \geq 2$, $n \in \mathbb{N}$ that

$$\begin{aligned}
|E_{k,n}(z)| &\leq \frac{|z| \cdot |1-z|}{2n}[2\|E'_{k-1,n}\|] \\
&+|E_{k-1,n}(z)| + \frac{|z| \cdot |1-z|}{2n} \cdot \frac{|z|^{k-3}(k-1)(k-2)}{n}(2k-3) \\
&\leq |E_{k-1,n}(z)| + \frac{|z| \cdot |1-z|}{2n}\left[2\|E'_{k-1,n}\| + \frac{2k(k-1)(k-2)}{n}\right] \\
&\leq |E_{k-1,n}(z)| + \frac{|z| \cdot |1-z|}{2n}\left[2(k-1)\|E_{k-1,n}\| + \frac{2k(k-1)(k-2)}{n}\right] \\
&\leq |E_{k-1,n}(z)| + \frac{|z| \cdot |1-z|}{2n}\left[2(k-1)\|\pi_{k-1,n} - e_{k-1}\|\right. \\
&\left. + 2(k-1)\left\|\frac{(k-1)(k-2)\left[e_{k-2} - e_{k-1}\right]}{2n}\right\| + \frac{2k(k-1)(k-2)}{n}\right],
\end{aligned}$$

where $\|\cdot\|$ denotes the uniform norm in $C(\overline{\mathbb{D}}_1)$.

Also, taking $r = 1$ in the inequality obtained in the proof of Theorem 1.1.2, (i), it follows

$$\|\pi_{k,n} - e_k\| \leq \frac{3}{n}(k-1)k.$$

As a consequence, we get

$$\begin{aligned}
|E_{k,n}(z)| \;\leq\; &|E_{k-1,n}(z)| + \frac{|z| \cdot |1-z|}{2n}\left[2(k-1)\frac{3(k-1)(k-2)}{n}\right. \\
&\left. + 2(k-1)\left\|\frac{(k-1)(k-2)[e_{k-2} - e_{k-1}]}{2n}\right\| + \frac{2k(k-1)(k-2)}{n}\right],
\end{aligned}$$

which by simple calculation implies

$$|E_{k,n}(z)| \leq |E_{k-1,n}(z)| + \frac{|z| \cdot |1-z|}{2n} \cdot \frac{10}{n} \cdot k(k-1)(k-2).$$

Since $E_{0,n}(z) = E_{1,n}(z) = E_{2,n}(z) = 0$, for any $z \in \mathbb{C}$, it follows that the last inequality is trivial for $k = 0, 1, 2$.

By writing the last inequality for $k = 3, 4, ...$, we easily obtain, step by step the following

$$|E_{k,n}(z)| \leq \frac{|z| \cdot |1-z|}{2n} \cdot \frac{10}{n} \cdot \sum_{j=3}^{k} j(j-1)(j-2) \leq \frac{|z| \cdot |1-z|}{2n} \cdot \frac{10}{n} \cdot k(k-1)(k-2)^2.$$

In conclusion,

$$\begin{aligned}
&\left|B_n(f)(z) - f(z) - \frac{z(1-z)}{2n}f''(z)\right| \\
&\leq \sum_{k=3}^{\infty} |c_k| \cdot |E_{k,n}(z)| \leq \frac{|z| \cdot |1-z|}{2n} \cdot \frac{10}{n} \cdot \sum_{k=3}^{\infty} |c_k| k(k-1)(k-2)^2.
\end{aligned}$$

APPROXIMATION BY COMPLEX BERNSTEIN AND CONVOLUTION TYPE OPERATORS

Note that since $f^{(4)}(z)=\sum_{k=4}^{\infty}c_k k(k-1)(k-2)(k-3)z^{k-4}$, and the series is absolutely convergent in $\overline{\mathbb{D}}_1$, it easily follows that $\sum_{k=3}^{\infty}|c_k|k(k-1)(k-2)^2<\infty$.

(ii) We will use the relationship obtained in the proof of Theorem 1.1.2, (i)

$$|\pi_{k,n}(z)-e_k(z)|\le \frac{3r(1+r)}{2n}\cdot k(k-1)r^{k-2},$$

for all $k,n\in\mathbb{N}$, $|z|\le r$, with $1\le r$.

Let us consider the relationship proved at the above point (i) given by

$$\begin{aligned}E_{k,n}(z) \;=\; &\frac{z(1-z)}{n}E'_{k-1,n}(z)+zE_{k-1,n}(z)\\ &+\frac{z^{k-2}(1-z)(k-1)(k-2)}{2n^2}[(k-2)-z(k-1)],\end{aligned}$$

for all $k\ge 2$, $n\in\mathbb{N}$ and $z\in\mathbb{C}$, and let us restrict it only for $|z|\le r$. For all $k,n\in\mathbb{N}$, $k\ge 2$ and $|z|\le r$, it implies

$$\begin{aligned}|E_{k,n}(z)| \;\le\; &\frac{r(1+r)}{n}|E'_{k-1,n}(z)|+r|E_{k-1,n}(z)|\\ &+\frac{(1+r)r^{k-2}(k-1)(k-2)}{2n^2}[(k-2)+r(k-1)].\end{aligned}$$

Now we will estimate $|E'_{k-1,n}(z)|$, for $k\ge 3$. Taking into account that $E_{k-1,n}(z)$ is a polynomial of degree $\le (k-1)$, we obtain

$$\begin{aligned}|E'_{k-1,n}(z)|&\le \frac{k-1}{r}\|E_{k-1,n}(z)\|_r\\ &\le \frac{k-1}{r}\left[\|\pi_{k-1,n}-e_{k-1}\|_r+\left\|\frac{(k-1)(k-2)(e_{k-1}-e_{k-2})}{2n}\right\|_r\right]\\ &\le \frac{k-1}{r}\left[\frac{3r(1+r)(k-1)(k-2)r^{k-3}}{2n}+\frac{r^{k-2}(r+1)(k-1)(k-2)}{2n}\right]\\ &\le \frac{k(k-1)(k-2)}{2n}\left[3(1+r)r^{k-3}+r^{k-3}(r+1)\right]\\ &\le \frac{2k(k-1)(k-2)(1+r)r^{k-3}}{n}.\end{aligned}$$

This implies

$$\frac{r(1+r)}{n}|E'_{k-1,n}(z)|\le \frac{2r(1+r)^2k(k-1)(k-2)r^{k-3}}{n^2},$$

and

$$\begin{aligned}|E_{k,n}(z)| \;\le\; & r|E_{k-1,n}(z)|+\frac{2r(1+r)^2k(k-1)(k-2)r^{k-3}}{n^2}\\ &+\frac{(1+r)(k-1)(k-2)r^{k-2}}{2n^2}\left[(k-2)+r(k-1)\right]=r|E_{k-1,n}(z)|\\ &+\frac{(1+r)(k-1)(k-2)r^{k-2}}{2n^2}\left[4k(1+r)+(k-2)+r(k-1)\right]\end{aligned}$$

$$= r|E_{k-1,n}(z)| + \frac{(1+r)(k-1)(k-2)r^{k-2}}{2n^2}\left[(5k-2)+r(5k-1)\right]$$
$$\leq r|E_{k-1,n}(z)| + \frac{(1+r)(k-1)(k-2)r^{k-2}5k(1+r)}{2n^2}$$
$$= r|E_{k-1,n}(z)| + \frac{5(1+r)^2k(k-1)(k-2)r^{k-2}}{2n^2}.$$

But $E_{0,n}(z) = E_{1,n}(z) = E_{2,n}(z) = 0$, for any $z \in \mathbb{C}$.

By writing the last inequality for $k = 3, 4, ...$, we easily obtain, step by step the following

$$|E_{k,n}(z)| \leq \frac{5(1+r)^2r^{k-2}}{2n^2}\left[\sum_{j=3}^{k} j(j-1)(j-2)\right]$$
$$\leq \frac{5(1+r)^2k(k-1)(k-2)^2r^{k-2}}{2n^2}.$$

As a conclusion, we obtain

$$\left|B_n(f)(z) - f(z) - \frac{z(1-z)}{2n}f''(z)\right| \leq \sum_{k=3}^{\infty} |c_k| \cdot |E_{k,n}(z)|$$
$$\leq \frac{5(1+r)^2}{2n^2}\sum_{k=3}^{\infty} |c_k|k(k-1)(k-2)^2r^{k-2}.$$

Note that since $f^{(4)}(z) = \sum_{k=4}^{\infty} c_k k(k-1)(k-2)(k-3)z^{k-4}$, and the series is absolutely convergent in $|z| \leq r$, it easily follows that $\sum_{k=3}^{\infty} |c_k|k(k-1)(k-2)^2r^{k-2} < \infty$. Therefore the theorem has been proved. □

Remark. By Gonska-Piţul-Raşa [102], p. 68, Proposition 7.2, for the real Bernstein polynomials

$$B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k}f(k/n), x \in [0,1]$$

attached to a function $f \in C^2[0,1]$, for all $x \in [0,1]$ and $n \in \mathbb{N}$ it holds

$$\left|B_n(f)(x) - f(x) - \frac{x(1-x)}{2n}f''(x)\right| \leq \frac{x(1-x)}{2n}\tilde{\omega}_1(f''; \frac{1}{3\sqrt{n}}),$$

where $\tilde{\omega}_1$ denotes the least concave majorant of the modulus of continuity ω_1 and

$$C^2[0,1] = \{f : [0,1] \to \mathbb{R}; f \text{ is twice continuously differentiable on } [0,1]\}.$$

Now, if $f \in C^3[0,1]$ then we immediately get that the best quantitative uniform estimate in the real Voronovskaja's result is of order $O(1/n^{3/2})$, which is essentially worst that the order $O(1/n^2)$ in Theorem 1.1.3. This suggest that in the real case, the order of approximation could be improved, for example that maybe $\tilde{\omega}_1$ could be replaced by ω_2.

In what follows we will prove that the orders of approximation in Theorem 1.1.2, (i) and (ii) are exactly $\frac{1}{n}$.

We present

Theorem 1.1.4. (Gal [79]) *Let* $R > 1$, $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ *and let us suppose that* $f : \mathbb{D}_R \to \mathbb{C}$ *is analytic in* $\mathbb{D}_R$, *that is we can write* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{D}_R$. *If* f *is not a polynomial of degree* ≤ 1, *then for any* $r \in [1, R)$ *we have*

$$\|B_n(f) - f\|_r \geq \frac{C_r(f)}{n}, n \in \mathbb{N},$$

where $\|f\|_r = \max\{|f(z)|; |z| \leq r\}$ *and the constant* $C_r(f)$ *depends only on* f *and* r.

Proof. For all $z \in \mathbb{D}_R$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} &B_n(f)(z) - f(z) \\ &\quad = \frac{1}{n}\left\{\frac{z(1-z)}{2}f''(z) + \frac{1}{n}\left[n^2\left(B_n(f)(z) - f(z) - \frac{z(1-z)}{2n}f''(z)\right)\right]\right\}. \end{aligned}$$

Since by hypothesis $f''(z)$ is not identical zero in $\mathbb{D}_R$, there exists $0 < r_0 < 1$ such that $M_0 = \inf_{|z|=r_0} |f''(z)| > 0$. Indeed, let us suppose the contrary. Then, choosing a sequence $0 < r_n < 1$, $n \in \mathbb{N}$ such that $r_n \searrow 0$, the continuity of f'' on the compact set $\{z \in \mathbb{C}; |z| = r_n\}$, implies that there exists z_n with $|z_n| = r_n$ and $f''(z_n) = 0$. It follows $z_n \to 0$ and by the continuity of f'' in $\mathbb{D}_R$ we get $f''(0) = 0$. The analyticity of f'' implies that 0 is an isolated zero, therefore there exists $r' > 0$ such that $f''(z) \neq 0$ for all $z \in \mathbb{D}_{r'}, z \neq 0$. But this is a contradiction because for sufficiently large n we have $z_n \in \mathbb{D}_{r'}$.

Now let $r \geq 1$ be arbitrary. We obviously have $\|B_n(f) - f\|_r \geq \|B_n(f) - f\|_{r_0}$ and by the Maximum Principle, there exists a point z_0 (depending on n, f and r_0) with $|z_0| = r_0$, such that $\|B_n(f) - f\|_{r_0} = |B_n(f)(z_0) - f(z_0)|$. We get

$$\begin{aligned} \|B_n(f) - f\|_r &\geq |B_n(f)(z_0) - f(z_0)| = \left|\frac{1}{n}\left\{\frac{z_0(1-z_0)}{2}f''(z_0)\right.\right. \\ &\quad \left.\left. + \frac{1}{n}\left[n^2\left(B_n(f)(z_0) - f(z_0) - \frac{z_0(1-z_0)}{2n}f''(z_0)\right)\right]\right\}\right| \\ &\geq \frac{1}{n}\left|\left|\frac{z_0(1-z_0)}{2}f''(z_0)\right| - \frac{1}{n}\left[n^2\left|B_n(f)(z_0) - f(z_0) - \frac{z_0(1-z_0)}{2n}f''(z_0)\right|\right]\right|. \end{aligned}$$

But $\left|\frac{z_0(1-z_0)}{2}f''(z_0)\right| \geq \frac{r_0(1-r_0)}{2}M_0 > 0$ and by Theorem 1.1.3 we have

$$\begin{aligned} &n^2\left|B_n(f)(z_0) - f(z_0) - \frac{z_0(1-z_0)}{2n}f''(z_0)\right| \\ &\quad \leq n^2\|B_n(f) - f - \frac{e_1(1-e_1)}{2n}f''\|_r \\ &\quad \leq n^2\frac{5K_r(f)(1+r)^2}{2n^2} = \frac{5K_r(f)(1+r)^2}{2}. \end{aligned}$$

Therefore, there exists an index n_0 depending only on f and r, such that for all $n \geq n_0$ we have

$$\left|\frac{z_0(1-z_0)}{2}f''(z_0)\right| - \frac{1}{n}\left[n^2\left|B_n(f)(z_0)-f(z_0)-\frac{z_0(1-z_0)}{2n}f''(z_0)\right|\right] \\ \geq \frac{r_0(1-r_0)}{4}M_0 > 0,$$

which immediately implies

$$\|B_n(f)-f\|_r \geq \frac{1}{n}\cdot\frac{r_0(1-r_0)}{4}M_0, \forall n \geq n_0.$$

For $n \in \{1,2,...,n_0-1\}$ we obviously have $\|B_n(f)-f\|_r \geq \frac{M_{r,n}(f)}{n}$ with $M_{r,n}(f) = n\cdot\|B_n(f)-f\|_r > 0$, which finally implies $\|B_n(f)-f\|_r \geq \frac{C_r(f)}{n}$ for all $n \in \mathbb{N}$, where $C_r(f) = \min\{M_{r,1}, M_{r,2}(f), ..., M_{r,n_0-1}(f), \frac{r_0(1-r_0)}{4}M_0\}$. □

Combining now Theorem 1.1.4 with Theorem 1.1.2, (i), we immediately get the following.

Corollary 1.1.5. (Gal [79]) *Let $R > 1$, $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ and let us suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$. If f is not a polynomial of degree ≤ 1, then for any $r \in [1,R)$ we have*

$$\|B_n(f)-f\|_r \sim \frac{1}{n}, n \in \mathbb{N},$$

where the constants in the equivalence depend on f and r.

In the case of simultaneous approximation we present the following.

Theorem 1.1.6. (Gal [79]) *Let $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ be with $R > 1$ and let us suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. Also, let $1 \leq r < r_1 < R$ and $p \in \mathbb{N}$ be fixed. If f is not a polynomial of degree $\leq \max\{1, p-1\}$, then we have*

$$\|B_n^{(p)}(f) - f^{(p)}\|_r \sim \frac{1}{n},$$

where the constants in the equivalence depend on f, r, r_1 and p.

Proof. Taking into account the upper estimate in Theorem 1.1.2, (ii), it remains to prove the lower estimate for $\|B_n^{(p)}(f) - f^{(p)}\|_r$. Firstly, denoting by Γ the circle of radius $r_1 >$ and center 0 (where $r_1 > r \geq 1$), we have the inequality $|v-z| \geq r_1 - r$ valid for all $|z| \leq r$ and $v \in \Gamma$.

As in the proof of Theorem 1.1.4, for all $v \in \Gamma$ and $n \in \mathbb{N}$ we have

$$B_n(f)(v) - f(v) \\ = \frac{1}{n}\left\{\frac{v(1-v)}{2}f''(v) + \frac{1}{n}\left[n^2\left(B_n(f)(v) - f(v) - \frac{v(1-v)}{2n}f''(v)\right)\right]\right\},$$

which replaced in the Cauchy's formula for derivatives implies

$$B_n^{(p)}(f)(z) - f^{(p)}(z) = \frac{1}{n}\left\{\frac{p!}{2\pi i}\int_\Gamma \frac{v(1-v)f''(v)}{2(v-z)^{p+1}}dv\right.$$

$$\left. + \frac{1}{n}\cdot\frac{p!}{2\pi i}\int_\Gamma \frac{n^2\left(B_n(f)(v) - f(v) - \frac{v(1-v)}{2n}f''(v)\right)}{(v-z)^{p+1}}dv\right\}$$

$$= \frac{1}{n}\left\{\left[\frac{z(1-z)}{2}f''(z)\right]^{(p)}\right.$$

$$\left. + \frac{1}{n}\cdot\frac{p!}{2\pi i}\int_\Gamma \frac{n^2\left(B_n(f)(v) - f(v) - \frac{v(1-v)}{2n}f''(v)\right)}{(v-z)^{p+1}}dv\right\}.$$

Passing now to absolute value, for all $|z| \le r$ and $n \in \mathbb{N}$ it follows

$$|B_n^{(p)}(f)(z) - f^{(p)}(z)| \ge \frac{1}{n}\left\{\left|\left[\frac{z(1-z)}{2}f''(z)\right]^{(p)}\right|\right.$$

$$\left. - \frac{1}{n}\left|\frac{p!}{2\pi}\int_\Gamma \frac{n^2\left(B_n(f)(v) - f(v) - \frac{v(1-v)}{2n}f''(v)\right)}{(v-z)^{p+1}}dv\right|\right\},$$

where by using Theorem 1.1.3, (ii), for all $|z| \le r$ and $n \in \mathbb{N}$ we get

$$\left|\frac{p!}{2\pi}\int_\Gamma \frac{n^2\left(B_n(f)(v) - f(v) - \frac{v(1-v)}{2n}f''(v)\right)}{(v-z)^{p+1}}dv\right|$$

$$\le \frac{p!}{2\pi}\cdot\frac{2\pi r_1 n^2}{(r_1-r)^{p+1}}\|B_n(f) - f - \frac{e_1(1-e_1)}{2n}f''\|_{r_1}$$

$$\le \frac{5K_{r_1}(f)(1+r_1)^2}{2}\cdot\frac{p! r_1}{(r_1-r)^{p+1}}.$$

Denoting now $F_p(z) = \left[\frac{z(1-z)}{2}f''(z)\right]^{(p)}$, by the hypothesis on f it follows that F_p is analytic and is not identically zero in $\mathbb{D}_R$. Reasoning exactly as in the proof of Theorem 1.1.4, there exists $0 < r_0 < 1$ such that $C_0 = \inf_{|z|=r_0}|F_p(z)| > 0$. Continuing exactly as in the proof of Theorem 1.1.4 (with $\|B_n(f) - f\|_r$ replaced by $\|B_n^{(p)}(f) - f^{(p)}\|_r$), finally there exists an index $n_0 \in \mathbb{N}$ depending on f, r, r_1 and p, such that for all $n \ge n_0$ we have

$$\|B_n^{(p)}(f) - f^{(p)}\|_r \ge \frac{1}{n}\cdot\frac{C_0}{2}.$$

The cases when $n \in \{1, 2, ..., n_0 - 1\}$ are similar with those in the proof of Theorem 1.1.4. □

Remark. Let us suppose that $f^{(p)} \in C[0,1]$, $p \in \mathbb{N}$. By taking $r = 1$ and $\lambda = 1$ in Xie [203], Theorem 2, we immediately obtain the following upper estimate for the derivatives of the real Bernstein polynomials attached to f, valid for all $n \ge n_p$

$$\|B_n^{(p)}(f) - f^{(p)}\| \le A_p[\omega_1(f^{(p)}; 1/n) + \omega_2^{\varphi}(f^{(p)}; 1/\sqrt{n}) + \|f^{(p)}\|/n],$$

where $\|\cdot\|$ denotes the uniform norm on $C[0,1]$, $n_p \in \mathbb{N}$ depends only on p, ω_1 denotes the uniform modulus of continuity, $\varphi(x) = \sqrt{x(1-x)}$ and ω_2^{φ} denotes the Ditzian-Totik second order modulus of smoothness defined in Ditzian-Totik [64].

Then, the above Theorem 1.1.6 suggests the following open question : for any $p \in \mathbb{N}$, there exist the positive constants C_p and n_p depending only on p, such that for all $n \geq n_p$

$$C_p[\omega_1(f^{(p)}; 1/n) + \omega_2^{\varphi}(f^{(p)}; 1/\sqrt{n}) + \|f^{(p)}\|/n] \leq \|B_n^{(p)}(f) - f^{(p)}\|.$$

The geometric properties of Bernstein polynomials are consequences of Theorem 1.1.2 and can be expressed by the following.

Theorem 1.1.7. (Gal [77], pp. 268-269, Theorem 3.4.2) *Let us suppose that $G \subset \mathbb{C}$ is open, such that $\overline{\mathbb{D}}_1 \subset G$ and $f : G \to \mathbb{C}$ is analytic in G.*

(i) If f is univalent in $\overline{\mathbb{D}}_1$, then there exists an index n_0 depending on f, such that for all $n \geq n_0$, the complex Bernstein polynomials $B_n(f)(z) = \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} f(k/n)$ are univalent in $\overline{\mathbb{D}}_1$.

(ii) If $f(0) = f'(0) - 1 = 0$ and f is starlike in $\overline{\mathbb{D}}_1$, that is

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \text{ for all } z \in \overline{\mathbb{D}}_1,$$

then there exists an index n_0 depending on f, such that for all $n \geq n_0$, the complex Bernstein polynomials are starlike in $\overline{\mathbb{D}}_1$.

If $f(0) = f'(0) - 1 = 0$ and f is starlike only in $\mathbb{D}_1$, then for any disk of radius $0 < r < 1$ and center 0 denoted by $\mathbb{D}_r$, there exists an index $n_0 = n_0(f, \mathbb{D}_r)$, such that for all $n \geq n_0$, the complex Bernstein polynomials $B_n(f)(z)$ are starlike in $\overline{\mathbb{D}}_r$, that is,

$$Re\left(\frac{zB_n'(f)(z)}{B_n(f)(z)}\right) > 0, \text{ for all } z \in \overline{\mathbb{D}}_r.$$

(iii) If $f(0) = f'(0) - 1 = 0$ and f is convex in $\overline{\mathbb{D}}_1$, that is

$$Re\left(\frac{zf''(z)}{f'(z)}\right) + 1 > 0, \text{ for all } z \in \overline{\mathbb{D}}_1,$$

then there exists an index n_0 depending on f, such that for all $n \geq n_0$, the complex Bernstein polynomials are convex in $\overline{\mathbb{D}}_1$.

If $f(0) = f'(0) - 1 = 0$ and f is convex only in $\mathbb{D}_1$, then for any disk of radius $0 < r < 1$ and center 0 denoted by $\mathbb{D}_r$, there exists an index $n_0 = n_0(f, \mathbb{D}_r)$, such that for all $n \geq n_0$, the complex Bernstein polynomials $B_n(f)(z)$ are convex in $\overline{\mathbb{D}}_r$, that is,

$$Re\left(\frac{zB_n''(f)(z)}{B_n'(f)(z)}\right) + 1 > 0, \text{ for all } z \in \overline{\mathbb{D}}_r.$$

(iv) If $f(0) = f'(0) - 1 = 0$, $f(z) \neq 0$, for all $z \in \overline{\mathbb{D}}_1 \setminus \{0\}$ and f is spirallike of type $\gamma \in (-\pi/2, \pi/2)$ in $\overline{\mathbb{D}}_1$, that is

$$Re\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right) > 0, \text{ for all } z \in \overline{\mathbb{D}}_1,$$

then there exists an index n_0 *depending on* f *and* γ*, such that for all* $n \geq n_0$ *we have* $B_n(f)(z) \neq 0$*, for all* $z \in \overline{\mathbb{D}}_1 \setminus \{0\}$*, and* $B_n(f)(z)$ *are spirallike of type* γ *in* $\overline{\mathbb{D}}_1$*.*

If $f(0) = f'(0) - 1 = 0$*,* $f(z) \neq 0$ *for all* $z \in \mathbb{D}_1 \setminus \{0\}$ *and* f *is spirallike of type* γ *only in* $\mathbb{D}_1$*, then for any disk of radius* $0 < r < 1$ *and center* 0 *denoted by* $\mathbb{D}_r$*, there exists an index* $n_0 = n_0(f, \mathbb{D}_r, \gamma)$*, such that for all* $n \geq n_0$*, the Bernstein polynomials* $B_n(f)(z) \neq 0$ *for all* $z \in \overline{\mathbb{D}}_r \setminus \{0\}$ *and they are spirallike of type* γ *in* $\overline{\mathbb{D}}_r$*, that is,*

$$Re\left(e^{i\gamma}\frac{zB_n'(f)(z)}{B_n(f)(z)}\right) > 0, \text{ for all } z \in \overline{\mathbb{D}}_r.$$

Proof. (i) It is immediate from the uniform convergence in Theorem 1.1.2 and a well-known result concerning sequences of analytic functions converging locally uniformly to an univalent function (see e.g. Kohr-Mocanu [118], p. 130, Theorem 4.1.17 or Graham-Kohr [105], Theorem 6.1.18).

For the proof of next points (ii), (iii) and (iv), let us observe that by Theorem 1.1.2, (i) and (ii) we get that for $n \to \infty$, we have $B_n(f)(z) \to f(z)$, $B_n'(f)(z) \to f'(z)$ and $B_n''(f)(z) \to f''(z)$, uniformly in $\overline{\mathbb{D}}_1$. In all what follows, denote $P_n(f)(z) = \frac{B_n(f)(z)}{nf(1/n)}$.

By $f(0) = f'(0) - 1 = 0$ and the univalence of f, we get $nf(1/n) \neq 0$, $P_n(f)(0) = \frac{f(0)}{nf(1/n)} = 0$, $P'(f)(0) = \frac{B_n'(f)(0)}{nf(1/n)} = 1$, $n \geq 2$, $nf(1/n) = \frac{f(1/n)-f(0)}{1/n}$ converges to $f'(0) = 1$ as $n \to \infty$, which means that for $n \to \infty$, we have $P_n(f)(z) \to f(z)$, $P_n'(f)(z) \to f'(z)$ and $P_n''(f)(z) \to f''(z)$, uniformly in $\overline{\mathbb{D}}_1$.

(ii) By hypothesis we get $|f(z)| > 0$ for all $z \in \overline{\mathbb{D}}_1$ with $z \neq 0$, which from the univalence of f in $\mathbb{D}_1$, implies that we can write $f(z) = zg(z)$, with $g(z) \neq 0$, for all $z \in \overline{\mathbb{D}}_1$, where g is analytic in $\mathbb{D}_1$ and continuous in $\overline{\mathbb{D}}_1$.

Writing $P_n(f)(z)$ in the form $P_n(f)(z) = zQ_n(f)(z)$, obviously $Q_n(f)(z)$ is a polynomial of degree $\leq n - 1$.

Let $|z| = 1$. We have

$$|f(z) - P_n(f)(z)| = |z| \cdot |g(z) - Q_n(f)(z)| = |g(z) - Q_n(f)(z)|,$$

which by the uniform convergence in $\overline{\mathbb{D}}_1$ of $P_n(f)$ to f and by the maximum modulus principle, implies the uniform convergence in $\overline{\mathbb{D}}_1$ of $Q_n(f)(z)$ to $g(z)$.

Since g is continuous in $\overline{\mathbb{D}}_1$ and $|g(z)| > 0$ for all $z \in \overline{\mathbb{D}}_1$, there exist an index $n_1 \in \mathbb{N}$ and $a > 0$ depending on g, such that $|Q_n(f)(z)| > a > 0$, for all $z \in \overline{\mathbb{D}}_1$ and all $n \geq n_0$.

Also, for all $|z| = 1$, we have

$$\begin{aligned}|f'(z) - P_n'(f)(z)| &= |z[g'(z) - Q_n'(f)(z)] + [g(z) - Q_n(f)(z)]| \\ &\geq \left|\ |z| \cdot |g'(z) - Q_n'(f)(z)| - |g(z) - Q_n(f)(z)|\ \right| \\ &= \left|\ |g'(z) - Q_n'(f)(z)| - |g(z) - Q_n(f)(z)|\ \right|,\end{aligned}$$

which from the maximum modulus principle, the uniform convergence of $P_n'(f)$ to f' and of $Q_n(f)$ to g, evidently implies the uniform convergence of $Q_n'(f)$ to g'.

Then, for $|z| = 1$, we get

$$\begin{aligned}\frac{zP_n'(f)(z)}{P_n(f)} &= \frac{z[zQ_n'(f)(z) + Q_n(f)(z)]}{zQ_n(f)(z)} \\ &= \frac{zQ_n'(f)(z) + Q_n(f)(z)}{Q_n(f)(z)} \to \frac{zg'(z) + g(z)}{g(z)} \\ &= \frac{f'(z)}{g(z)} = \frac{zf'(z)}{f(z)},\end{aligned}$$

which again from the maximum modulus principle, implies

$$\frac{zP_n'(f)(z)}{P_n(f)} \to \frac{zf'(z)}{f(z)}, \text{ uniformly in } \overline{\mathbb{D}}_1.$$

Since $Re\left(\frac{zf'(z)}{f(z)}\right)$ is continuous in $\overline{\mathbb{D}}_1$, there exists $\alpha \in (0, 1)$, such that

$$Re\left(\frac{zf'(z)}{f(z)}\right) \geq \alpha, \text{ for all } z \in \overline{\mathbb{D}}_1.$$

Therefore

$$Re\left[\frac{zP_n'(f)(z)}{P_n(f)(z)}\right] \to Re\left[\frac{zf'(z)}{f(z)}\right] \geq \alpha > 0$$

uniformly on $\overline{\mathbb{D}}_1$, i.e. for any $0 < \beta < \alpha$, there is n_0 such that for all $n \geq n_0$ we have

$$Re\left[\frac{zP_n'(f)(z)}{P_n(f)(z)}\right] > \beta > 0, \text{ for all } z \in \overline{\mathbb{D}}_1.$$

Since $P_n(f)(z)$ differs from $B_n(f)(z)$ only by a constant, this proves the first part in (ii).

For the second part, the proof is identical with the first part, with the only difference that instead of $\overline{\mathbb{D}}_1$, we reason for $\overline{\mathbb{D}}_r$.

(iv) Obviously we have

$$Re\left[e^{i\gamma}\frac{zP_n'(f)(z)}{P_n(f)(z)}\right] \to Re\left[e^{i\gamma}\frac{zf'(z)}{f(z)}\right],$$

uniformly in $\overline{\mathbb{D}}_1$. We also note that since f is univalent in $\overline{\mathbb{D}}_1$, by the above point (i), there exists n_1 such that $B_n(f)(z)$ is univalent in $\overline{\mathbb{D}}_1$ for all $n \geq n_1$, which by $B_n(f)(0) = 0$ implies $B_n(f)(z) \neq 0$, for all $z \in \overline{\mathbb{D}}_1 \setminus \{0\}$, $n \geq n_1$. For the rest, the proof is identical with that from the above point (ii).

(iii) For the first part, by hypothesis there is $\alpha \in (0, 1)$, such that

$$Re\left[\frac{zf''(z)}{f'(z)}\right] + 1 \geq \alpha > 0,$$

uniformly in $\overline{\mathbb{D}}_1$. It is not difficult to show that this is equivalent with the fact that for any $\beta \in (0, \alpha)$, the function $zf'(z)$ is starlike of order β in $\overline{\mathbb{D}}_1$ (see e.g. Mocanu-Bulboacă-Sălăgean [138], p. 77), which implies $f'(z) \neq 0$, for all $z \in \overline{\mathbb{D}}_1$, i.e. $|f'(z)| > 0$, for all $z \in \overline{\mathbb{D}}_1$. Also, by the same type of reasonings as those from the above point (ii), we get

$$Re\left[\frac{zP_n''(f)(z)}{P_n'(f)(z)}\right] + 1 \to Re\left[\frac{zf''(z)}{f'(z)}\right] + 1 \geq \alpha > 0,$$

uniformly in $\overline{\mathbb{D}}_1$. As a conclusion, for any $0 < \beta < \alpha$, there is n_0 depending on f, such that for all $n \geq n_0$ we have

$$Re\left[\frac{zP_n''(f)(z)}{P_n'(f)(z)}\right] + 1 > \beta > 0, \text{ for all } z \in \overline{\mathbb{D}}_1.$$

The proof of second part in (iii) is similar, which proves the theorem. □

1.1.2 *Bernstein-Faber Polynomials on Compact Sets*

In this subsection $G \subset \mathbb{C}$ will be considered a compact set such that $\tilde{\mathbb{C}} \setminus G$ is connected. In this case, according to the Riemann Mapping Theorem, a unique conformal mapping Ψ of $\tilde{\mathbb{C}} \setminus \overline{\mathbb{D}}_1$ onto $\tilde{\mathbb{C}} \setminus G$ exists so that $\Psi(\infty) = \infty$ and $\Psi'(\infty) > 0$.

By using the Faber polynomials $F_p(z)$ attached to G (see Definition 1.0.10), for $f \in A(\overline{G})$ we can introduce the Bernstein-Faber polynomials given by the formula

$$\mathcal{B}_n(f;\overline{G})(z) = \sum_{p=0}^{n} \binom{n}{p} \Delta_{1/n}^{p} F(0) \cdot F_p(z), z \in G, \ n \in \mathbb{N},$$

where

$$\Delta_h^p F(0) = \sum_{k=0}^{p} (-1)^{p-k} \binom{p}{k} F(kh), \ F(w) = \frac{1}{2\pi i} \int_{|u|=1} \frac{f(\Psi(u))}{u-w} du, \ w \in \mathbb{D}_1.$$

Here, since $F(1)$ is involved in $\Delta_{1/n}^{n} F(0)$ and therefore in the definition of $\mathcal{B}_n(f;G)(z)$ too, in addition we will suppose that F can be extended by continuity on the boundary $\partial\mathbb{D}_1$.

Remarks. 1) For $G = \overline{\mathbb{D}}_1$ it is easy to see that the above Bernstein-Faber polynomials one reduce to the classical complex Bernstein polynomials given by

$$B_n(f)(z) = \sum_{p=0}^{n} \binom{n}{p} \Delta_{1/n}^{p} f(0) z^p = \sum_{p=0}^{n} \binom{n}{p} z^p (1-z)^{n-p} f(p/n).$$

2) It is known that, for example, $\int_0^1 \frac{\omega_p(f\circ\Psi;u)_{\partial\mathbb{D}_1}}{u} du < \infty$ is a sufficient condition for the continuity on $\partial\mathbb{D}_1$ of F in the above definition of the Bernstein-Faber polynomials (see e.g. Gaier [76], p. 52, Theorem 6). Here $p \in \mathbb{N}$ is arbitrary fixed.

The first main result one refers to approximation on compact sets without any restriction on their boundaries and can be stated as follows.

Theorem 1.1.8. *Let G be a continuum (that is a connected compact subset of $\mathbb{C}$) and suppose that f is analytic in G, that is there exists $R > 1$ such that f is analytic in G_R. Here recall that G_R denotes the interior of the closed level curve Γ_R given* *by* $\Gamma_R = \{z; |\Phi(z)| = R\} = \{\Psi(w); |w| = R\}$ *(and that* $G \subset \overline{G}_r$ *for all* $1 < r < R$*). Also, we suppose that* F *given in the definition of Bernstein-Faber polynomials can be extended by continuity on* $\partial\mathbb{D}_1$*.*

For any $1 < r < R$ *the following estimate*

$$|\mathcal{B}_n(f;\overline{G})(z) - f(z)| \le \frac{C}{n},\ z \in \overline{G}_r,\ n \in \mathbb{N},$$

holds, where $C > 0$ *depends on* f*,* r *and* G_r *but it is independent of* n*.*

Proof. First we note that since G is a continuum then it follows that $\tilde{\mathbb{C}} \setminus G$ is simply connected. By the proof of Theorem 2, p. 52 in Suetin [186], for any fixed $1 < \beta < R$ we have $f(z) = \sum_{k=0}^{\infty} a_k(f)F_k(z)$ uniformly in $\overline{G}_\beta$, where $a_k(f)$ are the Faber coefficients and are given by $a_k(f) = \frac{1}{2\pi i}\int_{|u|=\beta} \frac{f[\Psi(u)]}{u^{k+1}}du$. Note here that $G \subset \overline{G}_\beta$.

First we will prove that

$$\mathcal{B}_n(f;\overline{G})(z) = \sum_{k=0}^{\infty} a_k(f)\mathcal{B}_n(F_k;\overline{G})(z),$$

for all $z \in G$. (Note here that by hypothesis we have $\overline{G} = G$). For this purpose, denote $f_m(z) = \sum_{k=0}^{m} a_k(f)F_k(z)$, $m \in \mathbb{N}$.

Since by the linearity of $\mathcal{B}_n$ we easily get

$$\mathcal{B}_n(f_m;\overline{G})(z) = \sum_{k=0}^{m} a_k(f)\mathcal{B}_n(F_k;\overline{G})(z),\ \text{for all } z \in G,$$

it suffices to prove that $\lim_{m\to\infty} \mathcal{B}_n(f_m;\overline{G})(z) = \mathcal{B}_n(f;\overline{G})(z)$, for all $z \in G$ and $n \in \mathbb{N}$.

First we have

$$\mathcal{B}_n(f_m;\overline{G})(z) = \sum_{p=0}^{n} \binom{n}{p} \Delta^p_{1/n} G_m(0) F_k(z),$$

where $G_m(w) = \frac{1}{2\pi i}\int_{|u|=1} \frac{f_m(\Psi(u))}{u-w}du$ and $F(w) = \frac{1}{2\pi i}\int_{|u|=1} \frac{f(\Psi(u))}{u-w}du$.

Note here that since by Gaier [76], p. 48, first relation before (6.17), we have

$$\mathcal{F}_k(w) = \frac{1}{2\pi i}\int_{|u|=1} \frac{F_k(\Psi(u))}{u-w}du = w^k,\ \text{for all } |w| < 1,$$

evidently that $\mathcal{F}_k(w)$ can be extended by continuity on $\partial\mathbb{D}_1$. This also immediately implies that $G_m(w) = \frac{1}{2\pi i}\int_{|u|=1} \frac{f_m(\Psi(u))}{u-w}du$ can be extended by continuity on $\partial\mathbb{D}_1$, which means that $\mathcal{B}_n(F_k;G)(z)$ and $\mathcal{B}_n(f_m;G)(z)$ are well defined.

Now, taking into account the Cauchy's theorem we also can write

$$G_m(w) = \frac{1}{2\pi i}\int_{|u|=\beta} \frac{f_m(\Psi(u))}{u-w}du \text{ and } F(w) = \frac{1}{2\pi i}\int_{|u|=\beta} \frac{f(\Psi(u))}{u-w}du.$$

For all $n, m \in \mathbb{N}$ and $z \in G$ it follows

$$\begin{aligned}
&|\mathcal{B}_n(f_m;\overline{G})(z) - \mathcal{B}_n(f;\overline{G})(z)| \\
&\leq \sum_{p=0}^{n} \binom{n}{p} |\Delta_{1/n}^{p}(G_m - F)(0)| \cdot |F_k(z)| \\
&\leq \sum_{p=0}^{n} \binom{n}{p} \sum_{j=0}^{p} \binom{p}{j} |(G_m - F)((p-j)/n)| \cdot |F_k(z)| \\
&\leq \sum_{p=0}^{n} \binom{n}{p} \sum_{j=0}^{p} \binom{p}{j} C_{j,p,\beta} \|f_m - f\|_{\overline{G}_\beta} \cdot |F_k(z)| \\
&\leq M_{n,p,\beta,G_\beta} \|f_m - f\|_{\overline{G}_\beta},
\end{aligned}$$

which by $\lim_{m\to\infty} \|f_m - f\|_{\overline{G}_\beta} = 0$ (see e.g. the proof of Theorem 2, p. 52 in Suetin [186]) implies the desired conclusion. Here $\|f_m - f\|_{\overline{G}_\beta}$ denotes the uniform norm of $f_m - f$ on $\overline{G}_\beta$.

Consequently we obtain

$$\begin{aligned}
|\mathcal{B}_n(f;\overline{G})(z) - f(z)| &\leq \sum_{k=0}^{\infty} |a_k(f)| \cdot |\mathcal{B}_n(F_k;\overline{G})(z) - F_k(z)| \\
&= \sum_{k=0}^{n} |a_k(f)| \cdot |\mathcal{B}_n(F_k;\overline{G})(z) - F_k(z)| \\
&\quad + \sum_{k=n+1}^{\infty} |a_k(f)| \cdot |\mathcal{B}_n(F_k;\overline{G})(z) - F_k(z)|.
\end{aligned}$$

Therefore it remains to estimate $|a_k(f)| \cdot |\mathcal{B}_n(F_k;\overline{G})(z) - F_k(z)|$, firstly for all $0 \leq k \leq n$ and secondly for $k \geq n+1$, where

$$\mathcal{B}_n(F_k;\overline{G})(z) = \sum_{p=0}^{n} \binom{n}{p} [\Delta_{1/n}^{p} \mathcal{F}_k(0)] \cdot F_p(z).$$

First it is useful to observe that by Gaier [76], p. 48, combined with the Cauchy's theorem, for any fixed $1 < \beta < R$ we have

$$\mathcal{F}_k(w) := \frac{1}{2\pi i} \int_{|u|=\beta} \frac{F_k[\Psi(u)]}{u - w} du = w^k = e_k(w), \text{ for all } |w| < \beta.$$

Denote

$$D_{n,p,k} = \binom{n}{p} \Delta_{1/n}^{p} e_k(0) = \binom{n}{p} [0, 1/n, ..., p/n; e_k] \cdot (p!)/n^p.$$

It follows

$$\mathcal{B}_n(F_k;\overline{G})(z) = \sum_{p=0}^{n} D_{n,p,k} \cdot F_p(z).$$

Since for the classical complex Bernstein polynomials attached to a disk of center in origin we can write $B_n(e_k)(z) = \sum_{p=0}^{n} D_{n,p,k} z^p$, since each e_k is convex of any order and $B_n(e_k)(1) = e_k(1) = 1$ for all k, it follows that all $D_{n,p,k} \geq 0$ and $\sum_{p=0}^{n} D_{n,p,k} = 1$, for all k and n. Also, note that $D_{n,k,k} = \frac{n(n-1)\ldots(n-k+1)}{n^k}$.

In the estimation of $|a_k(f)| \cdot |\mathcal{B}_n(F_k;\overline{G})(z) - F_k(z)|$ we distinguish two cases : 1) $0 \leq k \leq n$; 2) $k > n$.

Case 1. We have

$$|\mathcal{B}_n(F_k;\overline{G})(z) - F_k(z)| \leq |F_k(z)| \cdot |1 - D_{n,k,k}| + \sum_{p=0}^{k-1} D_{n,p,k} \cdot |F_p(z)|.$$

Fix now $1 < r < \beta$. By the inequality (13), p. 44 in Suetin [186] we have

$$|F_p(z)| \leq C(r)r^p, \text{ for all } z \in \overline{G}_r, p \geq 0,$$

which immediately implies

$$|\mathcal{B}_n(F_k;\overline{G})(z) - F_k(z)| \leq 2C(r)[1 - D_{n,k,k}]r^k \leq C(r)\frac{k(k-1)}{n}r^k,$$

for all $z \in \overline{G}_r$. Here we used the inequality $1 - \Pi_{i=1}^{k} x_i \leq \sum_{i=1}^{k}(1 - x_i)$ (valid if all $x_i \in [0,1]$) which implies the inequality

$$\begin{aligned} 1 - D_{n,k,k} &= 1 - \frac{n(n-1)\ldots(n-k+1)}{n^k} = 1 - \Pi_{i=1}^{k-1}\frac{n-i}{n} \\ &\leq \sum_{i=1}^{k-1}(1 - (n-i)/n) = \frac{1}{n}\sum_{i=0}^{k-1} i = \frac{k(k-1)}{2n}. \end{aligned}$$

Also by the above formula for $a_k(f)$ we easily obtain $|a_k(f)| \leq \frac{C(\beta,f)}{\beta^k}$, for all $k \geq 0$. Note that $C(r), C(\beta, f) > 0$ are constants independent of k.

For all $z \in \overline{G}_r$ and $k = 0, 1, 2, \ldots n$ it follows

$$|a_k(f)| \cdot |\mathcal{B}_n(F_k;\overline{G})(z) - F_k(z)| \leq \frac{C(r,\beta,f)}{n}k(k-1)\left[\frac{r}{\beta}\right]^k,$$

that is

$$\sum_{k=0}^{n} |a_k(f)| \cdot |\mathcal{B}_n(F_k;\overline{G})(z) - F_k(z)| \leq \frac{C(r,\beta,f)}{n}\sum_{k=2}^{n} k(k-1)d^k, \text{ for all } z \in \overline{G}_r,$$

where $0 < d = r/\beta < 1$. Also, clearly we have $\sum_{k=2}^{n} k(k-1)d^k \leq \sum_{k=2}^{\infty} k(k-1)d^k < \infty$ which finally implies that

$$\sum_{k=0}^{n} |a_k(f)| \cdot |\mathcal{B}_n(F_k;\overline{G})(z) - F_k(z)| \leq \frac{C^*(r,\beta,f)}{n}.$$

Case 2. We have

$$\begin{aligned} \sum_{k=n+1}^{\infty} |a_k(f)| \cdot |\mathcal{B}_n(F_k;\overline{G})(z) - F_k(z)| &\leq \sum_{k=n+1}^{\infty} |a_k(f)| \cdot |\mathcal{B}_n(F_k;\overline{G})(z)| \\ &+ \sum_{k=n+1}^{\infty} |a_k(f)| \cdot |F_k(z)|. \end{aligned}$$

By the estimates mentioned in the case 1), we immediately get

$$\sum_{k=n+1}^{\infty} |a_k(f)| \cdot |F_k(z)| \le C(r,\beta,f) \sum_{k=n+1}^{\infty} d^k, \text{ for all } z \in \overline{G}_r,$$

with $d = r/\beta$.

Also,

$$\begin{aligned}\sum_{k=n+1}^{\infty} |a_k(f)| \cdot |\mathcal{B}_n(F_k;\overline{G})(z)| &= \sum_{k=n+1}^{\infty} |a_k(f)| \cdot \left|\sum_{p=0}^{n} D_{n,p,k} \cdot F_p(z)\right| \\ &\le \sum_{k=n+1}^{\infty} |a_k(f)| \cdot \sum_{p=0}^{n} D_{n,p,k} \cdot |F_p(z)|.\end{aligned}$$

But for $p \le n < k$ and taking into account the estimates obtained in the Case 1) we get

$$|a_k(f)| \cdot |F_p(z)| \le C(r,\beta,f)\frac{r^p}{\beta^k} \le C(r,\beta,f)\frac{r^k}{\beta^k}, \text{ for all } z \in \overline{G}_r,$$

which implies

$$\begin{aligned}\sum_{k=n+1}^{\infty} |a_k(f)| \cdot |\mathcal{B}_n(F_k;\overline{G})(z) - F_k(z)| &\le C(r,\beta,f) \sum_{k=n+1}^{\infty}\sum_{p=0}^{n} D_{n,p,k}\left[\frac{r}{\beta}\right]^k \\ &= C(r,\beta,f) \sum_{k=n+1}^{\infty}\left[\frac{r}{\beta}\right]^k \\ &= C(r,\beta,f)\frac{d^{n+1}}{1-d},\end{aligned}$$

with $d = r/\beta$.

In conclusion, collecting the estimates in the Cases 1) and 2) we obtain

$$|\mathcal{B}_n(f;\overline{G})(z) - f(z)| \le \frac{C_1}{n} + C_2 d^{n+1} \le \frac{C}{n},\ z \in \overline{G}_r,\ n \in \mathbb{N}.$$

This proves the theorem. □

Remarks. 1) Simultaneous upper and lower estimates in Theorem 1.1.8 (that is a similar result to Corollary 1.1.5) can easily be obtained under some restrictions on the boundaries. For that purpose firstly we recall some useful concepts and results. The first important concept is that of *Faber set.* Thus, suppose that $G \subset \mathbb{C}$ is compact. If the Faber mapping T (given by Definition 1.0.10, (ii)) defined on the set of all polynomials $\mathcal{P}(\overline{\mathbb{D}}_1)$ and with values in the set of polynomials $\mathcal{P}(G)$ is continuous, then G is called a Faber set. In this case, T admits a unique extension to a linear and bounded mapping between the Banach spaces $A(\overline{\mathbb{D}}_1)$ and $A(\overline{G})$ (see the Remark after Definition 1.0.10, or for full details see e.g. the book of Gaier [76], pp. 48-49). For example, if G is a compact set which is a Jordan domain whose boundary Γ is a rectifiable curve of bounded rotation, then G is a Faber set (see e.g. Gaier [75], p. 51, Theorem 2). Also, Theorem 1 in Frerick-Müller [74] gives other sufficient conditions on the boundary of G which assures that the compact set G is a Faber set. As a consequence, a compact set G whose boundary consists of piecewise convex curves also is a Faber set (see Frerick-Müller [74], p. 429). By Lemma 1 in Frerick-Müller [74], if G is a compact Faber set then the Faber mapping $T : A(\overline{\mathbb{D}}_1) \to A(\overline{G})$ is injective.

2) Another important concept is that of *inverse Faber set*. Thus, according to Anderson-Clunie [22], p. 546, a Faber set G is called inverse Faber set if the Faber operator T is bijective, which implies that $T^{-1} : A(\overline{G}) \to A(\overline{\mathbb{D}}_1)$ given by (see Theorem 1.0.12 or e.g. Anderson-Clunie [22], relation (1.2))

$$T^{-1}(f)(\xi) = \frac{1}{2\pi i}\int_{|w|=1} \frac{f[\Psi(w)]}{w-\xi}dw,$$

also is linear and bounded. An important result is Theorem 2 in Anderson-Clunie [22], p. 548, which says that if G is the closure of a Jordan domain whose boundary Γ is rectifiable and of boundary rotation, and in addition Γ is free of cups, then G is an inverse Faber set. Let us also recall that if the compact set G is a Faber set, then for any $1 < r$, $\overline{G}_r$ is an inverse Faber set, where $\overline{G}_r$ denotes the closure of the Jordan domain bounded by the analytic simple curve $\Gamma_r = \{\Psi(w); |w| = r\}$ (see the Remark on page 434 in Frerick-Müller [74] or Anderson-Clunie [22]). Also, in this case by Theorem 3 in Frerick-Müller [74], for $f \in A(\overline{G}_r)$ we have $T^{-1}(f) \in A(\overline{\mathbb{D}}_r)$.

As a consequence of the considerations in the above two remarks, we can state the following result.

Theorem 1.1.9. *Let G be a compact Faber set such that $\tilde{\mathbb{C}} \setminus G$ is simply connected. If f is analytic on G, that is there exists $R > 1$ such that f is analytic in G_R and if f is not a polynomial of degree ≤ 1, then for any $1 < r < R$ we have*

$$\|\mathcal{B}_n(f;\overline{G}) - f\|_{\overline{G}_r} \sim \frac{1}{n},\ n \in \mathbb{N},$$

where the constants in the equivalence depend on f, r and G_r but are independent of n. Here $\|f\|_{\overline{G}_r} = \sup_{z\in\overline{G}_r} |f(z)|$.

Proof. According to the above considerations, there exists g analytic in $\overline{\mathbb{D}}_r$ such that $f = T(g)$, that is $g = T^{-1}(f)$ (therefore F can be extended by continuity on $\partial\mathbb{D}_1$. By hypothesis on f it follows that f cannot be of the form $f(z) = c_0F_0(z) + c_1F_1(z)$ where F_0 and F_1 are the Faber polynomials of degree 0 and 1 respectively and $c_0, c_1 \in \mathbb{C}$. This immediately implies that g is not a polynomial of degree ≤ 1.

First we have $B_n(T^{-1}(f)) = T^{-1}[\mathcal{B}_n(f;\overline{G})]$. Indeed,

$$B_n(T^{-1}(f))(z) = \sum_{p=0}^{n}\binom{n}{p}\Delta^p_{1/n}T^{-1}(f)(0)z^p = \sum_{p=0}^{n}\binom{n}{p}\Delta^p_{1/n}F(0)z^p,$$

APPROXIMATION BY COMPLEX BERNSTEIN AND CONVOLUTION TYPE OPERATORS

since $T^{-1}(f)(\xi)=\frac{1}{2\pi i}\int_{|w|=1}\frac{f[\Psi(w)]}{w-\xi}dw=F(\xi)$, and

$$\begin{aligned}T^{-1}[\mathcal{B}_n(f;\overline{G})](z)&=\frac{1}{2\pi i}\int_{|w|=1}\frac{\mathcal{B}_n(f;\overline{G})[\Psi(w)]}{w-z}dw\\&=\sum_{p=0}^{n}\binom{n}{p}\Delta_{1/n}^{p}F(0)\frac{1}{2\pi i}\int_{|w|=1}\frac{F_p[\Psi(w)]}{w-z}dw\\&=\sum_{p=0}^{n}\binom{n}{p}\Delta_{1/n}^{p}F(0)z^p,\end{aligned}$$

since according to Gaier [76], p. 48, first relation before (6.17), we have

$$\frac{1}{2\pi i}\int_{|w|=1}\frac{F_p[\Psi(w)]}{w-z}dw=z^p.$$

Then by Corollary 1.1.5 and by the linearity and continuity of T^{-1} we get

$$\begin{aligned}\frac{C}{n}\le\|B_n(g)-g\|_r&=\|B_n(g)-T^{-1}(f)\|_r=\|T^{-1}[\mathcal{B}_n(f;\overline{G})]-T^{-1}(f)\|_r\\&\le|||T^{-1}|||\cdot\|\mathcal{B}_n(f;\overline{G})-f\|_{\overline{G}_r}\le M\|\mathcal{B}_n(f;\overline{G})-f\|_{\overline{G}_r},\end{aligned}$$

which proves the lower estimate.

On the other hand we have $T[B_n(g)]=\mathcal{B}_n(T(g);\overline{G})$. Indeed,

$$T[B_n(g)](z)=\sum_{p=0}^{n}\Delta_{1/n}^{p}g(0)F_p(z),$$

and

$$\mathcal{B}_n(T(g);\overline{G})(z)=\sum_{p=0}^{n}\binom{n}{p}\Delta_{1/n}^{p}H(0)F_p(z),$$

where according to Gaier [76], p. 49. relation (6.17') we have

$$H(w)=\frac{1}{2\pi i}\int_{|u|=1}\frac{T(g)[\Psi(u)]}{u-w}du=g(w).$$

Therefore by the same Corollary 1.1.5 and by the linearity and continuity of T we obtain

$$\begin{aligned}\|\mathcal{B}_n(f;\overline{G})-f\|_{\overline{G}_r}&=\|\mathcal{B}_n(T(g);\overline{G})-T(g)\|_{\overline{G}_r}=\|T[B_n(g)]-T(g)\|_{\overline{G}_r}\\&\le|||T|||\cdot\|B_n(g)-g\|_r\le\frac{C}{n},\end{aligned}$$

which proves the upper estimate and the theorem. □

APPROXIMATION BY COMPLEX BERNSTEIN AND CONVOLUTION TYPE OPERATORS

1.2 Iterates of Bernstein Polynomials

First we deal with the approximation properties of the iterates of complex Bernstein polynomials and their relationship with the theory of the semigroups of operators.

For $R > 1$, let us define by $\mathbb{A}_R$ the space of all functions defined and analytic in the open disk of center 0 and radius R denoted by $\mathbb{D}_R$. Denoting $r_j = R - \frac{R-1}{j}$, $j \in \mathbb{N}$ and for $f \in \mathbb{A}_R$, $\|f\|_j = max\{|f(z)|; |z| \le r_j\}$, since $r_1 = 1$ and $r_j \nearrow R$, it is well-known that $\{\|\cdot\|_j, j \in \mathbb{N}\}$ is a countable family of increasing semi-norms on $\mathbb{A}_R$ and that $\mathbb{A}_R$ becomes a metrizable complete locally convex space (Fréchet space), with respect to the metric

$$d(f,g) = \sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{\|f-g\|_j}{1+\|f-g\|_j}, f, g \in \mathbb{A}_R.$$

It is well-known that $\lim_{n\to\infty} d(f_n, f) = 0$ is equivalent to the fact that the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly on compacts in $\mathbb{D}_R$. Details about the space $\mathbb{A}_R$ and the metric d can be found in e.g. Kohr-Mocanu [118], pp. 104-107.

Now, for $f \in \mathbb{A}_R$, that is of the form $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$, let us define the iterates of complex Bernstein polynomial $B_n(f)(z)$, by $B_n^{(1)}(f)(z) = B_n(f)(z)$ and $B_n^{(m)}(f)(z) = B_n[B_n^{(m-1)}(f)](z)$, for any $m \in \mathbb{N}$, $m \ge 2$. Since we have (see e.g. Lorentz [125], p. 88, proof of Theorem 4.1.1), $B_n(f)(z) = \sum_{k=0}^{\infty} c_k B_n(e_k)(z)$, by recurrence for all $m \ge 1$, we easily get that $B_n^{(m)}(f)(z) = \sum_{k=0}^{\infty} c_k B_n^{(m)}(e_k)(z)$, with $e_k(z) = z^k$.

The first main result of this section is the following.

Theorem 1.2.1. (Gal [78]) *Let $f \in \mathbb{A}_R$ with $R > 1$, that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$.*

(i) For any $n \in \mathbb{N}$, we have

$$\lim_{m\to\infty} d[B_n^{(m)}(f), B_1(f)] = 0;$$

(ii) If $\lim_{n\to\infty} \frac{m_n}{n} = 0$, then

$$\lim_{n\to\infty} d[B_n^{(m_n)}(f), f] = 0.$$

Moreover, for any fixed $q \in \mathbb{N}$, the following estimates hold

$$\|B_n^{(m)}(f) - f\|_q \le \frac{m}{n} \sum_{k=2}^{\infty} |c_k| k(k-1) r_q^k,$$

and

$$d[B_n^{(m)}(f), f] \le \frac{m}{n} \sum_{k=2}^{\infty} |c_k| k(k-1) r_q^k + \frac{1}{2^q},$$

where $\sum_{k=2}^{\infty} |c_k| k(k-1) r_q^k < \infty$.

(iii) If $\lim_{n\to\infty} \frac{m_n}{n} = \infty$, then

$$\lim_{n\to\infty} d[B_n^{(m_n)}(f), B_1(f)] = 0;$$

(iv) If $\lim_{n\to\infty}\frac{m_n}{n}=t\in(0,\infty)$*, then*

$$\lim_{n\to\infty} d[B_n^{(m_n)}(f), T(t)(f)]=0,$$

where $L(f)(z)=(1-z)f(0)+zf(1), z\in\mathbb{D}_R$*,*

$$T(t)(f)(z)=L(f)(z)+z(1-z)\int_0^1 G_t(z,y)[f(y)-L(f)(y)]dy, z\in\mathbb{D}_R,$$

$$G_t(z,y)=\sum_{k=2}^{\infty}\frac{k(2k-1)}{k-1}e^{-k(k-1)t/2}P_{k-2}^{(1,1)}(2z-1)P_{k-2}^{(1,1)}(2y-1),$$

$z\in\mathbb{D}_R, y\in[0,1]$*, and* $P_{k-2}^{(1,1)}(z)$*,* $|z|<R$*, are the Jacobi polynomials normalized to be* $k-1$ *at* $z=1$*.*

Proof. (i) Since from Karlin-Ziegler [114] it is known that if $m\to\infty$, then $B_n^{(m)}(f)(x)\to B_1(f)(x)$, uniformly on the interval $[0,1]$, according to the classical Vitali's result (see e.g. Kohr-Mocanu [118], p. 112, Theorem 3.2.10], it suffices to show that for any fixed $n\in\mathbb{N}$, the iterate sequence of polynomials $B_n^{(m)}(f)(z)$, $m=1,2,...$, is uniformly bounded with respect to $m\in\mathbb{N}$ in each $\overline{\mathbb{D}}_r$ with $1\le r<R$.

Let $n\in\mathbb{N}$ be fixed. According to He [107], p. 580, relationship (7), we can write

$$B_n(e_k)(z)=\sum_{j=1}^{k}S(k,j)\frac{n(n-1)...[n-(j-1)]}{n^k}z^j,$$

where $S(k,j)$ are the Stirling numbers of second kind. It is well-known that these numbers satisfy $S(k,j)\ge 0$, for all $j,k\in\mathbb{N}$ and

$$\sum_{j=1}^{k}S(k,j)n(n-1)...[n-(j-1)]=n^k,\ \text{for } k,n\in\mathbb{N}.$$

Let $|z|\le r$ with $r\ge 1$. Since $S(k,j)n(n-1)...[n-(j-1)]\ge 0$, for all $k,n,j\in\mathbb{N}$ with $1\le j\le k$, it follows

$$\begin{aligned}|B_n(e_k)(z)| &\le \sum_{j=1}^{k}S(k,j)\frac{n(n-1)...[n-(j-1)]}{n^k}|e_j(z)|\\ &\le \sum_{j=1}^{k}S(k,j)\frac{n(n-1)...[n-(j-1)]}{n^k}r^j\\ &\le \sum_{j=1}^{k}S(k,j)\frac{n(n-1)...[n-(j-1)]}{n^k}r^k=r^k.\end{aligned}$$

Applying now B_n to the above equality, we obtain

$$B_n^{(2)}(e_k)(z)=\sum_{j=1}^{k}S(k,j)\frac{n(n-1)...[n-(j-1)]}{n^k}B_n(e_j)(z),$$

APPROXIMATION BY COMPLEX BERNSTEIN AND CONVOLUTION TYPE OPERATORS

which from the last inequality implies

$$\begin{aligned}|B_n^{(2)}(e_k)(z)| &= \sum_{j=1}^{k} S(k,j)\frac{n(n-1)...[n-(j-1)]}{n^k}|B_n(e_j)(z)|\\ &\le \sum_{j=1}^{k} S(k,j)\frac{n(n-1)...[n-(j-1)]}{n^k}r^j \le r^k.\end{aligned}$$

Reasoning by recurrence, we easily get

$$|B_n^{(m)}(e_k)(z)| \le r^k,$$

for all $k, n, m \in \mathbb{N}$ and $z \in \overline{\mathbb{D}}_r$.

This implies

$$|B_n^{(m)}(f)(z)| \le \sum_{k=0}^{\infty} |c_k| r^k < +\infty,$$

for all $m, n \in \mathbb{N}$ and $z \in \overline{\mathbb{D}}_r$, which proves (i).

(ii) Since from the last inequality in (i), for each $r \in [1, R)$ the sequence $B_n^{(m)}(f)(z)$, $m, n = 1, 2, ...$, is in fact uniformly bounded in $\overline{\mathbb{D}}_r$ with respect to both $m, n \in \mathbb{N}$, and since by Kelisky-Rivlin [115], we have $B_n^{(m_n)}(f)(x) \to f(x)$ as $n \to \infty$, uniformly for $x \in [0, 1]$, it follows that the Vitali's convergence theorem implies the first convergence in (ii).

Since $B_n^{(m)}(f)(z) = \sum_{k=0}^{\infty} c_k B_n^{(m)}(e_k)(z)$, with $e_k(z) = z^k$, we get

$$|B_n^{(m)}(f)(z) - f(z)| \le \sum_{k=2}^{\infty} |c_k| \cdot |B_n^{(m)}(e_k)(z) - e_k(z)|.$$

But according to He [107], we have

$$B_n(e_k)(z) - e_k(z) = \sum_{j=1}^{k} S(k,j)\frac{n(n-1)...[n-(j-1)]}{n^k}e_j(z) - e_k(z).$$

Therefore,

$$\begin{aligned}B_n(e_k)(z) - e_k(z) &= \sum_{j=1}^{k-1} S(k,j)\frac{n(n-1)...[n-(j-1)]}{n^k}e_j(z)\\ &\quad +[(1-1/n)...(1-(k-1)/n)-1]e_k(z),\end{aligned}$$

which immediately implies

$$\begin{aligned}B_n^{(p)}[B_n(e_k)(z) - e_k(z)] &= \sum_{j=1}^{k-1} S(k,j)\frac{n(n-1)...[n-(j-1)]}{n^k}B_n^{(p)}(e_j)(z)\\ &\quad +[(1-1/n)...(1-(k-1)/n)-1]B_n^{(p)}(e_k)(z).\end{aligned}$$

Taking into account that by the proof of the above point (i), we have $|B_n^{(p)}(e_j)(z)| \leq r^j$, for all $p, n, j \in \mathbb{N}$ and $|z| \leq r$, it follows

$$\begin{aligned}
|B_n^{(p)}[B_n(e_k) - e_k](z)| &\leq \sum_{j=1}^{k-1} S(k,j) \frac{n(n-1)...[n-(j-1)]}{n^k} |B_n^{(p)}(e_j)(z)| \\
&\quad + [1-(1-1/n)...(1-(k-1)/n)] |B_n^{(p)}(e_k)(z)| \\
&\leq \sum_{j=1}^{k-1} S(k,j) \frac{n(n-1)...[n-(j-1)]}{n^k} r^j \\
&\quad + [1-(1-1/n)...(1-(k-1)/n)] r^k \\
&\leq 2[1-(1-1/n)...(1-(k-1)/n)] r^k.
\end{aligned}$$

But

$$B_n^{(m)}(e_k)(z) - e_k(z) = \sum_{p=0}^{m-1} B_n^{(p)}[B_n(e_k)(z) - e_k(z)],$$

which implies for all $|z| \leq r$ that

$$\begin{aligned}
|B_n^{(m)}(e_k) - e_k| &\leq \sum_{p=0}^{m-1} |B_n^{(p)}[B_n(e_k) - e_k](z)| \\
&\leq 2m[1-(1-1/n)...(1-(k-1)/n)] r^k,
\end{aligned}$$

and finally

$$|B_n^{(m)}(f)(z) - f(z)| \leq 2m \sum_{k=0}^{\infty} |c_k| \cdot [1-(1-1/n)...(1-(k-1)/n)] r^k.$$

Since $[1-(1-1/n)...(1-(k-1)/n)] \leq \frac{1}{n}[1+...+(k-1)] = \frac{k(k-1)}{2n}$, for all $k \in \mathbb{N}$, $k \geq 2$, by choosing $r = r_q$, we get the first required inequality in the statement. Note that $\sum_{k=2}^{\infty} |c_k| k(k-1) r^{k-2} < \infty$, since we have $f''(z) = \sum_{k=2}^{\infty} c_k k(k-1) z^{k-2}$, for all $|z| \leq r$.

The second estimate in (ii) is a direct consequence of the inequality

$$\begin{aligned}
d(f,g) &= \sum_{j=1}^{q} \frac{1}{2^j} \cdot \frac{\|f-g\|_j}{1+\|f-g\|_j} + \sum_{j=q+1}^{\infty} \frac{1}{2^j} \cdot \frac{\|f-g\|_j}{1+\|f-g\|_j} \\
&\leq \frac{\|f-g\|_q}{1+\|f-g\|_q} \sum_{j=1}^{q} \frac{1}{2^j} + \sum_{j=q+1}^{\infty} \frac{1}{2^j} \\
&\leq \frac{\|f-g\|_q}{1+\|f-g\|_q} + \frac{1}{2^q} \leq \|f-g\|_q + \frac{1}{2^q}.
\end{aligned}$$

(iii) Since $B_n^{(m_n)}(f)(x) \to B_1(f)(x)$, as $n \to \infty$, uniformly for $x \in [0,1]$ (see Kelisky-Rivlin [115]) the proof is similar with that of (ii).

(iv) First we prove that for any $t > 0$, the complex series $G_t(z, y)$ is uniformly and absolutely convergent for $|z|, |y| \le r$, with $r \ge 1$. Indeed, from the representation formula

$$P_n^{(1,1)}(z) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n+1}{k} \binom{n+1}{n-k} (z+1)^k (z-1)^{n-k}, |z| \le r,$$

we get

$$|P_n^{(1,1)}(z)| \le \left(\frac{r+1}{2}\right)^n \sum_{k=0}^{n} \binom{n+1}{k}\binom{n+1}{n-k} = \left(\frac{r+1}{2}\right)^n \binom{2n+2}{n}$$
$$\le \left(\frac{r+1}{2}\right)^n \left(\frac{(2n+2)e}{n}\right)^n \le \left(\frac{r+1}{2}\right)^n (4e)^n \le [6(r+1)]^n.$$

We used above the Vandermond's equality $\sum_{k=0}^{j} \binom{n}{k}\binom{m}{j-k} = \binom{n+m}{j}$ and the inequality $\binom{n}{k} \le \left(\frac{ne}{k}\right)^k$.

Now, since for $|z|, |y| \le r$ we get $|2z-1|, |2y-1| \le 2r+1$, denoting $\rho = 2r+1$, it follows

$$|G_t(z, y)| \le \sum_{k=2}^{\infty} \frac{k(2k-1)}{k-1} e^{-k(k-1)t/2} |P_{k-2}^{(1,1)}(2z-1)| \cdot |P_{k-2}^{(1,1)}(2y-1)|$$
$$\le \sum_{k=2}^{\infty} \frac{k(2k-1)}{k-1} e^{-k(k-1)t/2} [6\rho]^{2k-4}.$$

By applying the ratio test, the last series (of positive numbers) is convergent.

This shows that $T(t)(f)(z)$ is well defined for any $t > 0$ and all $|z| < R$.

In what follows, it suffices to prove that for any fixed $r \in [1, R)$ we have

$$\lim_{n\to\infty} \|B_n^{(m_n)}(f) - T(t)(f)\|_r = 0.$$

Since from Karlin-Ziegler [114], for $t > 0$ we have

$$\lim_{n\to\infty} B_n^{(m_n)}(f)(x) = L(f)(x) + x(1-x) \int_0^1 G_t(x, y)[f(y) - L(f)(y)]dy,$$

uniformly with respect to $x \in [0, 1]$, according to the Vitali's theorem, it suffices to prove that the sequence $(B_n^{(m_n)}(f)(z))_{n\in\mathbb{N}}$ is uniformly bounded in $\overline{\mathbb{D}_r}$. However this fact was proved by the last inequality at the above point (i) (see also the remark at the beginning of point (ii)). Therefore the theorem has been proved. □

Remarks. 1) The property (iv) in Theorem 1.2.1 suggests that for $f \in \mathbb{A}_R$, the limit of the iterates $B_n^{(m_n)}(f)(z)$ represents the semigroup of operators $T(t)(f)(z)$ defined on the locally convex space (Fréchet) $\mathbb{A}_R$.

2) The results in Theorem 1.2.1 extend some related results in the case of iterates of real Bernstein polynomials on $[0, 1]$ (see Karlin-Ziegler [114], Kelisky-Rivlin [115]).

In what follows we prove that the shape preserving properties for complex Bernstein polynomials in Theorem 1.1.7 hold for their iterates too.

In the proofs of these properties we need the following two auxiliary lemmas.

Lemma 1.2.2. (Gal [78]) *Let* $f \in \mathbb{A}_R$ *with* $R > 1$, *that is* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{D}_R$. *For any* $m \in \mathbb{N}$, *we have*

$$\lim_{n\to\infty} d[B_n^{(m)}(f), f] = 0,$$

that is the sequence $(B_n^{(m)}(f))_{n\in\mathbb{N}}$ *uniformly converges to* f *on compacts disks in* $\mathbb{D}_R$.

Proof. Note that Lemma 1.2.2 is a particular case of Theorem 1.2.1, (ii), for the constant sequence $m_n \equiv m$. □

Lemma 1.2.3. (Gal [78]) *Let* $f \in \mathbb{A}_R$, $R > 1$, *be satisfying* $f(0) = 0$. *For all* $m, n \in \mathbb{N}$ *we have* $[B_n^{(m)}(f)]'(0) = nB_n^{(m-1)}(f)(1/n)$ *and* $\lim_{n\to\infty}[n \cdot B_n^{(j)}(f)(1/n)] = f'(0)$, *for any fixed* j, *where by convention,* $B_n^{(0)}(f) = f$ *and* $[B_n^{(m)}(f)]'(0)$ *denotes the first derivative of* $B_n^{(m)}(f)(z)$ *at* 0.

Proof. We have

$$B_n^{(m)}(f)(z) = B_n[B_n^{(m-1)}(f)](z),$$

which by $B_n'(f)(0) = nf(1/n)$ implies

$$[B_n^{(m)}(f)]'(0) = nB_n^{(m-1)}(f)(1/n).$$

For the second part of lemma, let us observe that it suffices to prove it for real functions $f(x), x \in [0,1]$, in $C^2[0,1]$. Indeed, by $f(z) = U(x,y)+iV(x,y), z = x+iy$, where U and V have partial derivatives of any order, we get $f(x) = U(x,0) + iV(x,0) := g(x) + ih(x)$, for all $x \in [0,1]$, where g, h are continuously differentiable of any order. Also, we take into account that $B_n^{(m)}(\cdot)$ is a linear operator on $C[0,1]$, for any $n, m \in \mathbb{N}$.

We obviously have $nf(1/n) = \frac{f(1/n)-f(0)}{(1/n)} \to f'(0)$, as $n \to \infty$.

In what follows, we will use the well-known pointwise estimate for Bernstein polynomials when $f \in C^2[0,1]$, given by

$$|B_n(f)(x) - f(x)| \le C\|f''\|\frac{x(1-x)}{n}, \text{ for all } x \in [0,1], n \in \mathbb{N},$$

where $\|\cdot\|$ denotes the uniform norm in $C[0,1]$.

We get

$$\begin{aligned} nB_n(f)(1/n) &= \frac{B_n(f)(1/n) - B_n(f)(0)}{(1/n)} \\ &= \frac{B_n(f)(1/n) - f(1/n)}{(1/n)} + \frac{f(1/n) - f(0)}{(1/n)} \to f'(0), \end{aligned}$$

as $n \to \infty$, since by the above estimate we have

$$\left|\frac{B_n(f)(1/n) - f(1/n)}{(1/n)}\right| \le \frac{C\|f''\|}{n} \to 0, \text{ for } n \to \infty.$$

Then, we get

$$\begin{aligned}&nB_n^{(2)}(f)(1/n)\\&\quad=\frac{B_n[B_n(f)](1/n)-B_n(f)(1/n)}{(1/n)}+\frac{B_n(f)(1/n)-B_n(f)(0)}{(1/n)}\to f'(0),\end{aligned}$$

as $n\to\infty$. Indeed, by applying again the above pointwise estimate for f replaced by $B_n(f)$ and taking into account the inequality $\|B_n''(f)\|\le\|f''\|$ (which easily follows from e.g. Lorentz [125], p. 12, relation (2)), we obtain

$$\left|\frac{B_n[B_n(f)](1/n)-B_n(f)(1/n)}{(1/n)}\right|\le\frac{C\|B_n''(f)\|}{n}\le\frac{C\|f''\|}{n}\to 0,$$

as $n\to\infty$.

But $\lim_{n\to\infty}[n\cdot B_n^{(j)}(f)(1/n)]=f'(0)$, for any j, easily follows by mathematical induction, which proves the lemma. □

The main result is the following.

Theorem 1.2.4. (Gal [78]) *Let us suppose that $G\subset\mathbb{C}$ is open, so that $\overline{\mathbb{D}}_1\subset G$ and $f:G\to\mathbb{C}$ be analytic in G. Also, let $m\in\mathbb{N}$ be fixed.*

(i) If f is univalent in $\overline{\mathbb{D}}_1$, then there exists an index n_0 depending on f and m, so that the mth iterates $B_n^{(m)}(f)(z)$, be univalent in $\overline{\mathbb{D}}_1$, for all $n\ge n_0$.

(ii) If $f(0)=f'(0)-1=0$ and f is starlike in $\overline{\mathbb{D}}_1$, that is

$$Re\left(\frac{zf'(z)}{f(z)}\right)>0,\ \textit{for all } z\in\overline{\mathbb{D}}_1,$$

then there exists an index n_0 depending on f and m, so that the mth iterates $B_n^{(m)}(f)(z)$, be starlike in $\overline{\mathbb{D}}_1$, for all $n\ge n_0$.

If $f(0)=f'(0)-1=0$ and f is starlike only in $\mathbb{D}_1$, then for any disk of radius $0<r<1$ and center 0 denoted by $\mathbb{D}_r$, there exists an index $n_0=n_0(f,m,\mathbb{D}_r)$, so that the mth iterates $B_n^{(m)}(f)(z)$, be starlike in $\overline{\mathbb{D}}_r$ for all $n\ge n_0$, that is,

$$Re\left(\frac{z[B_n^{(m)}]'(f)(z)}{B_n^{(m)}(f)(z)}\right)>0,\ \textit{for all } z\in\overline{\mathbb{D}}_r.$$

(Here $[B_n^{(m)}]'(f)(z)$ denotes the first derivative of $B_n^{(m)}(f)(z)$.)

(iii) If $f(0)=f'(0)-1=0$ and f is convex in $\overline{\mathbb{D}}_1$, that is

$$Re\left(\frac{zf''(z)}{f'(z)}\right)+1>0,\ \textit{for all } z\in\overline{\mathbb{D}}_1,$$

then there exists an index n_0 depending on f and m, so that the mth iterates $B_n^{(m)}(f)(z)$, be convex in $\overline{\mathbb{D}}_1$, for all $n\ge n_0$.

If $f(0)=f'(0)-1=0$ and f is convex only in $\mathbb{D}_1$, then for any disk of radius $0<r<1$ and center 0 denoted by $\mathbb{D}_r$, there exists an index $n_0=n_0(f,m,\mathbb{D}_r)$, so that for all $n\ge n_0$, the mth iterates $B_n^{(m)}(f)(z)$ be convex in $\overline{\mathbb{D}}_r$, that is,

$$Re\left(\frac{z[B_n^{(m)}]''(f)(z)}{[B_n^{(m)}]'(f)(z)}\right)+1>0,\ \textit{for all } z\in\overline{\mathbb{D}}_r.$$

(iv) If $f(0)=f'(0)-1=0$, $f(z)\neq 0$, for all $z\in\overline{\mathbb{D}}_1\setminus\{0\}$ and f is spirallike of type $\gamma\in(-\pi/2,\pi/2)$ in $\overline{\mathbb{D}}_1$, that is

$$Re\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right)>0,\ \text{for all } z\in\overline{\mathbb{D}}_1,$$

then there exists an index n_0 depending on f, m and γ, so that the mth iterates $B_n^{(m)}(f)(z)\neq 0$, for all $z\in\overline{\mathbb{D}}_1\setminus\{0\}$, and $B_n^{(m)}(f)(z)$ be spirallike of type γ in $\overline{\mathbb{D}}_1$, for all $n\geq n_0$.

If $f(0)=f'(0)-1=0$, $f(z)\neq 0$ for all $z\in\mathbb{D}_1\setminus\{0\}$ and f is spirallike of type γ only in $\mathbb{D}_1$, then for any disk of radius $0<r<1$ and center 0 denoted by $\mathbb{D}_r$, there exists an index $n_0=n_0(f,m,\mathbb{D}_r,\gamma)$, so that for all $n\geq n_0$, the mth iterates $B_n^{(m)}(f)(z)\neq 0$ for all $z\in\overline{\mathbb{D}}_r\setminus\{0\}$ be spirallike of type γ in $\overline{\mathbb{D}}_r$, that is,

$$Re\left(e^{i\gamma}\frac{z[B_n^{(m)}]'(f)(z)}{B_n^{(m)}(f)(z)}\right)>0,\ \text{for all } z\in\overline{\mathbb{D}}_r.$$

Proof. (i) It is immediate from the uniform convergence in Lemma 1.2.2 and a well-known result concerning sequences of analytic functions converging locally uniformly to an univalent function (see e.g. Kohr-Mocanu [118], p. 130, Theorem 4.1.17).

For the proofs of the next points (ii), (iii) and (iv), let us make some general useful considerations. According to Lemma 1.2.2, combined with the Weierstrass's well-known result, it follows that as $n\to\infty$, uniformly in $\overline{\mathbb{D}}_1$ we have

$$B_n^{(m)}(f)(z)\to f(z), [B_n^{(m)}]'(f)(z)\to f'(z) \text{ and } [B_n^{(m)}]''(f)(z)\to f''(z).$$

In what follows we denote $C_{n,m}=[B_n^{(m)}(f)]'(0)$ and $P_n^{(m)}(f)(z)=\frac{B_n^{(m)}(f)(z)}{C_{n,m}}$.

Note here that $f'(0)=1$ combined with Lemma 1.2.3, implies that for any $m\in\mathbb{N}$, there exists $n(m,f)$ so that $C_{n,m}>0$, for all $n\geq n(m,f)$, (in fact we have $\lim_{n\to\infty}C_{n,m}=1$).

From $f(0)=0$ we get $B_n^{(m)}(f)(0)=0$ and $P_n^{(m)}(f)(0)=0$, while from the definition of $P_n^{(m)}$ we obtain $[P_n^{(m)}]'(f)(0)=1$.

By combining all of these facts with Lemma 1.2.2, we obtain that for $n\to\infty$, we have $P_n^{(m)}(f)(z)\to f(z)$, $[P_n^{(m)}]'(f)(z)\to f'(z)$ and $[P_n^{(m)}]''(f)(z)\to f''(z)$, uniformly in $\overline{\mathbb{D}}_1$.

(ii) By hypothesis we get $|f(z)|>0$ for all $z\in\overline{\mathbb{D}}_1$ with $z\neq 0$, which from the univalence of f in $\mathbb{D}_1$, implies that we can write $f(z)=zg(z)$, with $g(z)\neq 0$, for all $z\in\overline{\mathbb{D}}_1$, where g is analytic in $\mathbb{D}_1$ and continuous in $\overline{\mathbb{D}}_1$.

By writing $P_n^{(m)}(f)(z)$ in the form $P_n^{(m)}(f)(z)=zQ_{n,m}(f)(z)$, it is obvious that $Q_{n,m}(f)(z)$ is a polynomial of degree $\leq n-1$.

Let $|z|=1$. We have

$$|f(z)-P_n^{(m)}(f)(z)|=|z|\cdot|g(z)-Q_{n,m}(f)(z)|=|g(z)-Q_{n,m}(f)(z)|,$$

which by the uniform convergence in $\overline{\mathbb{D}}_1$ of $P_n^{(m)}(f)$ to f and by the maximum modulus principle implies the uniform convergence in $\overline{\mathbb{D}}_1$ of $Q_{n,m}(f)(z)$ to $g(z)$.

Since g is continuous in $\overline{\mathbb{D}}_1$ and $|g(z)| > 0$ for all $z \in \overline{\mathbb{D}}_1$, there exist an index $n_1 \in \mathbb{N}$ and $a > 0$ depending on g, so that $|Q_{n,m}(f)(z)| > a > 0$, for all $z \in \overline{\mathbb{D}}_1$ and all $n \geq n_0$.

Also, for all $|z| = 1$, we have

$$\begin{aligned}|f'(z) - [P_n^{(m)}]'(f)(z)| &= |z[g'(z) - Q'_{n,m}(f)(z)] + [g(z) - Q_{n,m}(f)(z)]| \\ &\geq \left|\quad |z||g'(z) - Q'_{n,m}(f)(z)| - |g(z) - Q_{n,m}(f)(z)| \quad\right| \\ &= \left|\quad |g'(z) - Q'_{n,m}(f)(z)| - |g(z) - Q_{n,m}(f)(z)| \quad\right|\end{aligned}$$

which from the maximum modulus principle, the uniform convergence of $[P_n^{(m)}]'(f)$ to f' and of $Q_{n,m}(f)$ to g, evidently implies the uniform convergence of $Q'_{n,m}(f)$ to g'.

Then, for $|z| = 1$, we get

$$\begin{aligned}\frac{z[P_n^{(m)}]'(f)(z)}{P_n^{(m)}(f)} &= \frac{z[zQ'_{n,m}(f)(z) + Q_{n,m}(f)(z)]}{zQ_{n,m}(f)(z)} \\ &= \frac{zQ'_{n,m}(f)(z) + Q_{n,m}(f)(z)}{Q_{n,m}(f)(z)} \to \frac{zg'(z) + g(z)}{g(z)} \\ &= \frac{f'(z)}{g(z)} = \frac{zf'(z)}{f(z)},\end{aligned}$$

which again from the maximum modulus principle, implies

$$\frac{z[P_n^{(m)}]'(f)(z)}{P_n^{(m)}(f)} \to \frac{zf'(z)}{f(z)}, \text{ uniformly in } \overline{\mathbb{D}}_1.$$

Since $Re\left(\frac{zf'(z)}{f(z)}\right)$ is continuous in $\overline{\mathbb{D}}_1$, there exists $\alpha \in (0,1)$, so that

$$Re\left(\frac{zf'(z)}{f(z)}\right) \geq \alpha, \text{ for all } z \in \overline{\mathbb{D}}_1.$$

Therefore,

$$Re\left[\frac{z[P_n^{(m)}]'(f)(z)}{P_n^{(m)}(f)(z)}\right] \to Re\left[\frac{zf'(z)}{f(z)}\right] \geq \alpha > 0$$

uniformly on $\overline{\mathbb{D}}_1$, i.e. for any $0 < \beta < \alpha$, there is n_0 so that for all $n \geq n_0$ we have

$$Re\left[\frac{z[P_n^{(m)}]'(f)(z)}{P_n^{(m)}(f)(z)}\right] > \beta > 0, \text{ for all } z \in \overline{\mathbb{D}}_1.$$

Since $P_n^{(m)}(f)(z)$ differs from $B_n^{(m)}(f)(z)$ only by a constant, this proves the first part in (ii).

For the second part, the proof is identical with the first part, with the only difference that instead of $\overline{\mathbb{D}}_1$, we reason for $\overline{\mathbb{D}}_r$.

(iv) Obviously we have

$$Re\left[e^{i\gamma}\frac{z[P_n^{(m)}]'(f)(z)}{P_n^{(m)}(f)(z)}\right] \to Re\left[e^{i\gamma}\frac{zf'(z)}{f(z)}\right],$$

uniformly in $\overline{\mathbb{D}}_1$.

We also note that since f is univalent in $\overline{\mathbb{D}}_1$, according to Lemma 1.2.2, there exists $n_1(m, f)$ so that $B_n^{(m)}(f)(z)$ be univalent in $\overline{\mathbb{D}}_1$ for all $n \geq n_1(m, f)$. Therefore, $B_n^{(m)}(f)(0) = 0$ implies $B_n^{(m)}(f)(z) \neq 0$, for all $z \in \overline{\mathbb{D}}_1 \setminus \{0\}$, $n \geq n_1(m, f)$. For the rest, the proof is identical with that from the above point (ii).

(iii) For the first part, by hypothesis there is $\alpha \in (0, 1)$, so that

$$Re\left[\frac{zf''(z)}{f'(z)}\right] + 1 \geq \alpha > 0,$$

uniformly in $\overline{\mathbb{D}}_1$. It is not difficult to show that this is equivalent with the fact that for any $\beta \in (0, \alpha)$, the function $zf'(z)$ is starlike of order β in $\overline{\mathbb{D}}_1$ (see e.g. Mocanu-Bulboacă-Sălăgean [138], p. 77), which implies that $f'(z) \neq 0$, for all $z \in \overline{\mathbb{D}}_1$, i.e. $|f'(z)| > 0$, for all $z \in \overline{\mathbb{D}}_1$. Also, by using the same type of reasonings as those mentioned in the above point (ii), we get

$$Re\left[\frac{z[P_n^{(m)}]''(f)(z)}{[P_n^{(m)}]'(f)(z)}\right] + 1 \to Re\left[\frac{zf''(z)}{f'(z)}\right] + 1 \geq \alpha > 0,$$

uniformly in $\overline{\mathbb{D}}_1$. As a conclusion, for any $0 < \beta < \alpha$, there is n_0 depending on f, so that for all $n \geq n_0$ we have

$$Re\left[\frac{z[P_n^{(m)}]''(f)(z)}{[P_n^{(m)}]'(f)(z)}\right] + 1 > \beta > 0, \text{ for all } z \in \overline{\mathbb{D}}_1.$$

The proof of second part in (iii) is similar, which proves the theorem. □

1.3 Generalized Voronovskaja Theorems for Bernstein Polynomials

It is well-known the fact that the classical Voronovskaja's theorem for real variable was generalized by Bernstein [42] as follows.

Theorem 1.3.1. (see e.g. Lorentz [125], p. 22-23) *If f is defined and bounded on $[0, 1]$ and the derivative $f^{(2p)}(x)$ exists at x, then we can write*

$$B_n(f)(x) = f(x) + \sum_{j=1}^{2p} \frac{f^{(j)}(x)}{j!} n^{-j} T_{n,j}(x) + \frac{\varepsilon_n}{n^p},$$

where $B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^k f(k/n)$ denotes the Bernstein polynomials, $T_{n,j}(x) = \sum_{i=0}^{n} (i - nx)^j \binom{n}{i} x^i (1 - x)^{n-i}$ and $\varepsilon_n \to 0$ as $n \to \infty$.

Remark. The classical Voronovskaja's theorem is recaptured for $k = 1$.

The goal of this section is to obtain a similar result to Theorem 1.3.1 for the complex Bernstein polynomials attached to analytic functions in compact disks with the centers in origin and radii ≥ 1. For the particular case $p = 1$ one recapture Theorem 1.1.3. Moreover, the analyticity of f will imply exact orders of approximation in the generalized Voronovskaja's theorems.

The first main result of this section is the following.

Theorem 1.3.2. (Gal [89]) *Let* $R>1$ *and let* $f:\mathbb{D}_R\to\mathbb{C}$ *be an analytic function, that is* $f(z)=\sum_{k=0}^{\infty}c_kz^k$. *Then for any* $1\le r<R$ *and any natural number* p *there exists a constant* $C_p>0$ *such that the following estimate*

$$\left|B_n(f)(z)-f(z)-\sum_{j=1}^{2p}\frac{f^{(j)}(z)}{j!}n^{-j}T_{n,j}(z)\right|\le\frac{C_{p,r}(f)}{n^{p+1}}$$

holds for all $z\in\mathbb{C}$ *with* $|z|\le r$ *and for all* $n\in\mathbb{N}$, *where*

$$C_{p,r}(f)=C_p\cdot\sum_{k=2p+1}^{\infty}|c_k|\,\frac{k!\cdot(k-2p)}{(k-2p-1)!}r^k<\infty.$$

Remark. Theorem 1.3.2 is the complex analogue of the Bernstein's result, with the quantitative estimate $\varepsilon_n\le 1/n$.

Proof of Theorem 1.3.2. Let e_k be the function defined by $e_k(z)=z^k$. Let us put $\pi_{k,n}=B_n(e_k)$. By Lemma 2.2 in Pop [155] one has

$$\pi_{k,n}(z)=\sum_{j=0}^{k}\frac{1}{j!n^j}T_{n,j}(z)\cdot\left(z^k\right)^{(j)},\tag{1.1}$$

where $\left(z^k\right)^{(j)}$ is the j-th derivative of the function $e_k(z)=z^k$. Note that

$$\left(z^k\right)^{(j)}=k(k-1)\ldots(k-j+1)z^{k-j}.\tag{1.2}$$

Let us introduce the polynomial $E_{k,n,p}$ by defining

$$E_{k,n,p}(z)=\pi_{k,n}(z)-e_k(z)-\sum_{j=1}^{2p}\frac{1}{j!n^j}T_{n,j}(z)\cdot\left(z^k\right)^{(j)}.\tag{1.3}$$

Since $T_{n,0}(z)=1$ we see by (1.1) and (1.2) that for $k\ge 2p+1$ we have

$$E_{k,n,p}(z)=\sum_{j=2p+1}^{k}\frac{1}{j!n^j}T_{n,j}(z)\cdot\left(z^k\right)^{(j)}=\sum_{j=2p+1}^{k}\frac{1}{n^j}\binom{k}{j}z^{k-j}T_{n,j}(z).\tag{1.4}$$

First we need the following auxiliary result.

Lemma 1.3.3. (Gal [89]) *For given* $p\in\mathbb{N}$ *and* $r\ge 1$ *there exists a constant* $C_p>0$ *such that for all* $k\ge 2p+1$, $|z|\le r$ *and* $n\in\mathbb{N}$ *the following estimate*

$$|E_{k,n,p}(z)|\le C_pr^k\frac{k!}{n^{p+1}(k-2p)!}(k-2p)^2$$

holds.

Proof of Lemma 1.3.3. We shall use mathematical induction over p. For $p=1$ this inequality has been proved in the proof of Theorem 1.1.3, (ii). Now suppose that the inequality is valid for p, and we want to prove it for $p+1$. At first we observe from (1.4) that $E_{k,n,p+1}(z)$ is equal to

$$E_{k,n,p}(z)\binom{k}{2p+1}\frac{z^{k-(2p+1)}}{n^{2p+1}}T_{n,2p+1}(z)\binom{k}{2p+2}\frac{z^{k-(2p+2)}}{n^{2p+2}}T_{n,2p+2}(z). \tag{1.5}$$

Let us define

$$R_{k,n,p}(z)=E_{k,n,p+1}(z)-zE_{k-1,n,p+1}(z)-\frac{z(1-z)}{n}E'_{k-1,n,p+1}(z). \tag{1.6}$$

Using (1.4), a simple computation shows that $R_{k,n,p}(z)$ is equal to

$$\sum_{j=2p+3}^{k}\frac{1}{n^j}\binom{k}{j}z^{k-j}T_{n,j}(z)-z\sum_{j=2p+3}^{k-1}\frac{1}{n^j}\binom{k-1}{j}z^{k-1-j}T_{n,j}(z) \tag{1.7}$$

$$-\frac{z(1-z)}{n}\sum_{j=2p+3}^{k-1}\frac{1}{n^j}\binom{k-1}{j}\frac{d}{dz}\left[z^{k-1-j}T_{n,j}(z)\right]. \tag{1.8}$$

Now let us rewrite (1.1) in the following trivial way

$$\sum_{j=2p+3}^{k}\frac{1}{n^j}\binom{k}{j}z^{k-j}T_{n,j}(z)=\pi_{k,n}(z)-\sum_{j=0}^{2p+2}\frac{1}{n^j}\binom{k}{j}z^{k-j}T_{n,j}(z) \tag{1.9}$$

and replace the summands in (1.7) and (1.8) by the corresponding expressions induced by (1.9). Then

$$\begin{aligned}R_{k,n,p}(z) &= \pi_{k,n}(z)-\sum_{j=0}^{2p+2}\frac{1}{n^j}\binom{k}{j}z^{k-j}T_{n,j}(z)-z\pi_{k-1,n}(z)\\ &\quad+\sum_{j=0}^{2p+2}\frac{1}{n^j}\binom{k-1}{j}z^{k-j}T_{n,j}(z)-\frac{z(1-z)}{n}\pi'_{k-1,n}(z)\\ &\quad+\frac{z(1-z)}{n}\sum_{j=0}^{2p+2}\frac{1}{n^j}\binom{k-1}{j}\frac{d}{dz}\left[z^{k-1-j}T_{n,j}(z)\right]'\\ &:= S_1-zS_2-\frac{z(1-z)}{n}S_3.\end{aligned}$$

Note that D. Andrica [24] has proved (see also the proof of Theorem 1.1.2, (i)) that

$$\pi_{k,n}(z)=z\pi_{k-1,n}(z)+\frac{z(1-z)}{n}\pi'_{k-1,n}(z),$$

which simplifies the above expression for $R_{k,n,p}(z)$ in an obvious way.

We want to deduce from the above formula that

$$|R_{k,n,p}(z)|\le C_p^*\frac{1}{n^{p+2}}r^k k(k-1)\ldots(k-2p-2), \tag{1.10}$$

for all $|z|\le r$, $n\in\mathbb{N}$ and $k\ge 2p+3$.

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For this purpose, we observe that in the above expression of $R_{k,n,p}(z)$ (with respect to S_1, S_2, S_3), the expression S_1 is equal to the left-hand side in (1.9), S_2 is equal to the left-hand side in (1.9) written for $k-1$ and S_3 is equal to the derivative with respect to z of the left-hand side in (1.9) written for $k-1$.

But since by Lorentz [125], p. 14, Theorem 1.5.1, $T_{n,j}(z)$ is a polynomial of degree $[j/2]$ with respect to n, it is clear from its form that the left-hand side in (1.9) contains only terms having at denominator $n^{j-[j/2]}$, $j \geq 2p+3$, that is only terms having at the denominator n^j with $j \geq p+2$. This immediately implies that $R_{k,n,p}(z)$ contains only terms having at denominator n^j with $j \geq p+2$. Now, since by the same Lorentz [125], p. 14, Theorem 1.5.1, $T_{n,j}(z)$ is a polynomial of degree j with respect to z, (by using again (1.9)) this immediately implies that for all $|z| \leq r$ we have an estimate of the form $|R_{n,k,p}(z)| \leq C_{p,k} r^k \frac{1}{n^{p+2}}$.

Therefore it remains to find out the form of the constant $C_{p,k}$. By the recurrence formula for $T_{n,j}(z)$ in Lorentz [125], p. 14, relation (3), it follows that $T_{n,j}(z)$ is a polynomial in n of degree $\leq [j/2]$ with at most $[j/2]$ terms containing the powers of n (where $j \leq 2p+2$), satisfying the estimate $|T_{n,j}(z)| \leq r^j A_j n^{[j/2]}$, for all $|z| \leq r$. Combining with the estimates $\binom{k}{j} \leq k(k-1)...(k-(2p+2)+1) \leq k(k-1)...(k-2p-2)$, $\binom{k-1}{j} \leq (k-1)(k-2)...(k-(2p+2)) \leq k(k-1)...(k-2p-2)$, $j = 1,,, 2p+2$, it easily follows that the modulus of all the nominators of the terms having at the denominators n^j with $j \geq p+2$, can be bounded by $C_p r^k k(k-1)...(k-2p-2)$, with a suitable chosen constant C_p depending only on p.

Now we shall estimate $E_{k,n,p+1}(z)$ by using (1.10). Indeed, by (1.6) we have

$$E_{k,n,p+1}(z) = zE_{k-1,n,p+1}(z) + \frac{z(1-z)}{n} E'_{k-1,n,p+1}(z) + R_{k,n,p}(z).$$

Let us denote $\|f\|_r = \sup_{|z|\leq r} |f(z)|$ and let us recall Bernstein's inequality

$$\left\|P'_j\right\|_r \leq \frac{j}{r} \|P_j\|_r,$$

valid for any polynomial P_j of degree $\leq j$. Since $|z(1-z)| \leq r(1+r) \leq 2r^2$ for all $|z| \leq r$ (recall that $1 \leq r$) and $E_{k-1,n,p+1}(z)$ is a polynomial of degree $\leq k$ we conclude that for $|z| \leq r$

$$|E_{k,n,p+1}(z)| \leq r|E_{k-1,n,p+1}(z)| + \frac{2r}{n} k |E_{k-1,n,p+1}(z)| + |R_{k,n,p}(z)|. \tag{1.11}$$

By equation (1.5) (applied to $k-1$ instead of k) one obtains

$$|E_{k-1,n,p+1}(z)| \leq |E_{k-1,n,p}(z)| + \binom{k-1}{2p+1} \frac{1}{n^{2p+1}} \left|z^{k-1-(2p+1)} T_{n,2p+1}(z)\right|$$

$$+ \binom{k-1}{2p+2} \frac{1}{n^{2p+2}} \left|z^{k-1-(2p+2)} T_{n,2p+2}(z)\right|.$$

We use the last inequality in order to estimate the middle term in (1.11) in the following way:

$$\frac{2r}{n} k |E_{k-1,n,p+1}(z)| \leq \frac{2rk}{n} \left[C_p r^{k-1} \frac{(k-1)...(k-2p)(k-2p-1)^2}{n^{p+1}} \right.$$

$$+A_{2p+1}r^{k-1}\frac{(k-1)...(k-(2p+1))}{n^{p+1}}+A_{2p}r^{k-1}\frac{(k-1)...(k-(2p+2))}{n^{p+2}}\Bigg].$$

From this we conclude that

$$|E_{k,n,p+1}(z)| \leq r\,|E_{k-1,n,p+1}(z)| + C_p r^k \frac{1}{n^{p+2}} k(k-1)\dots(k-2p-2).$$

Since by Lemma 2.2 in Pop [155] we have $E_{2p+2,n,p+1}(z) = 0$, from the above inequality by an inductive argument applied for $k = 2p+3, \dots$ we finally obtain

$$|E_{k,n,p+1}(z)| \leq C_p r^k \frac{1}{n^{p+2}} k(k-1)\dots(k-2p-1)(k-2p-2)^2,$$

which proves the lemma. □

Now we are in position to prove Theorem 1.3.2. Indeed, taking into account that by the estimate in Lemma 1.3.3 we have $E_{0,n,p}(z) = E_{1,n,p}(z) = \dots = E_{2p,n,p}(z) = 0$, it follows

$$\begin{aligned}&\left|B_n(f)(z) - f(z) - \sum_{j=1}^{2p} \frac{f^{(j)}(z)}{j!} n^{-j} T_{n,j}(z)\right| \\ &\leq \sum_{k=2p+1}^{\infty} |c_k| \cdot |E_{k,n,p}(z)| \\ &\leq \frac{C_p \cdot \sum_{k=2p+1}^{\infty} |c_k| k(k-1)...(k-2p+1)(k-2p)^2 r^k}{n^{p+1}},\end{aligned}$$

which proves the theorem. □

Remark. Analysing the proof of Lemma 1.3.3 and the reasonings for the proof of Theorem 1.3.2, in a similar way we can prove that in the case when $r = 1$ the pointwise estimate

$$\left|B_n(f)(z) - f(z) - \sum_{j=1}^{2p} \frac{f^{(j)}(z)}{j!} n^{-j} T_{n,j}(z)\right| \leq \frac{|z| \cdot |1-z| C_{p,1}(f)}{n^{p+1}}, \forall |z| \leq 1,$$

holds, where $C_{p,1}(f) = C_p \cdot \sum_{k=2p+1}^{\infty} |c_k| k(k-1)...(k-2p+1)(k-2p)^2 < \infty$.

Unlike the real case, for complex analytic functions the order of approximation in Theorem 1.3.2 is exactly $\frac{1}{n^{p+1}}$. More exactly, the second main result of this paper is the following.

Corollary 1.3.4. (Gal [89]) *Let* $R > 1$ *and let* $f : \mathbb{D}_R \to \mathbb{C}$ *be an analytic function, say* $f(z) = \sum_{k=0}^{\infty} c_k z^k$. *If* f *is not a polynomial of degree* $\leq 2p$ *then for any* $1 \leq r < R$ *and any natural number* p *we have*

$$\left\|B_n(f) - f - \sum_{j=1}^{2p} \frac{f^{(j)}}{j!} n^{-j} T_{n,j}\right\|_r \sim \frac{1}{n^{p+1}},\ n \in \mathbb{N},$$

where the constants in the equivalence depend only on f, r *and* p *and are independent of* n. *Here* $\|f\|_r = \sup_{|z| \leq r} |f(z)|$.

Proof. Taking into account Theorem 1.3.2, it remains to obtain the lower estimate for the quantity in the statement of Corollary 1.3.4. Thus, suppose that f is not a polynomial of degree $\leq 2p$. Keeping the notations in the previous section, since $f^{(s)}(z) = \sum_{k=s}^{\infty} c_k k(k-1)...(k-s+1)z^{k-s}$, by using (1.4) and simple calculations we easily obtain the identity

$$\begin{aligned}
&B_n(f)(z) - f(z) - \sum_{j=1}^{2p} \frac{f^{(j)}(z)}{j!} n^{-j} T_{n,j}(z) \\
&\quad = \sum_{k=2p+1}^{\infty} c_k E_{k,n,p}(z) \\
&\quad = \frac{1}{n^{p+1}} \left\{ \sum_{k=2p+1}^{\infty} c_k \left[\sum_{j=2p+1}^{k} \binom{k}{j} z^{k-j} n^{p+1-j} T_{n,j}(z) \right] \right\} \\
&\quad = \frac{1}{n^{p+1}} \left\{ \frac{T_{n,2p+1}(z)}{n^p (2p+1)!} f^{(2p+1)}(z) + \frac{T_{n,2p+2}(z)}{n^{p+1}(2p+2)!} f^{(2p+2)}(z) \right. \\
&\qquad \left. + \frac{1}{n} \left[n^{p+2} \sum_{k=2p+3}^{\infty} c_k E_{k,n,p+1}(z) \right] \right\}.
\end{aligned}$$

By the recurrence formula for $T_{n,j}(z)$ in Lorentz [125], p. 14, relation (3), it follows that $T_{n,j}(z)$ is a polynomial in n of degree $\leq [j/2]$ with at most $[j/2]$ terms containing the powers of n (where $j \leq 2p+2$). Also, by Lorentz [125], p. 14, Theorem 1.5.1, the coefficient of n^{p+1} in $T_{n,2p+2}(z)$ is $\frac{(2p+2)!}{2^{p+1}(p+1)!} \cdot [x(1-x)]^{p+1}$, while from the recurrence formula in Lorentz [125], p. 14, relation (3), it easily follows that the coefficient of n^p in $T_{n,2p+1}(z)$ is of the form $a_p(1-2z)[z(1-z)]^p$ with the constant $a_p > 0$ (a_p depends only on p). Therefore, it is easy to see that the sum $\frac{T_{n,2p+1}(z)}{n^p(2p+1)!} f^{(2p+1)}(z) + \frac{1}{n^{p+1}} \cdot \frac{T_{n,2p+2}(z)}{(2p+2)!} f^{(2p+2)}(z)$ can be written in the form

$$\begin{aligned}
&\frac{T_{n,2p+1}(z)}{n^p(2p+1)!} f^{(2p+1)}(z) + \frac{1}{n^{p+1}} \cdot \frac{T_{n,2p+2}(z)}{(2p+2)!} f^{(2p+2)}(z) \\
&\quad = \frac{a_p}{(2p+1)!}(1-2z)[z(1-z)]^p f^{(2p+1)}(z) \\
&\qquad + \frac{[z(1-z)]^{p+1}}{2^{p+1}(p+1)!} f^{(2p+2)}(z) \\
&\qquad + \frac{1}{n} F(z) f^{(2p+1)}(z) + \frac{1}{n} G(z) f^{(2p+2)}(z),
\end{aligned}$$

where the polynomials $F(z) := P_1(z) + \frac{1}{n}P_2(z) + ... + \frac{1}{n^{p-1}}P_p(z)$ and $G(z) := Q_1(z) + \frac{1}{n}Q_2(z) + ... + \frac{1}{n^p}Q_{p+1}(z)$ are bounded in any closed disk $|z| \leq r$ by constants depending on r and p but independent of n.

Replacing this form in the above identity and taking into account the inequalities

$$\|h+g\|_r \geq |\, \|h\|_r - \|g\|_r \,| \geq \|h\|_r - \|g\|_r,$$

we obtain

$$\left\|B_n(f) - f - \sum_{j=1}^{2p} \frac{f^{(j)}}{j!} n^{-j} T_{n,j}\right\|_r$$

$$\geq \frac{1}{n^{p+1}} \left\{ \left\| \frac{a_p}{(2p+1)!}(1-2e_1)[e_1(1-e_1)]^p f^{(2p+1)} + \frac{[e_1(1-e_1)]^{p+1}}{2^{p+1}(p+1)!} f^{(2p+2)} \right\|_r \right.$$

$$\left. - \frac{1}{n}\left[\left\| n^{p+2} \sum_{k=2p+3}^{\infty} c_k E_{k,n,p+1} + F f^{(2p+1)} + G f^{(2p+2)} \right\|_r \right] \right\}$$

$$:= \frac{1}{n^{p+1}} \left\{ \|U\|_r - \frac{1}{n}\left[\|V\|_r\right] \right\} \geq \frac{1}{n^{p+1}} \cdot \frac{1}{2} \|U\|_r,$$

for all $n \geq n_0$ (n_0 depends on f, p and r), under the conditions that $\|U\|_r > 0$ and if $\|V\|_r$ is bounded by a constant depending only on f, p and r. But this is exactly what happens, because from Theorem 1.3.2 (written for $p+1$) and from the above considerations on F and G it is immediate that $\|V\|_r$ is bounded by a constant depending only on f, p and r while by the fact that f is not a polynomial of degree $\leq 2p$ it follows $\|U\|_r > 0$. Indeed, for the last fact let us suppose the contrary. It follows that f must satisfy the differential equation (here recall that $a_p > 0$)

$$\frac{a_p}{(2p+1)!}(1-2z)[z(1-z)]^p f^{(2p+1)}(z) + \frac{[z(1-z)]^{p+1}}{2^{p+1}(p+1)!} f^{(2p+2)}(z) = 0, |z| \leq r.$$

Making the substitution $f^{(2p+1)}(z) := y(z)$ it follows that $y(z)$ necessarily is analytic in $\mathbb{D}_R$ (since f is supposed analytic there) and is solution of the first order differential equation

$$\frac{a_p}{(2p+1)!}(1-2z)[z(1-z)]^p y(z) + \frac{[z(1-z)]^{p+1}}{2^{p+1}(p+1)!} y'(z) = 0, |z| \leq r.$$

After simplification with $[z(1-z)]^p$, we get that $y(z)$ is an analytic function in $\mathbb{D}_R$ satisfying the differential equation

$$\frac{a_p}{(2p+1)!}(1-2z)y(z) + \frac{z(1-z)}{2^{p+1}(p+1)!} y'(z) = 0, |z| \leq r, z \neq 0, z \neq 1.$$

Writing $y(z)$ in the form $y(z) = \sum_{k=0}^{\infty} b_k z^k$, by comparison of coefficients, we easily obtain that $b_k = 0$, for all $k = 0, 1, ...,$ which implies that $y(z)$ is identical zero in $\overline{\mathbb{D}}_r \setminus \{0, 1\}$. Since y is analytic, it is continuous and therefore $y(0) = y(1) = 0$, which implies that $y(z) = 0$, for all $|z| \leq r$. But from the identity's theorem of analytic functions, it necessarily follows that $y(z) = 0$ for all $|z| < R$, obviously in contradiction with the hypothesis that f is not a polynomial of degree $\leq 2p$ in $\mathbb{D}_R$.

For $n \in \{1, ..., n_0 - 1\}$ we obviously have

$$\left\|B_n(f) - f - \sum_{j=1}^{2p} \frac{f^{(j)}}{j!} n^{-j} T_{n,j}\right\|_r \geq \frac{M_{r,n}(f)}{n^{p+1}},$$

with $M_{r,n}(f) = n^{p+1} \cdot \left\| B_n(f) - f - \sum_{j=1}^{2p} \frac{f^{(j)}}{j!} n^{-j} T_{n,j} \right\|_r > 0$, which finally implies

$$\left\| B_n(f) - f - \sum_{j=1}^{2p} \frac{f^{(j)}}{j!} n^{-j} T_{n,j} \right\|_r \geq \frac{C_{p,r}(f)}{n^{p+1}}, \text{ for all } n \in \mathbb{N},$$

where $C_{p,r}(f) = \min\{M_{r,1}(f), ..., M_{r,n_0-1}(f), \frac{1}{2}\|U\|_r\}$. This completes the proof.□

1.4 Butzer's Linear Combination of Bernstein Polynomials

In the paper of Butzer [51], were inductively introduced the operators $L_n^{[q]}(x)$ of real variable $x \in [0,1]$ by setting $L_n^{[0]}(f)(x) := B_n(f)(x)$ and

$$(2^q - 1) L_n^{[2q]}(f)(x) = 2^q L_{2n}^{[2q-2]}(f)(x) - L_n^{[2q-2]}(f)(x),$$

for $q \in \mathbb{N}$. For example, for $q = 1$ one obtains

$$L_n^{[2]}(f)(x) := 2L_{2n}^{[0]}(f)(x) - L_n^{[0]}(f)(x) = 2B_{2n}(f)(x) - B_n(f)(x).$$

In the same paper of Butzer [51], by using the generalized Voronovskaja's theorem of Bernstein [42] (that is Theorem 1.3.1), he proved that

$$|L_n^{[2q-2]}(f)(x) - f(x)| = O(n^{-q}).$$

The first main result of this section is the extension of Butzer's result to the case of complex variable and can be stated as follows.

Theorem 1.4.1. (Gal [89]) *For any analytic function $f : \mathbb{D}_R \to \mathbb{C}$ with $R > 1$, for each $1 \leq r < R$ and given natural number q there exists a constant $d_{q,r}(f) > 0$ such that the following estimate*

$$\left| L_n^{[2q-2]}(f)(z) - f(z) \right| \leq \frac{d_{q,r}(f)}{n^q},$$

is valid for all $|z| \leq r$ and $n \in \mathbb{N}$.

Proof. The proof of Theorem 1.4.1 is simple. Indeed, let us consider Butzer's linear combination of complex Bernstein polynomials defined by the recurrence

$$L_n^{[0]}(f)(z) = B_n(f)(z), (2^q - 1)L_n^{[2q]}(f)(z) = 2^q L_{2n}^{[2q-2]}(f)(z) - L_n^{[2q-2]}(f)(z),$$

where $z \in \mathbb{C}, q = 1, 2, ...$

Analysing the proofs of Lemma 1, Lemma 2 and Theorem 1 in Butzer [51] and taking into account Theorem 1.3.2, it is easy to see that their reasonings can analogously be used for the above linear combinations of complex Bernstein polynomials, so that finally we get the statement of Theorem 1.4.1. □

Remarks. 1) By the Remark after the proof of Lemma 1.3.3, it easily follows that in a similar way we get the pointwise estimate

$$|L_n^{[2q-2]}(f)(z) - f(z)| \leq d_{q,1}(f) \cdot \frac{|z| \cdot |1 - z|}{n^q}, \text{ for all } |z| \leq 1.$$

2) For $q = 2$, the estimate in Theorem 1.4.1 can easily be obtained with an explicit constant $d_{q,r}(f)$, by using the following estimate in Theorem 1.1.3, (ii) :

$$|B_n(f)(z) - f(z) - \frac{z(1-z)}{2n} f''(z)| \le \frac{5M_r(f)(1+r)^2}{2n^2}, r \ge 1, |z| \le r, n \in \mathbb{N},$$

where $M_r(f) = \sum_{k=3}^{\infty} |c_k| k(k-1)(k-2) r^{k-2}$.

Indeed, taking into account that $L_n^{[2]}(f)(z) = B_{2n}(f)(z) - B_n(f)(z)$ and that we can write the identity

$$L_n^{[2]}(f)(z) - f(z) = 2\left[B_{2n}(f)(z) - f(z) - \frac{z(1-z)}{4n} f''(z)\right] + \left[f(z) - B_n(f)(z) + \frac{z(1-z)}{2n} f''(z)\right],$$

we immediately obtain

$$\begin{aligned} &|L_n^{[2]}(f)(z) - f(z)| \\ &\le 2\left|B_{2n}(f)(z) - f(z) - \frac{z(1-z)}{4n} f''(z)\right| + \left|B_n(f)(z) - f(z) - \frac{z(1-z)}{2n} f''(z)\right| \\ &\le 2\frac{5M_r(f)(1+r)^2}{8n^2} + \frac{5M_r(f)(1+r)^2}{2n^2} = \frac{15(1+r)^2 M_r(f)}{4n^2}. \end{aligned}$$

3) For $q = 3$, in Theorem 1.4.1 similar reasonings can be applied. Indeed, first it easily follows that $L_n^{[4]}(f)(z) = \frac{8}{3} B_{4n}(f)(z) - 2B_{2n}(f)(z) + \frac{1}{3} B_n(f)(z)$ and that we can write the identity

$$\begin{aligned} &L_n^{[4]}(f)(z) - f(z) \\ &= \frac{8}{3}\left[B_{4n}(f)(z) - f(z) - \sum_{j=1}^{4} \frac{f^{(j)}(z)}{j!} (4n)^{-j} T_{4n,j}(z)\right] \\ &\quad -2\left[B_{2n}(f)(z) - f(z) - \sum_{j=1}^{4} \frac{f^{(j)}(z)}{j!} (2n)^{-j} T_{2n,j}(z)\right] \\ &\quad +\frac{1}{3}\left[B_n(f)(z) - f(z) - \sum_{j=1}^{4} \frac{f^{(j)}(z)}{j!} (n)^{-j} T_{n,j}(z)\right] \\ &\quad +\frac{f^{(4)}(z)[1 - 6z(1-z)]z(1-z)}{4! \cdot 8 \cdot n^3}. \end{aligned}$$

This identity follows from the identities $T_{n,1}(z) = 0$ and

$$\frac{f''(z)}{2!}\left[-\frac{8}{3} \cdot \frac{T_{4n,2}(z)}{16n^2} + 2 \cdot \frac{T_{2n,2}(z)}{4n^2} - \frac{1}{3} \cdot \frac{T_{n,2}(z)}{n^2}\right] = 0,$$

$$\frac{f'''(z)}{3!}\left[-\frac{8}{3} \cdot \frac{T_{4n,3}(z)}{64n^3} + 2 \cdot \frac{T_{2n,3}(z)}{8n^3} - \frac{1}{3} \cdot \frac{T_{n,3}(z)}{n^3}\right] = 0,$$

$$\frac{f^{(4)}(z)}{4!}\left[-\frac{8}{3} \cdot \frac{T_{4n,4}(z)}{256n^4} + 2 \cdot \frac{T_{2n,4}(z)}{16n^4} - \frac{1}{3} \cdot \frac{T_{n,4}(z)}{n^4}\right]$$

$$= -\frac{f^{(4)}(z)nz(1-z)[1-6z(1-z)]}{4!\cdot 8\cdot n^4}.$$

As a conclusion, an estimate with explicit constant in Theorem 1.3.2 for $p=2$ (which can be obtained by following the reasonings in the proof of Lemma 1.3.3 for $p=2$, but with explicit constants), will immediately give an explicit constant d for the estimate $|L_n^{[4]}(f)(z)-f(z)|\le d/n^3$.

4) A nice consequence of Corollary 1.3.4 and of Theorem 1.4.1 is that if f is not a polynomial of degree $\le q$ then the order of approximation in Theorem 1.4.1 is exactly $\frac{1}{n^q}$. For simplicity first we illustrate the particular cases $q=1,2,3$. Indeed, the case $q=1$ is contained in Corollary 1.1.5. In the $q=2$ case, taking into account the identity from the above Remark 2 and applying the reasonings in the proof of Corollary 1.3.4 for the case $p=1$, it follows that $L_n^{[2]}(f)(z)-f(z)$ can be written in the form

$$\begin{aligned}
&L_n^{[2]}(f)(z)-f(z)\\
&=2\frac{1}{(2n)^2}\left[\frac{1}{2n}A_{1,n}(f)(z)+\frac{T_{2n,3}(z)}{3!(2n)}f^{(3)}(z)+\frac{T_{2n,4}(z)}{4!(2n)^2}f^{(4)}(z)\right]\\
&\quad-\frac{1}{n^2}\left[\frac{1}{n}A_{2,n}(f)(z)+\frac{T_{n,3}(z)}{3!n}f^{(3)}(z)+\frac{T_{n,4}(z)}{4!n^2}f^{(4)}(z)\right]\\
&=\frac{1}{n^2}\left[\frac{1}{4n}A_{1,n}(f)(z)-\frac{1}{n}A_{2,n}(f)(z)+\frac{1}{2}\cdot\frac{T_{2n,3}(z)}{3!(2n)}f^{(3)}(z)\right.\\
&\quad\left.+\frac{1}{2}\cdot\frac{T_{2n,4}(z)}{4!(2n)^2}f^{(4)}(z)-\frac{T_{n,3}(z)}{3!n}f^{(3)}(z)-\frac{T_{n,4}(z)}{4!n^2}f^{(4)}(z)\right]\\
&=\frac{1}{n^2}\left[\frac{1}{n}A_{3,n}(f)(z)-\frac{(1-2z)z(1-z)f^{(3)}(z)}{2\cdot 3!}+\frac{1}{2}\cdot\frac{3z^2(1-z)^2f^{(4)}(z)}{2\cdot 4!}\right],
\end{aligned}$$

where for all $n\in\mathbb{N}$

$$\|A_{1,n}(f)\|_r, \|A_{2,n}(f)\|_r, \|A_{3,n}(f)\|_r \le C_r(f),$$

($C_r(f)$ is independent of n) and we used the formulas in Lorentz [125], p. 14, $T_{n,3}(z)=n(1-2z)z(1-z)$, $T_{n,4}(z)=nz(1-z)[3nz(1-z)+(1-6z(1-z))]$.

By using for

$$-\frac{(1-2z)z(1-z)f^{(3)}(z)}{12}+\frac{z^2(1-z)^2f^{(4)}(z)}{32}$$

similar reasonings with those in the proof of Corollary 1.3.4, case $p=1$, finally we easily obtain

$$\|L_n^{[2]}(f)(z)-f(z)\|_r\sim\frac{1}{n^2},\ n\in\mathbb{N},$$

where the constants in the equivalence depend only on f and r.

5) For the $q=3$ case we can apply similar reasonings. Indeed, applying to the three expressions between the brackets in the formula for $L_n^{[4]}(f)(z)-f(z)$ from the

above Remark 3, the reasonings in the proof of Corollary 1.3.4, the case $p = 2$, we obtain

$$\begin{aligned}
&L_n^{[4]}(f)(z) - f(z)\\
&= \frac{1}{n^3}\left[\frac{1}{n}A_{1,n}(f)(z) + \frac{8}{3\cdot 4^3}\cdot\frac{T_{4n,5}(z)}{5!\cdot(4n)^2}f^{(5)}(z) + \frac{8}{3\cdot 4^3}\cdot\frac{T_{4n,6}(z)}{6!\cdot(4n)^3}f^{(6)}(z)\right.\\
&\quad - 2\cdot\frac{1}{2^3}\cdot\frac{T_{2n,5}(z)}{5!\cdot(2n)^2}f^{(5)}(z) - 2\cdot\frac{1}{2^3}\cdot\frac{T_{2n,6}(z)}{6!\cdot(2n)^3}f^{(6)}(z)\\
&\quad \left.+ \frac{1}{3}\cdot\frac{T_{n,5}(z)}{5!\cdot n^2}f^{(5)}(z) + \frac{1}{3}\cdot\frac{T_{n,6}(z)}{6!\cdot n^3}f^{(6)}(z)\right] + \frac{f^{(4)}(z)[1-6z(1-z)]z(1-z)}{4!\cdot 8\cdot n^3}\\
&= \frac{1}{n^3}\left[\frac{1}{n}A_n(f)(z) + \frac{10z^2(1-z)^2(1-2z)f^{(5)}(z)}{8\cdot 5!} + \frac{15z^3(1-z)^3f^{(6)}(z)}{8\cdot 6!}\right.\\
&\quad \left.+ \frac{z(1-z)(1-6z(1-z))f^{(4)}(z)}{8\cdot 4!}\right],
\end{aligned}$$

where $\|A_{1,n}(f)\|_r, \|A_n(f)\|_r \le C_r(f)$ for all $n \in \mathbb{N}$ ($C_r(f)$ is independent of n) and we used the formula in Lorentz [125], p. 14, $T_{n,5}(z) = (1-2z)[10n^2z^2(1-z)^2 + n(z(1-z) - 12z^2(1-z)^2)]$ and the Theorem 1.5.1 in Lorentz [125], p. 14, for the formula of n^3 in the polynomial $T_{n,6}(z)$.

Now, by using for

$$\begin{aligned}
G(z) = {}& \frac{10z^2(1-z)^2(1-2z)f^{(5)}(z)}{8\cdot 5!} + \frac{15z^3(1-z)^3f^{(6)}(z)}{8\cdot 6!}\\
&+ \frac{z(1-z)(1-6z(1-z))f^{(4)}(z)}{8\cdot 4!}
\end{aligned}$$

similar reasonings with those for $U(z)$ in the proof of Corollary 1.3.4, finally we easily obtain

$$\|L_n^{[4]}(f)(z) - f(z)\|_r \sim \frac{1}{n^3},$$

where the constants in the equivalence depend only on f and r.

In what follows we present a proof of the equivalence result in approximation by $L_n^{[2q-2]}(f)(z)$ for general $q \ge 3$. Taking into account Theorem 1.4.1, therefore it suffices to prove the following lower estimate for general q.

Theorem 1.4.2. *Let $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ be with $R > 1$, $f : \mathbb{D}_R \to \mathbb{C}$ analytic in $\mathbb{D}_R$, i.e. $f(z) = \sum_{j=0}^{\infty} c_k z^k$, $z \in \mathbb{D}_R$, $1 \le r < R$ and $q \ge 3$.*

If f is not a polynomial of degree $\le q$ then for all $n \in \mathbb{N}$ we have

$$\|L_n^{[2q-2]}(f) - f\|_r \ge \frac{C}{n^q},$$

where the constant C is independent of n and depends on f, r and q.

The proof of Theorem 1.4.2 requires the following two auxiliary results.

Lemma 1.4.3. *Let $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ be with $R > 1$, $f : \mathbb{D}_R \to \mathbb{C}$ analytic in $\mathbb{D}_R$, and $1 \le r < R$. Also, let $L_n^{[q]}(f)(z)$ be the Butzer's polynomials defined by the recurrence in the previous section, $q \ge 2$.*

(i) We have the representation formula

$$L_n^{[2q-2]}(f)(z) = \sum_{i=0}^{q-1} \alpha_{i,q-1} B_{2^i n}(f)(z),$$

where $\alpha_{i,q-1} = (-1)^{q-1-i} S_{i,q-1}/\Pi_{k=1}^{q-1}(2^k - 1)$, $S_{0,q-1} = 1$ *by convention and*

$$S_{i,q-1} = \sum_{j_1, j_2, ..., j_i = 1,\ j_1 < ... < j_i} 2^{j_1} 2^{j_2} ... 2^{j_i},\ \text{for all } i \in \{1, ..., q-1\}.$$

(ii) The sums $S_{i,q}$ *satisfy the recurrence formula*

$$S_{i,q} = S_{i,q-1} + 2^q S_{i-1,q-1},\ \text{for all } i = 1, ..., q-1.$$

(iii) The numbers $\alpha_{i,q-1}$ *satisfy* $\sum_{i=0}^{q-1} \alpha_{i,q-1} = 1$ *and the recurrence*

$$\alpha_{i,q} = \frac{2^q}{2^q - 1}\alpha_{i-1,q-1} - \frac{1}{2^q - 1}\alpha_{i,q-1},\ \text{for all } i = 1, q-1,$$

where $\alpha_{0,q-1} = \frac{(-1)^{q-1}}{\Pi_{k=1}^{q-1}(2^k - 1)}$.

(iv) For all $i = 0, ..., q$ *we have*

$$\alpha_{i,q} = (-1)^{q-i} \frac{2^{i(i+1)/2}[2^{\binom{q}{i}} - 1]}{\prod_{j=1}^{q}(2^j - 1)}.$$

Proof. The proofs of (i) and (ii) can easily be done by mathematical induction after q. Also, (iii) is an immediate consequence of (ii) and (iv) is an immediate consequence of (iii). □

Lemma 1.4.4. *Let* $q \geq 3$.

(i) For all $j = 1, ..., q$, $z \in \mathbb{C}$ *and* $n \in \mathbb{N}$ *we have*

$$\sum_{i=0}^{q-1} \alpha_{i,q-1}(2^i)^{-j} T_{2^i n, j}(z) = 0.$$

(ii) For all $j = q+1, ..., 2q-2$, $z \in \mathbb{C}$ *and* $n \in \mathbb{N}$ *we have*

$$\sum_{i=0}^{q-1} \alpha_{i,q-1}(2^i)^{-j} T_{2^i n, j}(z) = n B_{q-1,j}(z),$$

where $B_{q-1,j}(z)$ *is a polynomial of degree* $\leq j$ *in* z.

Proof. By using the recurrence formulas in Lemma 1.4.3, (iii), the proofs of (i) and (ii) are immediate by mathematical induction with respect to q. The degree of the polynomials $B_{q-1,j}(z)$ with respect to z follows from Lorentz [125], p. 14, Theorem 1.5.1. □

Proof of Theorem 1.4.2. Taking into account the representation formula in Lemma 1.4.3 (i), we can write

$$L_n^{[2q-2]}(f)(z) - f(z) = \sum_{i=0}^{q-1} \alpha_{i,q-1}[B_{2^i n}(f)(z) - f(z)] =$$

$$\sum_{i=0}^{q-1} \alpha_{i,q-1} \left[B_{2^i n}(f)(z) - f(z) - \sum_{j=1}^{2q-2} \frac{f^{(j)}(z)}{j!}(2^i)^{-j} T_{2^i n,j}(z) \right] + R_{n,q-1}(f)(z),$$

where

$$\begin{aligned} R_{n,q-1}(f)(z) &= \sum_{i=0}^{q-1} \alpha_{i,q-1} \left[\sum_{j=1}^{2q-2} \frac{f^{(j)}(z)}{j!}(2^i n)^{-j} T_{2^i n,j}(z) \right] \\ &= \sum_{j=1}^{2q-2} \frac{f^{(j)}(z)}{j!} n^{-j} \left[\sum_{i=0}^{q-1} \alpha_{i,q-1}(2^i)^{-j} T_{2^i n,j}(z) \right]. \end{aligned}$$

Taking into account Lemma 1.4.4, (i) we have

$$R_{n,q-1}(f)(z) = \sum_{j=q+1}^{2q-2} \frac{f^{(j)}(z)}{j!} n^{-j} \left[\sum_{i=0}^{q-1} \alpha_{i,q-1}(2^i)^{-j} T_{2^i n,j}(z) \right],$$

which combined with Lemma 1.4.4, (ii) immediately implies

$$\begin{aligned} R_{n,q-1}(f)(z) &= \frac{1}{n^q} \left[\sum_{j=q+1}^{2q-2} \frac{f^{(j)}(z)}{j!} \cdot \frac{n B_{q-1,j}(z)}{n^{j-q}} \right] \\ &= \frac{1}{n^q} \left[\frac{f^{(q+1)}(z)}{(q+1)!} B_{q-1,q+1}(z) + \frac{1}{n} M_q(f)(z) \right], \end{aligned}$$

where $\|M_q(f)\|_r \le C$, with C independent of n.

Also, since from Theorem 1.5.1, p. 14 in Lorentz [125] we have that each $T_{2^i n,j}(z)$ is a polynomial in $z(1-z)$, this implies that $B_{q-1,q+1}(z)$ is of the form $B_{q-1,q+1}(z) = z(1-z)P_q(z)$, with degree $(P_q(z)) \le q-1$ and $P_q(0) \ne 0$.

On the other hand, reasoning exactly as in the proof of Corollary 1.3.4, case $q = p-1$, we get

$$\begin{aligned} & B_{2^i n}(f)(z) - f(z) - \sum_{j=1}^{2(q-1)} \frac{f^{(j)}(z)}{j!}(2^i n)^{-j} T_{2^i n,j}(z) \\ &= \frac{1}{(2^i n)^q} \left\{ \frac{T_{2^i n,2q-1}(z)}{(2^i n)^{q-1}(2q-1)!} f^{(2q-1)}(z) + \frac{T_{2^i n,2q}(z)}{(2q)!} f^{(2q)}(z) \right. \\ &\left. + \frac{1}{n} \left[(2^i n)^{q+1} \sum_{k=2q+1}^{\infty} c_k E_{k,2^i n,q}(z) \right] \right\}. \end{aligned}$$

Clearly hat the expressions $(2^i n)^{q+1} \sum_{k=2q+1}^{\infty} c_k E_{k,2^i n,q}(z)$ are bounded in $\mathbb{D}_r$ with bounds independent of n (see Corollary 1.3.4).

Also, since by Theorem 1.5.1, p. 14 in Lorentz [125] the coefficient of $(2^i n)^q$ in $T_{2^i n,2q}(z)$ is $\frac{(2q)!}{2^q \cdot q!}[z(1-z)]^q$, while from the recurrence formula in Lorentz [125], p. 14, relation (3), it easily follows that the coefficient of $(2^i n)^{q-1}$ in $T_{2^i n,2q-1}(z)$ is of the form $a_{q-1}(1-2z)[z(1-z)]^{q-1}$ with $a_{q-1} > 0$. Therefore, reasoning exactly as in the proof of Corollary 1.3.4 we can write

$$\begin{aligned}
&\frac{T_{2^i n,2q-1}(z)}{(2^i n)^{q-1}(2q-1)!} f^{(2q-1)}(z) + \frac{T_{2^i n,2q}(z)}{(2q)!} f^{(2q)}(z) \\
&= \frac{a_{q-1}}{(2q-1)!}(1-2z)[z(1-z)]^{q-1} f^{(2q-1)}(z) + \frac{[z(1-z)]^q}{2^q (q)!} f^{(2q)}(z) \\
&\quad + \frac{1}{n} F_i(z) f^{(2q-1)}(z) + \frac{1}{n} G_i(z) f^{(2q)}(z),
\end{aligned}$$

where $F_i(z)$ and $G_i(z)$ are polynomials bounded in $\mathbb{D}_r$ by constants independent of n.

Collecting all the above considerations in conclusion we obtain

$$\begin{aligned}
&L_n^{[2q-2]}(f)(z) - f(z) \\
&= \frac{1}{n^q}\left[\frac{f^{(q+1)}(z)}{(q+1)!} z(1-z)P_q(z) + \frac{2^q-1}{2^{q-1}} a_{q-1}(1-2z)[z(1-z)]^{q-1} f^{(2q-1)}(z)\right. \\
&\quad \left. + \frac{2^q-1}{2^{q-1}} \cdot \frac{[z(1-z)]^q}{2^q q!} f^{(2q)}(z) + \frac{1}{n} K_q(f)(z)\right],
\end{aligned}$$

where $\|K_q(f)\|_r \le C$ with C independent of n.

Denoting

$$\begin{aligned}
H_q(f)(z) &= \frac{f^{(q+1)}(z)}{(q+1)!} z(1-z)P_q(z) \\
&\quad + \frac{2^q-1}{2^{q-1}} a_{q-1}(1-2z)[z(1-z)]^{q-1} f^{(2q-1)}(z) \\
&\quad + \frac{2^q-1}{2^{q-1}} \cdot \frac{[z(1-z)]^q}{2^q q!} f^{(2q)}(z),
\end{aligned}$$

if $\|H_q(f)\|_r > 0$ then the expected lower estimate follows from the inequalities

$$\begin{aligned}
&\|L_n^{[2q-2]}(f) - f\|_r \\
&\ge \frac{1}{n^q}\left| \|H_q(f)\|_r - \frac{1}{n}\|K_q(f)\|_r \right| \\
&\ge \frac{1}{n^q}\left[\|H_q(f)\|_r - \frac{1}{n}\|K_q(f)\|_r \right] \ge \frac{\|H_q(f)\|_r}{2n^q},
\end{aligned}$$

for all $n \ge n_0$. From the reasonings in the same Corollary 1.3.4 we get the lower estimate of the same order for all $n \in \mathbb{N}$.

Now, to finish the proof it will be enough to show that if f is not a polynomial of degree $\le q$ then we have $\|H_q(f)\|_r > 0$. For this purpose, it will be enough to prove that the differential equation $H_q(f)(z) = 0$ for $z \in \mathbb{D}_r$ implies that f is a polynomial of degree $\le q$. Making the substitution $f^{(q+1)}(z) = y(z)$ it will be enough to prove that the differential equation in $z \in \mathbb{D}_r$

$$y(z)z(1-z)P_q(z) + A_q(1-2z)[z(1-z)]^{q-1}y^{(q-2)}(z) + B_q[z(1-z)]^q y^{(q-1)}(z) = 0,$$

has as analytic solution only $y(z) = 0$. Here $A_q, B_q > 0$ and $P_q(0) \neq 0$.

Simplifying with $z(1-z)$ supposed to be $\neq 0$, it follows the differential equation in $z \in \mathbb{D}_{r_1} \setminus \{0\}$

$$y(z)P_q(z) + A_q(1-2z)[z(1-z)]^{q-2}y^{(q-2)}(z) + B_q[z(1-z)]^{q-1}y^{(q-1)}(z) = 0,$$

with $r_1 < 1$. Passing now with $z \to 0$ in this equation we immediately obtain $y(0) = 0$. This means that we can write $y(z) = zh(z)$, with h analytic in $\mathbb{D}_{r_1}$. Calculating $y'(z) = h(z)+zh'(z)$ and $y''(z) = 2h'(z)+zh''(z)$, replacing in the above differential equation, simplifying with z^2 and then passing to limit in the simplified differential equation with $z \to 0$, we immediately obtain $h(0) = 0$. Therefore we can write $h(z) = zu(z)$ and $y(z) = z^2u(z)$, that is $y'(0) = 0$. Repeating the same reasonings for $y(z)$ written in this form, we arrive at the form $y(z) = z^3v(z)$, that is $y''(0) = 0$. Step by step by this kind of reasoning we will obtain $y^{(k)}(0) = 0$ for all $k = 0, 1, 2, ...,$. In conclusion we obtain $y(z) = 0$ for all $z \in \mathbb{D}_{r_1}$, which proves the Theorem 1.4.2. □

Remarks. 1) Let us suppose that $f \in C[0,1]$. By taking $\lambda = 1$ in Guo-Li-Liu [106], Theorem 2, we immediately obtain the following upper estimate valid for all $n \in \mathbb{N}$

$$\|L_n^{[2q-2]}(f) - f\| \le C\left[\frac{\|f\|}{n^q} + \omega_{2q}^{\varphi}(f; 1/\sqrt{n})\right],$$

where $\|\cdot\|$ denotes the uniform norm on $C[0,1]$, $C > 0$ is an absolute constant and $\omega_{2q}^{\varphi}(f; 1/\delta)$ denotes the Ditzian-Totik modulus of smoothness of order $2q$ with respect to the weight $\varphi(x) = \sqrt{x(1-x)}$.

Then, the equivalence results in the above Remarks 4 and 5 and Theorem 1.4.1 and 1.4.2 suggest the following open question : there exists an absolute constant $C' > 0$ such that for all $n \in \mathbb{N}$ we have

$$C'\left[\frac{\|f\|}{n^q} + \omega_{2q}^{\varphi}(f; 1/\sqrt{n})\right] \le \|L_n^{[2q-2]}(f) - f\|.$$

2) Analogously, Corollary 1.3.4 suggests equivalence result with respect to some suitable expressions involving Ditzian-Totik moduli of smoothness for the generalized Voronovskaja's theorem in the case of functions of real variable.

For $G \subset \mathbb{C}$ a compact set and $q \in \mathbb{N}$, define $\mathcal{L}_n^{[q]}(z)$, $z \in G$ by setting $\mathcal{L}_n^{[0]}\left(f; \overline{G}\right)(z) := \mathcal{B}_n(f; \overline{G})(z)$ and

$$(2^q - 1)\,\mathcal{L}_n^{[2q]}\left(f; \overline{G}\right)(z) = 2^q \mathcal{L}_{2n}^{[2q-2]}\left(f; \overline{G}\right)(z) - \mathcal{L}_n^{[2q-2]}\left(f; \overline{G}\right)(z).$$

Taking into account that these Butzer kind polynomials are linear combinations of Bernstein polynomials attached to G, by the above Theorems 1.4.1 and 1.4.2 and by similar reasonings with those in the proof of Theorem 1.1.9 (since T and T^{-1} are linear), we immediately obtain the following result.

Theorem 1.4.5. *Let G be a compact Faber set such that $\tilde{\mathbb{C}} \setminus G$ is simply connected. If f is analytic on G, that is there exists $R > 1$ such that f is analytic in G_R and if is not a polynomial of degree $\leq q$ then for any $1 < r < R$ we have*

$$\|\mathcal{L}_n^{[2q-2]}(f;\overline{G}) - f\|_{\overline{G}_r} \sim \frac{C}{n^q}, \text{ for all } n, q \in \mathbb{N},$$

where the constants in the equivalence depend on f, q, r and G_r but are independent of n. Here $\|f\|_{\overline{G}_r} = \sup_{z \in \overline{G}_r} |f(z)|$.

1.5 q-Bernstein Polynomials

In this section we present the approximation and shape preserving properties of the complex q-Bernstein polynomials. First we present upper estimates in approximation and we prove the Voronovskaja's convergence theorem in compact disks in $\mathbb{C}$, centered at origin, with quantitative estimate of this convergence. These results alow us to obtain the exact degrees in simultaneous approximation by complex q-Bernstein polynomials and their derivatives. Then we study the approximation properties of their iterates and finally we prove that the complex q-Bernstein polynomials preserve in the unit disk (beginning with an index) the starlikeness, convexity and spirallikeness. For $q = 1$, all these results become those proved for complex Bernstein polynomials in Sections 1.1 and 1.2.

Let $q > 0$. For any $n = 1, 2, ...$, define the q-integer $[n]_q := 1 + q + ... + q^{n-1}$, $[0]_q := 0$ and the q-factorial $[n]_q! := [1]_q[2]_q...[n]_q$, $[0]_q! := 1$. For $q = 1$ we obviously get $[n]_q = n$.

For integers $0 \leq k \leq n$, define

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Evidently, for $q = 1$ we get $[n]_1 = n$, $[n]_1! = n!$ and $\binom{n}{k}_1 = \binom{n}{k}$.

Now, for $f : [0, 1] \to \mathbb{C}$, the complex q-Bernstein polynomials are defined simply replacing z by x in the Phillips definition in [149], that is

$$B_{n,q}(f)(z) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \binom{n}{k}_q z^k \prod_{s=0}^{n-1-k} (1 - q^s z), n \in \mathbb{N}, z \in \mathbb{C}.$$

Here conventionally, the empty product is equal to 1. Also, note that for $q = 1$ we recapture the classical complex Bernstein polynomials.

First let us briefly expose the present situation of the main approximation results for the complex q-Bernstein polynomials. Thus, concerning the estimates in the convergence of complex q-Bernstein polynomials attached to analytic functions, by the next theorem and remarks we mention the following known results.

Theorem 1.5.1. (Ostrovska [146], Gal [87]) *Let $q > 0$, $R > 1$, $\mathbb{D}_R = \{z \in \mathbb{C}; |z| \le R\}$ and let us suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$, that is we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$.*

Then for the complex q-Bernstein polynomials we have the estimate

$$|B_{n,q}(f)(z) - f(z)| \le \frac{M_{r,q}(f)}{[n]_q}, \text{ for all } n \in \mathbb{N},$$

valid for all $n \in \mathbb{N}$ and $|z| \le r$, with $1 \le r < R$, where

$$0 < M_{r,q}(f) = 2\sum_{k=2}^{\infty}(k-1)[k-1]_q |c_k| r^k.$$

Moreover, $M_{r,q}(f) \le 2\sum_{k=2}^{\infty}(k-1)k|c_k|r^k := M_r(f) < \infty$, for all $r \in [1, R)$ and $q \in (0, 1]$, while if $q > 1$, then $M_{r,q}(f) < \infty$, for all $q < R$ and $r \in [1, \frac{R}{q})$.

Proof. Since only the case $q \ge 1$ (and $|z| \le 1$) was stated explicitly in Ostrovska [146], let us indicate below the proof in its full generality by using some results already proved in Ostrovska [146]. Indeed, first one easily observe that Lemma 3 in Ostrovska [146] is valid for all $q > 0$. Also, analysing the proof of Theorem 5 in Ostrovska [146] (which use Lemma 3), again it easily follows that its estimate is valid for any $q > 0$, and not only for $q \ge 1$. That is, for any $q > 0$, denoting $e_k(z) = z^k$, $k = 1, 2...$, $z \in \mathbb{C}$, for all $k, n \in \mathbb{N}$ and $|z| \le r$, we get the kind of estimate in Theorem 5 in Ostrovska [146], i.e.

$$|B_{n,q}(e_k)(z) - e_k(z)| \le 2r^k \frac{(k-1)[k-1]_q}{[n]_q},$$

which by $B_{n,q}(f)(z) - f(z) = \sum_{k=0}^{\infty} c_k(B_{n,q}(e_k)(z) - e_k(z))$, immediately implies the estimate.

Now, if $0 < q \le 1$, since $[k-1]_q \le k$, it is immediate that $M_{r,q}(f) = 2\sum_{k=2}^{\infty}(k-1)[k-1]_q|c_k|r^k \le 2\sum_{k=2}^{\infty}(k-1)k|c_k|r^k < \infty$.

If $q > 1$, then by the estimates $[k-1]_q \le [k]_q \le \frac{q^k}{q-1}$ and

$$M_{r,q}(f) = 2\sum_{k=2}^{\infty}(k-1)[k-1]_q|c_k|r^k \le \frac{2}{q-1}\sum_{k=2}^{\infty}(k-1)|c_k|r^k q^k,$$

it follows that $M_{r,q}(f) < \infty$ for $rq < R$, which proves the theorem. □

Remarks. 1) Let $0 < q \le 1$ be fixed. Since we have $\frac{1}{[n]_q} \to 1 - q$ as $n \to \infty$, by passing to limit with $n \to \infty$ in the estimate in Theorem 1.5.1 we don't obtain convergence of $B_n(f)(z)$ to $f(z)$. But this situation can be improved by choosing $0 < q_n < 1$ with $q_n \nearrow 1$ as $n \to \infty$. Indeed, since in this case $\frac{1}{[n]_{q_n}} \to 0$ as $n \to \infty$ (see Videnskii [194], formula (2.7)), from Theorem 1.5.1 we get that $B_{n,q_n}(f)(z) \to f(z)$, uniformly for $|z| \le r$, for any $1 \le r < R$.

2) If $q > 1$, since $\frac{1}{[n]_q} \le \frac{1}{n}$, then by Theorem 1.5.1 it follows that for any $r \ge 1$ with $rq < R$, we have $B_{n,q}(f)(z) \to f(z)$ as $n \to \infty$, uniformly for $|z| \le r$. In

fact, in this case by Theorem 6 in Ostrovska [146] (for upper estimate) and by Corollary 1 in Wang-Wu [198] we know much more : if f is not a linear function then $\|B_{n,q}(f)-f\|_r \sim q^{-n}$, for any $0<r<R/q$. Here $\|f\|_r = \sup\{|f(z)|; |z| \le r\}$.

3) It is worth mentioning other two interesting papers in the topic : approximation by complex q-Bernstein polynomials of the Cauchy kernel $1/(z-a)$ (see Ostrovska [147]) and of the logarithmic function (see Ostrovska [148]).

Now, concerning Voronovskaja-type results and approximation by iterates we can mention the following known results.

If $q \ge 1$ then qualitative Voronovskaja-type and saturation-type results for complex q-Bernstein polynomials were obtained in Wang-Wu [198].

If $0<q<1$ then for the real q-Bernstein polynomials, qualitative Voronovskaja-type and saturation results (see Wang [197]) and quantitative Voronovskaja's result (see Videnskii [195]) were recently obtained.

In this section we fulfil this gap for the complex case and obtain a Voronovskaja-type result with quantitative estimate for complex q-Bernstein polynomials with $0<q<1$. Compared with the quantitative result proved for the real q-Bernstein polynomials in Videnskii [195], our result is essentially better. Also, as an application of our quantitative Voronovskaja's result, the exact order in approximation by complex q_n-Bernstein polynomials with $0<q_n\le 1$ and $\lim_{n\to\infty} q_n = 1$ is obtained.

Taking into account that until present only iterates for real q-Bernstein polynomial were studied (see Ostrovska [146], Xiang-He-Yang [202]), also we fulfil this gap for the complex case, by obtaining approximation results for the iterates of complex q-Bernstein polynomials with $q>0$.

Also, suggested by the fact that for n sufficiently large the classical complex Bernstein polynomial $B_n(f)(z)$ preserves in the unit disk the starlikeness, convexity and spirallikeness, we will extend these kind of results to complex q_n-Bernstein polynomials, $B_{n,q_n}(f)(z), n \in \mathbb{N}$, with $0<q_n<1$ and $q_n \to 1$.

First we present the Voronovskaja-type formula.

Theorem 1.5.2. (Gal [87]) *Let* $0<q<1$, $R>1$, $\mathbb{D}_R = \{z \in \mathbb{C}; |z|<R\}$ *and let us suppose that* $f:\mathbb{D}_R \to \mathbb{C}$ *is analytic in* $\mathbb{D}_R$, *that is we can write* $f(z)=\sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{D}_R$.

(i) The following estimate holds :

$$\left|B_{n,q}(f)(z) - f(z) - \frac{z(1-z)}{2[n]_q} f''(z)\right| \le \frac{|z(1-z)|}{2[n]_q} \cdot \frac{9M(f)}{[n]_q},$$

for all $n \in \mathbb{N}, z \in \overline{\mathbb{D}}_1$, *where* $0 < M(f) = \sum_{k=3}^{\infty} |c_k| k(k-1)(k-2)^2 < \infty$.

(ii) Let $r \in [1,R)$. *Then*

$$\left|B_{n,q}(f)(z) - f(z) - \frac{z(1-z)}{2[n]_q} f''(z)\right| \le \frac{(1+r)}{2[n]_q} \cdot \frac{9K_r(f)}{[n]_q},$$

for all $n \in \mathbb{N}, |z| \le r$, *where* $K_r(f) = \sum_{k=3}^{\infty} |c_k| k(k-1)(k-2)^2 r^k < \infty$.

Proof. (i) Denoting $e_k(z) = z^k$, $k = 0, 1, ...$, and $\pi_{k,n,q}(z) = B_{n,q}(e_k)(z)$, we can write $B_{n,q}(f)(z) = \sum_{k=0}^{\infty} c_k \pi_{k,n,q}(z)$, which immediately implies

$$\left|B_{n,q}(f)(z) - f(z) - \frac{z(1-z)}{2[n]_q} f''(z)\right| \leq \sum_{k=2}^{\infty} |c_k| \cdot \left|\pi_{k,n,q}(z) - e_k(z) - \frac{z^{k-1}(1-z)k(k-1)}{2[n]_q}\right|,$$

for all $z \in \overline{\mathbb{D}}_1$, $n \in \mathbb{N}$.

Denote $D_q(f)(z) = \frac{f(z)-f(qz)}{z-qz}$, $q \neq 1$. In what follows, we prove the recurrence formula

$$\pi_{k+1,n,q}(z) = \frac{z(1-z)}{[n]_q} D_q[\pi_{k,n,q}](z) + z\pi_{k,n,q}(z),$$

for all $n \in \mathbb{N}$, $z \in \mathbb{C}$ and $k = 0, 1, 2, ...$

For $z = 0$ and $z = 1$, this recurrence is obviously satisfied. Therefore let us suppose $z \neq 0$ and $z \neq 1$.

Denoting

$$S_{k,n,q}(z) = \sum_{j=0}^{n} [j]_q^k \binom{n}{j}_q z^j \Pi_{s=0}^{n-1-j}(1 - q^s z),$$

and taking into account the formulas $q^j \frac{1-q^{n-j}}{1-q} = [n]_q - [j]_q$ and

$$D_q[f \cdot g](z) = g(z)D_q(f)(z) + f(qz)D_q(g)(z),\ D_q(z^j)(z) = [j]_q z^{j-1},$$

we obtain

$$\begin{aligned} &D_q[S_{k,n,q}](z) \\ &= \sum_{j=0}^{n} [j]_q^k \binom{n}{j}_q \{D_q(z^j)(z) \prod_{s=0}^{n-1-j}(1 - q^s z) + q^j z^j D_q(\Pi_{s=0}^{n-1-j}(1 - q^s z))\} \\ &= \frac{S_{k+1,n,q}(z)}{z} - \sum_{j=0}^{n} [j]_q^k \binom{n}{j}_q z^j \prod_{s=0}^{n-1-j}(1 - q^s z) \frac{q^j}{1-z} \cdot \frac{1-q^{n-j}}{1-q} \\ &= \frac{S_{k+1,n,q}(z)}{z} - \frac{[n]_q}{1-z} S_{k,n,q}(z) + \frac{S_{k+1,n,q}(z)}{1-z} = \frac{S_{k+1,n,q}(z)}{z(1-z)} - \frac{[n]_q}{1-z} S_{k,n,q}(z). \end{aligned}$$

Dividing now by $[n]_q^{k+1}$, the recurrence formula for $\pi_{k,n,q}(z)$ is immediate. Note that from this recurrence we easily obtain that degree $(\pi_{k,n,q}(z)) = k$.

Now, let us denote $E_{k,n,q}(z) = \pi_{k,n,q}(z) - e_k(z) - \frac{z^{k-1}(1-z)k(k-1)}{2[n]_q}$.

For all $k \geq 2$, $n \in \mathbb{N}$ and $z \in \overline{\mathbb{D}}_1$, the above recurrence leads us to

$$E_{k,n,q}(z) = \frac{z(1-z)}{[n]_q} D_q[E_{k-1,n,q}](z) + zE_{k-1,n,q}(z) + G_{k,n,q}(z),$$

where

$$\begin{aligned} G_{k,n,q}(z) &= \frac{z^{k-2}(1-z)}{2[n]_q} \cdot \frac{(k-1)(k-2)[k-2]_q}{[n]_q} \\ &\quad - \frac{z^{k-2}(1-z)}{2[n]_q} \cdot z \left(\frac{(k-1)[k-1]_q(k-2)}{[n]_q} + 2(k-1) - 2[k-1]_q \right), \end{aligned}$$

Taking into account that by the mean value theorem in complex analysis we have $|D_q(f)(z)| \leq \|f'\|_1$, where $\|\cdot\|_1$ denotes the uniform norm in $C(\overline{\mathbb{D}}_1)$, and by using the relationships $0 < 1-q < \frac{1}{[n]_q}$ and

$$\begin{aligned}(k-1)-[k-1]_q &= (1-q)+...+(1-q^{k-2})\\ &= (1-q)[1+(1+q)+...+(1+q+...q^{k-3})]\\ &\leq \frac{1}{[n]_q}(1+2+...+k-2) = \frac{(k-2)(k-1)}{2[n]_q},\end{aligned}$$

we obtain, for all $|z| \leq 1$, $k \geq 2$, $n \in \mathbb{N}$,

$$\begin{aligned}|E_{k,n,q}(z)| &\leq \frac{|z|\cdot|1-z|}{2[n]_q}[2\|E'_{k-1,n,q}\|_1] + |E_{k-1,n,q}(z)| + \frac{|z|\cdot|1-z|\cdot|z|^{k-3}}{2[n]_q}\\ &\cdot\left[\frac{(k-1)(k-2)[k-2]_q}{[n]_q} + \frac{(k-1)(k-2)[k-1]_q}{[n]_q} + 2(k-1) - 2[k-1]_q\right]\\ &\leq |E_{k-1,n,q}(z)| + \frac{|z|\cdot|1-z|}{2[n]_q}\\ &\cdot\left[2\|E'_{k-1,n,q}\|_1 + \frac{2(k-1)(k-2)[k-1]_q}{[n]_q} + \frac{(k-2)(k-1)}{[n]_q}\right]\\ &\leq |E_{k-1,n,q}(z)| + \frac{|z|\cdot|1-z|}{2[n]_q}\\ &\cdot\left[2(k-1)\|E_{k-1,n,q}\|_1 + \frac{2(k-1)(k-2)[k-1]_q}{[n]_q} + \frac{(k-2)(k-1)}{[n]_q}\right]\\ &\leq |E_{k-1,n,q}(z)| + \frac{|z|\cdot|1-z|}{2[n]_q}\,[2(k-1)\|\pi_{k-1,n,q} - e_{k-1}\|_1\\ &\quad + 2(k-1)^2(k-2)\frac{\|e_{k-2}(z)-e_{k-1}(z)\|_1}{2[n]_q} + \frac{2(k-1)(k-2)[k-1]_q}{[n]_q} +\\ &\quad + \left.\frac{(k-2)(k-1)}{[n]_q}\right] \leq \text{ (by the proof of Theorem 1.5.1)}\\ &\leq |E_{k-1,n,q}(z)| + \frac{|z|\cdot|1-z|}{2[n]_q}\left[2(k-1)\frac{2(k-2)[k-2]_q}{[n]_q}\right.\\ &\quad + \left.\frac{2(k-1)^2(k-2)}{[n]_q} + \frac{2(k-1)(k-2)[k-1]_q}{[n]_q} + \frac{(k-2)(k-1)}{[n]_q}\right]\\ &\leq |E_{k-1,n,q}(z)| + \frac{9|z|\cdot|1-z|k(k-1)(k-2)}{2[n]_q^2}.\end{aligned}$$

Therefore, we have obtained

$$|E_{k,n,q}(z)| \leq |E_{k-1,n,q}(z)| + \frac{9|z|\cdot|1-z|k(k-1)(k-2)}{2[n]_q^2}.$$

The last inequality is trivial for $k = 1, 2$, since $E_{1,n,q}(z) = E_{2,n,q}(z) = 0$, for any $z \in \mathbb{C}$.

Writing now the last inequality for $k = 3, 4, \ldots$, step by step we easily obtain

$$
\begin{aligned}
|E_{k,n,q}(z)| &\leq \frac{|z| \cdot |1-z|}{2[n]_q} \cdot \frac{9}{[n]_q} \cdot \sum_{j=3}^{k} j(j-1)(j-2) \\
&\leq \frac{|z| \cdot |1-z|}{2[n]_q} \cdot \frac{9}{[n]_q} \cdot k(k-1)(k-2)^2.
\end{aligned}
$$

In conclusion,

$$
\left| B_{n,q}(f)(z) - f(z) - \frac{z(1-z)}{2[n]_q} f''(z) \right| \leq \sum_{k=3}^{\infty} |c_k| \cdot |E_{k,n,q}(z)|
$$

$$
\leq \frac{|z| \cdot |1-z|}{2[n]_q} \cdot \frac{9}{[n]_q} \cdot \sum_{k=3}^{\infty} |c_k| k(k-1)(k-2)^2.
$$

Note that since $f^{(4)}(z) = \sum_{k=4}^{\infty} c_k k(k-1)(k-2)(k-3) z^{k-4}$ and the series is absolutely convergent in $\overline{\mathbb{D}}_1$, it easily follows that $\sum_{k=3}^{\infty} |c_k| k(k-1)(k-2)^2 < \infty$, which proves (i).

(ii) First we use the relationship in the proof of Theorem 1.5.1

$$
|\pi_{k,n,q}(z) - e_k(z)| \leq 2r^k \frac{(k-1)[k-1]_q}{[n]_q},
$$

for all $k, n \in \mathbb{N}$, $|z| \leq r$, with $1 \leq r < R$.

Denoting with $\| \cdot \|_r$ the norm in $C(\overline{\mathbb{D}}_r)$, where $\overline{\mathbb{D}}_r = \{z \in \mathbb{C}; |z| \leq r\}$, one observes that by a linear transformation, the Bernstein's inequality in the closed unit disk becomes $|P_k'(z)| \leq \frac{k}{r} \|P_k\|_r$, for all $|z| \leq r$, $r \geq 1$, which combined with the mean value theorem in complex analysis, implies $|D_q(P_k)(z)| \leq \|P_k'\|_r \leq \frac{k}{r} \|P_k\|_r$, for all $|z| \leq r$, where $P_k(z)$ is a complex polynomial of degree $\leq k$.

Now, taking into account the formula proved at the above point (i), given by

$$
\begin{aligned}
E_{k,n,q}(z) = {} & \frac{z(1-z)}{[n]_q} D_q[E_{k-1,n,q}](z) + z E_{k-1,n,q}(z) \\
& + \frac{z^{k-2}(1-z)}{2[n]_q} \left[\frac{(k-1)(k-2)[k-2]_q}{[n]_q} \right. \\
& \left. - z \left(\frac{(k-1)[k-1]_q (k-2)}{[n]_q} + 2(k-1) - [k-1]_q \right) \right],
\end{aligned}
$$

it follows for all $k, n \in \mathbb{N}$, $k \geq 2$ and $|z| \leq r$,

$$
|E_{k,n,q}(z)| \leq \frac{r(1+r)}{[n]_q} |D_q[E_{k-1,n,q}](z)| + r|E_{k-1,n,q}(z)| + \frac{(1+r)r^{k-2}}{2[n]_q} \cdot
$$

$$
\left[\frac{(k-1)(k-2)[k-2]_q}{[n]_q} + r \left(\frac{(k-1)(k-2)[k-1]_q}{[n]_q} + \frac{(k-2)(k-1)}{[n]_q} \right) \right]
$$

$$
\leq r|E_{k-1,n,q}(z)| + \frac{r(1+r)}{[n]_q} |D_q[E_{k-1,n,q}](z)| + \frac{3(1+r)r^{k-1} k(k-1)(k-2)}{2[n]_q^2}.
$$

Now, we will estimate $|D_q[E_{k-1,n,q}](z)|$, for $k \geq 3$. Taking into account that $E_{k-1,n,q}(z)$ is a polynomial of degree $\leq (k-1)$, we obtain

$$\begin{aligned}|D_q[E_{k-1,n,q}](z)| &\leq \frac{k-1}{r}\|E_{k-1,n,q}(z)\|_r \\ &\leq \frac{k-1}{r}\left[\|\pi_{k-1,n,q} - e_{k-1}\|_r + \|\frac{(k-1)(k-2)(e_{k-1}-e_{k-2})}{2[n]_q}\|_r\right] \\ &\leq \frac{k-1}{r}\left[\frac{2(k-2)[k-2]_q r^k}{[n]_q} + \frac{2r^{k-1}(k-1)(k-2)}{2[n]_q}\right] \\ &\leq \frac{3r^{k-1}k(k-1)(k-2)}{[n]_q}.\end{aligned}$$

This implies

$$\frac{r(1+r)}{[n]_q}|D_q[E_{k-1,n,q}](z)| \leq \frac{3(1+r)k(k-1)(k-2)r^k}{[n]_q^2},$$

and

$$\begin{aligned}|E_{k,n,q}(z)| &\leq r|E_{k-1,n,q}(z)| \\ &\quad + \frac{3(1+r)k(k-1)(k-2)r^k}{[n]_q^2} + \frac{3(1+r)k(k-1)(k-2)r^{k-1}}{2[n]_q^2} \\ &\leq r|E_{k-1,n,q}(z)| + \frac{9(1+r)k(k-1)(k-2)r^k}{2[n]_q^2}.\end{aligned}$$

But $E_{0,n}(z) = E_{1,n}(z) = E_{2,n}(z) = 0$, for any $z \in \mathbb{C}$. Writing now the last inequality for $k = 3, 4, \ldots$, step by step we easily obtain

$$\begin{aligned}|E_{k,n,q}(z)| &\leq \frac{9(1+r)r^k}{2[n]_q^2}\left[\sum_{j=3}^{k} j(j-1)(j-2)\right] \\ &\leq \frac{9(1+r)k(k-1)(k-2)^2 r^k}{2[n]_q^2}.\end{aligned}$$

As a conclusion, we obtain

$$\begin{aligned}\left|B_{n,q}(f)(z) - f(z) - \frac{z(1-z)}{2[n]_q}f''(z)\right| &\leq \sum_{k=3}^{\infty}|c_k| \cdot |E_{k,n}(z)| \\ &\leq \frac{9(1+r)}{2[n]_q^2}\sum_{k=3}^{\infty}|c_k|k(k-1)(k-2)^2 r^k.\end{aligned}$$

Note that since $f^{(4)}(z) = \sum_{k=4}^{\infty} c_k k(k-1)(k-2)(k-3)z^{k-4}$ and the series is absolutely convergent in $|z| \leq r$, it easily follows that $\sum_{k=3}^{\infty}|c_k|k(k-1)(k-2)^2 r^k < \infty$, which proves (ii). □

Remarks. 1) In the hypothesis on f in Theorem 1.5.2 and choosing $0 < q_n < 1$ with $q_n \nearrow 1$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty}[n]_{q_n}[B_{n,q_n}(f)(z) - f(z)] = \frac{z(1-z)f''(z)}{2},$$

uniformly in any compact disk included in the open disk of center 0 and radius R.

2) In Videnskii [194], Theorem 5.1, estimate (5.7), for the real q-Bernstein polynomials it is proved that for $f \in C^2[0,1]$, $x \in [0,1]$, and $0 < q_n < 1$ with $\lim_{n\to\infty} q_n = 1$, we have

$$\left|B_{n,q_n}(f)(x) - f(x) - \frac{f''(x)}{2} \cdot \frac{x(1-x)}{[n]_{q_n}}\right| \le \frac{Kx(1-x)}{[n]_{q_n}}\omega_1(f''; [n]_{q_n}^{-1/2}),$$

where ω_1 denotes the modulus of continuity. Obviously that the best order of approximation that can be obtained from this estimate is $O(1/[n]_{q_n}^{-3/2})$ (for $f \in C^m[0,1]$ with $m \ge 3$), while the order given by our Theorem 1.5.2 is $O(1/[n]_{q_n}^{-2})$, which is essentially better taking into account that as $n \to \infty$ we have $[n]_{q_n} \to \infty$.

In what follows we obtain the exact orders in approximation by complex q-Bernstein polynomials and their derivatives on compact disks.

In this sense, the first result is the following.

Theorem 1.5.3. (Gal [87]) *Let $0 < q_n \le 1$ be with $\lim_{n\to\infty} q_n = 1$, $R > 1$, $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ and let us suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$, that is we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. If f is not a polynomial of degree ≤ 1, then for any $r \in [1, R)$ we have*

$$\|B_{n,q_n}(f) - f\|_r \ge \frac{C_r(f)}{[n]_{q_n}}, n \in \mathbb{N},$$

where $\|f\|_r = \max\{|f(z)|; |z| \le r\}$ and the constant $C_r(f) > 0$ depends on f, r and on the sequence $(q_n)_{n\in\mathbb{N}}$ but it is independent of n.

Proof. For all $z \in \mathbb{D}_R$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}B_{n,q_n}(f)(z) - f(z) = \frac{1}{[n]_{q_n}} \Bigg\{ &\frac{z(1-z)}{2} f''(z) \\ &+ \frac{1}{[n]_{q_n}} \left[[n]_{q_n}^2 \left(B_{n,q_n}(f)(z) - f(z) - \frac{z(1-z)}{2[n]_{q_n}} f''(z) \right) \right] \Bigg\}.\end{aligned}$$

We will apply to this identity the following obvious property :

$$\|F + G\|_r \ge | \, \|F\|_r - \|G\|_r \, | \ge \|F\|_r - \|G\|_r.$$

Denoting $e_1(z) = z$ it follows

$$\|B_{n,q_n}(f) - f\|_r \ge$$

$$\frac{1}{[n]_{q_n}} \left\{ \left\| \frac{e_1(1-e_1)}{2} f'' \right\|_r - \frac{1}{[n]_{q_n}} \left[[n]_{q_n}^2 \left\| B_{n,q_n}(f) - f - \frac{e_1(1-e_1)}{2[n]_{q_n}} f'' \right\|_r \right] \right\}.$$

Taking into account that by hypothesis f is not a polynomial of degree ≤ 1 in $\mathbb{D}_R$, we get $\left\| \frac{e_1(1-e_1)}{2} f'' \right\|_r > 0$. Indeed, supposing the contrary it follows that $\frac{z(1-z)}{2} f''(z) = 0$ for all $z \in \overline{\mathbb{D}}_r$, which implies $f''(z) = 0$ for all $z \in \overline{\mathbb{D}}_r \setminus \{0, 1\}$. Since f is supposed to be analytic, from the identity's theorem of analytic (holomorphic) functions this necessarily implies that $f''(z) = 0$, for all $z \in \mathbb{D}_R$, i.e. that f is a polynomial of degree ≤ 1, which is a contradiction.

But by Theorem 1.5.2 we have

$$\begin{aligned}([n]_{q_n})^2 \left\| B_{n,q_n}(f) - f - \frac{e_1(1-e_1)}{2[n]_{q_n}} f'' \right\|_r &\leq ([n]_{q_n})^2 \frac{9K_r(f)(1+r)}{2([n]_{q_n})^2} \\ &= \frac{9K_r(f)(1+r)}{2}.\end{aligned}$$

Since by the Remark after the proof of Theorem 1.5.1 we have $\frac{1}{[n]_{q_n}} \to 0$ as $n \to \infty$, it follows that there exists an index n_0 depending only on f, r and on the sequence $(q_n)_n$, such that for all $n \geq n_0$ we have

$$\begin{aligned}&\left\| \frac{e_1(1-e_1)}{2} f'' \right\|_r - \frac{1}{[n]_{q_n}} \left[([n]_{q_n})^2 \left\| B_{n,q_n}(f) - f - \frac{e_1(1-e_1)}{2[n]_{q_n}} f'' \right\|_r \right] \\ &\qquad \geq \left\| \frac{e_1(1-e_1)}{2} f'' \right\|_r > 0,\end{aligned}$$

which immediately implies

$$\|B_{n,q_n}(f) - f\|_r \geq \frac{1}{[n]_{q_n}} \cdot \left\| \frac{e_1(1-e_1)}{4} f'' \right\|_r, \forall n \geq n_0.$$

For $1 \leq n \leq n_0 - 1$ we obviously have

$$\|B_{n,q_n}(f) - f\|_r \geq \frac{M_{r,n}(f)}{[n]_{q_n}},$$

with $M_{r,n}(f) = [n]_{q_n} \cdot \|B_{n,q_n}(f) - f\|_r > 0$, which finally implies

$$\|B_{n,q_n}(f) - f\|_r \geq \frac{C_r(f)}{[n]_{q_n}}, \text{ for all } n \in \mathbb{N},$$

where $C_r(f) = \min \left\{ M_{r,1}, M_{r,2}(f), ..., M_{r,n_0-1}(f), \left\| \frac{e_1(1-e_1)}{4} f'' \right\|_r \right\}$. □

Combining Theorem 1.5.3 with Theorem 1.5.1 we get the following.

Corollary 1.5.4. (Gal [87]) *Let Let* $0 < q_n \leq 1$ *be with* $\lim_{n\to\infty} q_n = 1$, $R > 1$, $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ *and let us suppose that* $f : \mathbb{D}_R \to \mathbb{C}$ *is analytic in* $\mathbb{D}_R$. *If* f *is not a polynomial of degree* ≤ 1, *then for any* $r \in [1, R)$ *we have*

$$\|B_{n,q_n}(f) - f\|_r \sim \frac{1}{[n]_{q_n}}, n \in \mathbb{N},$$

where the constants in the equivalence depend on f, r *and on the sequence* $(q_n)_n$ *but are independent of* n.

Proof. Since $q_n \leq 1$ for all $n \in \mathbb{N}$, by Theorem 1.5.1 it follows the upper estimate with the constant depending only on f and r (independent of the sequence $(q_n)_n$). Theorem 1.5.3 assures the lower estimate with the constant depending on f, r and on the sequence $(q_n)_n$, but independent of n. □

Remark. Theorem 1.5.3 and Corollary 1.5.4 in the case when $q_n = 1$ for all $n \in \mathbb{N}$ were obtained by Theorem 1.1.4 and Corollary 1.1.5.

In the case of approximation by the derivatives of complex q-Bernstein polynomials we can present the following new result which appears for the first time here.

Theorem 1.5.5. *Let* $0 < q_n \le 1$ *be with* $\lim_{n\to\infty} q_n = 1$, $R > 1$, $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ *and let us suppose that* $f : \mathbb{D}_R \to \mathbb{C}$ *is analytic in* $\mathbb{D}_R$, *i.e.* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{D}_R$. *Also, let* $1 \le r < r_1 < R$ *and* $p \in \mathbb{N}$ *be fixed. If* f *is not a polynomial of degree* $\le \max\{1, p-1\}$, *then we have*

$$\|B_{n,q_n}^{(p)}(f) - f^{(p)}\|_r \sim \frac{1}{[n]_{q_n}},$$

where the constants in the equivalence depend on f, r, r_1, p *and on the sequence* $(q_n)_n$, *but are independent of* n.

Proof. Denoting by Γ the circle of radius $r_1 >$ and center 0 (where $r_1 > r \ge 1$), by the Cauchy's formulas it follows that for all $|z| \le r$ and $n \in \mathbb{N}$ we have

$$B_{n,q_n}^{(p)}(f)(z) - f^{(p)}(z) = \frac{p!}{2\pi i}\int_\Gamma \frac{B_{n,q_n}(f)(v) - f(v)}{(v-z)^{p+1}}dv,$$

which by Theorem 1.5.1 and by the inequality $|v - z| \ge r_1 - r$ valid for all $|z| \le r$ and $v \in \Gamma$, immediately implies

$$\begin{aligned}\|B_{n,q_n}^{(p)}(f) - f^{(p)}\|_r &\le \frac{p!}{2\pi}\cdot\frac{2\pi r_1}{(r_1 - r)^{p+1}}\|B_{n,q_n}(f) - f\|_{r_1}\\ &\le M_{r_1}(f)\frac{p! r_1}{[n]_{q_n}(r_1 - r)^{p+1}}.\end{aligned}$$

It remains to prove the lower estimate for $\|B_{n,q_n}^{(p)}(f) - f^{(p)}\|_r$. For this purpose, as in the proof of Theorem 1.5.3, for all $v \in \Gamma$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}B_{n,q_n}(f)(v) - f(v) &= \frac{1}{[n]_{q_n}}\left\{\frac{v(1-v)}{2}f''(v)\right.\\ &\left.+ \frac{1}{[n]_{q_n}}\left[([n]_{q_n})^2\left(B_{n,q_n}(f)(v) - f(v) - \frac{v(1-v)}{2[n]_{q_n}}f''(v)\right)\right]\right\},\end{aligned}$$

which replaced in the above Cauchy's formula implies

$$\begin{aligned}B_{n,q_n}^{(p)}(f)(z) - f^{(p)}(z) &= \frac{1}{[n]_{q_n}}\left\{\frac{p!}{2\pi i}\int_\Gamma \frac{v(1-v)f''(v)}{2(v-z)^{p+1}}dv\right.\\ &\left.+ \frac{1}{[n]_{q_n}}\cdot\frac{p!}{2\pi i}\int_\Gamma \frac{n^2\left(B_{n,q_n}(f)(v) - f(v) - \frac{v(1-v)}{2[n]_{q_n}}f''(v)\right)}{(v-z)^{p+1}}dv\right\}\\ &= \frac{1}{[n]_{q_n}}\left\{\left[\frac{z(1-z)}{2}f''(z)\right]^{(p)}\right.\\ &\left.+ \frac{1}{[n]_{q_n}}\cdot\frac{p!}{2\pi i}\int_\Gamma \frac{([n]_{q_n})^2\left(B_{n,q_n}(f)(v) - f(v) - \frac{v(1-v)}{2[n]_{q_n}}f''(v)\right)}{(v-z)^{p+1}}dv\right\}.\end{aligned}$$

Passing now to $\|\cdot\|_r$ it follows

$$\left\|B^{(p)}_{n,q_n}(f)-f^{(p)}\right\|_r \geq \frac{1}{[n]_{q_n}}\left\{\left\|\left[\frac{e_1(1-e_1)}{2}f''\right]^{(p)}\right\|_r \right.$$
$$\left. -\frac{1}{[n]_{q_n}}\left\|\frac{p!}{2\pi}\int_\Gamma \frac{([n]_{q_n})^2\left(B_{n,q_n}(f)(v)-f(v)-\frac{v(1-v)}{2[n]_{q_n}}f''(v)\right)}{(v-e_1)^{p+1}}dv\right\|_r\right\},$$

where by using Theorem 1.5.2 and denoting $e_1(z)=z$ we get

$$\left\|\frac{p!}{2\pi}\int_\Gamma \frac{([n]_{q_n})^2\left(B_{n,q_n}(f)(v)-f(v)-\frac{v(1-v)}{2[n]_{q_n}}f''(v)\right)}{(v-e_1)^{p+1}}dv\right\|_r$$
$$\leq \frac{p!}{2\pi}\cdot\frac{2\pi r_1([n]_{q_n})^2}{(r_1-r)^{p+1}}\|B_{n,q_n}(f)-f-\frac{e_1(1-e_1)}{2[n]_{q_n}}f''\|_{r_1}$$
$$\leq \frac{5K_{r_1}(f)(1+r_1)^2}{2}\cdot\frac{p!r_1}{(r_1-r)^{p+1}}.$$

But by hypothesis on f we have $\left\|\left[\frac{e_1(1-e_1)}{2}f''\right]^{(p)}\right\|_r>0$. Indeed, supposing the contrary it follows that $\frac{z(1-z)}{2}f''(z)$ is a polynomial of degree $\leq p-1$. Now, if $p=1$ and $p=2$ then the analyticity of f obviously implies that f necessarily is a polynomial of degree $\leq 1=\max\{1,p-1\}$, which contradicts the hypothesis. If $p>2$ then the analyticity of f obviously implies that f necessarily is a polynomial of degree $\leq p-1=\max\{1,p-1\}$, which again contradicts the hypothesis.

In continuation reasoning exactly as in the proof of Theorem 1.5.3, we immediately get the desired conclusion. □

Remark. Theorem 1.5.5 in the case when $q_n=1$ for all $n\in\mathbb{N}$ was obtained by Theorem 1.1.6.

In what follows we consider the approximation properties for iterates. First we recall some considerations in Section 1.2. For $R>1$ let us define by $\mathbb{A}_R$ the space of all functions defined and analytic in the open disk of center 0 and radius R denoted by $\mathbb{D}_R$. Denoting $r_j=R-\frac{R-1}{j}$, $j\in\mathbb{N}$ and for $f\in A_R$, $\|f\|_j=max\{|f(z)|;|z|\leq r_j\}$, since $r_1=1$ and $r_j\nearrow R$ as $j\to\infty$, it is well-known that $\{\|\cdot\|_j, j\in\mathbb{N}\}$ it is a countable family of increasing semi-norms on $\mathbb{A}_R$ and that $\mathbb{A}_R$ becomes a metrizable complete locally convex space (Fréchet space), with respect to the metric

$$d(f,g)=\sum_{j=1}^{\infty}\frac{1}{2^j}\cdot\frac{\|f-g\|_j}{1+\|f-g\|_j}, f,g\in\mathbb{A}_R.$$

It is well-known that $\lim_{n\to\infty}d(f_n,f)=0$ is equivalent to the fact that the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly on compacts in $\mathbb{D}_R$.

Now, for $f\in\mathbb{A}_R$, that is of the form $f(z)=\sum_{k=0}^{\infty}c_kz^k$, for all $z\in\mathbb{D}_R$, let us define the iterates of complex q-Bernstein polynomial $B_{n,q}(f)(z)$, by $B^{(1)}_{n,q}(f)(z)=$

$B_{n,q}(f)(z)$ and $B^{(m)}_{n,q}(f)(z) = B_{n,q}[B^{(m-1)}_{n,q}(f)](z)$, for any $m \in \mathbb{N}$, $m \geq 2$. Since we have $B_{n,q}(f)(z) = \sum_{k=0}^{\infty} c_k B_{n,q}(e_k)(z)$, by recurrence for all $m \geq 1$, it easily follows $B^{(m)}_{n,q}(f)(z) = \sum_{k=0}^{\infty} c_k B^{(m)}_{n,q}(e_k)(z)$, with $e_k(z) = z^k$.

The main result is the following.

Theorem 1.5.6. (Gal [87]) *Let $f \in \mathbb{A}_R$, that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$.*

(i) Let $q \in (0,1)$. If $\lim_{n\to\infty} m_n = +\infty$, then

$$\lim_{n\to\infty} d[B^{(m_n)}_{n,q}(f), L_1(f)] = 0.$$

(ii) If $q \in (0,1]$ then for any fixed $s \in \mathbb{N}$, the following estimates hold

$$\|B^{(m)}_{n,q}(f) - f\|_s \leq \frac{2m}{[n]_q} \sum_{k=2}^{\infty} |c_k| k(k-1) r_s^k,$$

and

$$d[B^{(m)}_{n,q}(f), f] \leq \frac{2m}{[n]_q} \sum_{k=2}^{\infty} |c_k| k(k-1) r_s^k + \frac{1}{2^s},$$

where $\sum_{k=2}^{\infty} |c_k| k(k-1) r_s^k < \infty$.

(iii) Let $q \in (1,\infty)$. If $\lim_{n\to\infty} \frac{m_n}{[n]_q} = 0$, then

$$\lim_{n\to\infty} d[B^{(m_n)}_{n,q}(P), P] = 0,$$

for any polynomial P.

(iv) Let $q \in (1,\infty)$. If $1 \leq r < R$, then the following estimate holds for all $|z| \leq r$

$$|B^{(m)}_{n,q}(f)(z) - f(z)| \leq \frac{2m}{n} \sum_{k=2}^{\infty} |c_k| \left[\frac{q^k - 1 - k(q-1)}{(q-1)^2} + k(k-1)\right] r^k.$$

If, in addition, $q < R$, then since $\sum_{k=2}^{\infty} |c_k| \left[\frac{q^k-1-k(q-1)}{(q-1)^2} + k(k-1)\right] < \infty$, we obtain $B^{(m_n)}_{n,q}(f)(z) \to f(z)$, uniformly in $\overline{\mathbb{D}_1}$, for $\frac{m_n}{n} \to 0$ as $n \to \infty$.

(v) Let $q \in (1,\infty)$. If $\lim_{n\to\infty} \frac{m_n}{[n]_q} = \infty$, then

$$\lim_{n\to\infty} d[B^{(m_n)}_{n,q}(P), L_1(P)] = 0,$$

for any polynomial P.

Proof. (i) Let $0 < q < 1$ and $n \in \mathbb{N}$ be fixed. Denoting $s = min\{n,k\}$, by Lemma 3 in Ostrovska [146] we can write $B_{n,q}(e_k)(z) = \sum_{j=1}^{s} \alpha_{j,k,n,q} z^j$, where $\alpha_(j,k,n,q) \geq 0$, for all $j,k,n \in \mathbb{N}$ and $\sum_{j=1}^{s} \alpha_{j,k,n,q} = 1$, for $k,n \in \mathbb{N}$.

Let $|z| \leq r$ with $r \geq 1$. It follows

$$|B_{n,q}(e_k)(z)| \leq \sum_{j=1}^{s} \alpha_{j,k,n,q} |e_j(z)| \leq \sum_{j=1}^{s} \alpha_{j,k,n,q} r^j \leq \sum_{j=1}^{s} \alpha_{j,k,n,q} r^s = r^s \leq r^k.$$

Applying now above $B_{n,q}$, we obtain

$$B_{n,q}^{(2)}(e_k)(z)=\sum_{j=1}^{s}\alpha_{j,k,n,q}B_{n,q}(e_j)(z),$$

which from the last inequality implies

$$\begin{aligned}|B_{n,q}^{(2)}(e_k)(z)| &\leq \sum_{j=1}^{s}\alpha_{j,k,n,q}|B_{n,q}(e_j)(z)|\\ &\leq \sum_{j=1}^{s}\alpha_{j,k,n,q}r^j \leq \sum_{j=1}^{s}\alpha_{j,k,n,q}r^s = r^s \leq r^k.\end{aligned}$$

Reasoning by recurrence, we easily get $|B_{n,q}^{(m)}(e_k)(z)| \leq r^k$ for all $k, n, m \in \mathbb{N}$ and $z \in \overline{\mathbb{D}_r}$.

This implies

$$|B_{n,q}^{(m)}(f)(z)| = |\sum_{k=0}^{\infty}c_k B_{n,q}^{(m)}(e_k)(z)| \leq \sum_{k=0}^{\infty}|c_k|r^k < +\infty,$$

for all $m, n \in \mathbb{N}$ and $z \in \overline{\mathbb{D}_r}$, that is for each $r \in [1, R)$, the sequence $B_{n,q}^{(m)}(f)(z)$, $m, n = 1, 2, ...$, is uniformly bounded in $\overline{\mathbb{D}_r}$ with respect to both $m, n \in \mathbb{N}$.

Therefore, the sequence $B_{n,q}^{(m_n)}(f)(z)$, $n = 1, 2, ...$, is uniformly bounded in $\overline{\mathbb{D}_r}$ with respect to $n \in \mathbb{N}$.

Since by Theorem 8 in Ostrovska [146], for each $q \in (0, 1)$ and for $n \to \infty$ we have $B_{n,q}^{(m_n)}(f)(x) \to L_1(f)(x)$, for $x \in [0, 1]$, (even uniformly), the classical Vitali's convergence theorem implies the uniform convergence (as $n \to \infty$) on compacts in $\mathbb{D}_R$ of the sequence $B_{n,q}^{(m_n)}(f)(z)$ to $L_1(f)(z)$. Taking into account that the uniform convergence on compacts is equivalent to the convergence with respect to the metric d, (i) is proved.

(ii) Since $B_{n,q}^{(m)}(f)(z) = \sum_{k=0}^{\infty} c_k B_{n,q}^{(m)}(e_k)(z)$, with $e_k(z) = z^k$, we get

$$|B_{n,q}^{(m)}(f)(z) - f(z)| \leq \sum_{k=2}^{\infty}|c_k| \cdot |B_{n,q}^{(m)}(e_k)(z) - e_k(z)|.$$

To estimate $|B_{n,q}^{(m)}(e_k)(z) - e_k(z)|$, we have two possibilities : 1) $0 \leq k \leq n$; 2) $k > n \geq 1$.

Case 1). According to Lemma 3 in Ostrovska [146], we have

$$B_{n,q}(e_k)(z) - e_k(z) = \sum_{j=1}^{k}\alpha_{j,k,n,q}e_j(z) - e_k(z).$$

Therefore,

$$B_{n,q}(e_k)(z) - e_k(z) = \sum_{j=1}^{k-1}\alpha_{j,k,n,q}e_j(z) + [\alpha_{k,k,n,q} - 1]e_k(z),$$

which immediately implies

$$B^{(p)}_{n,q}[B_{n,q}(e_k)(z)-e_k(z)]=\sum_{j=1}^{k-1}\alpha_{j,k,n,q}B^{(p)}_{n,q}(e_j)(z)+[\alpha_{k,k,n,q}-1]B^{(p)}_{n,q}(e_k)(z).$$

Taking into account that by the proof of (i) we have $|B^{(p)}_{n,q}(e_j)(z)|\le r^j$, for all $p,n,j\in\mathbb{N}$ and $|z|\le r$ with $r\ge 1$, it follows

$$\begin{aligned}&|B^{(p)}_{n,q}[B_{n,q}(e_k)-e_k](z)|\\&\quad\le\sum_{j=1}^{k-1}\alpha_{j,k,n,q}|B^{(p)}_{n,q}(e_j)(z)|+|1-\alpha_{k,k,n,q}|\cdot|B^{(p)}_{n,q}(e_k)(z)|\\&\quad\le\sum_{j=1}^{k-1}\alpha_{j,k,n,q}r^j+|1-\alpha_{k,k,n,q}|r^k\le 2|1-\alpha_{k,k,n,q}|r^k.\end{aligned}$$

But

$$B^{(m)}_{n,q}(e_k)(z)-e_k(z)=\sum_{p=0}^{m-1}B^{(p)}_{n,q}[B_{n,q}(e_k)(z)-e_k(z)],$$

which implies, for all $|z|\le r$

$$|B^{(m)}_{n,q}(e_k)-e_k|\le\sum_{p=0}^{m-1}|B^{(p)}_{n,q}[B_{n,q}(e_k)-e_k](z)|\le 2m|1-\alpha_{k,k,n,q}|r^k.$$

Since by the same Lemma 3 in Ostrovska [146], we have

$$\alpha_{k,k,n,q}=\left(1-\frac{1}{[n]_q}\right)\ldots\left(1-\frac{[k-1]_q}{[n]_q}\right),$$

by using the inequality

$$1-\prod_{j=1}^{k}x_j\le\sum_{j=1}^{k}(1-x_j),0\le x_j\le 1,j=1,\ldots,k,$$

it follows

$$\begin{aligned}|1-\alpha_{k,k,n,q}|&\le\sum_{j=1}^{k-1}\frac{[j]_q}{[n]_q}=\frac{1}{[n]_q}\sum_{j=1}^{k-1}[j]_q\\&=\frac{1}{[n]_q}\sum_{j=0}^{k-2}q^j[k-(j+1)]\le\frac{1}{[n]_q}\sum_{j=0}^{k-2}[k-(j+1)]\\&=\frac{1}{[n]_q}[k(k-1)-k(k-1)/2]=\frac{k(k-1)}{2[n]_q}.\end{aligned}$$

Therefore

$$|B^{(m)}_{n,q}(e_k)-e_k|\le\frac{m}{[n]_q}k(k-1)r^k.$$

Case 2). For $1 \leq r < R$, $|z| \leq r$ and $k > n \geq 1$, we get

$$\begin{aligned}|B_{n,q}^{(m)}(e_k)(z) - e_k(z)| &\leq |B_{n,q}^{(m)}(e_k)(z)| + |e_k(z)| \\ &\leq 2r^k \leq 2\frac{k(k-1)}{n}r^k \leq 2\frac{k(k-1)}{[n]_q}r^k,\end{aligned}$$

since for $q \in (0,1]$ we have $[n]_q \leq n$.

As a conclusion, for both Cases 1) and 2), for $|z| \leq r$ we obtain

$$\begin{aligned}|B_{n,q}^{(m)}(f)(z) - f(z)| &\leq \sum_{k=1}^{\infty}|c_k| \cdot |B_{n,q}^{(m)}(e_k)(z) - e_k(z)| \\ &= \sum_{k=1}^{n}|c_k| \cdot |B_{n,q}^{(m)}(e_k)(z) - e_k(z)| \\ &\quad + \sum_{k=n+1}^{\infty}|c_k| \cdot |B_{n,q}^{(m)}(e_k)(z) - e_k(z)| \\ &\leq \frac{2m}{[n]_q}\sum_{k=2}^{\infty}|c_k|k(k-1)r^k.\end{aligned}$$

Now, by choosing $r = r_s$, we get the first required inequality in the statement. Note that $\sum_{k=2}^{\infty}|c_k|k(k-1)r^{k-2} < \infty$, since we have $f''(z) = \sum_{k=2}^{\infty}c_k k(k-1)z^{k-2}$, for all $|z| \leq r$.

The second estimate in (ii) is a direct consequence of the inequality

$$\begin{aligned}d(f,g) &= \sum_{j=1}^{s}\frac{1}{2^j}\cdot\frac{\|f-g\|_j}{1+\|f-g\|_j} + \sum_{j=s+1}^{\infty}\frac{1}{2^j}\cdot\frac{\|f-g\|_j}{1+\|f-g\|_j} \\ &\leq \frac{\|f-g\|_s}{1+\|f-g\|_s}\sum_{j=1}^{s}\frac{1}{2^j} + \sum_{j=s+1}^{\infty}\frac{1}{2^j} \\ &\leq \frac{\|f-g\|_s}{1+\|f-g\|_s} + \frac{1}{2^s} \leq \|f-g\|_s + \frac{1}{2^s}.\end{aligned}$$

This proves (ii).

(iii) The proof is similar with that of the point (i), by taking into account that from Theorem 10 in Ostrovska [146], for each $q \in (1,\infty)$, any polynomial P and for $\lim_{n\to\infty}\frac{m_n}{[n]_q} = 0$ we have $\lim_{n\to\infty}B_{n,q}^{(m_n)}(P)(x) = P(x)$, $x \in [0,1]$.

(iv) Let us suppose that $q \in (1,\infty)$ and $|z| \leq r$, with $1 \leq r < R$. We reason exactly as in the proof of the point (ii). First, to estimate $|B_{n,q}^{(m)}(e_k)(z) - e_k(z)|$, again we have two possibilities : 1) $0 \leq k \leq n$; 2) $k > n \geq 1$.

Case 1). Reasoning exactly as at the proof of the above point (ii), Case 1), by simple calculation we obtain

$$|B_{n,q}^{(m)}(e_k)(z) - e_k(z)| \leq \frac{2m}{[n]_q}\sum_{j=0}^{k-2}q^j[k-(j+1)]r^k = \frac{2m}{[n]_q}\cdot\frac{q^k-1-k(q-1)}{(q-1)^2}r^k.$$

Since for $q \in (1, \infty)$ we have $n \le [n]_q$, it follows

$$|B_{n,q}^{(m)}(e_k)(z) - e_k(z)| \le \frac{2m}{n} \cdot \frac{q^k - 1 - k(q-1)}{(q-1)^2} r^k.$$

Case 2). Reasoning exactly as the proof of the above point (ii), Case 2, we get

$$|B_{n,q}^{(m)}(e_k)(z) - e_k(z)| \le 2\frac{k(k-1)}{n} r^k.$$

As a conclusion, collecting the estimates in the Cases 1) and 2), we get

$$\begin{aligned}
|B_{n,q}^{(m)}(f)(z) - f(z)| &\le \sum_{k=1}^{\infty} |c_k| \cdot |B_{n,q}^{(m)}(e_k)(z) - e_k(z)| \\
&= \sum_{k=2}^{n} |c_k| \cdot |B_{n,q}^{(m)}(e_k)(z) - e_k(z)| \\
&\quad + \sum_{k=n+1}^{\infty} |c_k| \cdot |B_{n,q}^{(m)}(e_k)(z) - e_k(z)| \\
&\le \frac{2m}{n} \sum_{k=2}^{n} |c_k| \cdot \frac{q^k - 1 - k(q-1)}{(q-1)^2} r^k \\
&\quad + \frac{2}{n} \sum_{k=n+1}^{\infty} |c_k| \cdot k(k-1) r^k \\
&\le \frac{2m}{n} \sum_{k=2}^{\infty} |c_k| \left[\frac{q^k - 1 - k(q-1)}{(q-1)^2} + k(k-1) \right] r^k.
\end{aligned}$$

If, in addition we take $r = 1$ and $q < R$, then obviously $\sum_{k=2}^{\infty} |c_k| q^k < \infty$ and $\sum_{k=2}^{\infty} |c_k| k(k-1) < \infty$, which for $\frac{m_n}{n} \to 0$ as $n \to \infty$, by the above inequality implies that $B_{n,q}^{(m_n)}(f)(z) \to f(z)$, uniformly in $\overline{\mathbb{D}_1}$.

(v) Let $q \in (1, \infty)$ and $1 \le r < R$ be arbitrary. Taking into account that for any polynomial P, by Theorem 10 in Ostrovska [146], for $\lim_{n\to\infty} \frac{m_n}{[n]_q} = \infty$ as $n \to \infty$ we have $B_{n,q}^{(m_n)}(P)(x) \to L_1(P)(x)$, uniformly for $x \in [0, 1]$ and since by the above point (i), it is immediate that $B_{n,q}^{(m_n)}(P)(z)$, $n = 1, 2, ...$, is uniformly bounded in $\overline{\mathbb{D}_r}$ with respect to $n \in \mathbb{N}$, by Vitali's theorem it follows the uniform convergence on $\overline{\mathbb{D}_r}$ to $L_1(P)(z)$ of $B_{n,q}^{(m_n)}(P)(z)$ (as $n \to \infty$). Because r is arbitrary, it follows the convergence in d too (see the considerations at the beginning of this section). Note that for the Vitali's convergence result, the uniform convergence on $[0, 1]$ is not necessary, it suffices to have only pointwise convergence there. The theorem is proved. □

Finally we present the geometric properties for the complex q-Bernstein polynomials $B_{n,q_n}(f)(z)$, with $0 < q_n < 1$, and $q_n \to 1$ as $n \to \infty$.

Theorem 1.5.7. (Gal [87]) *Let us suppose that $G \subset \mathbb{C}$ is open, such that $\overline{\mathbb{D}_1} \subset G$ and $f : G \to \mathbb{C}$ is analytic in G. Also, let us consider $B_{n,q_n}(f)(z)$, with $0 < q_n < 1$, and $q_n \to 1$ as $n \to \infty$.*

If $f(0)=f'(0)-1=0$ and f is starlike (convex, spirallike of type η, respectively) in $\overline{\mathbb{D}}_1$, that is for all $z\in\overline{\mathbb{D}}_1$ (see e.g. Mocanu-Bulboacă-Sălăgean [138])

$$Re\left(\frac{zf'(z)}{f(z)}\right)>0\left(Re\left(\frac{zf''(z)}{f'(z)}\right)+1>0, Re\left(e^{i\eta}\frac{zf'(z)}{f(z)}\right)>0, resp.\right),$$

then there exists an index n_0 depending on f (and on η for spirallikeness), such that for all $n\ge n_0$, $B_{n,q_n}(f)(z)$, are starlike (convex, spirallike of type η, respectively) in $\overline{\mathbb{D}}_1$.

If $f(0)=f'(0)-1=0$ and f is starlike (convex, spirallike of type η, respectively) only in $\mathbb{D}_1$ (that is the corresponding inequalities hold only in $\mathbb{D}_1$), then for any disk of radius $0<r<1$ and center 0 denoted by $\mathbb{D}_r$, there exists an index $n_0=n_0(f,\mathbb{D}_r)$ (n_0 depends on η too in the case of spirallikeness), such that for all $n\ge n_0$, $B_{n,q_n}(f)(z)$, are starlike (convex, spirallike of type η, respectively) in $\overline{\mathbb{D}}_r$ (that is, the corresponding inequalities hold in $\overline{\mathbb{D}}_r$).

Proof. By Theorem 2 in Phillips [149] and by the classical Vitali's theorem, it follows that we have $B_{n,q_n}(f)(z)\to f(z)$, uniformly for $|z|\le 1$, which by the well-known Weierstrass's theorem implies $[B_{n,q_n}(f)]'(z)\to f'(z)$ and $[B_{n,q_n}(f)]''(z)\to f''(z)$, for $n\to\infty$, uniformly in $\overline{\mathbb{D}}_1$. In all what follows, denote $P_n(f)(z)=\frac{B_{n,q_n}(f)(z)}{[B_{n,q_n}(f)]'(0)}$, well defined for sufficiently large n. We easily get $P_n(f)(0)=0$, $P_n'(f)(0)=1$ for sufficiently large n, and $P_n(f)(z)\to f(z)$, $P_n'(f)(z)\to f'(z)$ and $P_n''(f)(z)\to f''(z)$, uniformly in $\overline{\mathbb{D}}_1$.

Suppose first that f is starlike in $\overline{\mathbb{D}}_1$. Then, by hypothesis we get $|f(z)|>0$ for all $z\in\overline{\mathbb{D}}_1$ with $z\neq 0$, which from the univalence of f in $\mathbb{D}_1$, implies that we can write $f(z)=zg(z)$, with $g(z)\neq 0$, for all $z\in\overline{\mathbb{D}_1}$, where g is analytic in $\mathbb{D}_1$ and continuous in $\overline{\mathbb{D}}_1$.

Writing $P_n(f)(z)$ in the form $P_n(f)(z)=zQ_n(f)(z)$, obviously $Q_n(f)(z)$ is a polynomial of degree $\le n-1$. Also, for $|z|=1$ we have

$$|f(z)-P_n(f)(z)|=|z|\cdot|g(z)-Q_n(f)(z)|=|g(z)-Q_n(f)(z)|,$$

which by the uniform convergence in $\overline{\mathbb{D}}_1$ of $P_n(f)$ to f and by the maximum modulus principle, implies the uniform convergence in $\overline{\mathbb{D}}_1$ of $Q_n(f)(z)$ to $g(z)$.

Since g is continuous in $\overline{\mathbb{D}}_1$ and $|g(z)|>0$ for all $z\in\overline{\mathbb{D}}_1$, there exist an index $n_1\in\mathbb{N}$ and $a>0$ depending on g, such that $|Q_n(f)(z)|>a>0$, for all $z\in\overline{\mathbb{D}}_1$ and all $n\ge n_0$. Also, for all $|z|=1$, we have

$$\begin{aligned}|f'(z)-P_n'(f)(z)|&=|z[g'(z)-Q_n'(f)(z)]+[g(z)-Q_n(f)(z)]|\\&\ge\big|\ \ |z|\cdot|g'(z)-Q_n'(f)(z)|-|g(z)-Q_n(f)(z)|\ \ \big|\\&=\big|\ \ |g'(z)-Q_n'(f)(z)|-|g(z)-Q_n(f)(z)|\ \ \big|,\end{aligned}$$

which from the maximum modulus principle, the uniform convergence of $P_n'(f)$ to f' and of $Q_n(f)$ to g, evidently implies the uniform convergence of $Q_n'(f)$ to g'.

Then, for $|z| = 1$, we get

$$\begin{aligned}\frac{zP_n'(f)(z)}{P_n(f)} &= \frac{z[zQ_n'(f)(z) + Q_n(f)(z)]}{zQ_n(f)(z)} \\ &= \frac{zQ_n'(f)(z) + Q_n(f)(z)}{Q_n(f)(z)} \to \frac{zg'(z) + g(z)}{g(z)} \\ &= \frac{f'(z)}{g(z)} = \frac{zf'(z)}{f(z)},\end{aligned}$$

which again from the maximum modulus principle, implies

$$\frac{zP_n'(f)(z)}{P_n(f)} \to \frac{zf'(z)}{f(z)}, \text{ uniformly in } \overline{\mathbb{D}}_1.$$

Since $Re\left(\frac{zf'(z)}{f(z)}\right)$ is continuous in $\overline{\mathbb{D}}_1$, there exists $\varepsilon \in (0,1)$, such that

$$Re\left(\frac{zf'(z)}{f(z)}\right) \geq \varepsilon, \text{ for all } z \in \overline{\mathbb{D}}_1.$$

Therefore

$$Re\left[\frac{zP_n'(f)(z)}{P_n(f)(z)}\right] \to Re\left[\frac{zf'(z)}{f(z)}\right] \geq \varepsilon > 0$$

uniformly on $\overline{\mathbb{D}}_1$, i.e. for any $0 < \rho < \varepsilon$, there is n_0 such that for all $n \geq n_0$ we have

$$Re\left[\frac{zP_n'(f)(z)}{P_n(f)(z)}\right] > \rho > 0, \text{ for all } z \in \overline{\mathbb{D}}_1.$$

Since $P_n(f)(z)$ differs from $B_{n,q_n}(f)(z)$ only by a constant, this proves the starlikeness of $B_{n,q_n}(f)(z)$, for sufficiently large n.

If f is supposed to be starlike only in $\mathbb{D}_1$, the proof is identical, with the only difference that instead of $\overline{\mathbb{D}}_1$, we reason for $\overline{\mathbb{D}}_r$.

The proofs in the cases when f is convex or spirallike of order η are similar and follow from the following uniform convergency (on $\overline{\mathbb{D}}_1$ or on $\overline{\mathbb{D}}_r$)

$$Re\left[\frac{zP_n''(f)(z)}{P_n'(f)(z)}\right] + 1 \to Re\left[\frac{zf''(z)}{f'(z)}\right] + 1$$

and

$$Re\left[e^{i\eta}\frac{zP_n'(f)(z)}{P_n(f)(z)}\right] \to Re\left[e^{i\eta}\frac{zf'(z)}{f(z)}\right].$$

The theorem is proved. □

1.6 Bernstein-Stancu Polynomials

In this section for two kinds of complex Bernstein-Stancu polynomials we study similar properties with those for the classical complex Bernstein polynomials and

APPROXIMATION BY COMPLEX BERNSTEIN AND CONVOLUTION TYPE OPERATORS

q-Bernstein polynomials in Sections 1.1, 1.2 and 1.5. More exactly we consider the following two kinds of polynomials :

$$S_n^{(\alpha,\beta)}(f)(z)=\sum_{k=0}^{n}\binom{n}{k}z^k(1-z)^{n-k}f[(k+\alpha)/(n+\beta)], z\in\mathbb{C},$$

where $0\le\alpha\le\beta$ are independent of n, (introduced and studied for the case of real variable in Stancu [173]) and

$$S_n^{<\gamma>}(f)(z)=\sum_{k=0}^{n}p_{n,k}^{<\gamma>}(z)f(k/n), z\in\mathbb{C},$$

(introduced and studied for the case of real variable in Stancu [174]) where $\gamma\ge 0$ may to depend on n and

$$\begin{aligned}&p_{n,k}^{<\gamma>}(z)\\&=\binom{n}{k}\frac{z(z+\gamma)...(z+(k-1)\gamma)(1-z)(1-z+\gamma)...(1-z+(n-k-1)\gamma)}{(1+\gamma)(1+2\gamma)...(1+(n-1)\gamma)}.\end{aligned}$$

Note that $S_n^{(0,0)}(f)(z)=S_n^{<0>}(f)(z)=B_n(f)(z)$.

We begin our study with $S_n^{(\alpha,\beta)}(f)(z)$. First we present upper estimates in simultaneous approximation.

Theorem 1.6.1. (Gal [80]) *Let $\mathbb{D}_R=\{z\in\mathbb{C};|z|<R\}$ be with $R>1$ and let us suppose that $f:\mathbb{D}_R\to\mathbb{C}$ is analytic in $\mathbb{D}_R$, i.e. $f(z)=\sum_{k=0}^{\infty}c_kz^k$, for all $z\in\mathbb{D}_R$. Suppose $0\le\alpha\le\beta$ and $1\le r<R$ are arbitrary fixed. For all $|z|\le r$ and $n\in\mathbb{N}$, we have*

$$|S_n^{(\alpha,\beta)}(f)(z)-f(z)|\le\frac{M_{2,r}^{(\beta)}(f)}{n+\beta},$$

where $0<M_{2,r}^{(\beta)}(f)=2r^2\sum_{j=2}^{\infty}j(j-1)|c_j|r^{j-2}+2\beta r\sum_{j=1}^{\infty}j|c_j|r^{j-1}<\infty$.

Also, if $1\le r<r_1<R$, then for all $|z|\le r$ and $n,p\in\mathbb{N}$, we have

$$|[S_n^{(\alpha,\beta)}(f)]^{(p)}(z)-f^{(p)}(z)|\le\frac{M_{2,r_1}^{(\beta)}(f)p!r_1}{(n+\beta)(r_1-r)^{p+1}}.$$

Proof. Denoting $e_k(z)=z^k$, we get $S_n^{(\alpha,\beta)}(f)(z)=\sum_{k=0}^{\infty}c_kS_n^{(\alpha,\beta)}(e_k)(z)$ and

$$|S_n^{(\alpha,\beta)}(f)(z)-f(z)|\le\sum_{k=0}^{\infty}|c_k|\cdot|S_n^{(\alpha,\beta)}(e_k)(z)-e_k(z)|.$$

To estimate $|S_n^{(\alpha,\beta)}(e_k)(z)-e_k(z)|$ for fixed $n\in\mathbb{N}$, we consider two possible cases : 1) $0\le k\le n$; 2) $k>n$.

Denoting by Δ^k the finite difference of order k, we will use the representation formula (see Stancu [173])

$$S_n^{(\alpha,\beta)}(f)(z)=\sum_{p=0}^{n}\binom{n}{p}\Delta_{1/(n+\beta)}^{p}f(\alpha/(n+\beta))e_p(z).$$

Case 1). If $k=0$, then obviously we have $S_n^{(\alpha,\beta)}(e_k)(z)-e_k(z)=0$. Therefore, let us suppose that $1\le k\le n$ and denote

$$\begin{aligned}C_{n,p,k}^{(\alpha,\beta)}&=\binom{n}{p}\Delta^p_{1/(n+\beta)}e_k(\alpha/(n+\beta))\\&=\binom{n}{p}[\alpha/(n+\beta),(\alpha+1)/(n+\beta),...,(\alpha+p)/(n+\beta);e_k](p!)/(n+\beta)^p.\end{aligned}$$

Since e_k is convex of any order, it follows $C_{n,p,k}\ge 0$ and since $S_n^{(\alpha,\beta)}(f)(1)=f[(n+\alpha)/(n+\beta)]$, we get $\sum_{p=0}^n C_{n,p,k}^{(\alpha,\beta)}=\frac{(n+\alpha)^k}{(n+\beta)^k}\le 1$.

For any $|z|\le r$ with $1\le r<R$, we can write

$$\begin{aligned}&|S_n^{(\alpha,\beta)}(e_k)(z)-e_k(z)|\\&\quad=|\sum_{p=0}^{k}C_{n,p,k}^{(\alpha,\beta)}e_p(z)-e_k(z)|=|[C_{n,k,k}^{(\alpha,\beta)}-1]e_k(z)+\sum_{p=0}^{k-1}C_{n,p,k}^{(\alpha,\beta)}e_p(z)|\\&\quad\le\left[1-\frac{n(n-1)...(n-(k-1))}{(n+\beta)^k}\right]r^k\\&\qquad+\left[\frac{(n+\alpha)^k}{(n+\beta)^k}-\frac{n(n-1)...(n-(k-1))}{(n+\beta)^k}\right]r^k\\&\quad=2\left[1-\frac{n(n-1)...(n-(k-1))}{(n+\beta)^k}\right]r^k+\left[\frac{(n+\alpha)^k}{(n+\beta)^k}-1\right]r^k\\&\quad\le 2\left[1-\frac{n(n-1)...(n-(k-1))}{(n+\beta)^k}\right]r^k\le\frac{1}{n+\beta}[2\beta k+k(k-1)]r^k.\end{aligned}$$

Here we have applied the formula (easily proved by mathematical induction)

$$1-\Pi_{j=1}^k x_j\le\sum_{j=1}^k(1-x_j),0\le x_j\le 1,j=1,...,k.$$

Case 2). For $1\le r<R$, $|z|\le r$ and $k>n\ge 1$, we get

$$\begin{aligned}|S_n^{(\alpha,\beta)}(e_k)(z)-e_k(z)|&\le|S_n^{(\alpha,\beta)}(e_k)(z)|+r^k\\&\le\sum_{p=0}^{n}C_{n,p,k}^{(\alpha,\beta)}r^p+r^k\le r^n+r^k\le 2r^k\\&\le 2nr^k\le 2(k-1)\cdot\frac{k+\beta}{n+\beta}r^k\\&=\frac{2k(k-1)+2\beta(k-1)}{n+\beta}r^k\le\frac{2k(k-1)+2\beta k}{n+\beta}r^k.\end{aligned}$$

Combining it with the above Case 1, we get the desired inequality.

For the simultaneous approximation, denoting by Γ the circle of radius $r_1>r$ and center 0, since for any $|z|\le r$ and $v\in\Gamma$, we have $|v-z|\ge r_1-r$, by the

Cauchy's formulas it follows that for all $|z| \leq r$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}|[S_n^{(\alpha,\beta)}(f)]^{(p)}(z) - f^{(p)}(z)| &= \frac{p!}{2\pi}\left|\int_\Gamma \frac{S_n^{(\alpha,\beta)}(f)(v) - f(v)}{(v-z)^{p+1}}dv\right| \\ &\leq \frac{M_{2,r_1}^{(\beta)}(f)}{n+\beta}\frac{p!}{2\pi}\frac{2\pi r_1}{(r_1-r)^{p+1}} \\ &= \frac{M_{2,r_1}^{(\beta)}(f)}{n+\beta}\frac{p! r_1}{(r_1-r)^{p+1}}.\end{aligned}$$

Finally, since by hypothesis, $f(z) = \sum_k^\infty c_k z^k$ is absolutely and uniformly convergent in $|z| \leq r$, for any $1 \leq r < R$, it is clear that $M_{2,r}^{(\beta)}(f) < \infty$. □

Remark. For $\alpha = \beta = 0$ in Theorem 1.6.1 the estimates in Theorem 1.1.2 for classical Bernstein polynomials are obtained.

A quantitative Voronovskaja-type formula follows.

Theorem 1.6.2. (Gal [80]) *Let $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ be with $R > 1$ and let us suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$, that is we can write $f(z) = \sum_{k=0}^\infty c_k z^k$, for all $z \in \mathbb{D}_R$. Let $0 \leq \alpha \leq \beta$. For all $|z| \leq 1$ and $n \in \mathbb{N}$, we have*

$$\begin{aligned}&\left|S_n^{(\alpha,\beta)}(f)(z) - f(z) + \frac{\beta z - \alpha}{n+\beta}f'(z) - \frac{nz(1-z)}{2(n+\beta)^2}f''(z)\right| \\ &\leq \frac{|z|\cdot|1-z|}{(n+\beta)^2}M_1^{(\alpha,\beta)}(f) + \frac{M_2^{(\alpha,\beta)}(f)}{(n+\beta)^2},\end{aligned}$$

where $0 < M_1^{(\alpha,\beta)}(f)$,

$$\begin{aligned}M_1^{(\alpha,\beta)}(f) = \sum_{k=2}^\infty |c_k| \Bigg[&\frac{9(k-1)^3(k-2)}{2} + \frac{(k-1)^2(k-2)^2}{2} + 4\beta(k-1)^3 \\ &+ \frac{3\beta(k-1)^2(k-2)}{2} + \frac{3\alpha(k-1)^2(k-2)}{2} + \beta k(k-1)^2(k-2)\Bigg] < \infty,\end{aligned}$$

$$0 < M_2^{(\alpha,\beta)}(f) = (\alpha+\beta)^2\sum_{k=2}^\infty |c_k|\frac{k(k-1)}{2} < \infty.$$

Proof. Denoting $e_k(z) = z^k$ and $\pi_{n,k}(z) = S_n^{(\alpha,\beta)}(e_k)(z)$, we obtain

$$\begin{aligned}&\left|S_n^{(\alpha,\beta)}(f)(z) - f(z) + \frac{\beta z - \alpha}{n+\beta}f'(z) - \frac{nz(1-z)}{2(n+\beta)^2}f''(z)\right| \\ &\leq \sum_{k=1}^\infty |c_k|\left|\pi_{n,k}(z) - e_k(z) + \frac{z^{k-1}(\beta z - \alpha)k}{n+\beta} - \frac{nz^{k-1}(1-z)k(k-1)}{2(n+\beta)^2}\right|.\end{aligned}$$

Differentiating the sum $s_{n,k}(z) = \sum_{j=0}^{n}(j+\alpha)^k\binom{n}{j}z^j(1-z)^{n-j}$ and then dividing the formula by $(n+\beta)^{k+1}$, by simple calculation we get the recurrence formula

$$\pi_{n,k+1}(z) = \frac{z(1-z)}{n+\beta}\pi'_{n,k}(z) + \frac{\alpha+nz}{n+\beta}\pi_{n,k}(z), z \in \mathbb{C}.$$

Denoting $G_{n,k}(z) = \pi_{n,k}(z) - e_k(z) + \frac{z^{k-1}(\beta z-\alpha)k}{n+\beta} - \frac{nz^{k-1}(1-z)k(k-1)}{2(n+\beta)^2}$, the above recurrence for $\pi_{n,k}(z)$ implies

$$\begin{aligned}G_{n,k}(z) &= \frac{z(1-z)}{n+\beta}\pi'_{n,k-1}(z) + \frac{\alpha+nz}{n+\beta}\pi_{n,k-1}(z) - e_k(z)\\ &\quad + \frac{z^{k-1}(\beta z-\alpha)k}{n+\beta} - \frac{nz^{k-1}(1-z)k(k-1)}{2(n+\beta)^2},\end{aligned}$$

which by simple calculation implies the following recurrence formula for $G_{n,k}(z)$ (valid for all $k \geq 2$ since $G_{n,0}(z) = G_{n,1}(z) = 0$)

$$G_{n,k}(z) = \frac{z(1-z)}{n+\beta}G'_{n,k-1}(z) + \frac{\alpha+nz}{n+\beta}G_{n,k-1}(z) + A,$$

where

$$\begin{aligned}A :=& \frac{z^{k-1}(1-z)(k-1)}{n+\beta} - \frac{z^{k-2}(1-z)(k-1)[(k-2)(\beta z-\alpha)+\beta z]}{(n+\beta)^2}\\ &+ \frac{z^{k-2}(1-z)(k-1)(k-2)[(k-2)-(k-1)z]}{2(n+\beta)}\left(\frac{n+\beta}{(n+\beta)^2} - \frac{\beta}{(n+\beta)^2}\right)\\ &+ \frac{\alpha+nz}{n+\beta}z^{k-1} - \frac{\alpha+nz}{(n+\beta)^2}z^{k-2}(\beta z-\alpha)(k-1)\\ &+ \frac{\alpha+nz}{2(n+\beta)}z^{k-2}(1-z)(k-1)(k-2)\left[\frac{n+\beta}{(n+\beta)^2} - \frac{\beta}{(n+\beta)^2}\right] - \frac{n+\beta}{n+\beta}z^k\\ &+ \frac{z^{k-1}(\beta z-\alpha)k}{n+\beta} - \frac{z^{k-1}(1-z)k(k-1)}{2(n+\beta)} + \frac{\beta z^{k-1}(1-z)k(k-1)}{2(n+\beta)^2}.\end{aligned}$$

In what follows, we will write the expression A in the form $A := T_1 + T_2 + T_3$, where T_1 is the sum of all the terms containing $(n+\beta)$ at the denominator, T_2 is the sum of all the terms containing $(n+\beta)^2$ at the denominator and T_3 is the sum of all the terms containing $(n+\beta)^3$ at the denominator. Therefore, by writing (for T_2) $\alpha + nz = (\alpha - \beta z) + (n+\beta)z$, we obtain

$$\begin{aligned}T_1 &= \frac{z^{k-1}}{n+\beta}[(1-z)(k-1) + (\alpha+nz) - z(n+\beta) + k(\beta z-\alpha)\\ &\quad -k(k-1)(1-z)/2] = \frac{z^{k-1}(k-1)}{n+\beta}[z(k/2+\beta-1) + 1 - \alpha - k/2],\end{aligned}$$

$$\begin{aligned}&T_2\\ &= \frac{z^{k-2}(k-1)}{(n+\beta)^2}\left[\frac{(1-z)(k-2)z(n+\beta)}{2} + \frac{(1-z)(k-2)z(-2\beta-(k-1))}{2}\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{(1-z)(k-2)[3\alpha+(k-2)]}{2}-(\beta z-\alpha)(n+\beta)z+(\beta z-\alpha)^2\Big]\\
&=-T_1+\frac{z^{k-2}(k-1)}{2(n+\beta)^2}\left[(1-z)z(k-2)[-2\beta-(k-1)]\right.\\
&\quad\left.+(1-z)(k-2)[3\alpha+(k-2)]+2(\beta z-\alpha)^2\right]\\
&=-T_1+\frac{z^{k-1}(1-z)(k-1)(k-2)}{2(n+\beta)^2}[-2\beta-(k-1)]\\
&\quad+\frac{z^{k-2}(1-z)(k-1)(k-2)[3\alpha+(k-2)]}{2(n+\beta)^2}+\frac{z^{k-2}(k-1)}{(n+\beta)^2}(\beta z-\alpha)^2,
\end{aligned}
$$

$$
\begin{aligned}
T_3&=-\frac{\beta z^{k-2}(1-z)(k-1)(k-2)[(k-2)-(k-1)z]}{2(n+\beta)^3}\\
&\quad-\frac{\beta(\alpha+nz)z^{k-2}(1-z)(k-1)(k-2)}{2(n+\beta)^2}\\
&=-\frac{\beta z^{k-2}(1-z)(k-1)(k-2)}{2(n+\beta)^3}[z(n-(k-1))+\alpha+(k-2)].
\end{aligned}
$$

First, we observe that in the sum $T_1+T_2+T_3$, the terms containing $(n+\beta)$ at the denominator cancel. Now, we will estimate $|A|=|T_1+T_2+T_3|$ for $|z|\le 1$.

For all $k\ge 2$ we obtain

$$
\begin{aligned}
|A|&\le\frac{|z|\cdot|1-z|(k-1)(k-2)[2\beta+(k-1)]}{2(n+\beta)^2}\\
&\quad+\frac{|z|\cdot|1-z|(k-1)(k-2)[3\alpha+(k-2)]}{2(n+\beta)^2}+\frac{(k-1)(\beta+\alpha)^2}{(n+\beta)^2}\\
&\quad+\frac{|z|\cdot|1-z|\beta(k-1)(k-2)[n+\alpha+2k-3]}{2(n+\beta)^3}
\end{aligned}
$$

$$
\begin{aligned}
&=|z|\cdot|1-z|\left\{\frac{(k-1)(k-2)[2\beta+(k-1)]}{2(n+\beta)^2}+\frac{(k-1)(k-2)[3\alpha+k-2]}{2(n+\beta)^2}\right.\\
&\quad\left.+\frac{\beta(k-1)(k-2)[n+\alpha+2k-3]}{2(n+\beta)^3}\right\}+\frac{(k-1)(\beta+\alpha)^2}{(n+\beta)^2}
\end{aligned}
$$

$$
\begin{aligned}
&\le\frac{|z|\cdot|1-z|}{(n+\beta)^2}\left\{(k-1)^2(k-2)/2+2\beta(k-1)(k-2)/2\right.\\
&\quad+(k-1)\left[(k-2)^2/2+\frac{3\alpha(k-2)}{2}\right]+\frac{\beta(k-1)(k-2)}{2}\\
&\quad\left.+\frac{\beta(\alpha-\beta)(k-1)(k-2)}{2(n+\beta)}+\frac{2\beta k(k-1)(k-2)}{2(n+\beta)}\right\}+\frac{(k-1)(\beta+\alpha)^2}{(n+\beta)^2}
\end{aligned}
$$

$$\leq \frac{|z|\cdot|1-z|}{(n+\beta)^2}\big\{(k-1)^2(k-2)/2+\beta(k-1)(k-2)+(k-1)(k-2)^2/2$$
$$+\,3\alpha(k-1)(k-2)/2+\frac{\beta(k-1)(k-2)}{2}+\beta k(k-1)(k-2)\Big\}$$
$$+\frac{(k-1)(\alpha+\beta)^2}{(n+\beta)^2}=\frac{|z|\cdot|1-z|}{(n+\beta)^2}\Big\{(k-1)^2(k-2)/2+\frac{3\beta(k-1)(k-2)}{2}$$
$$+\,(k-1)(k-2)^2/2+3\alpha(k-1)(k-2)/2+\beta k(k-1)(k-2)\big\}$$
$$+\frac{(k-1)(\alpha+\beta)^2}{(n+\beta)^2}.$$

Therefore, denoting by $\|\cdot\|$ the uniform norm in the closed unit disk and estimating for $|z|\leq 1$ the absolute value of $G_{n,k}(z)$ in the formula of recurrence, also by using the Bernstein's inequality (since $G_{n,k}(z)$ is a polynomial of degree k) and the estimate for $\|\pi_{n,k}-e_k\|$ in the proof of Theorem 1.6.1, we obtain

$$|G_{n,k}(z)| \leq \frac{|z|\cdot|1-z|}{n+\beta}\|G'_{n,k-1}\|+\frac{n+\alpha}{n+\beta}|G_{n,k-1}(z)|+|A|$$
$$\leq |G_{n,k-1}(z)|+\frac{|z|\cdot|1-z|(k-1)}{n+\beta}\|G_{n,k-1}\|+|A|$$
$$\leq |G_{n,k-1}(z)|+\frac{|z|\cdot|1-z|(k-1)}{n+\beta}\Big[\|\pi_{n,k-1}-e_{k-1}\|$$
$$+\,\frac{(k-1)(\alpha+\beta)}{n+\beta}+\frac{n(k-1)(k-2)}{(n+\beta)^2}\Big]+|A|\leq |G_{n,k-1}(z)|$$
$$+\frac{|z|\cdot|1-z|(k-1)}{n+\beta}\Big[\frac{2(k-1)(k-2)+2\beta(k-1)}{n+\beta}$$
$$+\,\frac{(k-1)(\alpha+\beta)}{n+\beta}+\frac{(k-1)(k-2)}{n+\beta}\Big]+|A|\leq |G_{n,k-1}(z)|$$
$$+\frac{|z|\cdot|1-z|(k-1)^2}{n+\beta}\cdot\frac{2(k-2)+2\beta+(\alpha+\beta)+(k-2)}{n+\beta}$$
$$+|A|\leq |G_{n,k-1}(z)|+\frac{4|z|\cdot|1-z|(k-1)^2}{(n+\beta)^2}(k-2+\beta)+|A|$$
$$\leq |G_{n,k-1}(z)|+\frac{4|z|\cdot|1-z|(k-1)^2}{(n+\beta)^2}(k-2+\beta)$$
$$+\frac{|z|\cdot|1-z|}{(n+\beta)^2}\Big\{(k-1)^2(k-2)/2+\frac{3\beta(k-1)(k-2)}{2}$$
$$+\,(k-1)(k-2)^2/2+3\alpha(k-1)(k-2)/2+\beta k(k-1)(k-2)\big\}$$
$$+\frac{(k-1)(\alpha+\beta)^2}{(n+\beta)^2}:=|G_{n,k-1}(z)|+A(n,k,\alpha,\beta)(z).$$

That is, in conclusion for all $|z|\leq 1$ we can write

$$|G_{n,k}(z)|\leq |G_{n,k-1}(z)|+A(n,k,\alpha,\beta)(z).$$

Since $G_{n,1}(z)=0$, reasoning by recurrence for $k=2,3,...$, we immediately obtain

$$\begin{aligned}&|G_{n,k}(z)|\\ &\le \frac{|z|\cdot|1-z|}{(n+\beta)^2}\sum_{j=2}^{k}\Big[9(j-1)^2(j-2)/2+4\beta(j-1)^2+(j-1)(j-2)^2/2\\ &\quad+\frac{3\beta(j-1)(j-2)}{2}+3\alpha(j-1)(j-2)/2+\beta j(j-1)(j-2)\Big]\\ &\quad+\frac{(\alpha+\beta)^2}{(n+\beta)^2}\sum_{j=2}^{k}(j-1)\\ &\le \frac{|z|\cdot|1-z|}{(n+\beta)^2}\Big[9(k-1)^3(k-2)/2+(k-1)^2(k-2)^2/2+4\beta(k-1)^3\\ &\quad+\frac{3\beta(k-1)^2(k-2)}{2}+3\alpha(k-1)^2(k-2)/2+\beta k(k-1)^2(k-2)\Big]\\ &\quad+\frac{k(k-1)(\alpha+\beta)^2}{2(n+\beta)^2}.\end{aligned}$$

As a conclusion, the desired estimate is immediate from

$$\left|S_n^{(\alpha,\beta)}(f)(z)-f(z)+\frac{\beta z-\alpha}{n+\beta}f'(z)-\frac{nz(1-z)}{2(n+\beta)^2}f''(z)\right|\le\sum_{k=1}^{\infty}|c_k|\cdot|G_{n,k}(z)|. \qquad \square$$

Remarks. 1) Taking now $\alpha=\beta=0$ in Theorem 1.6.2 we get the Voronovskaja's theorem with an upper estimate for the classical Bernstein polynomials in Theorem 1.1.3.

2) Following exactly the lines in the proof of Theorem 1.6.2 it is immediate that in fact for any $1\le r<R$ we have an upper estimate of the form

$$\left\|S_n^{(\alpha,\beta)}(f)-f+\frac{\beta e_1-\alpha}{n+\beta}f'-\frac{ne_1(1-e_1)}{2(n+\beta)^2}f''\right\|_r\le\frac{M_r^{(\alpha,\beta)}(f)}{(n+\beta)^2},$$

where the constant $M_r^{(\alpha,\beta)}(f)>0$ is independent of n and depends on f, r, α and β.

In what follows we prove that the degrees of approximation in Theorem 1.6.1 in fact are exact. Since the particular case $\alpha=\beta=0$ (that is the case of classical Bernstein polynomials) was already considered by Theorem 1.1.4, Corollary 1.1.5 and Theorem 1.1.6, in the next Theorem 1.6.3, Corollary 1.6.4 and Theorem 1.6.5 it will be excluded.

First we present :

Theorem 1.6.3. (Gal [81]) *Let $R>1$, $0\le\alpha\le\beta$ with $\alpha+\beta>0$, $\mathbb{D}_R=\{z\in\mathbb{C};|z|<R\}$ and let us suppose that $f:\mathbb{D}_R\to\mathbb{C}$ is analytic in $\mathbb{D}_R$, that is we can write $f(z)=\sum_{k=0}^{\infty}c_kz^k$, for all $z\in\mathbb{D}_R$. If f is not a polynomial of degree 0 and $1\le r<R$, then we have*

$$\|S_n^{(\alpha,\beta)}(f)-f\|_r\ge\frac{C_r^{(\alpha,\beta)}(f)}{n+\beta}, n\in\mathbb{N},$$

where the constant $C_r^{(\alpha,\beta)}(f)$ depends only on f, r, α and β.

Proof. For all $z \in \mathbb{D}_R$ and $n \in \mathbb{N}$ we have

$$S_n^{(\alpha,\beta)}(f)(z) - f(z) = \frac{1}{n+\beta}\left\{-(\beta z-\alpha)f'(z) + \frac{z(1-z)}{2}f''(z)\right.$$

$$+\frac{1}{n+\beta}\left[(n+\beta)^2\left(S_n^{(\alpha,\beta)}(f)(z) - f(z) + \frac{\beta z-\alpha}{n+\beta}f'(z) - \frac{nz(1-z)}{2(n+\beta)^2}f''(z)\right)\right.$$

$$\left.\left. - \frac{\beta z(1-z)}{2}f''(z)\right]\right\}.$$

Note that in the case $\alpha = \beta = 0$ in Corollary 1.1.5, necessarily f was supposed to be not a polynomial of degree ≤ 1.

In what follows we will apply to the above identity the following obvious property :

$$\|F+G\|_r \geq |\,\|F\|_r - \|G\|_r\,| \geq \|F\|_r - \|G\|_r.$$

It follows

$$\|S_n^{(\alpha,\beta)}(f) - f\|_r \geq \frac{1}{n+\beta}\left\{\left\|-(\beta e_1-\alpha)f' + \frac{e_1(1-e_1)}{2}f''\right\|_r\right.$$

$$-\frac{1}{n+\beta}\cdot\left[\left\|(n+\beta)^2\left(S_n^{(\alpha,\beta)}(f) - f + \frac{\beta e_1-\alpha}{n+\beta}f' - \frac{ne_1(1-e_1)}{2(n+\beta)^2}f''\right)\right.\right.$$

$$\left.\left.\left. - \frac{\beta e_1(1-e_1)}{2}f''\right\|_r\right]\right\}.$$

Since by Remark 2 after the proof of Theorem 1.6.2 we have

$$\left\|(n+\beta)^2\left(S_n^{(\alpha,\beta)}(f) - f + \frac{\beta e_1-\alpha}{n+\beta}f' - \frac{ne_1(1-e_1)}{2(n+\beta)^2}f''\right) - \frac{\beta e_1(1-e_1)}{2}f''\right\|_r$$

$$\leq M_r^{(\alpha,\beta)}(f) + \beta\|f''\|_r,$$

and denoting $H(z) = -(\beta z-\alpha)f'(z) + \frac{z(1-z)}{2}f''(z)$, if we prove that $\|H\|_r > 0$, then it is clear that there exists an index n_0 depending only on f, α and β, such that

$$\|S_n^{(\alpha,\beta)}(f) - f\|_r \geq \frac{1}{n+\beta}\cdot\frac{\|H\|_r}{2}, \forall n \geq n_0.$$

For $n \in \{1, 2, ..., n_0-1\}$ we have $\|S_n^{(\alpha,\beta)}(f) - f\|_r \geq \frac{A_{n,r}^{(\alpha,\beta)}(f)}{n+\beta}$ with $A_{n,r}^{(\alpha,\beta)}(f) = (n+\beta)\cdot\|S_n^{(\alpha,\beta)}(f) - f\|_r > 0$, which finally implies $\|S_n^{(\alpha,\beta)}(f) - f\|_r \geq \frac{C_r^{(\alpha,\beta)}(f)}{n+\beta}$ for all $n \in \mathbb{N}$, with $C_r^{(\alpha,\beta)}(f) = \min\left\{A_{1,r}^{(\alpha,\beta)}, A_{2,r}^{(\alpha,\beta)}(f), ..., A_{n_0-1,r}^{(\alpha,\beta)}(f), \frac{\|H\|_r}{2}\right\}$.

Therefore it remains to show that $\|H\|_r > 0$. Indeed, suppose that $\|H\|_r = 0$. We have two possibilities : 1) $0 = \alpha < \beta$ or 2) $0 < \alpha \leq \beta$.

Case 1). We obtain $H(z) = -\beta z f'(z) + \frac{z(1-z)}{2} f''(z) = 0$, for all $|z| \le r$ and denoting $y(z) = f'(z)$, it follows that $y(z)$ is an analytic function in $\mathbb{D}_R$, solution of the differential equation $-\beta z y(z) + \frac{z(1-z)}{2} y'(z) = 0, |z| \le r$, which after simplification with $z \ne 0$ becomes $-\beta y(z) + \frac{(1-z)}{2} y'(z) = 0, |z| \le r$. Now, seeking $y(z)$ in the form $y(z) = \sum_{k=0}^{\infty} b_k z^k$ and replacing it in the differential equation, by the identification of the coefficients we easily obtain $b_k = 0$ for all $k = 0, 1, ...,$. Therefore $y(z) = 0$ for all $|z| \le r$, which by the identity's theorem on analytic (holomorphic) functions implies $y(z) = 0$ for all $z \in \mathbb{D}_R$ and the contradiction that f is a polynomial of degree ≤ 0.

Case 2). Denoting $y(z) = f'(z)$ by hypothesis it follows that $y(z)$ is an analytic function in $\mathbb{D}_R$ solution of the differential equation $(-\beta z + \alpha) y(z) + \frac{z(1-z)}{2} y'(z) = 0, |z| \le r$.

Taking $z = 0$ it follows $\alpha y(0) = 0$, which means $y(0) = 0$. Seeking $y(z)$ in the form $y(z) = \sum_{k=1}^{\infty} b_k z^k$ and replacing it in the differential equation, by the identification of the coefficients we easily obtain $b_k = 0$ for all $k = 1, 2, ...,$, which finally leads to the contradiction that f is a constant. □

Combining now Theorem 1.6.3 with Theorem 1.6.1 we immediately get the following.

Corollary 1.6.4. (Gal [81]) *Let* $R > 1$, $0 \le \alpha \le \beta$ *with* $\alpha + \beta > 0$, $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ *and let us suppose that* $f : \mathbb{D}_R \to \mathbb{C}$ *is analytic in* $\mathbb{D}_R$. *If* f *is not a polynomial of degree* 0 *and* $1 \le r < R$, *then we have*

$$\|S_n^{(\alpha,\beta)}(f) - f\|_r \sim \frac{1}{n+\beta}, n \in \mathbb{N},$$

where the constants in the equivalence depend on f, r, α *and* β.

In the case of simultaneous approximation we present :

Theorem 1.6.5. (Gal [81]) *Let* $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ *be with* $R > 1$, $0 \le \alpha \le \beta$ *with* $\alpha + \beta > 0$ *and let us suppose that* $f : \mathbb{D}_R \to \mathbb{C}$ *is analytic in* $\mathbb{D}_R$, *i.e.* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{D}_R$. *Also, let* $1 \le r < r_1 < R$ *and* $p \in \mathbb{N}$ *be fixed. If* f *is not a polynomial of degree* $\le p - 1$, *then we have*

$$\|[S_n^{(\alpha,\beta)}(f)]^{(p)} - f^{(p)}\|_r \sim \frac{1}{n+\beta},$$

where the constants in the equivalence depend on f, α, β, r, r_1 *and* p.

Proof. Taking into account Theorem 1.6.1, it remains to prove the lower estimate for $\|[S_n^{(\alpha,\beta)}(f)]^{(p)} - f^{(p)}\|_r$ only. Denoting by Γ the circle of radius $r_1 > r$ (with $r \ge 1$) and center 0, by the Cauchy's formulas it follows that for all $|z| \le r$ and $n \in \mathbb{N}$ we have

$$[S_n^{(\alpha,\beta)}(f)]^{(p)}(z) - f^{(p)}(z) = \frac{p!}{2\pi i} \int_{\Gamma} \frac{S_n^{(\alpha,\beta)}(f)(v) - f(v)}{(v-z)^{p+1}} dv,$$

where we have the inequality $|v - z| \ge r_1 - r$ valid for all $|z| \le r$ and $v \in \Gamma$.

As in the proof of Theorem 1.6.3 (keeping the notation for H), for all $v \in \Gamma$ and $n \in \mathbb{N}$ we have

$$S_n^{(\alpha,\beta)}(f)(v) - f(v) = \frac{1}{n+\beta}\{H(v)$$
$$+\frac{1}{n+\beta}\left[(n+\beta)^2\left(S_n^{(\alpha,\beta)}(f)(v) - f(v) + \frac{\beta v-\alpha}{n+\beta}f'(v) - \frac{nv(1-v)}{2(n+\beta)^2}f''(v)\right)\right.$$
$$\left.\left. - \frac{\beta v(1-v)}{2}f''(v)\right]\right\},$$

which replaced in the above Cauchy's formula implies

$$[S_n^{(\alpha,\beta)}(f)]^{(p)}(z) - f^{(p)}(z) = \frac{1}{n+\beta}\left\{H^{(p)}(z) + \frac{1}{n+\beta}\cdot\right.$$
$$\left[\frac{p!}{2\pi i}\int_\Gamma \frac{(n+\beta)^2\left(S_n^{(\alpha,\beta)}(f)(v) - f(v) + \frac{\beta v-\alpha}{n+\beta}f'(v) - \frac{nv(1-v)}{2(n+\beta)^2}f''(v)\right)}{(v-z)^{p+1}}dv\right.$$
$$\left.\left. - \frac{p!}{2\pi i}\int_\Gamma \frac{\beta v(1-v)}{2(v-z)^{p+1}}f''(v)dv\right]\right\}.$$

Passing now to absolute value, for all $|z| \le r$ and $n \in \mathbb{N}$ it follows

$$|[S_n^{(\alpha,\beta)}(f)]^{(p)}(z) - f^{(p)}(z)| \ge \frac{1}{n+\beta}\left\{|H^{(p)}(z)| - \frac{1}{n+\beta}\cdot\right.$$
$$\left[\left|\frac{p!}{2\pi i}\int_\Gamma \frac{(n+\beta)^2\left(S_n^{(\alpha,\beta)}(f)(v) - f(v) + \frac{\beta v-\alpha}{n+\beta}f'(v) - \frac{nv(1-v)}{2(n+\beta)^2}f''(v)\right)}{(v-z)^{p+1}}dv\right.\right.$$
$$\left.\left.\left. - \frac{p!}{2\pi i}\int_\Gamma \frac{\beta v(1-v)}{2(v-z)^{p+1}}f''(v)dv\right|\right]\right\},$$

where by using the Remark 2 after the proof of Theorem 1.6.2, for all $|z| \le r$ and $n \in \mathbb{N}$ we get

$$\left|\frac{p!}{2\pi i}\int_\Gamma \frac{(n+\beta)^2\left(S_n^{(\alpha,\beta)}(f)(v) - f(v) + \frac{\beta v-\alpha}{n+\beta}f'(v) - \frac{nv(1-v)}{2(n+\beta)^2}f''(v)\right)}{(v-z)^{p+1}}dv-\right.$$

$$\left.\frac{p!}{2\pi i}\int_\Gamma \frac{\beta v(1-v)}{2(v-z)^{p+1}}f''(v)dv\right| \le \frac{p!}{2\pi}\cdot\frac{2\pi r_1 M_{r_1}^{(\alpha,\beta)}}{(r_1-r)^{p+1}} + \frac{p!}{2\pi}\cdot\frac{2\pi r_1\beta r_1(1+r_1)\|f''\|_{r_1}}{2(r_1-r)^{p+1}}.$$

Denoting now $F_p(z) = H^{(p)}(z)$, we prove that $\|F_p\|_r > 0$. Indeed, if we suppose that $\|F_p\|_r = 0$ then it follows that f satisfies the differential equation

$$-\beta z f'(z) + \frac{z(1-z)}{2}f''(z) = Q_{p-1}(z), \forall |z| \le r,$$

where $Q_{p-1}(z)$ is a polynomial of degree $\le p-1$. Simplifying with z, making the substitution $y(z) = f'(z)$, searching $y(z)$ in the form $y(z) = \sum_{k=0}^{\infty} b_k z^k$ and then replacing in the differential equation, by simple calculations we easily obtain that

$b_k = 0$ for all $k \geq p-1$, that is $y(z)$ is a polynomial of degree $\leq p-2$. This implies the contradiction that f is a polynomial of degree $\leq p-1$.

Continuing exactly as in the proof of Theorem 1.6.3 (with $\|S_n^{(\alpha,\beta)}(f) - f\|_r$ replaced by $\|[S_n^{(\alpha,\beta)}(f)]^{(p)} - f^{(p)}\|_r$), finally there exists an index $n_0 \in \mathbb{N}$ depending on f, r, r_1 and p, such that for all $n \geq n_0$ we have

$$\|[S_n^{(\alpha,\beta)}(f)]^{(p)} - f^{(p)}\|_r \geq \frac{1}{n} \cdot \frac{C_0}{2}.$$

Also, the cases when $n \in \{1, 2, ..., n_0 - 1\}$ are similar with those in the proof of Theorem 1.6.3. □

Now defining the m-th iterates by ${}^m S_n^{(\alpha,\beta)}(f)(z)$, first we prove the following

Theorem 1.6.6. (Gal [80]) *Let $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ be with $R > 1$ and let us suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$, that is we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. Let $0 \leq \alpha \leq \beta$ and $1 \leq r < R$. Then, uniformly in $|z| \leq r$, $\forall n \in \mathbb{N}$, we have*

$$\lim_{m\to\infty} {}^m S_n^{(0,\beta)}(f)(z) = f(0), \quad \lim_{m\to\infty} {}^m S_n^{(\alpha,\alpha)}(f)(z) = f(1),$$

$\lim_{m\to\infty} {}^m S_n^{(\alpha,\beta)}(f)(z) = b_0$, *where b_0 is of the form*

$$b_0 = \sum_{j=1}^{n} d_j f((j+\alpha)/(n+\beta)), \text{ with } d_j \geq 0, j = 0, ..., n, \sum_{j=1}^{n} d_j = 1,$$

all the values $d_j, j = 0, ..., n$ being independent of f.

Proof. By Theorem 2 in Gonska-Piţul-Raşa [103], for any $n \in \mathbb{N}$ we have

$$\lim_{m\to\infty} {}^m S_n^{(0,\beta)}(f)(x) = f(0), \quad \lim_{m\to\infty} {}^m S_n^{(\alpha,\alpha)}(f)(x) = f(1),$$

$$\lim_{m\to\infty} {}^m S_n^{(\alpha,\beta)}(f)(x) = \sum_{j=1}^{n} d_j f[(j+\alpha)/(n+\beta)],$$

uniformly with respect to $x \in [0,1]$. From the classical Vitali's result, it suffices to show that for any fixed $n \in \mathbb{N}$, the sequence $({}^m S_n^{(\alpha,\beta)}(f)(z))_{m\in\mathbb{N}}$ is uniformly bounded for $|z| \leq r$.

We obviously have ${}^m S_n^{(\alpha,\beta)}(f)(z) = \sum_{k=0}^{\infty} c_k \cdot {}^m S_n^{(\alpha,\beta)}(e_k)(z)$.

But from the proof of Theorem 1.6.1 (both Cases 1) and 2)) it easily follows that $|S_n^{(\alpha,\beta)}(e_k)(z)| \leq r^k$, for all $k, n \in \mathbb{N}$, $|z| \leq r$, which implies

$$|{}^2 S_n^{(\alpha,\beta)}(e_k)(z)| = \left| \sum_{p=0}^{min\{n,k\}} C_{n,p,k}^{(\alpha,\beta)} S_n^{(\alpha,\beta)}(e_p)(z) \right| \leq r^k,$$

and by recurrence it easily follows $|{}^m S_n^{(\alpha,\beta)}(e_k)(z)| \leq r^k$, for all $m, k, n \in \mathbb{N}$.

This implies that

$$|{}^m S_n^{(\alpha,\beta)}(f)(z)| \leq \sum_{k=0}^{\infty} |c_k| \cdot |{}^m S_n^{(\alpha,\beta)}(e_k)(z)| \leq \sum_{k=0}^{\infty} |c_k| r^k < \infty,$$

for all $m, n \in \mathbb{N}$, which proves the theorem. □

Also, the following quantitative result holds.

Theorem 1.6.7. (Gal [80]) *Let* $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ *be with* $R > 1$ *and let us suppose that* $f : \mathbb{D}_R \to \mathbb{C}$ *is analytic in* $\mathbb{D}_R$*, that is we can write* $f(z) = \sum_{k=0}^{\infty} c_k z^k$*, for all* $z \in \mathbb{D}_R$*. Let* $0 \le \alpha \le \beta$ *and* $1 \le r < R$*. Then, for all* $|z| \le r$*, we have*

$$|{}^m S_n^{(\alpha,\beta)}(f)(z) - f(z)| \le \frac{2m}{n+\beta} \sum_{k=1}^{\infty} |c_k| \cdot [\beta k + k(k-1)] r^k.$$

Proof. We easily see that

$$|{}^m S_n^{(\alpha,\beta)}(f)(z) - f(z)| \le \sum_{k=1}^{\infty} |c_k| \cdot |{}^m S_n^{(\alpha,\beta)}(e_k)(z) - e_k(z)|.$$

We have two possibilities : 1) $0 \le k \le n$; 2) $k > n$.

Case 1). We successively get

$$\begin{aligned} S_n^{(\alpha,\beta)}(e_k)(z) - e_k(z) &= \sum_{j=1}^{k} C_{n,j,k}^{(\alpha,\beta)} e_j(z) - e_k(z) \\ &= [C_{n,k,k}^{(\alpha,\beta)} - 1] e_k(z) + \sum_{j=1}^{k-1} C_{n,j,k}^{(\alpha,\beta)} e_j(z), \end{aligned}$$

$$\begin{aligned} {}^p S_n^{(\alpha,\beta)}[S_n^{(\alpha,\beta)}(e_k)(z) - e_k(z)] = {} & [C_{n,k,k}^{(\alpha,\beta)} - 1] \cdot {}^p S_n^{(\alpha,\beta)}(e_k)(z) \\ & + \sum_{j=0}^{k-1} C_{n,j,k}^{(\alpha,\beta)} \cdot {}^p S_n^{(\alpha,\beta)}(e_j)(z), \end{aligned}$$

$$\begin{aligned} |{}^p S_n^{(\alpha,\beta)}[S_n^{(\alpha,\beta)}(e_k)(z) - e_k(z)]| \le {} & |1 - C_{n,k,k}^{(\alpha,\beta)}| \cdot |{}^p S_n^{(\alpha,\beta)}(e_k)(z)| \\ & + |1 - C_{n,k,k}^{(\alpha,\beta)}| \cdot \max_{j=0,\ldots,k-1} \{|{}^p S_n^{(\alpha,\beta)}(e_j)(z)|\}. \end{aligned}$$

But by the proofs of Theorems 1.6.1 and 1.6.6, for all $p, n, k \in \mathbb{N}$ we have

$$|1 - C_{n,k,k}^{(\alpha,\beta)}| \le \frac{1}{n+\beta} \left[\frac{k(k-1)}{2} + \beta k \right], |{}^p S_n^{(\alpha,\beta)}(e_k)(z)| \le r^k,$$

which implies

$$\begin{aligned} |{}^p S_n^{(\alpha,\beta)}[S_n^{(\alpha,\beta)}(e_k)(z) - e_k(z)]| &\le 2|1 - C_{n,k,k}^{(\beta)}| r^k \\ &= \frac{1}{n+\beta} [2\beta k + k(k-1)] r^k, \end{aligned}$$

and

$$\begin{aligned} |{}^m S_n^{(\alpha,\beta)}(e_k)(z) - e_k(z)| &= \left| \sum_{p=0}^{m-1} \cdot {}^p S_n^{(\alpha,\beta)}[S_n^{(\alpha,\beta)}(e_k)(z) - e_k(z)] \right| \\ &\le \frac{m}{n+\beta} [2\beta k + k(k-1)] r^k. \end{aligned}$$

Case 2). As in the proof of Theorem 1.6.1, Case 2), for all $k > n$ we get

$$|{}^mS_n^{(\alpha,\beta)}(e_k)(z) - e_k(z)| \le 2r^k \le \frac{2k(k-1)+2\beta k}{n+\beta}r^k.$$

As a conclusion, from both Cases 1) and 2), we obtain

$$\begin{aligned}
|{}^mS_n^{(\alpha,\beta)}(f)(z) - f(z)| &\le \sum_{k=1}^{\infty} |c_k| \cdot |{}^mS_n^{(\alpha,\beta)}(e_k)(z) - e_k(z)| \\
&= \sum_{k=1}^{n} |c_k| \cdot |{}^mS_n^{(\alpha,\beta)}(e_k)(z) - e_k(z)| \\
&\quad + \sum_{k=n+1}^{\infty} |c_k| \cdot |{}^mS_n^{(\alpha,\beta)}(e_k)(z) - e_k(z)| \\
&\le \sum_{k=1}^{n} |c_k| \frac{m}{n+\beta}[2\beta k + k(k-1)]r^k \\
&\quad + \sum_{k=n+1}^{\infty} |c_k| \frac{2\beta k + 2k(k-1)}{n+\beta} r^k \\
&\le \frac{2m}{n+\beta} \sum_{k=1}^{\infty} |c_k| \cdot [\beta k + k(k-1)]r^k,
\end{aligned}$$

which proves the theorem. □

Corollary 1.6.8. (Gal [80]) *Suppose* $\frac{m_n}{n} \to 0$ *when* $n \to \infty$. *Then*

$${}^{m_n}S_n^{(\alpha,\beta)}(f)(z) \to f(z),$$

uniformly with respect to $|z| \le r$, *for any* $1 \le r < R$.

Proof. It is immediate by passing to limit with $n \to \infty$ in Theorem 1.6.7. □

Remark. Theorem 1.6.7 and Corollary 1.6.8 are new even for the case of real functions of one real variable, since they are not covered by Gonska-Kacsó-Piţul [101], Gonska-Piţul-Raşa [103].

In what follows we present some similar properties for the complex Bernstein-Stancu polynomials $S_n^{<\gamma>}(f)(z)$. The first results concern the approximation properties.

Theorem 1.6.9. (Gal [82]) *Let* $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ *be with* $R > 1$ *and let us suppose that* $f : \mathbb{D}_R \to \mathbb{C}$ *is analytic in* $\mathbb{D}_R$, *that is we can write* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{D}_R$.

Let $0 \le \gamma$ *which can be dependent on* n *and* $1 \le r < R$. *Then, for all* $|z| \le r$ *and* $n \in \mathbb{N}$, *we have*

$$|S_n^{<\gamma>}(f)(z) - f(z)| \le M_{2,r,n}^{<\gamma>}(f),$$

where $0 < M_{2,r,n}^{<\gamma>}(f) = \frac{2}{n}\sum_{j=2}^{\infty} j(j-1)|c_j|r^j + \frac{\gamma(r+1)}{6r}\sum_{j=2}^{\infty} j(j-1)(2j-1)|c_j|r^j < \infty$.

Also, if $1 \le r < r_1 < R$, *then for all* $|z| \le r$ *and* $n, p \in \mathbb{N}$, *we have*

$$|[S_n^{<\gamma>}(f)]^{(p)}(z) - f^{(p)}(z)| \le \frac{M_{2,r_1,n}^{<\gamma>}(f)p!r_1}{(r_1 - r)^{p+1}}.$$

Proof. Since $S_n^{<\gamma>}(f)(z) = \sum_{k=0}^{\infty} c_k S_n^{<\gamma>}(e_k)(z)$, we get

$$|S_n^{<\gamma>}(f)(z) - f(z)| \le \sum_{k=0}^{\infty} |c_k| \cdot |S_n^{<\gamma>}(e_k)(z) - e_k(z)|.$$

To estimate $|S_n^{<\gamma>}(e_k)(z) - e_k(z)|$ for any fixed $n \in \mathbb{N}$, we will consider two possible cases : 1) $0 \le k \le n$; 2) $k > n$.

We will use the well-known representation (see Stancu [174])

$$S_n^{<\gamma>}(f)(z) = \sum_{p=0}^{n} \binom{n}{p} \frac{z(z+\gamma)...(z+(p-1)\gamma)}{(1+\gamma)...(1+(p-1)\gamma)} \Delta_{1/n}^p f(0).$$

Denoting

$$D_{n,p,k} = \binom{n}{p} \Delta_{1/n}^p e_k(0) = \binom{n}{p} [0, 1/n, ..., p/n; e_k](p!)/n^p,$$

since e_k is convex of any order, it follows that all $D_{n,p,k} \ge 0$ and

$$S_n^{<\gamma>}(e_k)(z) = \sum_{p=0}^{min\{n,k\}} D_{n,p,k} \frac{z(z+\gamma)...(z+(p-1)\gamma)}{(1+\gamma)...(1+(p-1)\gamma)}.$$

Also, since $S_n^{<\gamma>}(f)(1) = f(1)$, we get $\sum_{p=0}^{n} D_{n,p,k} = \sum_{p=0}^{min\{n,k\}} D_{n,p,k} = 1$.

Note that since for any $j = 0, 1, ...,$ we have $\frac{r+j\gamma}{1+j\gamma} \le r$, for all $0 \le p \le min\{n,k\} \le k$ and $|z| \le r$ we obtain

$$\frac{|z(z+\gamma)...(z+(p-1)\gamma)|}{(1+\gamma)...(1+(p-1)\gamma)} \le r\frac{r+\gamma}{1+\gamma} \cdot ... \frac{r+(p-1)\gamma}{1+(p-1)\gamma} \le r^p \le r^k,$$

which for all $|z| \le r$ and $n, k \in \mathbb{N}$, immediately implies

$$|S_n^{<\gamma>}(e_k)(z)| \le r^k \sum_{p=0}^{min\{n,k\}} D_{n,p,k} = r^k.$$

Case 1). If $k = 0$, then obviously we have $S_n^{<\gamma>}(e_k)(z) - e_k(z) = 0$. Therefore, let us suppose that $1 \le k \le n$. By using the representation in Stancu [174], we obtain

$$\begin{aligned}|S_n^{<\gamma>}(e_k)(z) \; - \; e_k(z)| & \\ \le & \left|\frac{n(n-1)...(n-(k-1))}{n^k} \cdot \frac{z(z+\gamma)...(z+(k-1)\gamma)}{(1+\gamma)...(1+(k-1)\gamma)} - z^k\right| \\ & + \sum_{p=0}^{k-1} D_{n,p,k} \left|\frac{z(z+\gamma)...(z+(p-1)\gamma)}{(1+\gamma)...(1+(p-1)\gamma)}\right| \\ := & E_{n,k}^{<\gamma>}(z) + F_{n,k}^{<\gamma>}(z).\end{aligned}$$

For $|z| \leq r$ it follows

$$F_{n,k}^{<\gamma>}(z) \leq r^k \sum_{p=0}^{k-1} D_{n,p,k} = r^k[1 - D_{n,k,k}]$$
$$= r^k[1 - \frac{n(n-1)...(n-(k-1))}{n^k}] \leq r^k \frac{k(k-1)}{2n}.$$

Here we have applied the inequality $1 - \Pi x_i \leq \sum(1 - x_i)$, with all $0 \leq x_i \leq 1$.

Also,

$$\begin{aligned} E_{n,k}^{<\gamma>}(z) \leq & \left| \frac{n(n-1)...(n-(k-1))}{n^k} \frac{z(z+\gamma)...(z+(k-1)\gamma)}{(1+\gamma)...(1+(k-1)\gamma)} \right. \\ & \left. - \frac{z(z+\gamma)...(z+(k-1)\gamma)}{(1+\gamma)...(1+(k-1)\gamma)} \right| + \left| \frac{z(z+\gamma)...(z+(k-1)\gamma)}{(1+\gamma)...(1+(k-1)\gamma)} - z^k \right| \\ \leq & \left| \frac{z(z+\gamma)...(z+(k-1)\gamma)}{(1+\gamma)...(1+(k-1)\gamma)} \right| \cdot \left| 1 - \frac{n(n-1)...(n-(k-1))}{n^k} \right| \\ & + \left| \frac{z(z+\gamma)...(z+(k-1)\gamma)}{(1+\gamma)...(1+(k-1)\gamma)} - z^k \right| \\ \leq & \, r^k \frac{k(k-1)}{2n} + \left| \frac{z(z+\gamma)...(z+(k-1)\gamma)}{(1+\gamma)...(1+(k-1)\gamma)} - z^k \right|. \end{aligned}$$

For any fixed $|z| \leq r$, let us denote $g_k(\alpha)(z) = \frac{z(z+\alpha)...(z+(k-1)\alpha)}{(1+\alpha)...(1+(k-1)\alpha)}$, where $\alpha \geq 0$. Then, by applying the mean value theorem, there is $\xi \in [0, \gamma]$ such that

$$\left| \frac{z(z+\gamma)...(z+(k-1)\gamma)}{(1+\gamma)...(1+(k-1)\gamma)} - z^k \right| = |g_k(\gamma)(z) - g_k(0)(z)| \leq \gamma \cdot \max \left| \frac{dg_k(\xi)(z)}{d\alpha} \right|.$$

But denoting $u_j(\alpha)(z) = \frac{z+j\alpha}{1+j\alpha}$, we have $g_k(\alpha)(z) = z\Pi_{j=1}^{k-1} u_j(\alpha)(z)$ and

$$\begin{aligned} \frac{dg_k(\alpha)(z)}{d\alpha} &= z \sum_{j=1}^{k-1} \left(\frac{z+j\alpha}{1+j\alpha} \right)'_\alpha \cdot \prod_{i=1, i\neq j}^{k-1} \frac{z+i\alpha}{1+i\alpha} \\ &= z \sum_{j=1}^{k-1} \frac{j(1-z)}{(1+j\alpha)^2} \prod_{i=1, i\neq j}^{k-1} \frac{z+i\alpha}{1+i\alpha}. \end{aligned}$$

Since $\frac{j}{(1+j\xi)^2} \leq j^2$, passing to modulus (for $0 \leq \xi \leq \gamma$ and $|z| \leq r$), we obtain

$$\left| \frac{dg_k(\xi)(z)}{d\alpha} \right| \leq r(r+1) \sum_{j=1}^{k-1} j^2 r^{k-2} = (r+1) r^{k-1} \frac{k(k-1)(2k-1)}{6}.$$

It follows

$$E_{n,k}^{<\gamma>}(z) \leq r^k \frac{k(k-1)}{2n} + \gamma(r+1) r^{k-1} \frac{k(k-1)(2k-1)}{6}.$$

Collecting all the above estimates, we get for all $|z| \leq r$

$$\begin{aligned} |S_n^{<\gamma>}(e_k)(z) - e_k(z)| \leq & \, r^k \frac{k(k-1)}{2n} + r^k \frac{k(k-1)}{2n} \\ & + \gamma(r+1) r^{k-1} \frac{k(k-1)(2k-1)}{6} \\ = & \, r^k \left[\frac{k(k-1)}{n} + \gamma \cdot \frac{r+1}{r} \cdot \frac{k(k-1)(2k-1)}{6} \right]. \end{aligned}$$

Case 2). We have

$$\begin{aligned}|S_n^{<\gamma>}(e_k)(z) - e_k(z)| &\le |S_n^{<\gamma>}(e_k)(z)| + |e_k(z)| \\ &\le \sum_{p=0}^{n} D_{n,p,k}\left|\frac{z(z+\gamma)...(z+(p-1)\gamma)}{(1+\gamma)...(1+(p-1)\gamma)}\right| + |e_k(z)|.\end{aligned}$$

Reasoning as in the above Case 1), we get

$$|S_n^{<\gamma>}(e_k)(z) - e_k(z)| \le r^n + r^k \le 2r^k \le \frac{2(k-1)k}{n} r^k.$$

Collecting all the results in the Cases 1) and 2), we immediately obtain for all $|z| < r$ and $k = 0, 1, 2, ...$,

$$|S_n^{<\gamma>}(e_k)(z) - e_k(z)| \le r^k \left[\frac{2k(k-1)}{n} + \gamma \cdot \frac{r+1}{r} \cdot \frac{k(k-1)(2k-1)}{6}\right],$$

which implies the corresponding estimate in statement.

For the simultaneous approximation, denoting by Γ the circle of radius $r_1 > r$ and center 0, since for any $|z| \le r$ and $v \in \Gamma$, we have $|v - z| \ge r_1 - r$, by the Cauchy's formulas it follows that for all $|z| \le r$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}|[S_n^{<\gamma>}(f)]^{(p)}(z) - f^{(p)}(z)| &= \frac{p!}{2\pi}\left|\int_{\Gamma} \frac{S_n^{<\gamma>}(f)(v) - f(v)}{(v-z)^{p+1}} dv\right| \\ &\le M_{2,r_1,n}^{<\gamma>}(f)\frac{p!}{2\pi}\frac{2\pi r_1}{(r_1 - r)^{p+1}} \\ &= M_{2,r_1,n}^{<\gamma>}(f)\frac{p! r_1}{(r_1 - r)^{p+1}}.\end{aligned}$$

□

Remark. For $\gamma = 0$ we get the results for the classical complex Bernstein polynomials in Theorem 1.1.2.

Now, defining the m-th iterates by ${}^m S_n^{<\gamma>}(f)(z)$, first we prove the following qualitative result.

Theorem 1.6.10. (Gal [82]) *Let $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ be with $R > 1$ and let us suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$, that is we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. Let $0 \le \gamma$. Uniformly in $|z| \le r$, where $1 \le r < R$, we have*

$$\lim_{m\to\infty} {}^m S_n^{<\gamma>}(f)(z) = (1-z)f(0) + zf(1), \forall n \in \mathbb{N}.$$

Proof. From Agratini-Rus [4], Remark 2 after Theorem 9, p. 165, for any $n \in \mathbb{N}$, we have $\lim_{m\to\infty} {}^m S_n^{<\gamma>}(f)(x) = (1-x)f(0) + xf(1)$, uniformly with respect to $x \in [0,1]$. From the classical Vitali's result, it suffices to show that for any fixed $n \in \mathbb{N}$, the sequence $({}^m S_n^{<\gamma>}(f)(z))_{m\in\mathbb{N}}$ is uniformly bounded for $|z| \le r$.

We have ${}^m S_n^{<\gamma>}(f)(z) = \sum_{k=0}^{\infty} c_k \cdot {}^m S_n^{<\gamma>}(e_k)(z)$. We will prove that for all $n, m, k \in \mathbb{N}$ and $|z| \le r$, we have $|{}^m S_n^{<\gamma>}(e_k)(z)| \le r^k$.

Indeed, for $m = 1$ it easily follows (also see the proof of Theorem 1.6.9) from the representation formula

$$S_n^{<\gamma>}(e_k)(z) = \sum_{j=0}^{n} D_{n,j,k} \frac{z(z+\gamma)...(z+(j-1)\gamma)}{(1+\gamma)...(1+(j-1)\gamma)}$$
$$= \sum_{j=0}^{min\{n,k\}} D_{n,j,k} \frac{z(z+\gamma)...(z+(j-1)\gamma)}{(1+\gamma)...(1+(j-1)\gamma)},$$

with $D_{n,j,k} \geq 0$ and $\sum_{j=0}^{n} D_{n,j,k} = \sum_{j=0}^{min\{n,k\}} D_{n,j,k} = 1$.

Denote $h_j(z) = z(z+\gamma)...(z+(j-1)\gamma) = \sum_{i=0}^{j} c_i^{(j)} e_i(z)$, where $c_i^{(j)} \geq 0$, $c_j^{(j)} = 1$ and $\sum_{i=0}^{j} c_i^{(j)} = h_j(1) = (1+\gamma)...(1+(j-1)\gamma)$.

By the linearity of $S_n^{<\gamma>}$, we get

$$|^2 S_n^{<\gamma>}(e_k)| = \left| \sum_{j=0}^{min\{n,k\}} D_{n,j,k} \frac{1}{(1+\gamma)...(1+(j-1)\gamma)} \cdot \sum_{i=0}^{j} c_i^{(j)} S_n^{<\gamma>}(e_i)(z) \right|$$
$$\leq \sum_{j=0}^{min\{n,k\}} D_{n,j,k} \frac{1}{(1+\gamma)...(1+(j-1)\gamma)} \cdot \sum_{i=0}^{j} c_i^{(j)} r^j \leq r^k,$$

and by mathematical induction it easily follows that for all $n, m, k \in \mathbb{N}$ we have

$$|^m S_n^{<\gamma>}(e_k)(z)| \leq r^k, \text{ for all } |z| \leq r.$$

This implies that

$$|^m S_n^{<\gamma>}(f)(z)| \leq \sum_{k=0}^{\infty} |c_k| \cdot |^m S_n^{<\gamma>}(e_k)(z)| \leq \sum_{k=0}^{\infty} |c_k| r^k < \infty,$$

for all $m, n \in \mathbb{N}$, which proves the theorem. □

The following quantitative result is not correspondent to the above qualitative one.

Theorem 1.6.11. (Gal [82]) *Let $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ be with $R > 1$ and let us suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$, that is we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$.*

Let $0 \leq \gamma$, $1 \leq r < R$ and $D_{n,k,k} = \frac{n(n-1)...(n-(k-1))}{n^k}$. Then, for all $|z| \leq r$ we have

$$|^m S_n^{<\gamma>}(f)(z) - f(z)|$$
$$\leq m \sum_{k=2}^{\infty} |c_k| \left[\frac{2k(k-1)}{n} + \left(1 - \frac{D_{n,k,k}}{(1+\gamma)...(1+(k-1)\gamma)}\right) + \gamma(k-1)^2 \right] r^k.$$

Proof. From the proof of Theorem 1.6.10, it follows that for all $n, m, k \in \mathbb{N}$ and $|z| \leq r$, we have $|^m S_n^{<\gamma>}(e_k)(z)| \leq r^k$. Also

$$|^m S_n^{<\gamma>}(f)(z) - f(z)| \leq \sum_{k=2}^{\infty} |c_k| \cdot |^m S_n^{<\gamma>}(e_k)(z) - e_k(z)|.$$

We have two possibilities : 1) $2 \le k \le n$; 2) $k > n$.

Case 1). With the notations for $g_j(\alpha)(z)$ in the proof of Theorem 1.6.9 and for $h_j(z), c_i^{(j)}$ in the proof of Theorem 1.6.10, we can write

$$\begin{aligned}
&|{}^m S_n^{<\gamma>}(e_k)(z) - e_k(z)| \\
&= \left|\sum_{p=0}^{m-1} {}^p S_n^{<\gamma>}[S_n^{<\gamma>}(e_k)(z) - e_k(z)]\right| \\
&= \left|\sum_{p=0}^{m-1} {}^p S_n^{<\gamma>}\left[\sum_{j=1}^{k} D_{n,j,k}\cdot g_j(\gamma)(z) - e_k(z)\right]\right| \\
&= \left|\sum_{p=0}^{m-1} \left[\sum_{j=1}^{k} D_{n,j,k}\cdot {}^p S_n^{<\gamma>}(g_j(\gamma))(z) -^p S_n^{<\gamma>}(e_k)(z)\right]\right| \\
&\le \sum_{p=0}^{m-1}\sum_{j=1}^{k-1} D_{n,j,k}|{}^p S_n^{<\gamma>}(g_j(\gamma))(z)| \\
&\quad + \sum_{p=0}^{m-1} |D_{n,k,k} \cdot^p S_n^{<\gamma>}(g_k(\gamma))(z) -^p S_n^{<\gamma>}(e_k)(z)| \\
&= \sum_{p=0}^{m-1}\sum_{j=1}^{k-1} D_{n,j,k}|{}^p S_n^{<\gamma>}(g_j(\gamma))(z)| \\
&\quad + \sum_{p=0}^{m-1} \left|\frac{D_{n,k,k}}{(1+\gamma)...(1+(k-1)\gamma)} \cdot^p S_n^{<\gamma>}[\sum_{i=0}^{k} c_i^{(k)} e_i(z)] -^p S_n^{<\gamma>}(e_k)(z)\right| \\
&\le \sum_{p=0}^{m-1}\sum_{j=1}^{k-1} D_{n,j,k}|{}^p S_n^{<\gamma>}(g_j(\gamma))(z)| \\
&\quad + \sum_{p=0}^{m-1} \left|\frac{D_{n,k,k}}{(1+\gamma)...(1+(k-1)\gamma)} \cdot^p S_n^{<\gamma>}(e_k)(z) -^p S_n^{<\gamma>}(e_k)(z)\right| \\
&\quad + \sum_{p=0}^{m-1} \frac{D_{n,k,k}}{(1+\gamma)...(1+(k-1)\gamma)} \left|\sum_{i=0}^{k-1} c_i^{(k)} \cdot^p S_n^{<\gamma>}(e_i)(z)\right| \\
&:= T_1 + T_2 + T_3.
\end{aligned}$$

Reasoning exactly as in the proof of Theorem 1.6.10, we easily get for all j, p and $|z| \le r$ that

$$|{}^p S_n^{<\gamma>}(g_j(\gamma))(z)| \le r^j.$$

Taking into account the formula for $1 - D_{n,k,k}$ in the proof of Theorem 1.6.9, we get

$$T_1 \le \sum_{p=0}^{m-1}\sum_{j=1}^{k-1} r^k D_{n,j,k} = mr^k[1 - D_{n,k,k}] \le mr^k \frac{k(k-1)}{2n}.$$

Also,

$$T_2 = \sum_{p=0}^{m-1} |{}^p S_n^{<\gamma>}(e_k)(z)| \left[1 - \frac{D_{n,k,k}}{(1+\gamma)...(1+(k-1)\gamma)}\right]$$
$$\leq mr^k \left[1 - \frac{D_{n,k,k}}{(1+\gamma)...(1+(k-1)\gamma)}\right].$$

Finally,

$$T_3 \leq \sum_{p=0}^{m-1} \frac{D_{n,k,k}}{(1+\gamma)...(1+(k-1)\gamma)}[(1+\gamma)...(1+(k-1)\gamma) - 1]r^k$$
$$= mr^k D_{n,k,k} \left[1 - \frac{1}{(1+\gamma)...(1+(k-1)\gamma)}\right].$$

But, taking into account the inequalities $D_{n,k,k} \leq 1$ and

$$1 - \Pi_{j=1}^{k-1} x_j \leq \sum_{j=1}^{k-1}(1 - x_j), 0 \leq x_j \leq 1, j = 1, ..., k-1,$$

applied for $x_j = \frac{1}{1+j\gamma}$, we obtain

$$D_{n,k,k} \left[1 - \frac{1}{(1+\gamma)...(1+(k-1)\gamma)}\right] \leq \sum_{j=1}^{k-1}[1 - 1/(1+j\gamma)] = \sum_{j=1}^{k-1} \frac{j\gamma}{1+j\gamma}$$
$$\leq (k-1) \cdot \frac{\gamma(k-1)}{1+\gamma(k-1)} \leq \gamma(k-1)^2.$$

Collecting all these inequalities, we obtain

$$|{}^m S_n^{<\gamma>}(e_k)(z) - e_k(z)|$$
$$\leq mr^k \left[\frac{k(k-1)}{2n} + \left(1 - \frac{D_{n,k,k}}{(1+\gamma)...(1+(k-1)\gamma)}\right) + \gamma(k-1)^2\right].$$

Case 2). We get

$$|{}^m S_n^{<\gamma>}(e_k)(z) - e_k(z)| \leq |{}^m S_n^{<\gamma>}(e_k)(z)| + |e_k(z)|$$
$$\leq 2r^k \leq \frac{2k(k-1)}{n} r^k.$$

As a conclusion, from both Cases 1) and 2), we obtain

$$|{}^m S_n^{<\gamma>}(f)(z) - f(z)|$$
$$\leq \sum_{k=2}^{\infty} |c_k| \cdot |{}^m S_n^{<\gamma>}(e_k)(z) - e_k(z)|$$
$$= \sum_{k=2}^{n} |c_k| \cdot |{}^m S_n^{<\gamma>}(e_k)(z) - e_k(z)| + \sum_{k=n+1}^{\infty} |c_k| \cdot |{}^m S_n^{<\gamma>}(e_k)(z) - e_k(z)|$$
$$\leq \sum_{k=2}^{n} |c_k| mr^k \left[\frac{k(k-1)}{2n} + \left(1 - \frac{D_{n,k,k}}{(1+\gamma)...(1+(k-1)\gamma)}\right) + \gamma(k-1)^2\right]$$
$$+ \sum_{k=n+1}^{\infty} |c_k| r^k \frac{2k(k-1)}{n}$$
$$\leq m \sum_{k=2}^{\infty} |c_k| r^k \left[\frac{2k(k-1)}{n} + \left(1 - \frac{D_{n,k,k}}{(1+\gamma)...(1+(k-1)\gamma)}\right) + \gamma(k-1)^2\right] r^k,$$

which proves the theorem. □

Remark. For $\gamma = 0$ we get some results for classical complex Bernstein polynomials in Section 1.2.

Corollary 1.6.12. (Gal [82]) *(i) Let $1 \leq r < R$. For $\gamma := \gamma_n = 1/n$ and $|z| \leq r$ we have the estimate*

$$|^m S_n^{<\gamma_n>}(f)(z) - f(z)| \leq \frac{m}{n} \sum_{k=2}^{\infty} |c_k| \left[2k(k-1) + 2(k-1)^3 + (k-1)^2\right] r^k.$$

(ii) If $\gamma := \gamma_n = 1/n$ and $\frac{m_n}{n} \to 0$ as $n \to \infty$, then $^{m_n} S_n^{<\gamma_n>}(f)(z) \to f(z)$, uniformly with respect $|z| \leq r$.

Proof. (i) Taking $\gamma = 1/n$ we obtain for all $k \geq 2$

$$\begin{aligned} 1 - \frac{D_{n,k,k}}{(1+\gamma)...(1+(k-1)\gamma)} &= 1 - \Pi_{j=1}^{k-1} \frac{n-j}{n+j} \\ &\leq \sum_{j=1}^{k-1} \left[1 - \frac{n-j}{n+j}\right] = 2 \sum_{j=1}^{k-1} \frac{j}{j+n} \\ &\leq 2(k-1) \frac{k-1}{n+(k-1)} \leq 2 \frac{(k-1)^3}{n}, \end{aligned}$$

which replaced in Theorem 1.6.11 gives

$$|^m S_n^{<\gamma_n>}(f)(z) - f(z)| \leq \frac{m}{n} \sum_{k=2}^{\infty} |c_k| \left[2k(k-1) + 2(k-1)^3 + (k-1)^2\right] r^k.$$

(ii) It is immediate by passing to limit with $n \to \infty$ in the estimate proved in (i). □

Remark. The results in Theorem 1.6.11 and Corollary 1.6.12 are new even for the case of real functions of one real variable, since they are not covered by those in Gonska-Kacsó-Piţul [101] or Gonska-Piţul-Raşa [103], whose estimates one refer to the difference $|^m L_n(f)(x) - B_1(f)(x)|$, with $B_1(f)(x) = f(0) + [f(1) - f(0)]x$ and $^m L_n(f)$ representing the mth iterate of the positive linear operator $L_n(f)$.

Finally we present the geometric properties of $S_n^{<\gamma>}(f)(z)$.

Theorem 1.6.13. (Gal [82]) *Let us suppose that $G \subset \mathbb{C}$ is open, such that $\overline{\mathbb{D}}_1 \subset G$ and $f : G \to \mathbb{C}$ is analytic in G. Also, let us consider $(S_n^{<\gamma(n)>}(f)(z))_{n \in \mathbb{N}}$, where we suppose that $lim_{n\to\infty}\gamma(n) = 0$.*

If $f(0) = f'(0) - 1 = 0$ and f is starlike (convex, spirallike of type η, respectively) in $\overline{\mathbb{D}}_1$, that is for all $z \in \overline{\mathbb{D}}_1$ (see e.g. Mocanu, P. T., Bulboacă, T. and Sălăgean [138])

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \left(Re\left(\frac{zf''(z)}{f'(z)}\right) + 1 > 0, Re\left(e^{i\eta}\frac{zf'(z)}{f(z)}\right) > 0, resp.\right),$$

then there exists an index n_0 *depending on* f *(and on* η *for spirallikeness), such that for all* $n \geq n_0$, $S_n^{<\gamma(n)>}(f)(z)$, *are starlike (convex, spirallike of type* η*, respectively) in* $\overline{\mathbb{D}}_1$.

If $f(0) = f'(0) - 1 = 0$ *and* f *is starlike (convex, spirallike of type* η*, respectively) only in* $\mathbb{D}_1$ *(that is the corresponding inequalities hold only in* $\mathbb{D}_1$*), then for any disk of radius* $0 < r < 1$ *and center* 0 *denoted by* $\mathbb{D}_r$*, there exists an index* $n_0 = n_0(f, \mathbb{D}_r)$ *(*n_0 *depends on* η *too in the case of spirallikeness), such that for all* $n \geq n_0$, $S_n^{<\gamma(n)>}(f)(z)$*, are starlike (convex, spirallike of type* η*, respectively) in* $\overline{\mathbb{D}}_r$ *(that is, the corresponding inequalities hold in* $\overline{\mathbb{D}}_r$*).*

Proof. By Theorem 1.6.9 it follows that we have $S_n^{<\gamma(n)>}(f)(z) \to f(z)$, uniformly for $|z| \leq 1$, which by the well-known Weierstrass's theorem implies $[S_n^{<\gamma(n)>}(f)]'(z) \to f'(z)$ and $[S_n^{<\gamma(n)>}(f)]''(z) \to f''(z)$, for $n \to \infty$, uniformly in $\overline{\mathbb{D}}_1$. In all what follows, denote $P_n(f)(z) = \frac{S_n^{<\gamma(n)>}(f)(z)}{[S_n^{<\gamma(n)>}(f)]'(0)}$, well defined for sufficiently large n. We easily get $P_n(f)(0) = 0$, $P_n'(f)(0) = 1$ for sufficiently large n, and $P_n(f)(z) \to f(z)$, $P_n'(f)(z) \to f'(z)$ and $P_n''(f)(z) \to f''(z)$, uniformly in $\overline{\mathbb{D}}_1$.

Suppose first that f is starlike in $\overline{\mathbb{D}}_1$. Then, by hypothesis we get $|f(z)| > 0$ for all $z \in \overline{\mathbb{D}}_1$ with $z \neq 0$, which from the univalence of f in $\mathbb{D}_1$, implies that we can write $f(z) = zg(z)$, with $g(z) \neq 0$, for all $z \in \overline{\mathbb{D}}_1$, where g is analytic in $\mathbb{D}_1$ and continuous in $\overline{\mathbb{D}}_1$.

Writing $P_n(f)(z)$ in the form $P_n(f)(z) = zQ_n(f)(z)$, obviously $Q_n(f)(z)$ is a polynomial of degree $\leq n-1$. Also, for $|z| = 1$ we have

$$|f(z) - P_n(f)(z)| = |z| \cdot |g(z) - Q_n(f)(z)| = |g(z) - Q_n(f)(z)|,$$

which by the uniform convergence in $\overline{\mathbb{D}}_1$ of $P_n(f)$ to f and by the maximum modulus principle, implies the uniform convergence in $\overline{\mathbb{D}}_1$ of $Q_n(f)(z)$ to $g(z)$.

Since g is continuous in $\overline{\mathbb{D}}_1$ and $|g(z)| > 0$ for all $z \in \overline{\mathbb{D}}_1$, there exist an index $n_1 \in \mathbb{N}$ and $a > 0$ depending on g, such that $|Q_n(f)(z)| > a > 0$, for all $z \in \overline{\mathbb{D}}_1$ and all $n \geq n_0$. Also, for all $|z| = 1$, we have

$$\begin{aligned}|f'(z) - P_n'(f)(z)| &= |z[g'(z) - Q_n'(f)(z)] + [g(z) - Q_n(f)(z)]| \\ &\geq |\quad |z| \cdot |g'(z) - Q_n'(f)(z)| - |g(z) - Q_n(f)(z)| \quad | \\ &= |\quad |g'(z) - Q_n'(f)(z)| - |g(z) - Q_n(f)(z)| \quad |,\end{aligned}$$

which from the maximum modulus principle, the uniform convergence of $P_n'(f)$ to f' and of $Q_n(f)$ to g, evidently implies the uniform convergence of $Q_n'(f)$ to g'.

Then, for $|z| = 1$, we get

$$\begin{aligned}\frac{zP_n'(f)(z)}{P_n(f)} &= \frac{z[zQ_n'(f)(z) + Q_n(f)(z)]}{zQ_n(f)(z)} \\ &= \frac{zQ_n'(f)(z) + Q_n(f)(z)}{Q_n(f)(z)} \to \frac{zg'(z) + g(z)}{g(z)} \\ &= \frac{f'(z)}{g(z)} = \frac{zf'(z)}{f(z)},\end{aligned}$$

which again from the maximum modulus principle, implies

$$\frac{zP_n'(f)(z)}{P_n(f)} \to \frac{zf'(z)}{f(z)}, \text{ uniformly in } \overline{\mathbb{D}}_1.$$

Since $Re\left(\frac{zf'(z)}{f(z)}\right)$ is continuous in $\overline{\mathbb{D}}_1$, there exists $\varepsilon \in (0,1)$, such that

$$Re\left(\frac{zf'(z)}{f(z)}\right) \geq \varepsilon, \text{ for all } z \in \overline{\mathbb{D}}_1.$$

Therefore

$$Re\left[\frac{zP_n'(f)(z)}{P_n(f)(z)}\right] \to Re\left[\frac{zf'(z)}{f(z)}\right] \geq \varepsilon > 0$$

uniformly on $\overline{\mathbb{D}}_1$, i.e. for any $0 < \rho < \varepsilon$, there is n_0 such that for all $n \geq n_0$ we have

$$Re\left[\frac{zP_n'(f)(z)}{P_n(f)(z)}\right] > \rho > 0, \text{ for all } z \in \overline{\mathbb{D}}_1.$$

Since $P_n(f)(z)$ differs from $S_n^{<\gamma(n)>}(f)(z)$ only by a constant, this proves the starlikeness of $S_n^{<\gamma(n)>}(f)(z)$, for sufficiently large n.

If f is supposed to be starlike only in $\mathbb{D}_1$, the proof is identical, with the only difference that instead of $\overline{\mathbb{D}}_1$, we reason for $\overline{\mathbb{D}}_r$.

The proofs in the cases when f is convex or spirallike of order η are similar and follow from the following uniform convergences (on $\overline{\mathbb{D}}_1$ or on $\overline{\mathbb{D}}_r$)

$$Re\left[\frac{zP_n''(f)(z)}{P_n'(f)(z)}\right] + 1 \to Re\left[\frac{zf''(z)}{f'(z)}\right] + 1$$

and

$$Re\left[e^{i\eta}\frac{zP_n'(f)(z)}{P_n(f)(z)}\right] \to Re\left[e^{i\eta}\frac{zf'(z)}{f(z)}\right]. \qquad \square$$

Remark. If f is univalent in $\overline{\mathbb{D}}_1$, then from the uniform convergence in Theorem 1.6.9 and a well-known result in complex analysis, concerning sequences of analytic functions converging locally uniformly to an univalent function, it is immediate that for sufficiently large n, the complex polynomials $S_n^{<\gamma(n)>}(f)(z)$ (where $\gamma(n) \to 0$, for $n \to \infty$), must be univalent in $\overline{\mathbb{D}}_1$.

At the end of this section we will extend the Bernstein-Stancu polynomials $S_n^{(\alpha,\beta)}(f)(z)$ and some of their approximation results to compact subsets $G \subset \mathbb{C}$.

For this purpose, in what follows $G \subset \mathbb{C}$ we will be considered a compact set such that $\tilde{\mathbb{C}} \setminus G$ is connected. In this case, according to the Riemann Mapping Theorem, a unique conformal mapping Ψ of $\tilde{\mathbb{C}} \setminus \overline{\mathbb{D}}_1$ onto $\tilde{\mathbb{C}} \setminus G$ exists so that $\Psi(\infty) = \infty$ and $\Psi'(\infty) > 0$.

By using the Faber polynomials $F_p(z)$ attached to G (see Definition 1.0.10), for $f \in A(\overline{G})$ and $0 \leq \alpha \leq \beta$ we can introduce the Bernstein-Stancu-Faber polynomials given by the formula

$$\mathcal{S}_n^{(\alpha,\beta)}(f;\overline{G})(z) = \sum_{p=0}^{n} \binom{n}{p} \Delta^p_{1/(n+\beta)} F(\alpha/(n+\beta)) \cdot F_p(z), z \in G, \ n \in \mathbb{N},$$

where $0 \le \alpha \le \beta$, $F(w) = \frac{1}{2\pi i}\int_{|u|=1} \frac{f(\Psi(u))}{u-w}du$, $w \in \mathbb{D}_1$, and

$$\Delta_h^p F(\alpha/(n+\beta)) = \sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k}F(\alpha/(n+\beta)+kh).$$

Here, in the case when $\alpha = \beta$ since $F(1)$ is involved in $\Delta^n_{1/(n+\beta)}F(\alpha/(n+\beta))$ and therefore in the definition of $\mathcal{S}_n^{(\alpha,\beta)}(f;G)(z)$ too, in addition we will suppose that F can be extended by continuity on the boundary $\partial\mathbb{D}_1$.

Remarks. 1) For $G = \overline{\mathbb{D}}_1$ it is easy to see that the above Bernstein-Stancu-Faber polynomials one reduce to the classical complex Bernstein-Stancu polynomials given by

$$\begin{aligned} S_n^{(\alpha,\beta)}(f)(z) &= \sum_{p=0}^{n}\binom{n}{p}\Delta^p_{1/(n+\beta)}f(\alpha/(n+\beta))z^p \\ &= \sum_{p=0}^{n}\binom{n}{p}z^p(1-z)^{n-p}f[(p+\alpha)/(n+\beta)]. \end{aligned}$$

2) It is known that, for example, $\int_0^1 \frac{\omega_p(f\circ\Psi;u)_{\partial\mathbb{D}_1}}{u}du < \infty$ is a sufficient condition for the continuity on $\partial\mathbb{D}_1$ of F in the above definition of the Bernstein-Stancu-Faber polynomials (see e.g. Gaier [76], p. 52, Theorem 6). Here $p \in \mathbb{N}$ is arbitrary fixed.

3) In the case when $\alpha = \beta = 0$, $\mathcal{S}_n^{(\alpha,\beta)}(f;\overline{G})(z)$ becomes $\mathcal{B}_n(f;\overline{G})(z)$.

The first main result one refers to approximation on compact sets without any restriction on their boundaries and can be stated as follows.

Theorem 1.6.14. *Let G be a continuum (that is a connected compact subset of $\mathbb{C}$) and suppose that f is analytic in G, that is there exists $R > 1$ such that f is analytic in G_R. Here recall that G_R denotes the interior of the closed level curve Γ_R given by $\Gamma_R = \{z; |\Phi(z)| = R\} = \{\Psi(w); |w| = R\}$ (and that $G \subset \overline{G}_r$ for all $1 < r < R$). Also, we suppose that F given in the definition of Bernstein-Stancu-Faber polynomials can be extended by continuity on $\partial\mathbb{D}_1$.*

For any $1 < r < R$ the following estimate

$$|\mathcal{S}_n^{(\alpha,\beta)}(f;\overline{G})(z) - f(z)| \le \frac{C}{n},\ z \in \overline{G}_r,\ n \in \mathbb{N},$$

holds, where $C > 0$ depends on f, α, β, r and G_r but it is independent of n.

Proof. First we note that since G is a continuum then it follows that $\tilde{\mathbb{C}} \setminus G$ is simply connected. By the proof of Theorem 2, p. 52 in Suetin [186], for any fixed $1 < \eta < R$ we have $f(z) = \sum_{k=0}^{\infty} a_k(f)F_k(z)$ uniformly in $\overline{G}_\eta$, where $a_k(f)$ are the Faber coefficients and are given by $a_k(f) = \frac{1}{2\pi i}\int_{|u|=\eta}\frac{f[\Psi(u)]}{u^{k+1}}du$. Note here that $G \subset \overline{G}_\eta$.

First we will prove that

$$\mathcal{S}_n^{(\alpha,\beta)}(f;\overline{G})(z) = \sum_{k=0}^{\infty} a_k(f)\mathcal{S}_n^{(\alpha,\beta)}(F_k;\overline{G})(z),$$

APPROXIMATION BY COMPLEX BERNSTEIN AND CONVOLUTION TYPE OPERATORS

for all $z \in G$. (Note here that by hypothesis we have $\overline{G} = G$). For this purpose, denote $f_m(z) = \sum_{k=0}^{m} a_k(f)F_k(z)$, $m \in \mathbb{N}$.

Since by the linearity of $\mathcal{S}_n(\alpha, \beta)$ we easily get

$$\mathcal{S}_n^{(\alpha,\beta)}(f_m; \overline{G})(z) = \sum_{k=0}^{m} a_k(f)\mathcal{S}_n^{(\alpha,\beta)}(F_k; \overline{G})(z), \text{ for all } z \in G,$$

it suffices to prove that $\lim_{m\to\infty} \mathcal{S}_n^{(\alpha,\beta)}(f_m; \overline{G})(z) = \mathcal{S}_n^{(\alpha,\beta)}(f; \overline{G})(z)$, for all $z \in G$ and $n \in \mathbb{N}$.

First we have

$$\mathcal{S}_n^{(\alpha,\beta)}(f_m; \overline{G})(z) = \sum_{p=0}^{n} \binom{n}{p} \Delta^p_{1/(n+\beta)} G_m(\alpha/(n+\beta))F_k(z),$$

where $G_m(w) = \frac{1}{2\pi i} \int_{|u|=1} \frac{f_m(\Psi(u))}{u-w} du$ and $F(w) = \frac{1}{2\pi i} \int_{|u|=1} \frac{f(\Psi(u))}{u-w} du$.

Note here that since by Gaier [76], p. 48, first relation before (6.17), we have

$$\mathcal{F}_k(w) = \frac{1}{2\pi i} \int_{|u|=1} \frac{F_k(\Psi(u))}{u-w} du = w^k, \text{ for all } |w| < 1,$$

evidently that $\mathcal{F}_k(w)$ can be extended by continuity on $\partial\mathbb{D}_1$. This also immediately implies that $G_m(w) = \frac{1}{2\pi i} \int_{|u|=1} \frac{f_m(\Psi(u))}{u-w} du$ can be extended by continuity on $\partial\mathbb{D}_1$, which means that $\mathcal{S}_n^{(\alpha,\beta)}(F_k; G)(z)$ and $\mathcal{S}_n^{(\alpha,\beta)}(f_m; G)(z)$ are well defined.

Now, taking into account the Cauchy's theorem we also can write

$$G_m(w) = \frac{1}{2\pi i} \int_{|u|=\eta} \frac{f_m(\Psi(u))}{u-w} du \text{ and } F(w) = \frac{1}{2\pi i} \int_{|u|=\eta} \frac{f(\Psi(u))}{u-w} du.$$

For all $n, m \in \mathbb{N}$ and $z \in G$ it follows

$$\begin{aligned}
&|\mathcal{S}_n^{(\alpha,\beta)}(f_m; \overline{G})(z) - \mathcal{S}_n^{(\alpha,\beta)}(f; \overline{G})(z)| \\
&\leq \sum_{p=0}^{n} \binom{n}{p} |\Delta^p_{1/(n+\beta)}(G_m - F)(\alpha/(n+\beta))| \cdot |F_k(z)| \\
&\leq \sum_{p=0}^{n} \binom{n}{p} \sum_{j=0}^{p} \binom{p}{j} |(G_m - F)(\alpha/(n+\beta) + (p-j)/(n+\beta))| \cdot |F_k(z)| \\
&\leq \sum_{p=0}^{n} \binom{n}{p} \sum_{j=0}^{p} \binom{p}{j} C_{j,p,\eta,\alpha,\beta} \|f_m - f\|_{\overline{G}_\eta} \cdot |F_k(z)| \\
&\leq M_{n,p,\eta,\alpha,\beta,G_\eta} \|f_m - f\|_{\overline{G}_\eta},
\end{aligned}$$

which by $\lim_{m\to\infty} \|f_m - f\|_{\overline{G}_\eta} = 0$ (see e.g. the proof of Theorem 2, p. 52 in Suetin [186]) implies the desired conclusion. Here $\|f_m - f\|_{\overline{G}_\eta}$ denotes the uniform norm of $f_m - f$ on $\overline{G}_\eta$.

Consequently we obtain

$$|\mathcal{S}_n^{(\alpha,\beta)}(f;\overline{G})(z)-f(z)| \le \sum_{k=0}^{\infty}|a_k(f)|\cdot|\mathcal{S}_n^{(\alpha,\beta)}(F_k;\overline{G})(z)-F_k(z)|$$
$$=\sum_{k=0}^{n}|a_k(f)|\cdot|\mathcal{S}_n^{(\alpha,\beta)}(F_k;\overline{G})(z)-F_k(z)|$$
$$+\sum_{k=n+1}^{\infty}|a_k(f)|\cdot|\mathcal{S}_n^{(\alpha,\beta)}(F_k;\overline{G})(z)-F_k(z)|.$$

Therefore it remains to estimate $|a_k(f)|\cdot|\mathcal{S}_n^{(\alpha,\beta)}(F_k;\overline{G})(z)-F_k(z)|$, firstly for all $0\le k\le n$ and secondly for $k\ge n+1$, where

$$\mathcal{S}_n^{(\alpha,\beta)}(F_k;\overline{G})(z)=\sum_{p=0}^{n}\binom{n}{p}[\Delta^p_{1/(n+\beta)}\mathcal{F}_k(\alpha/(n+\beta))]\cdot F_p(z).$$

First it is useful to observe that by Gaier [76], p. 48, combined with the Cauchy's theorem, for any fixed $1<\eta<R$ we have

$$\mathcal{F}_k(w):=\frac{1}{2\pi i}\int_{|u|=\eta}\frac{F_k[\Psi(u)]}{u-w}du=w^k=e_k(w),\ \text{for all } |w|<\eta.$$

Denote

$$D_{n,p,k}^{(\alpha,\beta)}=\binom{n}{p}\Delta^p_{1/(n+\beta)}e_k(\alpha/(n+\beta))$$

$$=\binom{n}{p}[\alpha/(n+\beta),(\alpha+1)/(n+\beta),...,(\alpha+p)/(n+\beta);e_k]\cdot(p!)/(n+\beta)^p.$$

It follows

$$\mathcal{S}_n^{(\alpha,\beta)}(F_k;\overline{G})(z)=\sum_{p=0}^{n}D_{n,p,k}^{(\alpha,\beta)}\cdot F_p(z).$$

Since $S_n^{(\alpha,\beta)}(f)(1)=f[(n+\alpha)/(n+\beta)]$ and since each e_k is convex of any order, it follows $D_{n,p,k}^{(\alpha,\beta)}>0$ and, by taking $f(z)=e_k(z)$ we get $\sum_{p=0}^n D_{n,p,k}^{(\alpha,\beta)}=\frac{(n+\alpha)^k}{(n+\beta)^k}\le 1$, for all k and n.

In the estimation of $|a_k(f)|\cdot|\mathcal{S}_n^{(\alpha,\beta)}(F_k;\overline{G})(z)-F_k(z)|$ we distinguish two cases : 1) $0\le k\le n$; 2) $k>n$.

Case 1. We have

$$|\mathcal{S}_n^{(\alpha,\beta)}(F_k;\overline{G})(z)-F_k(z)|\le|F_k(z)|\cdot|1-D_{n,k,k}^{(\alpha,\beta)}|+\sum_{p=0}^{k-1}D_{n,p,k}^{(\alpha,\beta)}\cdot|F_p(z)|.$$

Fix now $1<r<\eta$. By the inequality (13), p. 44 in Suetin [186] we have

$$|F_p(z)|\le C(r)r^p,\ \text{for all } z\in\overline{G}_r,\ p\ge 0,$$

which immediately implies

$$\begin{aligned}
&|\mathcal{S}_n^{(\alpha,\beta)}(F_k;\overline{G})(z)-F_k(z)|\\
\leq\ &\left[1-\frac{n(n-1)...(n-(k-1))}{(n+\beta)^k}\right]C(r)r^k\\
&+\left[\frac{(n+\alpha)^k}{(n+\beta)^k}-\frac{n(n-1)...(n-(k-1))}{(n+\beta)^k}\right]C(r)r^k\\
=\ &2\left[1-\frac{n(n-1)...(n-(k-1))}{(n+\beta)^k}\right]C(r)r^k+\left[\frac{(n+\alpha)^k}{(n+\beta)^k}-1\right]C(r)r^k\\
\leq\ &2\left[1-\frac{n(n-1)...(n-(k-1))}{(n+\beta)^k}\right]C(r)r^k\leq\frac{1}{n+\beta}\left[2\beta k+k(k-1)\right]C(r)r^k.
\end{aligned}$$

Here we used the inequality $1-\Pi_{i=1}^k x_i\leq\sum_{i=1}^k(1-x_i)$, valid if all $x_i\in[0,1]$. Also by the above formula for $a_k(f)$ we easily obtain $|a_k(f)|\leq\frac{C(\eta,f)}{\eta^k}$, for all $k\geq 0$. Note that $C(r),C(\eta,f)>0$ are constants independent of k.

For all $z\in\overline{G}_r$ and $k=0,1,2,...n$ it follows

$$|a_k(f)|\cdot|\mathcal{S}_n^{(\alpha,\beta)}(F_k;\overline{G})(z)-F_k(z)|\leq\frac{C(r,\eta,f)}{n+\beta}\left[2\beta k+k(k-1)\right]\cdot\left[\frac{r}{\eta}\right]^k,$$

that is

$$\sum_{k=0}^n|a_k(f)|\cdot|\mathcal{S}_n^{(\alpha,\beta)}(F_k;\overline{G})(z)-F_k(z)|\leq\frac{C(r,\eta,f)}{n+\beta}\sum_{k=1}^n[2\beta k+k(k-1)]d^k,$$

for all $z\in\overline{G}_r$, where $0<d=r/\eta<1$.

Also, clearly we have $\sum_{k=1}^n[2\beta k+k(k-1)]d^k\leq\sum_{k=1}^\infty[2\beta k+]k(k-1)d^k<\infty$ which finally implies that

$$\sum_{k=0}^n|a_k(f)|\cdot|\mathcal{S}_n^{(\alpha,\beta)}(F_k;\overline{G})(z)-F_k(z)|\leq\frac{C^*(r,\eta,\beta,f)}{n}.$$

Case 2. We have

$$\begin{aligned}
\sum_{k=n+1}^\infty|a_k(f)|\cdot|\mathcal{S}_n^{(\alpha,\beta)}(F_k;\overline{G})(z)-F_k(z)|\leq&\sum_{k=n+1}^\infty|a_k(f)|\cdot|\mathcal{S}_n^{(\alpha,\beta)}(F_k;\overline{G})(z)|\\
&+\sum_{k=n+1}^\infty|a_k(f)|\cdot|F_k(z)|.
\end{aligned}$$

By the estimates mentioned in the case 1), we immediately get

$$\sum_{k=n+1}^\infty|a_k(f)|\cdot|F_k(z)|\leq C(r,\eta,f)\sum_{k=n+1}^\infty d^k,\ \text{for all } z\in\overline{G}_r,$$

with $d=r/\beta$.

Also,

$$\sum_{k=n+1}^{\infty}|a_k(f)|\cdot|\mathcal{S}_n^{(\alpha,\beta)}(F_k;\overline{G})(z)| = \sum_{k=n+1}^{\infty}|a_k(f)|\cdot\left|\sum_{p=0}^{n}D_{n,p,k}^{(\alpha,\beta)}\cdot F_p(z)\right|$$
$$\leq \sum_{k=n+1}^{\infty}|a_k(f)|\cdot\sum_{p=0}^{n}D_{n,p,k}^{(\alpha,\beta)}\cdot|F_p(z)|.$$

But for $p \leq n < k$ and taking into account the estimates obtained in the Case 1) we get

$$|a_k(f)|\cdot|F_p(z)| \leq C(r,\eta,f)\frac{r^p}{\eta^k} \leq C(r,\beta,f)\frac{r^k}{\eta^k},\ \text{for all } z \in \overline{G}_r,$$

which implies

$$\sum_{k=n+1}^{\infty}|a_k(f)|\cdot|\mathcal{S}_n^{(\alpha,\beta)}(F_k;\overline{G})(z) - F_k(z)| \leq C(r,\eta,f)\sum_{k=n+1}^{\infty}\sum_{p=0}^{n}D_{n,p,k}^{(\alpha,\beta)}\left[\frac{r}{\beta}\right]^k$$
$$= C(r,\beta,f)\sum_{k=n+1}^{\infty}\left[\frac{r}{\beta}\right]^k$$
$$= C(r,\beta,f)\frac{d^{n+1}}{1-d},$$

with $d = r/\beta$.

In conclusion, collecting the estimates in the Cases 1) and 2) we obtain

$$|\mathcal{S}_n^{(\alpha,\beta)}(f;\overline{G})(z) - f(z)| \leq \frac{C_1}{n+\beta} + C_2 d^{n+1} \leq \frac{C}{n},\ z \in \overline{G}_r,\ n \in \mathbb{N}.$$

This proves the theorem. □

Remark. In the case when $\alpha = \beta = 0$ we recapture Theorem 1.1.8.

As a consequence of the Remarks 1 and 2 after the proof of Theorem 1.1.8, we can state the following result.

Theorem 1.6.15. *Let G be a compact Faber set such that $\tilde{\mathbb{C}}\setminus G$ is simply connected. If f is analytic on G, that is there exists $R > 1$ such that f is analytic in G_R and if f is not a polynomial of degree ≤ 1, then for any $1 < r < R$ we have*

$$\|\mathcal{S}_n^{(\alpha,\beta)}(f;\overline{G}) - f\|_{\overline{G}_r} \sim \frac{1}{n},\ n \in \mathbb{N},$$

where the constants in the equivalence depend on f, α, β, r and G_r but are independent of n. Here $\|f\|_{\overline{G}_r} = \sup_{z\in\overline{G}_r}|f(z)|$.

Proof. According to Remark 2 after the proof of Theorem 1.1.8, there exists g analytic in $\overline{\mathbb{D}}_r$ such that $f = T(g)$, that is $g = T^{-1}(f)$ (therefore F can be extended by continuity on $\partial\mathbb{D}_1$. By hypothesis on f it follows that f cannot be of the form $f(z) = c_0F_0(z)+c_1F_1(z)$ where F_0 and F_1 are the Faber polynomials of degree 0 and

1 respectively and $c_0, c_1 \in \mathbb{C}$. This immediately implies that g is not a polynomial of degree ≤ 1.

First we have $S_n^{(\alpha,\beta)}(T^{-1}(f)) = T^{-1}[\mathcal{S}_n^{(\alpha,\beta)}(f;\overline{G})]$. Indeed,

$$\begin{aligned} S_n^{(\alpha,\beta)}(T^{-1}(f))(z) &= \sum_{p=0}^{n} \binom{n}{p} \Delta^p_{1/(n+\beta)} T^{-1}(f)(\alpha/(n+\beta)) z^p \\ &= \sum_{p=0}^{n} \binom{n}{p} \Delta^p_{1/(n+\beta)} F(\alpha/(n+\beta)) z^p, \end{aligned}$$

since $T^{-1}(f)(\xi) = \frac{1}{2\pi i}\int_{|w|=1} \frac{f[\Psi(w)]}{w-\xi} dw = F(\xi)$, and

$$\begin{aligned} &T^{-1}[\mathcal{S}_n^{(\alpha,\beta)}(f;\overline{G})](z) \\ &= \frac{1}{2\pi i}\int_{|w|=1} \frac{\mathcal{S}_n^{(\alpha,\beta)}(f;\overline{G})[\Psi(w)]}{w-z} dw \\ &= \sum_{p=0}^{n} \binom{n}{p} \Delta^p_{1/(n+\beta)} F(\alpha/(n+\beta)) \frac{1}{2\pi i}\int_{|w|=1} \frac{F_p[\Psi(w)]}{w-z} dw \\ &= \sum_{p=0}^{n} \binom{n}{p} \Delta^p_{1/(n+\beta)} F(\alpha/(n+\beta)) z^p, \end{aligned}$$

since according to Gaier [76], p. 48, first relation before (6.17), we have

$$\frac{1}{2\pi i}\int_{|w|=1} \frac{F_p[\Psi(w)]}{w-z} dw = z^p.$$

Then by Theorem 1.6.3 (see also Corollary 1.6.4) and by the linearity and continuity of T^{-1} we get

$$\begin{aligned} \frac{C}{n} \leq \|S_n^{(\alpha,\beta)}(g) - g\|_r &= \|S_n^{(\alpha,\beta)}(g) - T^{-1}(f)\|_r \\ &= \|T^{-1}[\mathcal{S}_n^{(\alpha,\beta)}(f;\overline{G})] - T^{-1}(f)\|_r \\ &\leq \||T^{-1}\|| \cdot \|\mathcal{S}_n^{(\alpha,\beta)}(f;\overline{G}) - f\|_{\overline{G}_r} \\ &\leq M\|\mathcal{S}_n^{(\alpha,\beta)}(f;\overline{G}) - f\|_{\overline{G}_r}, \end{aligned}$$

which proves the lower estimate.

On the other hand we have $T[S_n^{(\alpha,\beta)}(g)] = \mathcal{S}_n^{(\alpha,\beta)}(T(g);\overline{G})$. Indeed,

$$T[S_n^{(\alpha,\beta)}(g)](z) = \sum_{p=0}^{n} \Delta^p_{1/(n+\beta)} g(\alpha/(n+\beta)) F_p(z),$$

and

$$\mathcal{S}_n^{(\alpha,\beta)}(T(g);\overline{G})(z) = \sum_{p=0}^{n} \binom{n}{p} \Delta^p_{1/(n+\beta)} H(\alpha/(n+\beta)) F_p(z),$$

where according to Gaier [76], p. 49. relation (6.17') we have

$$H(w) = \frac{1}{2\pi i}\int_{|u|=1} \frac{T(g)[\Psi(u)]}{u-w} du = g(w).$$

Therefore by the same Corollary 1.6.4 and by the linearity and continuity of T we obtain

$$\begin{aligned}\|\mathcal{S}_n^{(\alpha,\beta)}(f;\overline{G}) - f\|_{\overline{G}_r} &= \|\mathcal{S}_n^{(\alpha,\beta)}(T(g);\overline{G}) - T(g)\|_{\overline{G}_r}\\ &= \|T[S_n^{(\alpha,\beta)}(g)] - T(g)\|_{\overline{G}_r}\\ &\leq |||T||| \cdot \|S_n^{(\alpha,\beta)}(g) - g\|_r \leq \frac{C}{n},\end{aligned}$$

which proves the upper estimate and the theorem. □

Remark. For $\alpha = \beta = 0$, we recapture Theorem 1.1.9.

1.7 Bernstein-Kantorovich Type Polynomials

In this section we extend some results in the previous sections to the following complex Kantorovich variants of these polynomials defined (for the case of real variable) by Kantorovich [112]

$$K_n(f)(z) = (n+1)\sum_{k=0}^{n} p_{n,k}(z)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt,$$

and (for the case of real variable) Bărbosu [36]

$$K_n^{(\alpha,\beta)}(f)(z) = (n+1+\beta)\sum_{k=0}^{n} p_{n,k}(z)\int_{(k+\alpha)/(n+1+\beta)}^{(k+1+\alpha)/(n+1+\beta)} f(t)dt.$$

For our purpose will be useful the results expressed by the following.

Theorem 1.7.1. *Let* $F(z) = \int_0^z f(t)dt$.

(i) (see e.g. Lorentz [125], p. 30) Denoting by $B_n(f)(z)$ *the Bernstein polynomials, we have*

$$K_n(f)(z) = B'_{n+1}(F)(z), z \in \mathbb{C}.$$

(ii) (Gal [91]) Denoting $S_n^{(\alpha,\beta)}(f)(z) = \sum_{k=0}^{n}\binom{n}{k}z^k(1-z)^{n-k}f[(k+\alpha)/(n+\beta)], z \in \mathbb{C}$, *the Bernstein-Stancu polynomials studied in Section 1.6, where* $0 \leq \alpha \leq \beta$ *are independent of* n, *we have*

$$K_n^{(\alpha,\beta)}(f)(z) = \frac{n+1+\beta}{n+1}\left[S_{n+1}^{(\alpha,\beta)}(F)\right]'(z), z \in \mathbb{C}.$$

Proof. (ii) It is immediate by the formula

$$\begin{aligned}&[S_{n+1}^{(\alpha,\beta)}(F)]'(z)\\ &= (n+1+\beta)\sum_{k=0}^{n} p_{n,k}(z)\left[F\left(\frac{k+\alpha+1}{n+\beta+1}\right) - F\left(\frac{k+\alpha}{n+1+\beta}\right)\right]\\ &\quad -\beta\sum_{k=0}^{n} p_{n,k}(z)\left[F\left(\frac{k+\alpha+1}{n+\beta+1}\right) - F\left(\frac{k+\alpha}{n+1+\beta}\right)\right]\\ &= K_n^{(\alpha,\beta)}(f)(z) - \frac{\beta}{n+1+\beta}K_n^{(\alpha,\beta)}(f)(z).\end{aligned}$$

□

Now, as a consequence of Theorem 1.7.1, (i) and Theorem 1.1.6, we immediately get the following.

Corollary 1.7.2. (Gal [91]) *Let $f : \mathbb{D}_R \to \mathbb{C}$ be analytic in $\mathbb{D}_R$ with $R > 1$ and $1 \leq r < R$.*

(i) If f is not a polynomial of degree ≤ 0 then for all $n \in \mathbb{N}$ we have

$$\|K_n(f) - f\|_r \sim \frac{1}{n},$$

where the constants in the equivalence depend only on f and r.

(ii) If f is not a polynomial of degree $\leq \max\{1, p-1\}$ then for all $p, n \in \mathbb{N}$ we have

$$\|K_n^{(p)}(f) - f^{(p)}\|_r \sim \frac{1}{n},$$

with the constants in the equivalence depending only on f, r and p.

Proof. We combine Theorem 1.7.1, (i) with Theorem 1.1.6.

(i) We get

$$\|K_n(f) - f\|_r = \|B'_{n+1}(F) - F'\|_r \sim \frac{1}{n+1},$$

if F is not a polynomial of degree $\leq \max\{1, 1\} = 1$, which ends the proof.

(ii) We obtain

$$\|K_n^{(p)}(f) - f^{(p)}\|_r = \|B_{n+1}^{(p+1)}(F) - F^{(p+1)}\|_r \sim \frac{1}{n+1},$$

if F is not a polynomial of degree $\leq \max\{1, p\} = p$, which ends the proof. □

As a consequence of Theorems 1.7.1, (ii) and Theorem 1.6.5, we also get the following.

Corollary 1.7.3. (Gal [91]) *Let $f : \mathbb{D}_R \to \mathbb{C}$ be analytic in $\mathbb{D}_R$ with $R > 1$, $1 \leq r < R$ and $0 \leq \alpha \leq \beta$, $\alpha + \beta > 0$.*

(i) If f is not identical 0, then for all $n \in \mathbb{N}$ we have

$$\|K_n^{(\alpha,\beta)}(f) - f\|_r \sim \frac{1}{n+\beta},$$

where the constants in the equivalence depend only on f, r, α and β.

(ii) If f is not a polynomial of degree $\leq p-1$ then for all $p, n \in \mathbb{N}$ we have

$$\|[K_n^{(\alpha,\beta)}(f)]^{(p)} - f^{(p)}\|_r \sim \frac{1}{n},$$

with the constants in the equivalence depending only on f, r, α, β and p.

Proof. We combine Theorem 1.7.1, (ii) with Theorem 1.6.5.

(i) We get

$$\|K_n^{(\alpha,\beta)}(f) - f\|_r = \|[S_{n+1}^{(\alpha,\beta)}(F)]' - F'\|_r \sim \frac{1}{n+\beta},$$

if F is not a polynomial of degree ≤ 0, which ends the proof.

(ii) We obtain

$$\|[K_n^{(\alpha,\beta)}(f)]^{(p)} - f^{(p)}\|_r = \|[S_{n+1}^{(\alpha,\beta)}(F)]^{(p+1)} - F^{(p+1)}\|_r \sim \frac{1}{n+\beta},$$

if F is not a polynomial of degree $\leq p$, which ends the proof. □

Upper estimates with explicit constants in approximation by these kind of polynomials and in Voronovskaja-type formula can be derived as follows. First we consider the case of $K_n(f)(z)$ polynomials.

Theorem 1.7.4. (Gal [91]) *Let* $f : \mathbb{D}_R \to \mathbb{C}$ *be analytic in* $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ *with* $R > 1$, *i.e.* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{D}_R$. *Suppose* $1 \le r < r_1 < R$. *Then for all* $|z| \le r$ *and* $n, p \in \mathbb{N}$, *we have :*

(i)

$$|K_n^{(p)}(f)(z) - f^{(p)}(z)| \le \frac{C_{2,r_1}(f)(p+1)!r_1}{(n+1)(r_1-r)^{p+2}},$$

where $0 < C_{2,r_1}(f) = 2\sum_{j=2}^{\infty}(j-1)|c_{j-1}|r_1^j < \infty$ *;*

(ii)

$$\begin{aligned}|K_n(f)(z) - f(z) - \frac{1-2z}{2(n+1)} \cdot f'(z) - \frac{z(1-z)}{2(n+1)} \cdot f''(z)| \\ \le C_{r_1,n+1}(f) \cdot \frac{r_1}{(r_1-r)^2},\end{aligned}$$

where

$$C_{r_1,n}(f) = \frac{5(1+r_1)^2}{2n} \cdot \frac{\sum_{k=3}^{\infty}|c_{k-1}|(k-1)(k-2)^2 r_1^{k-2}}{n}.$$

Proof. (i) Combining Theorem 1.7.1, (i) with Theorem 1.1.2, (ii), we obtain

$$|K_n^{(p)}(f)(z) - f^{(p)}(z)| = |B_{n+1}^{(p+1)}(F)(z) - F^{(p+1)}(z)| \le \frac{M_{2,r_1}(F)(p+1)!r_1}{(n+1)(r_1-r)^{p+2}},$$

where $0 < M_{2,r_1}(F) = 2\sum_{j=2}^{\infty} j(j-1)|C_j|r_1^j < \infty$ and $F(z) = \sum_{k=0}^{\infty} C_k z^k$, $z \in \mathbb{D}_R$. But we also get

$$F(z) = \int_0^z [\sum_{k=0}^{\infty} c_k t^k]dt = \sum_{k=0}^{\infty} \frac{c_k}{k+1} z^{k+1} = \sum_{k=1}^{\infty} \frac{c_{k-1}}{k} z^k,$$

which implies $C_k = \frac{c_{k-1}}{k}$ and $C_{2,r_1}(f) = 2\sum_{j=2}^{\infty}(j-1)|c_{j-1}|r_1^j$.

(ii) Replacing in Theorem 1.1.3, (ii), n by $n+1$, r by r_1 and f by F, for all $|z| \le r_1$ and $n \in \mathbb{N}$, we obtain

$$\left|B_{n+1}(F)(z) - F(z) - \frac{z(1-z)}{2(n+1)}F''(z)\right| \le \frac{5(1+r_1)^2}{2(n+1)} \cdot \frac{M_{r_1}(F)}{n+1},$$

where

$$\begin{aligned}M_{r_1}(F) &= \sum_{k=3}^{\infty}|C_k|k(k-1)(k-2)^2 r_1^{k-2} = \sum_{k=3}^{\infty}|c_{k-1}|(k-1)(k-2)^2 r_1^{k-2} \\ &:= A_{r_1}(f).\end{aligned}$$

Here again we wrote $F(z) = \sum_{k=0}^{\infty} C_k z^k$, for all $z \in \mathbb{D}_R$.

Now, denoting $C_{r_1,n}(f) = \frac{5(1+r_1)^2}{2n} \cdot \frac{A_{r_1}(f)}{n}$, by Γ the circle of radius $r_1 > r$ and center 0, and $E_n(F)(z) = B_{n+1}(F)(z) - F(z) - \frac{z(1-z)}{2(n+1)}F''(z)$, since for any $|z| \le r$ and $v \in \Gamma$, we have $|v - z| \ge r_1 - r$, by the Cauchy's formula it follows that for all $|z| \le r$ and $n \in \mathbb{N}$, we obtain

$$\begin{aligned}|E'_n(F)(z)| &= \frac{1}{2\pi}\left|\int_\Gamma \frac{E_n(f)(z)}{(v-z)^2}dv\right| \le C_{r_1,n+1}(f)\frac{1}{2\pi}\frac{2\pi r_1}{(r_1-1)^2} \\ &= C_{r_1,n+1}(f) \cdot \frac{r_1}{(r_1-r)^2}.\end{aligned}$$

But by Theorem 1.7.1, (i) we obtain

$$E'_n(F)(z) = K_n(f)(z) - f(z) - \frac{1-2z}{2(n+1)} \cdot f'(z) - \frac{z(1-z)}{2(n+1)} \cdot f''(z),$$

which proves the theorem. □

In the case of $K_n^{(\alpha,\beta)}(f)(z)$ polynomials we have the following.

Theorem 1.7.5. (Gal [91]) *Let* $f : \mathbb{D}_R \to \mathbb{C}$ *be analytic in* $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ *with* $R > 1$, *i.e.* $f(z) = \sum_{k=0}^\infty c_k z^k$, *for all* $z \in \mathbb{D}_R$. *Suppose* $1 \le r < r_1 < R$. *Then for all* $|z| \le r$ *and* $n, p \in \mathbb{N}$, *we have :*

(i)

$$|[K_n^{(\alpha,\beta)}(f)]^{(p)}(z) - f^{(p)}(z)| \le \frac{C_{2,r_1}^{(\beta)}(f)(p+1)!r_1}{(n+1)(r_1-r)^{p+2}} + \frac{\beta}{n+1}\|f\|_r,$$

where $0 < C_{2,r_1}^{(\beta)}(f) = 2\sum_{j=2}^\infty (j-1)|c_{j-1}|r_1^j + 2\beta\sum_{j=1}^\infty |c_{j-1}|r_1^j < \infty$ *;*

(ii)

$$\left|K_n^{(\alpha,\beta)}(f)(z) - f(z) + \left(\frac{\beta z - \alpha}{n+1} - \frac{1-2z}{2(n+\beta+1)}\right)f'(z)\right.$$

$$\left. - \frac{z(1-z)}{2(n+\beta+1)}f''(z)\right| \le \frac{C(f, r_1, \alpha, \beta)}{(n+1)(n+\beta+1)} \cdot \frac{r_1}{(r_1-r)^2},$$

where $C(f, r_1, \alpha, \beta)$ *is a positive constant depending only on* f, r_1, α *and* β.

Proof. (i) Combining Theorem 1.7.1, (ii) with Theorem 1.6.1, for all $|z| \le r$ we obtain

$$|[K_n^{(\alpha,\beta)}(f)]^{(p)}(z) - f^{(p)}(z)| = \left|\frac{n+1+\beta}{n+1}[S_{n+1}^{(\alpha,\beta)}(F)]^{(p+1)}(z) - F^{(p+1)}(z)\right|$$

$$\le \left|\frac{n+1+\beta}{n+1}|[S_{n+1}^{(\alpha,\beta)}(F)]^{(p+1)}(z) - F^{(p+1)}(z)| + \frac{\beta}{n+1}|F^{(p+1)}(z)|\right|$$

$$\le \frac{n+1+\beta}{n+1} \cdot \frac{M_{2,r_1}^{(\beta)}(F)(p+1)!r_1}{(n+\beta+1)(r_1-r)^{p+2}} + \frac{\beta}{n+1} \cdot |f^{(p)}(z)|$$

$$\leq \frac{M_{2,r_1}^{(\beta)}(F)(p+1)!r_1}{(n+1)(r_1-r)^{p+2}} + \frac{\beta}{n+1} \cdot \|f^{(p)}\|_r,$$

where $\|f^{(p)}\|_r = \sup\{|f^{(p)}(z)|; |z| \leq r\}$ and reasoning exactly as in the proof of Theorem 1.7.4, (i), we get

$$\begin{aligned} M_{2,r_1}^{(\beta)}(F) &= 2\sum_{j=2}^{\infty} j(j-1)|C_j|r_1^j + 2\beta\sum_{j=1}^{\infty} j|C_j|r_1^j \\ &= 2\sum_{j=2}^{\infty}(j-1)|c_{j-1}|r_1^j + 2\beta\sum_{j=1}^{\infty}|c_{j-1}|r_1^j := C_{2,r_1}^{(\beta)}(f). \end{aligned}$$

(ii) Replacing in Remark 2 after Theorem 1.6.2 n by $n+1$, r by r_1 and f by F, for all $|z| \leq r_1$ and $n \in \mathbb{N}$, we obtain

$$\left| S_{n+1}^{(\alpha,\beta)}(F)(z) - F(z) + \frac{\beta z - \alpha}{n+\beta+1}F'(z) - \frac{(n+1)z(1-z)}{2(n+\beta+1)^2}F''(z) \right|$$

$$\leq \frac{C(f, r_1, \alpha, \beta)}{(n+\beta+1)^2},$$

where the positive constant $C(f, r_1, \alpha, \beta)$ depends only on f, r, α and β. Let us denote

$$E_n(F)(z) = S_{n+1}^{(\alpha,\beta)}(F)(z) - F(z) + \frac{\beta z - \alpha}{n+\beta+1}F'(z) - \frac{(n+1)z(1-z)}{2(n+\beta+1)^2}F''(z).$$

If Γ is the circle of radius $r_1 > r$ and center 0, and since for any $|z| \leq r$ and $v \in \Gamma$, we have $|v - z| \geq r_1 - r$, by the Cauchy's formula it follows that for all $|z| \leq r$ and $n \in \mathbb{N}$, we obtain as in the proof of Theorem 1.7.4, (ii)

$$|E_n'(F)(z)| \leq C(f, r_1, \alpha, \beta) \cdot \frac{r_1}{(r_1-r)^2} \cdot \frac{1}{(n+\beta+1)^2}.$$

But by Theorem 1.7.1, (ii) we obtain

$$\begin{aligned} E_n'(F)(z) \;=\; & \frac{n+1}{n+1+\beta}K_n^{(\alpha,\beta)}(f)(z) - f(z) + \frac{1}{n+\beta+1}[(\beta z - \alpha)f(z)]' \\ & - \frac{n+1}{2(n+\beta+1)^2}[(z-z^2)f'(z)]' = \frac{n+1}{n+\beta+1} \\ & \cdot \left[K_n^{(\alpha,\beta)}(f)(z) - f(z) + f'(z)\left(\frac{\beta z - \alpha}{n+1} - \frac{1-2z}{2(n+\beta+1)}\right) f'(z) \right. \\ & \left. - \frac{z(1-z)}{2(n+\beta+1)}f''(z) \right], \end{aligned}$$

which immediately proves the theorem. □

In what follows we will prove an equivalence result for approximation in Voronovskaja's theorem in the case of complex Bernstein-Kantorovich polynomials $K_n(f)(z)$, analogous to that for complex Bernstein polynomials contained by Corollary 1.3.4.

Theorem 1.7.6. *Let $R > 1$ and let $f : \mathbb{D}_R \to \mathbb{C}$ be an analytic function, say $f(z) = \sum_{k=0}^{\infty} c_k z^k$. If f is not a polynomial of degree $\leq 2p-1$ then for any $1 \leq r < R$ and any natural number p we have*

$$\left\| K_n(f) - f - \sum_{j=1}^{2p} \frac{(n+1)^{-j}}{j!}[f^{(j)}T_{n+1,j} + f^{(j-1)}T'_{n+1,j}] \right\|_r \sim \frac{1}{n^{p+1}}, \ n \in \mathbb{N},$$

where the constants in the equivalence depend only on f, r and p and are independent of n. Recall here that $T_{n+1,j}(z) = \sum_{i=0}^{n+1}(i-nz)^j \binom{n+1}{i} z^i(1-z)^{n+1-i}$.

Proof. Denoting $F(z) = \int_0^z f(t)dt$, by Theorem 1.3.2 we get

$$\left\| B_{n+1}(F) - F - \sum_{j=1}^{2p} \frac{F^{(j)}}{j!}(n+1)^{-j}T_{n+1,j} \right\|_r \leq \frac{C_{p,r}(f)}{(n+1)^{p+1}}.$$

Let $1 \leq r < r_1 < R$ and denote by γ the circle of radius $r_1 > r$ and center 0 and

$$H(z) = B_{n+1}(F)(z) - F(z) - \sum_{j=1}^{2p} \frac{F^{(j)}(z)}{j!}(n+1)^{-j}T_{n+1,j}(z).$$

Since for any $|z| \leq r$ and $v \in \gamma$ we have $|v - z| \geq r_1 - r$, by the Cauchy's formula we get $H'(z) = \frac{1}{2\pi i}\int_\gamma \frac{H(v)}{(v-z)^2}dv$, which implies

$$\|H'\|_r \leq \frac{\|H\|_{r_1}}{2\pi} \cdot \frac{2\pi r_1}{(r_1-r)^2} \leq \frac{C_{p,r}(F)r_1}{(r_1-r)^2} \cdot \frac{1}{(n+1)^{p+1}} \leq \frac{C_{p,r,r_1}(F)}{n^{p+1}},$$

which is exactly the upper estimate in the statement of Theorem 1.7.6. Note that if $f(z) = \sum_{k=0}^{\infty} c_k z^k$ then

$$F(z) = \sum_{k=0}^{\infty} \frac{c_k}{k+1} z^{k+1} = \sum_{j=1}^{\infty} \frac{c_{j-1}}{j} z^j := \sum_{j=1}^{\infty} C_j^* z^j,$$

where $C_j^* = \frac{c_{j-1}}{j}$, for all $j = 1, 2, ...,$.

So it remains to prove the lower estimate. In the proof of Corollary 1.3.4, write the first identity for $B_{n+1}(F)$ and F and then take the first derivative. It follows

$$K_n(f)(z) - f(z) - \sum_{j=1}^{2p} \frac{(n+1)^{-j}}{j!}[f^{(j)}(z)T_{n+1,j} + f^{j-1}(z)T'_{n+1,j}(z)]$$
$$= \frac{1}{(n+1)^{p+1}} \left\{ \frac{T'_{n+1,2p+1}(z)}{(n+1)^p(2p+1)!}F^{(2p+1)}(z) + \frac{T_{n+1,2p+1}(z)}{(n+1)^p(2p+1)!}F^{(2p+2)}(z) \right.$$
$$+ \frac{T'_{n+1,2p+2}(z)}{(n+1)^{p+1}(2p+2)!}F^{(2p+2)}(z) + \frac{T_{n+1,2p+2}(z)}{(n+1)^{p+1}(2p+2)!}F^{(2p+3)}(z)$$
$$\left. + \frac{1}{n+1}\left[(n+1)^{p+2}\sum_{k=2p+3}^{\infty} C_k^* E'_{k,n+1,p+1}(z)\right] \right\}.$$

Here $E_{k,n+1,p}(z)$ does not depend on F and it is a polynomial of degree $\le k$ in z. By the Bernstein's inequality and by Lemma 1.3.3 it follows

$$\|E'_{k,n+1,p+1}\|_r \le \frac{k}{r}\|E_{k,n+1,p+1}\|_r \le C_p r^{k-1}\frac{(k+1)!(k-2(p+1))^2}{(n+1)^{p+2}(k-2(p+1))!}.$$

By the same Lemma 1.3.3 we get that $(n+1)^{p+2}\sum_{k=2p+3}^{\infty} C_k^* E'_{k,n+1,p+1}(z)$ is bounded in $\mathbb{D}_r$ by a constant independent of n.

It remains to deal with the expression

$$A := \frac{T'_{n+1,2p+1}(z)}{(n+1)^p(2p+1)!}F^{(2p+1)}(z) + \frac{T_{n+1,2p+1}(z)}{(n+1)^p(2p+1)!}F^{(2p+2)}(z)$$
$$+ \frac{T'_{n+1,2p+2}(z)}{(n+1)^{p+1}(2p+2)!}F^{(2p+2)}(z) + \frac{T_{n+1,2p+2}(z)}{(n+1)^{p+1}(2p+2)!}F^{(2p+3)}(z).$$

By using the recurrence formula in Lorentz [125], p. 14, relation (3)

$$T_{n+1,j+1}(z) = z(1-z)[T'_{n+1,j}(z) + (n+1)jT_{n+1,j-1}(z)],$$

for $j = 2p+1$ and $j = 2p+2$, we obtain

$$T'_{n+1,2p+1}(z) = \frac{T_{n+1,2p+2}(z)}{z(1-z)} - (n+1)(2p+1)T_{n+1,2p}(z),$$

and

$$T'_{n+1,2p+2}(z) = \frac{T_{n+1,2p+3}(z)}{z(1-z)} - (n+1)(2p+2)T_{n+1,2p+1}(z).$$

Replacing these in A, exactly as in the proof of Corollary 1.3.4 it follows that in the expression of A only the terms independent of n matter for the lower estimate. Simple calculation shows that these terms are given by

$$\begin{aligned} G(f)(z) = &\frac{a_p[(1-2z)(z(1-z))^p]' f^{(2p)}(z)}{(2p+1)!} \\ &+\frac{a_{p+1}(1-2z)[z(1-z)]^{p+1}}{z(1-z)(2p+2)!}f^{(2p+1)}(z) \\ &+\frac{[z(1-z)]^{p+1}}{2^{p+1}(p+1)!}f^{(2p+2)}(z), \end{aligned}$$

where $a_p, a_{p+1} > 0$. In order to obtain the lower estimate of order $\frac{1}{n^{p+1}}$, reasoning as in the proof of Corollary 1.3.4 it suffices to prove that if f is not a polynomial of degree $\le 2p-1$ then $\|G(f)\|_r > 0$. Making the substitution $f^{(2p)}(z) = y(z)$ it follows that it suffices to prove that if $y(z)$ is not identical zero then $\|G(y)\|_r > 0$. For this purpose we reason as in the proof of Corollary 1.3.4. Thus we can show that the only solution of the differential equation

$$\frac{a_p[(1-2z)(z(1-z))^p]' y(z)}{(2p+1)!} + \frac{a_{p+1}(1-2z)[z(1-z)]^{p+1}}{z(1-z)(2p+2)!}y'(z)$$
$$+ \frac{[z(1-z)]^{p+1}}{2^{p+1}(p+1)!}y''(z) = 0$$

is $y(z) = 0$ for all $z \in \mathbb{D}_r$. Writing $y(z) = \sum_{k=0}^{\infty} b_k z^k$ and reasoning as in the proof of Corollary 1.3.4 , we easily obtain step by step that $b_0 = 0$, $b_1 = 0$, so on, $b_k = 0$ for all k. We omit here the calculation details which are simple. The theorem is proved. □

At the end of this section, concerning the mth iterates ${}^mK_n^{(\alpha,\beta)}(f)(z)$, we can prove the following result.

Theorem 1.7.7. (Gal [91]) *Let* $f : \mathbb{D}_R \to \mathbb{C}$ *be analytic in* $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ *with* $R > 1$, *i.e.* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{D}_R$. *Suppose* $1 \le r < r_1 < R$. *Then for all* $|z| \le r$ *and* $n, p \in \mathbb{N}$, *we have*

$$\begin{aligned}&|[{}^mK_n^{(\alpha,\beta)}(f)]^{(p)}(z) - f^{(p)}(z)|\\ &\quad\le \frac{2m}{n+1+\beta}\sum_{k=1}^{\infty}|c_{k-1}|\cdot|\beta+(k-1)|r^k\cdot\frac{(p+1)!r_1}{(r_1-r)^{p+1}}.\end{aligned}$$

Proof. First we easily observe that

$${}^mK_n^{(\alpha,\beta)}(f)(z) = \frac{d}{dz}[{}^mS_{n+1}^{(\alpha,\beta)}(F)](z),$$

where $F(z) = \int_0^z f(t)dt = \sum_{k=0}^{\infty} C_k z^k$. Taking into account Theorem 1.6.7, the Cauchy's formula and reasoning exactly as in the proofs of Theorem 1.7.4, (i) and 1.7.5, (i), it follows

$$\begin{aligned}&|[{}^mK_n^{(\alpha,\beta)}(f)]^{(p)}(z) - f^{(p)}(z)|\\ &= |[{}^mS_{n+1}^{(\alpha,\beta)}(F)]^{(p+1)}(z) - F^{(p+1)}(z)|\\ &\le \frac{2m}{n+1+\beta}\sum_{k=1}^{\infty}|C_k|\cdot|\beta k+k(k-1)|r^k\cdot\frac{(p+1)!r_1}{(r_1-r)^{p+1}}\\ &= \frac{2m}{n+1+\beta}\sum_{k=1}^{\infty}|c_{k-1}|\cdot|\beta+(k-1)|r^k\cdot\frac{(p+1)!r_1}{(r_1-r)^{p+1}},\end{aligned}$$

which proves the theorem. □

Remark. 1) For $\beta = 0$ in Theorem 1.7.7 we get corresponding results for the iterates of classical complex Kantorovich polynomials. Note that in the real case, some asymptotic results for iterates of Kantorovich polynomials were obtained in Nagel [143].

2) If $\frac{m_n}{n} \to 0$ when $n \to \infty$, then by Theorem 1.7.7 it is immediate that

$$[{}^{m_n}K_n^{(\alpha,\beta)}(f)]^{(p)}(z) \to f^{(p)}(z),$$

uniformly with respect to $|z| \le 1$, for any $1 \le r < R$.

1.8 Favard-Szász-Mirakjan Operators

In this section we obtain quantitative estimates of the convergence and of the Voronovskaja's theorem in compact disks, for complex Favard-Szász-Mirakjan operators attached to an analytic function in a disk of radius $R > 1$ and center 0. The section is divided in two parts. In the first part of it, the analytic function satisfies some suitable exponential-type growth condition, while in the second part it does not satisfy such of conditions. But in the second case, the price paid is that the uniform convergence and the estimates hold in closed disks of radii $< \frac{R}{2}$ only.

Also, we will prove that beginning with an index, these operators preserve the starlikeness, convexity and spirallikeness in the unit disk.

If $f : [0, \infty) \to \mathbb{R}$ then it is well-known that the Favard-Szász-Mirakjan operators are given by (see Favard [72], Szász [188], Mirakjan [137]) $S_n(f)(x) = e^{-nx} \sum_{j=0}^{\infty} \frac{(nx)^j}{j!} f(j/n)$, $x \in [0, \infty)$, where for the convergence of $S_n(f)(x)$ to $f(x)$, usually f is supposed to be of exponential growth, that is $|f(x)| \le Ce^{Bx}$, for all $x \in [0, +\infty)$, with $C, B > 0$ (see Favard [72]). Also, concerning quantitative estimates in approximation of $f(x)$ by $S_n(f)(x)$, in e.g. Totik [191], it is proved that under some additional assumptions on f, we actually have $|S_n(f)(x) - f(x)| \le \frac{C}{n}$, for all $x \in \mathbb{R}_+$, $n \in \mathbb{N}$.

The complex Favard-Szász-Mirakjan operator is obtained from the real version, simply replacing the real variable x by the complex one z, that is

$$S_n(f)(z) = e^{-nz} \sum_{j=0}^{\infty} \frac{(nz)^j}{j!} f(j/n).$$

Let us note that in our results, the domain of definition of the approximated function $f : \overline{\mathbb{D}}_R \bigcup [R, \infty) \to \mathbb{C}$ seem to be rather strange. However, the analyticity of f on $\mathbb{R}$ on $\mathbb{D}_R$ assures the representation $f(z) = \sum_{k=0}^{\infty} c_k z^k$, which is essential in the proof of quantitative estimates in any $\overline{\mathbb{D}}_r$ with $1 \le r < R$ (while on $[0, \infty)$ the well known estimates in the case of real variable can be used).

Probably a more natural domain of definition for f would be a strip around the OX-axis, but in this case the representation $f(z) = \sum_{k=0}^{\infty} c_k z^k$ fails, fact which produces the failure of the methods of proofs in this case.

In this first part, supposing that $f : [0, +\infty) \to \mathbb{C}$ of exponential growth, can be prolonged to an analytic function in an open disk (with center in origin) by keeping exponential growth, we obtain quantitative estimates in closed disks with center in origin, similar in form with that in the real case in Totik [191] mentioned above.

Also, we recall that the first result concerning the convergence of complex $S_n(f)(z)$ to $f(z)$ belonging to a class of analytic functions satisfying a suitable exponential-type growth condition in a parabolic domain, was proved in Dressel-Gergen-Purcell [65], but without any estimate of the approximation error.

The first main result of this section can be summarized by the following.

Theorem 1.8.1. (Gal [83]) *Let* $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ *be with* $1 < R < +\infty$ *and suppose that* $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ *is continuous in* $[R, +\infty) \cup \overline{\mathbb{D}}_R$*, analytic in* $\mathbb{D}_R$*, i.e.* $f(z) = \sum_{k=0}^{\infty} c_k z^k$*, for all* $z \in \mathbb{D}_R$*, and that there exist* $M, C, B > 0$ *and* $A \in (\frac{1}{R}, 1)$*, with the property* $|c_k| \le M \frac{A^k}{k!}$*, for all* $k = 0, 1, ...,$ *(which implies* $|f(z)| \le Me^{A|z|}$ *for all* $z \in \mathbb{D}_R$*) and* $|f(x)| \le Ce^{Bx}$*, for all* $x \in [R, +\infty)$*.*

(i) Let $1 \le r < \frac{1}{A}$ *be arbitrary fixed. For all* $|z| \le r$ *and* $n \in \mathbb{N}$*, we have*

$$|S_n(f)(z) - f(z)| \le \frac{C_{r,A}}{n},$$

where $C_{r,A} = \frac{M}{2r}\sum_{k=2}^{\infty}(k+1)(rA)^k < \infty$.

(ii) If $1 \le r < r_1 < \frac{1}{A}$ *are arbitrary fixed, then for all* $|z| \le r$ *and* $n, p \in \mathbb{N}$,

$$|S_n^{(p)}(f)(z) - f^{(p)}(z)| \le \frac{p! r_1 C_{r_1,A}}{n(r_1 - r)^{p+1}},$$

where $C_{r_1,A}$ *is given as at the above point (i).*

Proof. (i) According to Theorem 2 in Lupaş [127], we can write

$$S_n(f)(z) = \sum_{j=0}^{\infty}[0, 1/n, ..., j/n; f]z^j,$$

where $[0, 1/n, ..., j/n; f]$ denotes the divided difference of f on the knots $0, 1/n, ..., j/n$. Note that the above formula was proved in Lupaş [127] for functions of real variable, but the formula holds in complex setting too, since only algebraic calculations were used (see the proof of Theorem 2 in Lupaş [127]).

Taking in this representation formula $e_k(z) = z^k$, we get that $T_{n,k}(z) := S_n(e_k)(z)$ is a polynomial of degree $\le k$, $k = 0, 1, 2, ...$, and $T_{n,0}(z) = 1, T_{n,1}(z) = z$, for all $z \in \mathbb{C}$. Also, differentiating $T_{n,k}(z)$ with respect to $z \ne 0$, we get

$$\begin{aligned} T'_{n,k}(z) &= \sum_{j=0}^{\infty}\frac{j^k}{n^k}\left[-ne^{-nz}\frac{(nz)^j}{j!} + e^{-nz}jn\frac{(nz)^{j-1}}{j!}\right] \\ &= -nT_{n,k}(z) + \sum_{j=0}^{\infty}\frac{j^{k+1}}{n^{k+1}}e^{-nz}\frac{n}{z}\frac{(nz)^j}{j!} \\ &= -nT_{n,k}(z) + \frac{n}{z}T_{n,k+1}(z), \end{aligned}$$

which implies

$$T_{n,k+1}(z) = \frac{z}{n}T'_{n,k}(z) + zT_{n,k}(z),$$

for all $z \in \mathbb{C}$, $k \in \{0, 1, 2, ..., \}$, $n \in \mathbb{N}$. From this it is immediate the recurrence formula

$$T_{n,k}(z) - z^k = \frac{z}{n}[T_{n,k-1}(z) - z^{k-1}]' + z[T_{n,k-1}(z) - z^{k-1}] + \frac{k-1}{n}z^{k-1},$$

for all $z \in \mathbb{C}$, $k, n \in \mathbb{N}$.

Now, let $1 \le r < R$. Denoting with $\|\cdot\|_r$ the norm in $C(\overline{\mathbb{D}}_r)$, where $\overline{\mathbb{D}}_r = \{z \in \mathbb{C}; |z| \le r\}$, by a linear transformation, the Bernstein's inequality in the closed unit disk becomes $|P'_k(z)| \le \frac{k}{r}\|P_k\|_r$, for all $|z| \le r$, where $P_k(z)$ is a polynomial of degree $\le k$. Therefore, from the above recurrence formula, we get

$$\begin{aligned} \|T_{n,k} - e_k\|_r \le \; &\frac{r}{n} \cdot \|T_{n,k-1} - e_{k-1}\|_r \frac{k-1}{r} \\ &+ r\|T_{n,k-1} - e_{k-1}\|_r + \frac{r^{k-1}(k-1)}{n}, \end{aligned}$$

which implies the recurrence

$$\|T_{n,k}-e_k\|_r \le \left(r+\frac{k-1}{n}\right)\|T_{n,k-1}-e_{k-1}\|_r+\frac{r^{k-1}(k-1)}{n}.$$

In what follows we prove by mathematical induction with respect to k (with $n \ge 1$ supposed to be fixed, arbitrary), that this recurrence implies

$$\|T_{n,k}-e_k\|_r \le \frac{(k+1)!}{2n}r^{k-1}, \text{ for all } k \ge 2, n \ge 1.$$

Indeed, for $k=2$ and $n \in \mathbb{N}$, the left-hand side is $\frac{r}{n}$ while the right-hand side is $\frac{3r}{n}$. Supposing now that it is true for k, the above recurrence implies

$$\|T_{n,k+1}-e_{k+1}\|_r \le \left(r+\frac{k}{n}\right)\frac{(k+1)!}{2n}r^{k-1}+\frac{r^k k}{n}.$$

It remains to prove that

$$\left(r+\frac{k}{n}\right)\frac{(k+1)!}{2n}r^{k-1}+\frac{r^k k}{n} \le \frac{(k+2)!}{2n}r^k,$$

or, after simplifications, equivalently to

$$\left(r+\frac{k}{n}\right)(k+1)!+2rk \le (k+2)!r.$$

It is easy to see that this last inequality holds true for all $k \ge 2$ and $n \in \mathbb{N}$.

Now, from the hypothesis on f (that is $|f(x)| \le max\{M, C\}e^{max\{A,B\}x}$, for all $x \in \mathbb{R}_+$), it follows that (see e.g. Dressel-Gergen-Purcell [65], pp. 1171-1172 and p. 1178) $S_n(f)(z)$ is analytic in $\mathbb{D}_R$. Therefore, it is easy to see that we can write

$$S_n(f)(z)=\sum_{k=0}^{\infty}c_k S_n(e_k)(z)=\sum_{k=0}^{\infty}c_k T_{n,k}(z), \text{ for all } z \in \mathbb{D}_R,$$

which from the hypothesis on c_k, immediately implies for all $|z| \le r$

$$|S_n(f)(z)-f(z)| \le \sum_{k=2}^{\infty}|c_k|\cdot|T_{k,n}(z)-e_k(z)| \le \sum_{k=2}^{\infty}M\frac{A^k}{k!}\frac{(k+1)!}{2n}r^{k-1}$$
$$=\frac{M}{2nr}\sum_{k=2}^{\infty}(k+1)(rA)^k=\frac{C_{r,A}}{n},$$

where $C_{r,A}=\frac{M}{2r}\sum_{k=2}^{\infty}(k+1)(rA)^k<\infty$, for all $1 \le r < \frac{1}{A}$, taking into account that the series $\sum_{k=2}^{\infty}u^{k+1}$ and therefore its derivative $\sum_{k=2}^{\infty}(k+1)u^k$, are uniformly and absolutely convergent in any compact disk included in the open unit disk.

(ii) Denoting by γ the circle of radius $r_1 > r$ and center 0, since for any $|z| \le r$ and $v \in \gamma$, we have $|v-z| \ge r_1-r$, by the Cauchy's formulas it follows that for all $|z| \le r$ and $n \in \mathbb{N}$, we have

$$|S_n^{(p)}(f)(z)-f^{(p)}(z)|=\frac{p!}{2\pi}\left|\int_\gamma \frac{S_n(f)(v)-f(v)}{(v-z)^{p+1}}dv\right| \le \frac{C_{r_1,A}}{n}\frac{p!}{2\pi}\frac{2\pi r_1}{(r_1-r)^{p+1}}$$
$$=\frac{C_{r_1,A}}{n}\frac{p!r_1}{(r_1-r)^{p+1}},$$

which proves (ii) and the theorem. □

Remark. Let us show in detail the relationship

$$S_n(f)(z) = \sum_{k=0}^{\infty} c_k S_n(e_k)(z),$$

used in the proof of Theorem 1.8.1, (i) as follows. For this purpose, for any $m \in \mathbb{N}$ let us define

$$f_m(z) = \sum_{j=0}^{m} c_j z^j \text{ if } |z| \le r \text{ and } f_m(x) = f(x) \text{ if } x \in (r, +\infty).$$

From the hypothesis on f it is clear that for any $m \in \mathbb{N}$ it follows $|f_m(x)| \le C_m e^{B_m x}$, for all $x \in [0, +\infty)$. This implies that for each fixed $m, n \in \mathbb{N}$ and z,

$$\begin{aligned} |S_n(f_m)(z)| &\le |e^{-nz}| \sum_{j=0}^{\infty} \frac{(n|z|)^j}{j!} |f_m(j/n)| \\ &\le C_m |e^{-nz}| \sum_{j=0}^{\infty} \frac{(n|z|)^j}{j!} e^{B_m j/n} < \infty, \end{aligned}$$

since by the ratio criterium the last series is convergent. Therefore $S_n(f_m)(z)$ is well-defined.

Denoting

$$f_{m,k}(z) = c_k e_k(z) \text{ if } |z| \le r \text{ and } f_{m,k}(x) = \frac{f(x)}{m+1} \text{ if } x \in (r, \infty),$$

it is clear that each $f_{m,k}$ is of exponential growth on $[0, \infty)$ and that $f_m(z) = \sum_{k=0}^{m} f_{m,k}(z)$. Since from the linearity of S_n we have

$$S_n(f_m)(z) = \sum_{k=0}^{m} c_k S_n(e_k)(z), \text{ for all } |z| \le r,$$

it suffices to prove that $\lim_{m\to\infty} S_n(f_m)(z) = S_n(f)(z)$ for any fixed $n \in \mathbb{N}$ and $|z| \le r$. But this is immediate from $\lim_{m\to\infty} \|f_m - f\|_r = 0$, from $\|f_m - f\|_{B[0,+\infty)} \le \|f_m - f\|_r$ and from the inequality

$$|S_n(f_m)(z) - S_n(f)(z)| \le |e^{-nz}| \cdot e^{n|z|} \cdot \|f_m - f\|_{B[0,\infty)} \le M_{r,n} \|f_m - f\|_r,$$

valid for all $|z| \le r$. Here $\| \cdot \|_{B[0,+\infty)}$ denotes the uniform norm on $C[0, +\infty)$-the space of all real-valued bounded functions on $[0, +\infty)$.

In what follows we obtain the Voronovskaja-type formula with a quantitative estimate for the complex Favard-Szász-Mirakjan operator.

Theorem 1.8.2. (Gal [83]) *Suppose that the hypothesis on the function f and the constants R, M, C, B, A in the statement of Theorem 1.8.1 hold and let $1 \le r < \frac{1}{A}$ be arbitrary fixed.*

(i) The following upper estimate in the Voronovskaja-type formula holds

$$\left| S_n(f)(z) - f(z) - \frac{z}{2n} f''(z) \right| \le \frac{3MA|z|}{r^2 n^2} \sum_{k=2}^{\infty} (k+1)(rA)^{k-1},$$

for all $n \in \mathbb{N}, |z| \leq r$.

(ii) We have the following equivalence in the Voronovskaja's formula

$$\left\| S_n(f) - f - \frac{e_1}{2n} f'' \right\|_r \sim \frac{1}{n^2},$$

where the constants in the equivalence depend on f *and* r *but are independent of* n.

Proof. (i) Denoting $e_k(z) = z^k$, $k = 0, 1, ...,$ and $T_{n,k}(z) = S_n(e_k)(z)$, by the proof of Theorem 1.8.1, (i), we can write $S_n(f)(z) = \sum_{k=0}^{\infty} c_k T_{n,k}(z)$, which immediately implies

$$\left| S_n(f)(z) - f(z) - \frac{z}{2n} f''(z) \right| \leq \sum_{k=2}^{\infty} |c_k| \cdot \left| T_{n,k}(z) - e_k(z) - \frac{z^{k-1}k(k-1)}{2n} \right|$$

for all $z \in \mathbb{D}_R$, $n \in \mathbb{N}$.

By the recurrence relationship in the proof of Theorem 1.8.1, (i), satisfied by $T_{n,k}(z)$, denoting $E_{k,n}(z) = T_{n,k}(z) - e_k(z) - \frac{z^{k-1}k(k-1)}{2n}$, we immediately get the new recurrence

$$E_{k,n}(z) = \frac{z}{n} E'_{k-1,n}(z) + z E_{k-1,n}(z) + \frac{z^{k-2}(k-1)(k-2)^2}{2n^2},$$

for all $k \geq 2$, $n \in \mathbb{N}$ and $z \in \mathbb{D}_R$.

This implies, for all $|z| \leq r$, $k \geq 2$, $n \in \mathbb{N}$,

$$\begin{aligned}
&|E_{k,n}(z)| \\
&\leq \frac{|z|}{2n}[2\|E'_{k-1,n}\|_r] + |z| \cdot |E_{k-1,n}(z)| + \frac{|z|}{2n} \cdot \frac{r^{k-3}(k-1)(k-2)^2}{n} \\
&\leq r|E_{k-1,n}(z)| + \frac{|z|}{2n}[2\|E'_{k-1,n}\|_r + \frac{r^{k-3}(k-1)(k-2)^2}{n}] \\
&\leq r|E_{k-1,n}(z)| + \frac{|z|}{2n}\left[\frac{2(k-1)}{r}\|E_{k-1,n}\|_r + \frac{r^{k-3}(k-1)(k-2)^2}{n}\right] \\
&\leq r|E_{k-1,n}(z)| + \frac{|z|}{2n}\left[\frac{2(k-1)}{r}\|T_{n,k-1} - e_{k-1}\|_r\right. \\
&\quad \left. + \frac{2(k-1)}{r} \cdot \frac{r^{k-2}(k-1)(k-2)}{2n} + \frac{r^{k-3}(k-1)(k-2)^2}{n}\right] \leq r|E_{k-1,n}(z)| \\
&\quad + \frac{|z|}{2n}\left[\frac{2(k-1)}{r} \cdot \frac{r^{k-2}k!}{2n} + \frac{2(k-1)}{r} \cdot \frac{r^{k-2}(k-1)(k-2)}{2n}\right. \\
&\quad \left. + \frac{r^{k-3}(k-1)(k-2)^2}{n}\right] \leq r|E_{k-1,n}(z)| + \frac{3|z|r^{k-3}}{2n^2}(k-1)k! \\
&\leq r|E_{k-1,n}(z)| + \frac{3|z|r^{k-3}}{2n^2}(k+1)!,
\end{aligned}$$

that is

$$|E_{k,n}(z)| \leq r|E_{k-1,n}(z)| + \frac{3|z|r^{k-3}}{2n^2}(k+1)!, \text{ for all } |z| \leq r.$$

Taking $k = 2, 3, ...,$ in this last inequality, step by step we obtain

$$|E_{k,n}(z)| \le \frac{3|z|r^{k-3}}{2n^2} \sum_{j=3}^{k+1} j! \le \frac{3|z|r^{k-3}(k+1)!}{n^2},$$

which implies

$$\begin{aligned}\left|S_n(f)(z) - f(z) - \frac{z}{2n} f''(z)\right| &\le \sum_{k=2}^{\infty} |c_k| \cdot |E_{k,n}(z)|| \\ &\le \frac{3M|z|}{n^2} \sum_{k=2}^{\infty} \frac{A^k}{k!}(k+1)! r^{k-3} \\ &\le \frac{3MA|z|}{r^2 n^2} \sum_{k=2}^{\infty} (k+1)(rA)^{k-1},\end{aligned}$$

for all $|z| \le r$, where for $rA < 1$ we obviously have $\sum_{k=2}^{\infty}(k+1)(rA)^{k-1} < \infty$.

(ii) Taking into account the above point (i) it will be enough to prove the lower estimate. For this purpose we will use the ideas in the proof of Corollary 1.3.4. More exactly, let us consider the expression which appear in the generalized Voronovskaja's formula for the complex Favard-Szász-Mirakjan operators, that is

$$Q_{n,p}(f)(z) = S_n(f)(z) - f(z) - \sum_{j=1}^{2p} \frac{1}{j! n^j} A_{n,j}(z) f^{(j)}(z),$$

where

$$A_{n,j} = n^j S_n[(\cdot - z)^j](z) = e^{-nz} \sum_{k=0}^{\infty} \frac{(nz)^k}{k!}(k - nz)^j.$$

The idea in the proof of Corollary 1.3.4 is that in order to get the lower estimate $\|Q_{n,p}(f)\|_r \ge \frac{C}{n^{p+1}}$ we need first to prove the upper estimate $\|Q_{n,p+1}(f)\|_r \le \frac{C}{n^{p+2}}$. In what follows for simplicity we will consider above the case $p = 1$. Therefore, first we need an upper estimate for $|Q_{n,2}(f)(z)|$.

According to Lemma 1.2 in Pop [156], $A_{n,j}(z)$ is a polynomial of degree $[j/2]$. Also, from Lemmas 1.2 and 1.3 and Consequence 1.2 in Pop [156] we easily get $A_{n,0}(z) = 1$, $A_{n,1}(z) = 0$, $A_{n,2}(z) = nz$, $A_{n,3}(z) = nz$, $A_{n,4}(z) = 3n^2z^2 + nz$, which replaced in the expression of $Q_{n,2}(f)(z)$ will mean that we need to prove an upper estimate of the form (valid for all $|z| \le r$)

$$\left|S_n(f)(z) - f(z) - \frac{z}{2n} f''(z) - \frac{z}{6n^2} f^{(3)}(z) - \frac{3nz^2 + z}{24n^3} f^{(4)}(z)\right| \le \frac{C}{n^3}.$$

Since $S_n(f)(z) = \sum_{k=0}^{\infty} c_k T_{n,k}(z)$ and denoting

$$\begin{aligned}E_{k,n,2}(z) \,=\, & T_{n,k}(z) - e_k(z) - \frac{k(k-1)z^{k-1}}{2n} - \frac{k(k-1)(k-2)z^{k-2}}{6n^2} \\ & - \frac{k(k-1)(k-2)(k-2)}{24n^3}(3nz^2 + z)z^{k-4},\end{aligned}$$

we can write

$$\left|S_n(f)(z) - f(z) - \frac{z}{2n}f''(z) - \frac{z}{6n^2}f^{(3)}(z) - \frac{3nz^2+z}{24n^3}f^{(4)}(z)\right|$$

$$\leq \sum_{k=5}^{\infty} |c_k| \cdot |E_{k,n,2}(z)|,$$

by taking into account that by simple calculation we get $E_{k,n,2}(z) = 0$ for all$k = 0, 1, 2, 3, 4$. Also, clearly $E_{k,n,2}(z)$ is a polynomial of degree $\leq k$.

Now, if we denote $E_{k,n,1}(z) = T_{n,k}(z) - e_k - \frac{k(k-1)z^{k-1}}{2n}$, by the proof of the above point (i) we can write the recurrence formula

$$E_{k,n,1}(z) = \frac{z}{n}E'_{k-1,n,1}(z) + zE_{k-1,n,1}(z) + \frac{z^{k-2}(k-1)(k-2)^2}{2n^2}.$$

On the other hand, simple calculation lead us to the formula

$$E_{k,n,e}(z) = E_{k,n,1}(z) - P_{k,n}(z),$$

where

$$P_{k,n}(z) = \frac{k(k-1)(k-2)(3k-5)}{24n^2}z^{k-2} + \frac{k(k-1)(k-2)(k-3)}{24n^3}z^{k-3}.$$

The recurrence formula for $E_{k,n,1}(z)$ implies the following recurrence formula for $E_{k,n,2}(z)$

$$E_{k,n,2}(z) = \frac{z}{n}E'_{k-1,n,2}(z) + zE_{k-1,n,2}(z) + R_{k,n}(z),$$

where

$$\begin{aligned} R_{k,n}(z) \, = \, & \frac{(k-1)(k-2)(k-3)(k-4)(3k-5)}{24n^3}z^{k-3} \\ & + \frac{(k-1)(k-2)(k-3)(k-4)^2}{24n^4}z^{k-4}. \end{aligned}$$

From this recurrence, applying the Bernstein's inequality and the estimate for $\|E_{k-1,n,1}\|_r \leq \frac{3r^{k-3}k!}{n^2}$ from the above point (i), for all $|z| \leq r$, $n \in \mathbb{N}$ and $k \geq 5$ we obtain

$$\begin{aligned} & \|E_{k,n,2}\|_r \\ & \leq r\|E_{k-1,n,2}(z)\|_r + \frac{r}{n}\|E'_{k-1,n,2}\|_r + \|R_{k,n}\|_r \leq r\|E_{k-1,n,2}\|_r \\ & \quad + \frac{r}{n} \cdot \frac{k-1}{r}\|E_{k-1,n,1}\|_r + \frac{r}{n} \cdot \frac{k-3}{r}\|P_{k,n}\|_r + \|R_{k,n}\|_r \leq r\|E_{k-1,n,2}\|_r \\ & \quad + \frac{k-1}{n} \cdot \cdot \frac{3r^{k-3}k!}{n^2} + \frac{k-3}{n} \cdot \frac{r^{k-3}}{24n^2}(k-1)(k-2)(k-3)(3k-8) \\ & \quad + \frac{k-3}{n} \cdot \frac{r^{k-4}}{24n^3}(k-1)(k-2)(k-3)(k-4) \end{aligned}$$

$$\begin{aligned}
&+\frac{r^{k-3}}{24n^3}(k-1)(k-2)(k-3)(k-4)(3k-5)\\
&+\frac{r^{k-4}}{24n^4}(k-1)(k-2)(k-3)(k-4)^2 \le r\|E_{k-1,n,2}\|_r + \frac{3r^{k-3}(k+1)!}{n^3}\\
&+\frac{3r^{k-3}(k+1)!}{24n^3} + \frac{r^{k-4}k!}{24n^4} + \frac{3r^{k-3}k!}{24n^3} + \frac{r^{k-4}k!}{24n^4}\\
\le{}& r\|E_{k-1,n,2}\|_r + \frac{3r^{k-3}(k+1)!}{n^3} + \frac{3r^{k-3}(k+1)!}{2n^3}\\
&+\frac{r^{k-3}k!}{2n^3} + \frac{3r^{k-3}k!}{2n^3} + \frac{r^{k-3}k!}{2n^3}\\
={}& r\|E_{k-1,n,2}\|_r + \frac{7r^{k-3}(k+1)!}{n^3}.
\end{aligned}$$

Therefore we have obtained that for all $n \in \mathbb{N}$ and $k = 5, 6, ...$

$$\|E_{k,n,2}\|_r \le r\|E_{k-1,n,2}\|_r + \frac{7r^{k-3}(k+1)!}{n^3}.$$

Taking here step by step $k = 5, 6, ...$ and taking into account that $E_{k,n,2}(z) = 0$ for $k = 0, 1, 2, 3, 4$, we easily obtain

$$\|E_{k,n,2}\|_r \le \frac{7r^{k-3}}{n^3}\sum_{j=5}^{k}(j+1)! \le \frac{7r^{k-3}(k+2)!}{n^3}, k = 5, 6, ...,$$

which implies

$$\begin{aligned}
\left\|S_n(f) - f - \frac{e_1}{2n}f'' - \frac{e_1}{6n^2}f^{(3)} - \frac{3ne_2+e_1}{24n^3}f^{(4)}\right\|_r &\le \sum_{k=5}^{\infty}|c_k|\cdot\|E_{k,n,2}\|_r\\
&\le \frac{C_r(f)}{n^3}.
\end{aligned}$$

Now, by similar reasonings with those in the case of Bernstein polynomials in Lemma 2.2 in Pop [155], we easily obtain

$$S_n(e_k)(z) = \sum_{j=0}^{k}\frac{1}{j!n^j}A_{n,j}(z)(z^k)^{(j)}.$$

Denoting for arbitrary $p \in \mathbb{N}$

$$E_{k,n,p}(z) = S_n(e_k)(z) - e_k(z) - \sum_{j=1}^{2p}\frac{1}{n^j j!}A_{n,j}(z)(z^k)^{(j)},$$

the above formula for $S_n(e_k)(z)$ implies that

$$E_{k,n,p}(z) = \sum_{j=2p+1}^{k}\frac{1}{n^j}\binom{k}{j}z^{k-j}A_{n,j}(z),$$

and by direct calculation we arrive at

$$
\begin{aligned}
S_n(f)(z) \;-\; f(z) - \sum_{j=1}^{2p} \frac{f^{(j)}(z)}{j!} n^{-j} A_{n,j}(z) &= \sum_{k=2p+1}^{\infty} E_{k,n,p}(z) \\
&= \frac{1}{n^{p+1}} \left\{ \sum_{k=2p+1}^{\infty} c_k \left[\sum_{j=2p+1}^{k} \binom{k}{j} z^{k-j} n^{p+1-j} A_{n,j}(z) \right] \right\} \\
&= \frac{1}{n^{p+1}} \left\{ \frac{A_{n,2p+1}(z)}{n^p (2p+1)!} f^{(2p+1)}(z) + \frac{A_{n,2p+2}(z)}{n^{p+1}(2p+2)!} f^{(2p+2)}(z) \right. \\
&\quad \left. + \frac{1}{n} \left[n^{p+2} \sum_{k=2p+3}^{\infty} c_k E_{k,n,p+1}(z) \right] \right\}.
\end{aligned}
$$

By Corollary 1.3 in Pop [156], $A_{n,j}(z)$ is a polynomial of degree $\leq [j/2]$ and by e.g. Agratini [3], p. 237, Lemma 3.9.4, is a polynomial of degree $\leq j$ in z. Therefore we can write

$$
\frac{A_{n,2p+1}(z)}{n^p (2p+1)!} f^{(2p+1)}(z) + \frac{A_{n,2p+2}(z)}{n^{p+1}(2p+2)!} f^{(2p+2)}(z)
$$

$$
= P_{2p+1}(z) f^{(2p+1)}(z) + P_{2p+2}(z) f^{(2p+2)}(z) + \frac{1}{n} F(z) f^{(2p+1)}(z) + \frac{1}{n} G(z) f^{(2p+2)}(z),
$$

where $F(z)$ and $G(z)$ are bounded polynomials on $\mathbb{D}_r$ by constants depending on r and p nut independent of n.

In what follows we will find the form of $P_{2p+1}(z)$ and $P_{2p+2}(z)$. First, by taking $p(z) = z$ in e.g. Lemma 1.3 in Pop [156] we get the recurrence formula

$$
A_{n,j+1}(z) = z[A'_{n,j}(z) + nj A_{n,j-1}(z)],
$$

which immediately implies $A_{n,0}(z) = 1$, $A_{n,1}(z) = 1$, $A_{n,2}(z) = nz$, $A_{n,3}(z) = nz$, $A_{n,4}(z) = 3n^2z^2 + nz$, $A_{n,5}(z) = 10n^2z^2 + nz$, $A_{n,6}(z) = 15n^3z^3 + 25n^2z^2$, $A_{n,7}(z) = 105n^3z^3 + 56n^2z^2$, and so on. By mathematical induction it easily follows that the coefficient of $n^{[(2p+1)/2]} = n^p$ in $A_{n,2p+1}(z)$ is of the form $c_p z^p$ with $c_p > 0$, while the coefficient of $n^{[(2p+2)/2]} = n^{p+1}$ in $A_{n,2p+2}(z)$ is of the form $d_p z^{p+1}$ with $d_p > 0$. Therefore we have $P_{2p+1}(z) = c_p z^p$ and $P_{2p+2}(z) = d_p z^{p+1}$.

Denoting now $U(f)(z) = P_{2p+1}(z) f^{(2p+1)}(z) + P_{2p+2}(z) f^{(2p+2)}(z)$, we will prove that if f is not a polynomial of degree $\leq 2p$ then $\|U(f)\|_r > 0$. Indeed, suppose that for such an f we have $\|U(f)\|_r = 0$, that is the following differential equation holds

$$
P_{2p+1}(z) f^{(2p+1)}(z) + P_{2p+2}(z) f^{(2p+2)}(z) = 0, \; z \in \mathbb{D}_r.
$$

Making the substitution $f^{(2p+1)}(z) = y(z)$ and replacing the above found form for $P_{2p+1}(z)$ and $P_{2p+2}(z)$, we obtain

$$
c_p z^p y(z) + d_p z^{p+1} y'(z) = 0, \; z \in \mathbb{D}_r.
$$

Simplifying with $z^p \neq 0$ we obtain

$$c_p y(z) + d_p z y'(z) = 0, \ z \in \mathbb{D}_r \setminus \{0\}.$$

Passing here with $z \to 0$ we get $y(0) = 0$. Writing $y(z) = \sum_{k=1}^{\infty} a_k z^k$ and replacing in the above differential, by the identification of the coefficients we easily obtain that $a_k = 0$ for all $k \geq 1$, that is $y(z) = 0$, for all $z \in \mathbb{D}_r$ and therefore f necessarily is a polynomial of degree $\leq 2p$, a contradiction.

Therefore for f a polynomial of degree $\leq 2p$, the supposition that $\|U(f)\|_r = 0$ is false, that is in this case we have $\|U(f)\|_r > 0$.

From this point, reasoning exactly as in the proof of Corollary 1.3.4 for $p = 1$ we obtain

$$\|S_n(f) - f - \sum_{j=1}^{2} \frac{f^{(j)}}{j!} n^{-j} A_{n,j}\|_r \geq \frac{C_r(f)}{n^2},$$

which proves the point (ii) and the theorem. □

The next result shows that the order of approximation in Theorem 1.8.1 is exactly $\frac{1}{n}$.

Corollary 1.8.3. (Gal [83]) *In the hypothesis of Theorem 1.8.1, if f is not a polynomial of degree ≤ 1 in the case (i) and if f is not a polynomial of degree $\leq p$, ($p \geq 1$) in the case (ii), then $\frac{1}{n}$ is in fact the exact order of approximation.*

Proof. Applying the norm $\|\cdot\|_r$ to the identity

$$S_n(f)(z) - f(z) = \frac{1}{n}\left\{\frac{z}{2}f''(z) + \frac{1}{n}\left[n^2\left(S_n(f)(z) - f(z) - \frac{z}{2n}f''(z)\right)\right]\right\},$$

it follows

$$\|S_n(f) - f\|_r \geq \frac{1}{n}\left\{\left\|\frac{e_1}{2}f''\right\|_r - \frac{1}{n}\left[n^2\left\|S_n(f) - f - \frac{e_1}{2n}f''\right\|_r\right]\right\}.$$

If f is not a polynomial of degree ≤ 1 then evidently $\left\|\frac{e_1}{2}f''\right\|_r > 0$, which combined with the estimate in Theorem 1.8.2 immediately implies that $\|S_n(f) - f\|_r \geq \frac{C}{n}$, for all $n \geq n_0$, with $C > 0$ independent of n. Since for $n = 1, 2, ..., n_0 - 1$ the inequality $\|S_n(f) - f\|_r \geq \frac{C_1}{n}$ is trivial with a constant $C_1 > 0$ and taking into account the upper estimate in Theorem 1.8.1, (i), we get the desired conclusion.

Now, replacing $S_n(f)(z) - f(z)$ in the above identity, to the Cauchy formula in the proof of Theorem 1.8.1, (ii) and then applying the norm $\|\cdot\|_r$ to the integral identity, we get

$$\|S_n^{(p)}(f) - f^{(p)}\|_r$$

$$\geq \frac{1}{n}\left\{\left\|\left[\frac{e_1}{2}f''\right]^{(p)}\right\|_r - \frac{1}{n}\left\|\frac{p!}{2\pi}\int_{\Gamma}\frac{n^2\left(S_n(f)(v) - f(v) - \frac{v}{2n}f''(v)\right)}{(v-e_1)^{p+1}}dv\right\|_r\right\},$$

which combined again with Theorem 1.8.2 and taking into account that $\left\|\left[\frac{e_1}{2}f''\right]^{(p)}\right\|_r > 0$ (since f is not a a polynomial of degree $\leq p$), as above leads us to the same conclusion. □

In the second part of this section the growth conditions of exponential-type on f will be omitted. The only condition imposed to f is to be bounded on $[0, \infty)$, case when it is clear that the complex Favard-Szász-Mirakjan operators given by $S_n(f)(z) = e^{-nz}\sum_{j=0}^{\infty}\frac{(nz)^j}{j!}f(j/n)$ are well defined for all $z \in \mathbb{C}$.

In this sense, the first result is expressed by the following.

Theorem 1.8.4. (Gal [84]) *For $2 < R < +\infty$ let $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ be bounded on $[0, +\infty)$ and analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$.*

(i) If $1 \le r < \frac{R}{2}$ then for all $|z| \le r$ and $n \in \mathbb{N}$ it follows

$$|S_n(f)(z) - f(z)| \le \frac{C_{r,f}}{n},$$

with $C_{r,f} = 6\sum_{k=2}^{\infty}|c_k|(k-1)(2r)^{k-1} < \infty$.

(ii) If $1 \le r < r_1 < \frac{R}{2}$ then for all $|z| \le r$ and $n, p \in \mathbb{N}$ it follows

$$|S_n^{(p)}(f)(z) - f^{(p)}(z)| \le \frac{p! r_1 C_{r_1,f}}{n(r_1 - r)^{p+1}},$$

where $C_{r_1,f}$ is as above.

Proof. (i) From the proof of Theorem 1.8.1, (i) we can write

$$S_n(f)(z) = \sum_{j=0}^{\infty}[0, 1/n, ..., j/n; f]z^j,$$

where $[0, 1/n, ..., j/n; f]$ denotes the divided difference of f on the knots $0, 1/n, ..., j/n$.

For $f(z) = e_k(z) = z^k$ and applying the mean value theorem for divided differences, for all $|z| \le r$ and $k, n \in \mathbb{N}$ it follows

$$\begin{aligned}|S_n(e_k)(z)| &\le \sum_{j=0}^{k}|[0, 1/n, ..., j/n; e_k]|r^j\\ &= \sum_{j=1}^{k}\frac{k(k-1)...(k-j+1)}{j!}r^{k-j}r^j\\ &\le r^k\sum_{j=0}^{k}\binom{k}{j} = (2r)^k.\end{aligned}$$

Denoting $T_{n,k}(z) := S_n(e_k)(z)$ clearly that it is a polynomial of degree $\le k$, $k = 0, 1, 2, ...,$, $T_{n,0}(z) = 1, T_{n,1}(z) = z$, for all $z \in \mathbb{C}$ and by the proof of Theorem 1.8.1, (i) the recurrence formula

$$T_{n,k}(z) - z^k = \frac{z}{n}[T_{n,k-1}(z) - z^{k-1}]' + z[T_{n,k-1}(z) - z^{k-1}] + \frac{k-1}{n}z^{k-1},$$

holds for all $z \in \mathbb{C}$, $k, n \in \mathbb{N}$.

Applying as in the proof of Theorem 1.8.1, (i) the Bernstein's inequality, from the above recurrence formula, for all $|z| \le r$ we get

$$\begin{aligned}|T_{n,k} - e_k| &\le \frac{r}{n} \cdot \|T_{n,k-1} - e_{k-1}\|_r \frac{k-1}{r} + r|T_{n,k-1} - e_{k-1}| + \frac{r^{k-1}(k-1)}{n} \\ &\le \frac{k-1}{n}[\|T_{n,k-1}\|_r + r^{k-1}] + r|T_{n,k-1} - e_{k-1}| + \frac{r^{k-1}(k-1)}{n} \\ &\le \frac{k-1}{n}[(2r)^{k-1} + r^{k-1}] + r|T_{n,k-1} - e_{k-1}| + \frac{r^{k-1}(k-1)}{n} \\ &\le r|T_{n,k-1} - e_{k-1}| + \frac{3(k-1)(2r)^{k-1}}{n}.\end{aligned}$$

Since $T_{n,1}(z) = e_1(z)$, for $k = 2$ the above inequality implies $|T_{n,2}(z) - z^2| \le \frac{3r}{n}2^1$, for $k = 3$ it implies $|T_{n,3}(z) - z^3| \le \frac{3r^2}{n}(1 \cdot 2^1 + 2 \cdot 2^2)$, and step by step for all $|z| \le r$ we finally obtain

$$\begin{aligned}|T_{n,k}(z) - z^k| &\le \frac{3r^{k-1}}{n}\left(\sum_{j=1}^{k-1} j2^j\right) = \frac{3r^{k-1}}{n}\left[(k-2)2^k + 2\right] \\ &\le \frac{6(k-1)}{n}(2r)^{k-1}.\end{aligned}$$

Here the formula $\sum_{j=1}^{k-1} j2^j = (k-2)2^k + 2$ can easily be proved by mathematical induction.

Since $S_n(f)(z) = \sum_{k=0}^{\infty} c_k T_{n,k}(z)$, we get

$$|S_n(f)(z) - f(z)| \le \sum_{k=2}^{\infty} |c_k| \cdot |T_{n,k}(z) - z^k| \le \frac{C_{r,f}}{n},$$

which proves (i).

(ii) Denote by γ the circle of radius $r_1 > r$ and center 0. For any $|z| \le r$ and $v \in \gamma$, we have $|v - z| \ge r_1 - r$ and by the Cauchy's formulas for all $|z| \le r$ and $n \in \mathbb{N}$ it follows

$$\begin{aligned}|S_n^{(p)}(f)(z) - f^{(p)}(z)| &= \frac{p!}{2\pi}\left|\int_\gamma \frac{S_n(f)(v) - f(v)}{(v-z)^{p+1}} dv\right| \le \frac{C_{r_1,f}}{n}\frac{p!}{2\pi}\frac{2\pi r_1}{(r_1 - r)^{p+1}} \\ &= \frac{C_{r_1,f}}{n}\frac{p! r_1}{(r_1 - r)^{p+1}},\end{aligned}$$

which proves (ii) and the theorem. □

Remark. A proof of the relationship $S_n(f)(z) = \sum_{k=0}^{\infty} c_k S_n(e_k)(z)$ used in the proof of Theorem 1.8.4, (i) is as follows. For any $m \in \mathbb{N}$ define

$$f_m(z) = \sum_{j=0}^{m} c_j z^j \text{ if } |z| \le r \text{ and } f_m(x) = f(x) \text{ if } x \in (r, +\infty).$$

From the hypothesis on f it is clear that each f_m is bounded on $[0,+\infty)$, which implies that

$$|S_n(f_m)(z)| \le |e^{-nz}|\sum_{j=0}^{\infty}\frac{(n|z|)^j}{j!}M(f_m) = |e^{-nz}|\cdot e^{n|z|}M(f_m) < \infty,$$

that is all $S_n(f_m)(z)$ are well-defined.

Denoting

$$f_{m,k}(z) = c_k e_k(z) \text{ if } |z| \le r \text{ and } f_{m,k}(x) = \frac{f(x)}{m+1} \text{ if } x \in (r,\infty),$$

it is clear that each $f_{m,k}$ is bounded on $[0,\infty)$ and that $f_m(z) = \sum_{k=0}^{m} f_{m,k}(z)$. Since from the linearity of S_n we have

$$S_n(f_m)(z) = \sum_{k=0}^{m} c_k S_n(e_k)(z), \text{ for all } |z| \le r,$$

it suffices to prove that $\lim_{m\to\infty} S_n(f_m)(z) = S_n(f)(z)$ for any fixed $n \in \mathbb{N}$ and $|z| \le r$. But this is immediate from $\lim_{m\to\infty}\|f_m - f\|_r = 0$, from $\|f_m - f\|_{B[0,+\infty)} \le \|f_m - f\|_r$ and from the inequality

$$|S_n(f_m)(z) - S_n(f)(z)| \le |e^{-nz}|\cdot e^{n|z|}\cdot\|f_m - f\|_{B[0,\infty)} \le M_{r,n}\|f_m - f\|_r,$$

valid for all $|z| \le r$. Here $\|\cdot\|_{B[0,+\infty)}$ denotes the uniform norm on $C[0,+\infty)$-the space of all real-valued bounded functions on $[0,+\infty)$.

Also, the following Voronovskaja-type formula holds.

Theorem 1.8.5. (Gal [84]) *For $2 < R < +\infty$ let $f : [R,+\infty)\cup\overline{\mathbb{D}}_R \to \mathbb{C}$ be bounded on $[0,+\infty)$ and analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. Also, let $1 \le r < \frac{R}{2}$.*

(i) For all $|z| \le r$ and $n \in \mathbb{N}$ it follows

$$|S_n(f)(z) - f(z) - \frac{z}{2n}f''(z)| \le M_{r,f}\cdot\frac{|z|}{n^2},$$

with $M_{r,f} = 26\sum_{k=3}^{\infty}|c_k|(k-1)^2(k-2)(2r)^{k-3} < \infty$.

(ii) For all $n \in \mathbb{N}$ we have

$$\|S_n(f) - f - \frac{e_1}{2n}f''\|_r \sim \frac{1}{n^2},$$

where the constants in the equivalence depend on f and r but are independent of n.

Proof. (i) Keeping the notations in the proof of Theorem 1.8.2 (i) we have

$$\left|S_n(f)(z) - f(z) - \frac{z}{2n}f''(z)\right| \le \sum_{k=2}^{\infty}|c_k|\cdot|E_{k,n}(z)|,$$

where it is necessary to obtain a suitable estimate for $|E_{k,n}(z)|$. For this purpose we use the recurrence in the proof of Theorem 1.8.2 (i) given by

$$E_{k,n}(z) = \frac{z}{n}E'_{k-1,n}(z) + zE_{k-1,n}(z) + \frac{z^{k-2}(k-1)(k-2)^2}{2n^2},$$

for all $k \geq 2$, $n \in \mathbb{N}$ and $z \in \mathbb{D}_R$.

This implies, for all $|z| \leq r$, $k \geq 2$, $n \in \mathbb{N}$,

$$\begin{aligned}
&|E_{k,n}(z)| \\
&\leq \frac{|z|}{2n}[2\|E'_{k-1,n}\|_r] + |z| \cdot |E_{k-1,n}(z)| + \frac{|z|}{2n} \cdot \frac{r^{k-3}(k-1)(k-2)^2}{n} \\
&\leq r|E_{k-1,n}(z)| + \frac{|z|}{2n}\left[2\|E'_{k-1,n}\|_r + \frac{r^{k-3}(k-1)(k-2)^2}{n}\right] \leq r|E_{k-1,n}(z)| \\
&\quad + \frac{|z|}{2n}\left[\frac{2(k-1)}{r}\|E_{k-1,n}\|_r + \frac{r^{k-3}(k-1)(k-2)^2}{n}\right] \leq r|E_{k-1,n}(z)| \\
&\quad + \frac{|z|}{2n}\left[\frac{2(k-1)}{r}\|T_{n,k-1} - e_{k-1}\|_r + \frac{2(k-1)}{r} \cdot \frac{r^{k-2}(k-1)(k-2)}{2n}\right. \\
&\quad \left. + \frac{r^{k-3}(k-1)(k-2)^2}{n}\right] \leq r|E_{k-1,n}(z)| \\
&\quad + \frac{|z|}{2n}\left[\frac{2(k-1)}{r} \cdot \frac{6(k-2)}{n} \cdot (2r)^{k-2} + \frac{2(k-1)}{r} \cdot \frac{r^{k-2}(k-1)(k-2)}{2n}\right. \\
&\quad \left. + \frac{r^{k-3}(k-1)(k-2)^2}{n}\right] \leq r|E_{k-1,n}(z)| + \frac{26|z|(k-1)^2(k-2)(2r)^{k-3}}{2n^2},
\end{aligned}$$

that is

$$|E_{k,n}(z)| \leq r|E_{k-1,n}(z)| + \frac{13|z|(k-1)^2(k-2)(2r)^{k-3}}{n^2}.$$

For $k = 1, 2$ we get $E_{k,n}(z) = 0$, for $k = 3$ in this last inequality we obtain $|E_{3,n}(z)| \leq \frac{13|z|r^0}{n^2}(3-1)^2(3-2)2^0$ and for $k = 4$ it follows

$$|E_{4,n}(z)| \leq \frac{13|z|r^1}{n^2}[(3-1)^2(3-2)2^0 + (4-1)^2(4-2)2^1].$$

Then step by step finally we arrive at

$$\begin{aligned}
|E_{k,n}(z)| &\leq \frac{13|z|r^{k-3}}{n^2}\left(\sum_{j=3}^{k}(j-1)^2(j-2)2^{j-3}\right) \\
&\leq \frac{13|z|r^{k-3}}{n^2} \cdot (k-1)^2(k-2)\sum_{j=3}^{k} 2^{j-3} \\
&= \frac{13|z|r^{k-3}}{n^2} \cdot (k-1)^2(k-2)(2^{k-2}-1) \\
&\leq \frac{26|z|(2r)^{k-3}}{n^2} \cdot (k-1)^2(k-2).
\end{aligned}$$

In conclusion it follows

$$\left|S_n(f)(z) - f(z) - \frac{z}{2n}f''(z)\right| \leq \sum_{k=2}^{\infty}|c_k| \cdot |E_{k,n}(z)|$$

$$\leq \frac{26|z|}{n^2}\sum_{k=3}^{\infty}|c_k|(k-1)^2(k-2)(2r)^{k-3} = M_{r,f}\cdot\frac{|z|}{n^2},$$

which proves the point (i).

(ii) The proof is similar to the proof of Theorem 1.8.2 (ii), with the only difference that for $|E_{k,n,1}(z)| := |E_{k,n}(z)|$ we use the upper estimate in the above point (i) and for the other terms appearing to be estimated we use upper estimates in accordance with this one. □

Now we are in position to prove that the order of approximation in Theorem 1.8.4 is exactly $\frac{1}{n}$.

Thus we have :

Theorem 1.8.6. (Gal [84]) *Let* $2 < R < +\infty$, $1 \leq r < \frac{R}{2}$ *and* $f : [R, +\infty)\cup\overline{\mathbb{D}}_R \to \mathbb{C}$ *be bounded on* $[0, +\infty)$ *and analytic in* $\mathbb{D}_R$, *that is* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{D}_R$.

If f *is not a polynomial of degree* ≤ 1, *then the estimate*

$$\|S_n(f) - f\|_r \geq \frac{C_r(f)}{n}, n \in \mathbb{N},$$

holds, where the constant $C_r(f)$ *depends on* f *and* r *but is independent of* n.

Proof. For all $z \in \mathbb{D}_R$ and $n \in \mathbb{N}$ we get

$$S_n(f)(z) - f(z)$$

$$= \frac{1}{n}\left\{\frac{z}{2}f''(z) + \frac{1}{n}\left[n^2\left(S_n(f)(z) - f(z) - \frac{z}{2n}f''(z)\right)\right]\right\}.$$

We apply to this identity the following property :

$$\|F + G\|_r \geq |\, \|F\|_r - \|G\|_r \,| \geq \|F\|_r - \|G\|_r.$$

We get

$$\|S_n(f) - f\|_r \geq \frac{1}{n}\left\{\left\|\frac{e_1}{2}f''\right\|_r - \frac{1}{n}\left[n^2\left\|S_n(f) - f - \frac{e_1}{2n}f''\right\|_r\right]\right\}.$$

Because by hypothesis f is not a polynomial of degree ≤ 1 in $\mathbb{D}_R$, it follows $\left\|\frac{e_1}{2}f''\right\|_r > 0$. Indeed, supposing the contrary it follows that $\frac{z}{2}f''(z) = 0$ for all $z \in \overline{\mathbb{D}}_r$, that is $f''(z) = 0$ for all $z \in \overline{\mathbb{D}}_r \setminus \{0\}$. Since f is analytic, from the identity theorem on analytic functions this implies that $f''(z) = 0$, for all $z \in \mathbb{D}_R$, that is f is a polynomial of degree ≤ 1, which is contradiction with the hypothesis.

Now, by Theorem 1.8.5 we have

$$n^2\left\|S_n(f) - f - \frac{e_1}{2n}f''\right\|_r \leq 26r\sum_{k=3}^{\infty}|c_k|(k-1)^2(k-2)(2r)^{k-3} < \infty.$$

Consequently, there exists n_1 (depending only on f and r) such that for all $n \geq n_1$ we have

$$\left\|\frac{e_1}{2}f''\right\|_r - \frac{1}{n}\left[n^2\left\|S_n(f) - f - \frac{e_1}{2n}f''\right\|_r\right] \geq \frac{1}{2}\left\|\frac{e_1}{2}f''\right\|_r,$$

which implies

$$\|S_n(f) - f\|_r \geq \frac{1}{n} \cdot \frac{1}{2} \left\| \frac{e_1}{2} f'' \right\|_r, \forall n \geq n_1.$$

For $n \in \{1, ..., n_1 - 1\}$ we clearly have $\|S_n(f) - f\|_r \geq \frac{M_{r,n}(f)}{n}$ with $M_{r,n}(f) = n \cdot \|S_n(f) - f\|_r > 0$, which finally implies

$$\|S_n(f) - f\|_r \geq \frac{C_r(f)}{n},$$

for all n, with $C_r(f) = \min\{M_{r,1}(f), ..., M_{r,n_1-1}(f), \frac{1}{2} \left\| \frac{e_1}{2} f'' \right\|_r\}$. □

From Theorem 1.8.6 and Theorem 1.8.4, (i) we immediately obtain the following consequence.

Corollary 1.8.7. (Gal [84]) *Let $2 < R < +\infty$ and $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ be bounded on $[0, +\infty)$ and analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$.*

If $1 \leq r < \frac{R}{2}$ is arbitrary fixed and if f is not a polynomial of degree ≤ 1, then the estimate

$$\|S_n(f) - f\|_r \sim \frac{1}{n}, n \in \mathbb{N},$$

holds, where the constants in the equivalence depend only on f and r.

Regarding the simultaneous approximation of the function and its derivatives we can present :

Theorem 1.8.8. (Gal [84]) *Let $2 < R < +\infty$ and $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ be bounded on $[0, +\infty)$ and analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$.*

If $1 \leq r < r_1 < \frac{R}{2}$, $p \in \mathbb{N}$ and if f is not a polynomial of degree $\leq p$, then we have

$$\|S_n^{(p)}(f) - f^{(p)}\|_r \sim \frac{1}{n},$$

where the constants in the equivalence depend only on f, r, r_1 and p.

Proof. The upper estimate is exactly Theorem 1.8.4, (ii), therefore it remains to prove the lower estimate. Denote by Γ the circle of radius r_1 and center 0 (where $\frac{R}{2} > r_1 > r \geq 1$). By the Cauchy's formulas for all $|z| \leq r$ and $n \in \mathbb{N}$ it follows

$$S_n^{(p)}(f)(z) - f^{(p)}(z) = \frac{p!}{2\pi i} \int_\Gamma \frac{S_n(f)(v) - f(v)}{(v - z)^{p+1}} dv,$$

where $|v - z| \geq r_1 - r$ for all $|z| \leq r$ and $v \in \Gamma$.

As in the proof of Theorem 1.8.6, for all $v \in \Gamma$ and $n \in \mathbb{N}$ we get

$$S_n(f)(v) - f(v)$$

$$= \frac{1}{n} \left\{ \frac{v}{2} f''(v) + \frac{1}{n} \left[n^2 \left(S_n(f)(v) - f(v) - \frac{v}{2n} f''(v) \right) \right] \right\},$$

which replaced in the Cauchy's formula implies

$$S_n^{(p)}(f)(z) - f^{(p)}(z) = \frac{1}{n} \left\{ \frac{p!}{2\pi i} \int_\Gamma \frac{v f''(v)}{2(v - z)^{p+1}} dv \right.$$

$$+\frac{1}{n}\cdot\frac{p!}{2\pi i}\int_{\Gamma}\frac{n^2\left(S_n(f)(v)-f(v)-\frac{v}{2n}f''(v)\right)}{(v-z)^{p+1}}dv\Bigg\}$$

$$=\frac{1}{n}\left\{\left[\frac{z}{2}f''(z)\right]^{(p)}+\frac{1}{n}\cdot\frac{p!}{2\pi i}\int_{\Gamma}\frac{n^2\left(S_n(f)(v)-f(v)-\frac{v}{2n}f''(v)\right)}{(v-z)^{p+1}}dv\right\}.$$

Passing to the norm $\|\cdot\|_r$, for all $n\in\mathbb{N}$ we obtain

$$\|S_n^{(p)}(f)-f^{(p)}\|_r$$

$$\geq\frac{1}{n}\left\{\left\|\left[\frac{e_1}{2}f''\right]^{(p)}\right\|_r-\frac{1}{n}\left\|\frac{p!}{2\pi}\int_{\Gamma}\frac{n^2\left(S_n(f)(v)-f(v)-\frac{v}{2n}f''(v)\right)}{(v-z)^{p+1}}dv\right\|_r\right\},$$

where by Theorem 1.8.5, for all $n\in\mathbb{N}$ it follows

$$\left\|\frac{p!}{2\pi}\int_{\Gamma}\frac{n^2\left(S_n(f)(v)-f(v)-\frac{v}{2n}f''(v)\right)}{(v-z)^{p+1}}dv\right\|_r$$

$$\leq\frac{p!}{2\pi}\cdot\frac{2\pi r_1 n^2}{(r_1-r)^{p+1}}\left\|S_n(f)-f-\frac{e_1}{2n}f''\right\|_{r_1}$$

$$\leq 26r_1\sum_{k=3}^{\infty}|c_k|(k-1)^2(k-2)(2r_1)^{k-3}\cdot\frac{p!r_1}{(r_1-r)^{p+1}}.$$

Now, by hypothesis on f we have $\left\|\left[\frac{e_1}{2}f''\right]^{(p)}\right\|_r>0$. Indeed, supposing the contrary it follows that $\frac{z}{2}f''(z)$ is a polynomial of degree $\leq p-1$, which by the analyticity of f obviously implies that f is a polynomial of degree $\leq p$, in contradiction with the hypothesis.

For the rest of the proof, reasoning exactly as in the proof of Theorem 1.8.6, we immediately get the required conclusion. □

Remarks. 1) Since the boundedness of f on $[0,\infty)$ in the Theorems 1.8.4, 1.8.5, 1.8.6, Corollary 1.8.7 and Theorem 1.8.8 is used only for the existence of the complex Favard-Szász-Mirakjan operator, taking into account the Remark after the proof of Theorem 1.8.1 it follows that in the above mentioned results it can replaced by the condition of exponential growth $|f(z)|\leq Me^{Bx}$, for all $x\in[0,\infty)$.

2) The domain of approximation $[R,+\infty)\cup\overline{\mathbb{D}}_R$ in the previous results of this section, seem less usual. More natural could be, for example, a strip of the form

$$T_R=\{z=x+iy\in\mathbb{C};x\in\mathbb{R}\text{ and }|y|\leq R\}.$$

In what follows we will obtain a weighted approximation result by the complex Favard-Szász-Mirakjan operator $S_n(f)(z)$ in such a strip T_R.

First we need the following.

Lemma 1.8.9. *For fixed arbitrary $z_0\in\mathbb{C}$, let us denote $e_1(z)=z$ and*

$$T_{n,k,z_0}(z):=S_n((e_1-z_0)^k)(z)=e^{-nz}\sum_{j=0}^{\infty}\frac{(nz)^j}{j!}\left(\frac{j}{n}-z_0\right)^k.$$

(i) For all $n \in \mathbb{N}$, $k \in \mathbb{N}\bigcup\{0\}$ *and* $z \in \mathbb{C}$ *we have*

$$T_{n,k+1,z_0}(z) = \frac{z}{n}T'_{n,k,z_0}(z) + (z-z_0)T_{n,k,z_0}(z).$$

(ii) For all $n \in \mathbb{N}$, $k \in \mathbb{N}$ *with* $k \geq 2$ *and* $z \in \mathbb{C}$ *we have*

$$\begin{aligned} T_{n,k,z_0}(z) - (z-z_0)^k &= \frac{z}{n}\left[T_{n,k-1,z_0}(z) - (z-z_0)^{k-1}\right]' \\ &\quad + (z-z_0)[T_{n,k-1,z_0}(z) - (z-z_0)^{k-1}] \\ &\quad + \frac{k-1}{n}z(z-z_0)^{k-2}. \end{aligned}$$

(iii) For all $n, k \in \mathbb{N}$ *and* $z, z_0 \in \mathbb{C}$ *with* $|z-z_0| \leq r$ *we have*

$$|T_{n,k,z_0}(z)| \leq r^k(3+2|z_0|)^k.$$

Proof. (i) By differentiating T_{n,k,z_0} we easily get the required recurrence formula.

(ii) It is an immediate consequence of (i).

(iii) By the proof of Theorem 1.8.1 (i) we have the representation in Lupaş [127]

$$S_n(f)(z) = \sum_{j=0}^{\infty}[0, 1/n, ..., j/n; f]z^j,$$

which immediately implies

$$\begin{aligned} S_n((e_1-z_0)^p)(z) &= \sum_{j=0}^{p}[0, 1/n, ..., j/n; (e_1-z_0)^p]z^j \\ &= \sum_{j=0}^{p}\left[\sum_{k=j}^{p}[0, 1/n, ..., k/n; (e_1-z_0)^p]\binom{k}{j}z_0^{k-j}\right](z-z_0)^j. \end{aligned}$$

For $|z-z_0| \leq r$ we obtain

$$\begin{aligned} |S_n((e_1-z_0)^p)(z)| &\leq \sum_{j=0}^{p}\left[\sum_{k=j}^{p}\binom{k}{j}\frac{p(p-1)...(p-k+1)}{k!}|z_0|^{k-j}r^{p-k}\right]r^j \\ &\leq r^p\sum_{j=0}^{p}\left[\sum_{k=j}^{p}\binom{k}{j}\binom{p}{k}|z_0|^{k-j}\right] \\ &\leq r^p\sum_{j=0}^{p}(1+|z_0|)^{p-j}\left[\sum_{k=j}^{p}\binom{k}{j}\binom{p}{k}\right] \\ &= r^p\sum_{j=0}^{p}(1+|z_0|)^{p-j}\binom{p}{j}2^{p-j} = r^p(3+2|z_0|)^p, \end{aligned}$$

where we used the formula (see e.g. Tomescu [189], p. 11, Exercise 1.5, 2))

$$\sum_{k=j}^{p}\binom{k}{j}\binom{p}{k} = \binom{p}{j}2^{p-j}.$$

□

Corollary 1.8.10. *(i) For all $n, k \in \mathbb{N}$, $z_0 \in \mathbb{C}$ and $r \geq 1$ we have the estimate*

$$\|T_{n,k,z_0} - (e_1 - z_0)^k\|_{\overline{\mathbb{D}}(z_0,r)} \leq \frac{5}{6nr}(k-1)k(2k-1)[r(3+2|z_0|)]^k.$$

(ii) Let $r \geq 1$. Suppose that $f : T_r \to \mathbb{C}$ is analytic in the strip T_r and that f satisfies the conditions

$$|f^k(x_0)| \leq \frac{M}{k^3},$$

for all $k \in \mathbb{N}\bigcup\{0\}$ and all $x_0 \in \mathbb{R}$, where $M > 0$ is independent of x_0 and k. Denoting the weight $w_r(x) = e^{-r(3+2|x|)}$, $x \in \mathbb{R}$ and the weighted norm on T_r by

$$\|f\|_{T_r,w_r} = \sup_{x\in\mathbb{R}} w_r(x)\|f - S_n(f)\|_{\overline{\mathbb{D}}(x,r)},$$

we have

$$\|f - S_n(f)\|_{T_r,w_r} \leq \frac{15M}{6nr},$$

for all $n \in \mathbb{N}$.

Proof. (i) First we estimate $|T_{n,k,z_0}(z) - (z-z_0)^k|$. By the recurrence in Lemma 1.8.9 (i) it easily follows that $T_{n,k,z_0}(z)$ is a polynomial in z of degree $\leq k$. We will use the following generalization of the Bernstein's inequality due to Pommerenke [151]

$$\|P_n'\|_K \leq \frac{en^2}{2cap(K)}\|P_n\|_K \leq \frac{2n^2}{cap(K)}\|P_n\|_K,$$

where $P_n(z)$ is a polynomial of degree n, K is a continuum in $\mathbb{C}$ and $cap(K)$ denotes the capacity of K.

By Lemma 1.8.9 (ii) and (iii) combined with the above Bernstein-type inequality applied for $K = \overline{\mathbb{D}}(z_0, r) = \{z \in \mathbb{C}; |z - z_0| \leq r\}$, since $cap(K) = r$ we obtain

$$\begin{aligned}
&\|T_{n,k,z_0} - (e_1 - z_0)^k\|_{\overline{\mathbb{D}}(z_0,r)} \\
\leq\ & \frac{(r+|z_0|)}{n}\|T_{n,k-1,z_0} - (e_1-z_0)^{k-1}\|_{\overline{\mathbb{D}}(z_0,R)} \cdot \frac{2(k-1)^2}{r} \\
&+r\|T_{n,k-1,z_0} - (e_1-z_0)^{k-1}\|_{\overline{\mathbb{D}}(z_0,r)} + \frac{k-1}{n}(r+|z_0|)r^{k-2} \\
\leq\ & \frac{r+|z_0|}{n}\left[\|T_{n,k-1,z_0}\|_{\overline{\mathbb{D}}(z_0,R)} + r^{k-1}\right]\frac{2(k-1)^2}{r} \\
&+r\|T_{n,k-1,z_0} - (e_1-z_0)^{k-1}\|_{\overline{\mathbb{D}}(z_0,r)} + \frac{k-1}{n}(r+|z_0|)r^{k-2} \\
\leq\ & \frac{r+|z_0|}{n}\left[r^{k-1}(3+2|z_0|)^{k-1} + r^{k-1}\right]\frac{2(k-1)^2}{r} \\
&+r\|T_{n,k-1,z_0} - (e_1-z_0)^{k-1}\|_{\overline{\mathbb{D}}(z_0,r)} + \frac{k-1}{n}(r+|z_0|)r^{k-2},
\end{aligned}$$

which finally leads to the inequality

$$\begin{aligned}\|T_{n,k,z_0} - (e_1-z_0)^k\|_{\overline{\mathbb{D}}(z_0,r)} &\le r\|T_{n,k-1,z_0}-(e_1-z_0)^{k-1}\|_{\overline{\mathbb{D}}(z_0,r)}\\ &+\frac{r(3+2|z_0|)2(k-1)^2}{n}[2r^{k-2}(3+2|z_0|)^{k-1}]\\ &+\frac{k-1}{n}r^{k-2}r(3+2|z_0|)\\ &\le r\|T_{n,k-1,z_0}-(e_1-z_0)^{k-1}\|_{\overline{\mathbb{D}}(z_0,r)}+\frac{5(k-1)^2[r(3+2|z_0|)]^k}{nr}.\end{aligned}$$

For $k=1$ this inequality obviously one reduces to $0 \le 0$. Therefore, let $k \ge 2$.

For $k=2$ we easily obtain

$$\|T_{n,2,z_0}-(e_1-z_0)^2\|_{\overline{\mathbb{D}}(z_0,r)} \le \frac{5}{nr}\{1^2[r(3+2|z_0|)]^2\}.$$

For $k=3$ it follows

$$\begin{aligned}\|T_{n,3,z_0}-(e_1-z_0)^3\|_{\overline{\mathbb{D}}(z_0,r)} &\le r\cdot\frac{5}{nr}\{1^2[r(3+2|z_0|)]^2\}+\frac{5\cdot 2^2[r(3+2|z_0|)]^3}{nr}\\ &\le \frac{5}{nr}[r(3+2|z_0|)]^3(1^2+2^2),\end{aligned}$$

and reasoning by recurrence finally we arrive at

$$\begin{aligned}\|T_{n,k,z_0}-(e_1-z_0)^k\|_{\overline{\mathbb{D}}(z_0,r)} &\le \frac{5}{nr}[r(3+2|z_0|)]^k(1^2+2^2+...+(k-1)^2)\\ &= \frac{5}{6nr}(k-1)k(2k-1)[r(3+2|z_0|)]^k,\end{aligned}$$

which proves (i).

(ii) For arbitrary fixed $x \in \mathbb{R}$, since f is analytic in the strip T_r, we have the Taylor expansion

$$f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(x)}{k!}(z-x)^k,$$

valid for all $z \in \mathbb{C}$ with $|z-x| \le r$. First we prove that

$$S_n(f)(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(x)}{k!}S_n((e_1-x)^k)(z),$$

for all $z \in \mathbb{C}$ with $|z-x| \le r$. For this purpose let us define $f_m(z)=\sum_{k=0}^{m}\frac{f^{(k)}(x)}{k!}(z-x)^k$, if $|z-x| \le r$ and $f_m(z)=f(z)$ if $z \in (-\infty, x-r]\bigcup[x+r,+\infty)$. Since by hypothesis f is bounded on $\mathbb{R}$, reasoning exactly as in the Remark after the proof of Theorem 1.8.4 we easily get the desired property.

Therefore, taking into account the above point (i), for all $|z-x| \le r$ we can write

$$\begin{aligned}|f(z)-S_n(f)(z)| &\le \frac{5}{6nr}\cdot\sum_{k=2}^{\infty}\frac{|f^{(k)}(x)|}{k!}(k-1)k(2k-1)[r(3+2|x|)]^k\\ &\le \frac{5M}{6nr}\cdot\sum_{k=2}^{\infty}\frac{(k-1)k(2k-1)}{k^3}\cdot\frac{[r(3+2|x|)]^k}{k!}\\ &\le \frac{15M}{6nr}\cdot\sum_{k=2}^{\infty}\frac{[r(3+2|x|)]^k}{k!},\end{aligned}$$

that is

$$\|f - S_n(f)\|_{\overline{\mathbb{D}}(x,r)} \leq \frac{15M}{6nr} \cdot \sum_{k=2}^{\infty} \frac{[r(3+2|x|)]^k}{k!}.$$

Multiplying this inequality by $w_r(x) = 1/\{\sum_{k=0}^{\infty}[r(3+2|x|)]^k/k!\}$ and passing to supremum with $x \in \mathbb{R}$ we easily obtain the weighted inequality in the statement. The corollary is proved. □

1.9 Baskakov Operators

The aim of the present section is to extend some kinds of results in the previous sections to complex Baskakov operators.

For x real and ≥ 0, the original formula of the Baskakov operator is given by (see Baskakov [35])

$$Z_n(f)(x) = (1+x)^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k f(k/n).$$

Denoting

$$W_n(f)(x) = \sum_{j=0}^{\infty} \frac{n(n+1)...(n+j-1)}{n^j} [0, 1/n, ..., j/n; f] x^j, x \geq 0,$$

(where for $j = 0$ we take $n(n+1)...(n+j-1) = 1$), according to Lupaş [128], Theorem 2, $Z_n(f)(x) = W_n(f)(x)$, for all $x \geq 0$ (under the hypothesis on f that $Z_n(f)(x)$ is well defined). But if x is not positive then $W_n(f)(x)$ and $Z_n(f)(x)$ do not necessarily coincide. For example, if $x = -1/2$ then we easily get that for all $n \in \mathbb{N}$, $Z_n(f)(-1/2)$ represents the sum of a divergent series even for the simplest function $f(x) = 1$, for all x, while clearly $W_n(f)(-1/2) = 1$, for $f(x) = 1$ and all $n \in \mathbb{N}$.

Consequently, the complex versions of these two operators denoted by

$$Z_n(f)(z) = (1+z)^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{z}{1+z}\right)^k f(k/n),$$

and

$$W_n(f)(z) = \sum_{j=0}^{\infty} \frac{n(n+1)...(n+j-1)}{n^j} [0, 1/n, ..., j/n; f] z^j,$$

do not necessarily coincide for all $z \in \mathbb{C}$. Because of this reason in this section they will be studied separately, under different hypothesis on f and $z \in \mathbb{C}$.

Remarks. 1) A sufficient condition for the existence of the operator $W_n(f)(z)$, $z \in \mathbb{C}$, can be expressed by the fact that f has all its derivatives bounded in $[0, \infty)$

by the same constant $M > 0$. Indeed, in this case, for $r > 0$, $z \in \mathbb{C}$ and $|z| \le r$, we get

$$|W_n(f)(z)| \le M \sum_{k=0}^{\infty} \frac{n(n+1)...(n+k-1)r^k}{n^k k!}, \text{ for all } |z| \le r.$$

Denoting $a_k(n,r) = \frac{n(n+1)...(n+k-1)r^k}{n^k k!}$, we have $\frac{a_{k+1}(n,r)}{a_k(n,r)} = \frac{r(n+k)}{n(1+k)}$. Since as $k \to \infty$ we have $\frac{a_{k+1}(n,r)}{a_k(n,r)} \searrow \frac{r}{n}$, then for a fixed $n_0 \in \mathbb{N}$ and $r < \frac{n_0}{2}$, there exists k_0 such that for all $k > k_0$ we have $\frac{a_{k+1}(n_0,r)}{a_k(n_0,r)} < \frac{2r}{n_0}$. By the ratio test and by the inequality $\frac{a_{k+1}(n,r)}{a_k(n,r)} < \frac{a_{k+1}(n_0,r)}{a_k(n_0,r)}$ for all $n > n_0$, we immediately get that $W_n(f)(z)$ is well-defined and analytic for all $n > n_0$ and $|z| \le \frac{n_0}{2}$.

2) A sufficient condition for the existence of the complex operator $Z_n(f)(z)$ can be stated as follows. Suppose for example that $z \in \mathbb{C}$ satisfies $Re z \ge 0$, $|z| \le r$ and that $|f(x)| \le M$ for all $x \in [0, \infty)$. Then $1 + z \ne 0$ and for all $n \in \mathbb{N}$ it follows

$$\begin{aligned}|Z_n(f)(z)| &\le M|1+z|^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{|z|}{|1+z|}\right)^k \\ &\le M|1+z|^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \rho^k.\end{aligned}$$

where $\rho = \sqrt{\frac{r^2}{1+r^2}} < 1$. Indeed, for $z = x + iy$ with $x \ge 0$ and $\sqrt{x^2+y^2} \le r$ first we get (since $h(t) = t/(1+t)$ is increasing for $t \ge 0$)

$$\left(\frac{|z|}{|1+z|}\right)^2 = \frac{x^2+y^2}{1+2x+(x^2+y^2)} \le \frac{x^2+y^2}{1+(x^2+y^2)} \le \frac{r^2}{1+r^2}.$$

By the ratio test applied to the series $\sum_{k=0}^{\infty} \binom{n+k-1}{k} \rho^k := \sum_{k=0}^{\infty} a_k$, we get that for any fixed $n \in \mathbb{N}$ there exists k_0 such that $\frac{a_{k+1}}{a_k} = \rho \frac{k+n}{k+1} < \rho' < 1$, for all $k \ge k_0$. This implies that $|Z_n(f)(z)| < \infty$ and therefore $Z_n(f)(z)$ is analytic as function of z as above. Also, as we will see later, for $z \in \mathbb{C}$ as above the operator $Z_n(f)(z)$ exists under more general conditions on f.

3) As in the case of complex Favard-Szász-Mirakjan operators, for the complex Baskakov operators too we note that the domain of definition of the approximated function $f : \overline{\mathbb{D}}_R \bigcup [R, \infty) \to \mathbb{C}$ seems to be rather strange. But the analyticity of f on $\mathbb{D}_R$ assures the representation $f(z) = \sum_{k=0}^{\infty} c_k z^k$, which is essential in the proof of quantitative estimates in any $\overline{\mathbb{D}}_r$ with $1 \le r < R$. On the other hand, on $[0, \infty)$ the well-known estimates in the case of real variable can be used.

A more natural domain of definition for f would be a strip around the OX-axis. Unfortunately, in this case the representation $f(z) = \sum_{k=0}^{\infty} c_k z^k$ fails and we cannot use the methods of proofs in this case.

Concerning the complex operator $W_n(f)(z)$, upper estimates in simultaneous approximation, Voronovskaja's result with a quantitative estimate and exact estimates in simultaneous approximation for these operators are obtained. The hypothesis on f in these cases consist in exponential growth in a compact disk and in the boundedness (by the same constant) on $[0, \infty)$ of the derivatives of all orders of f.

At the end of this section we present similar results for the complex operator $Z_n(f)(z)$, under different hypothesis on f and z. More exactly, the disks can be replaced by semidisks and the boundedness of all derivatives of f on $[0, \infty)$ can be replaced by the weaker condition that f is of exponential growth on $[0, \infty)$.

For future research would be of interest to find larger classes of functions for which similar results with those stated in the next Theorems 1.9.1-1.9.9 hold.

The first main result of this section can be summarized by the following.

Theorem 1.9.1. (Gal [90]) *For* $n_0 \in \mathbb{N}$ *and* $R > 0$ *with* $3 \le n_0 < 2R < +\infty$ *let* $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ *be with all its derivatives bounded in* $[0, \infty)$ *by the same positive constant, analytic in* $\mathbb{D}_R$*, that is* $f(z) = \sum_{k=0}^{\infty} c_k z^k$*, for all* $z \in \mathbb{D}_R$*, and suppose that there exist* $M > 0$ *and* $A \in (\frac{1}{R}, 1)$*, with the property* $|c_k| \le M \frac{A^k}{k!}$*, for all* $k = 0, 1, \ldots$*, (this implies* $|f(z)| \le M e^{A|z|}$ *for all* $z \in \mathbb{D}_R$*).*

(i) Let $1 \le r < \min\{\frac{n_0}{2}, \frac{1}{A}\}$*. For all* $|z| \le r$ *and* $n > n_0$ *the estimate*

$$|W_n(f)(z) - f(z)| \le \frac{C_{r,A,M}}{n},$$

holds, with $C_{r,A,M} = 6M \sum_{k=2}^{\infty} (k+1)(k-1)(rA)^k < \infty$*.*

(ii) In the case of simultaneous approximation by complex Baskakov operators, we have : if $1 \le r < r_1 < \min\{\frac{n_0}{2}, \frac{1}{A}\}$ *are fixed, then for all* $|z| \le r$*,* $p \in \mathbb{N}$ *and* $n > n_0$ *the estimate*

$$|W_n^{(p)}(f)(z) - f^{(p)}(z)| \le \frac{p! r_1 C_{r_1,A,M}}{n(r_1 - r)^{p+1}},$$

holds, with $C_{r_1,A,M}$ *is given as above.*

Proof. (i) Denoting $e_k(z) = z^k$, $T_{n,k}(z) := W_n(e_k)(z)$, clearly that $T_{n,k}(z)$ is a polynomial of degree $\le k$, $k = 0, 1, 2, \ldots$, and $T_{n,0}(z) = 1, T_{n,1}(z) = z$, for all $z \in \mathbb{C}$. Also, for all $z \in \mathbb{C}$ and $n, p \in \mathbb{N}$ the following recurrence holds :

$$T_{n,p+1}(z) = \frac{z(1+z)}{n} T'_{n,p}(z) + z T_{n,p}(z).$$

Indeed, simple calculation shows that this recurrence is equivalent to

$$[0, 1/n, \ldots, j/n; e_{p+1}] = \frac{j}{n}[0, 1/n, \ldots, j/n; e_p] + [0, 1/n, \ldots, (j-1)/n; e_p],$$

which is an immediate consequence of the well-known relation (see e.g. Stancu [172], p. 256, Exercise 4.9)

$$[x_0, \ldots, x_m; f \cdot g] = \sum_{i=0}^{m} [x_0, \ldots, x_i; f] \cdot [x_i, \ldots, x_m; g]$$

applied here for $m = j$, $f = e_p$, $g = e_1$ and $x_i = i/n$.

From the above recurrence we obtain

$$\begin{aligned}T_{n,p}(z)-z^p &= \frac{z(1+z)}{n}[T_{n,p-1}(z)-z^{p-1}]' \\ &\quad +z[T_{n,p-1}(z)-z^{p-1}]+\frac{z^{p-1}(1+z)(p-1)}{n},\end{aligned}$$

for all $z\in\mathbb{C}$, $p,n\in\mathbb{N}$.

In what follows we will use the Bernstein's inequality on $\overline{\mathbb{D}}_r=\{z\in\mathbb{C};|z|\le r\}$.

Thus, passing to modulus for $|z|\le r$, $r\ge 1$, from the above recurrence formula, we obtain

$$\begin{aligned}|T_{n,p}(z)-e_p(z)| &\le \frac{(p-1)(1+r)}{n}\|T_{n,p-1}-e_{p-1}\|_r+r|T_{n,p-1}(z)-e_{p-1}(z)| \\ &\quad +\frac{r^{p-1}(p-1)(1+r)}{n}\le r|T_{n,p-1}(z)-e_{p-1}(z)| \\ &\quad +\frac{(p-1)(1+r)}{n}[\|T_{n,p-1}\|_r+r^{p-1}]+\frac{r^{p-1}(p-1)(1+r)}{n} \\ &\le r|T_{n,p-1}(z)-e_{p-1}(z)|+\frac{2(p-1)r}{n}[\|T_{n,p-1}\|_r+r^{p-1}] \\ &\quad +\frac{2r^p(p-1)}{n}.\end{aligned}$$

Since for any $p\in\mathbb{N}$ we have

$$T_{n,p}(z)=\sum_{k=0}^{p}\frac{n(n+1)...(n+k-1)}{n^k}[0,1/n,...,k/n;e_p]z^k,$$

by using the mean value theorem in complex analysis we get

$$\begin{aligned}\|T_{n,p}(z)\|_r &\le \sum_{k=1}^{p}\frac{n(n+1)...(n+k-1)}{n^k}\cdot\frac{p(p-1)...(p-k+1)}{k!}r^{p-k}r^k \\ &= r^p\sum_{k=1}^{p}\binom{p}{k}\frac{n+1}{n}\cdot ...\cdot\frac{n+k-1}{n}\le r^p\sum_{k=1}^{p}\binom{p}{k}k! \\ &= r^p\sum_{k=1}^{p}p(p-1)...(p-k+1)\le r^p p\cdot p!\le r^p(p+1)!.\end{aligned}$$

Replacing in the above inequality, for all $|z|\le r$ it follows

$$\begin{aligned}|T_{n,p}(z)-e_p(z)| &\le r|T_{n,p-1}(z)-e_{p-1}(z)|+\frac{2(p-1)r}{n}\cdot[r^{p-1}p!+r^{p-1}] \\ &\quad +\frac{2r^p(p-1)}{n}\le r|T_{n,p-1}(z)-e_{p-1}(z)|+\frac{6(p+1)!r^p}{n}.\end{aligned}$$

Starting from $p=2,3,...$, and reasoning by mathematical induction with respect to p we get

$$\|T_{n,p}-e_p\|_r\le\frac{6r^p}{n}\left[\sum_{j=2}^{p}(p+1)!\right]\le\frac{6r^p(p+1)!(p-1)}{n}.$$

From the next Remark 1 after the present proof we have

$$W_n(f)(z) = \sum_{k=0}^{\infty} c_k W_n(e_k)(z) = \sum_{k=0}^{\infty} c_k T_{n,k}(z),$$

which from the hypothesis on c_k, implies that for all $|z| \le r$ and $n > n_0$, where $1 \le r < \min\{\frac{n_0}{2}, \frac{1}{A}\}$, we have

$$\begin{aligned}|W_n(f)(z) - f(z)| &\le \sum_{k=2}^{\infty} |c_k| \cdot |T_{k,n}(z) - e_k(z)| \\ &\le \sum_{k=2}^{\infty} M\frac{A^k}{k!} \cdot \frac{6r^k(k+1)!(k-1)}{n} \\ &= \frac{6M}{n}\sum_{k=2}^{\infty}(k+1)(k-1)(rA)^k = \frac{C_{r,A,M}}{n},\end{aligned}$$

where $C_{r,A,M} = 6M\sum_{k=2}^{\infty}(k+1)(k-1)(rA)^k < \infty$, for all $1 \le r < \frac{1}{A}$, because the series $\sum_{k=2}^{\infty} u^{k+1}$ and therefore its derivative $\sum_{k=2}^{\infty}(k+1)ku^{k-1}$ too, are uniformly and absolutely convergent in any compact disk included in the open unit disk.

(ii) Denote by γ the circle of radius $r_1 > r$ and center 0. Since for $|z| \le r$ and $v \in \gamma$, we have $|v - z| \ge r_1 - r$, by the Cauchy's formulas for all $|z| \le r$ and $n \in \mathbb{N}$, we obtain

$$|W_n^{(p)}(f)(z) - f^{(p)}(z)| = \frac{p!}{2\pi}\left|\int_{\gamma}\frac{W_n(f)(v) - f(v)}{(v-z)^{p+1}}dv\right|$$

$$\le \frac{C_{r_1,A,M}}{n}\frac{p!}{2\pi}\frac{2\pi r_1}{(r_1 - r)^{p+1}} = \frac{C_{r_1,A,M}}{n}\frac{p!r_1}{(r_1-r)^{p+1}},$$

which proves the theorem. □

Remarks. 1) The proof of the relation $W_n(f)(z) = \sum_{k=0}^{\infty} c_k W_n(e_k)(z)$ used in the proof of Theorem 1.9.1, (i) is indicated bellow. First we show that under the hypothesis in the statement of Theorem 1.9.1 we can write $f(x) = \sum_{k=0}^{\infty} c_k x^k$, for all $x \in [0,\infty)$, where the series is uniformly convergent in any compact interval $[0,b]$. Indeed, by hypothesis f is infinitely differentiable on $[0,\infty)$ with all the derivatives bounded by the same constant. Then, if we consider the Taylor series of f at 0 (here it is not important that $f : [0,\infty) \to \mathbb{C}$ since we can write the decomposition $f(x) = F(x) + iG(x)$ with $F, G : [0,\infty) \to \mathbb{R}$ and instead of f we reason on F and G), that is $\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$, with $x \ge 0$, then by the Lagrange form of the remainder it follows that $f(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$ for all $x \ge 0$, where the Taylor series uniformly converge on each compact interval $[0,b]$. Since for $z \in \mathbb{D}_R$ we have $f(z) = \sum_{k=0}^{\infty} c_k z^k$, obviously we obtain $f(x) = \sum_{k=0}^{\infty} c_k x^k$, for all $x \ge 0$.

For $m \in \mathbb{N}$ define

$$f_m(z) = \sum_{j=0}^{m} c_j z^j \text{ if } |z| \le r \text{ and } f_m(x) = \sum_{j=0}^{m} c_j x^j \text{ if } x \in (r, +\infty).$$

Clearly that on $[0, \infty)$, each f_m is infinitely differentiable with all its derivatives bounded by the same constant. Also, by the linearity of W_n we get

$$W_n(f_m)(z) = \sum_{k=0}^{m} c_k W_n(e_k)(z), \text{ for all } |z| \leq r.$$

For $n \in \mathbb{N}$, reasoning as in the proof of Theorem 1.9.1, (i) it follows that for all $|z| \leq r$ we have $|W_n(e_k)(z)| \leq r^k (k+1)!$, which implies

$$\begin{aligned} |W_n(f_m)(z)| &\leq \sum_{k=0}^{m} |c_k| \cdot |W_n(e_k)(z)| \leq \sum_{k=0}^{\infty} |c_k| \cdot |W_n(e_k)(z)| \\ &\leq M \sum_{k=0}^{\infty} (k+1)(rA)^k := M_r(f), \end{aligned}$$

for all $m \in \mathbb{N}$, $|z| \leq r$, $n > n_0$ with $1 \leq r < \min\{\frac{n_0}{2}, \frac{1}{A}\}$.

From Vitali's result (Theorem 1.0.1) it suffices to show that for any $n > n_0 \geq 3$ there exists $0 < x_0 < 1$ (depending on n) such that

$$\lim_{m \to \infty} W_n(f_m)(x) = W_n(f)(x), \text{ for } x \in [0, x_0].$$

Indeed, for $x_0 \geq 0$ we have $W_n(f)(x_0) = Z_n(f)(x_0)$ and denoting $\rho_0 = \frac{x_0}{1+x_0} \in [0, 1)$, by the hypothesis on f we get

$$\begin{aligned} &|W_n(f)(x_0) - W_n(f_m)(x_0)| \\ &= \left| (1+x_0)^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{x_0}{1+x_0} \right)^k \left[\sum_{j=m+1}^{\infty} c_j \left(\frac{k}{n} \right)^j \right] \right| \\ &\leq (1+x_0)^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \rho_0^k \sum_{j=m+1}^{\infty} |c_j| \frac{k^j}{n^j} \\ &= (1+x_0)^{-n} \frac{1}{n^{m+1}} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \rho_0^k \sum_{j=m+1}^{\infty} |c_j| \frac{k^j}{n^{j-m-1}} \\ &\leq (1+x_0)^{-n} \frac{1}{n^{m+1}} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \rho_0^k \sum_{j=m+1}^{\infty} |c_j| k^j \\ &\leq M(1+x_0)^{-n} \frac{1}{n^{m+1}} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \rho_0^k \sum_{j=m+1}^{\infty} \frac{(kA)^j}{j!} \\ &= \frac{M(1+x_0)^{-n}}{n^{m+1}} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \rho_0^k \left(e^{kA} - \sum_{j=0}^{m} \frac{(kA)^j}{j!} \right) \\ &\leq \frac{M(1+x_0)^{-n}}{n^{m+1}} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \rho_0^k \frac{e^{kA}(kA)^{m+1}}{(m+1)!} \\ &\leq \frac{M(1+x_0)^{-n}}{n^{m+1}} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \rho_0^k e^{2kA} \\ &= \frac{M(1+x_0)^{-n}}{n^{m+1}} \sum_{k=0}^{\infty} \binom{n+k-1}{k} (\rho_0 e^{2A})^k. \end{aligned}$$

Choose $x_0 > 0$ with $\rho_0 e^{2A} := \rho(x_0) < \frac{1}{n}$, where $n > n_0 \geq 3$. Because the function $h(t) = \frac{t}{1+t}$ is continuous and $h(0) = 0$, this is always possible. We get

$$(1+x_0)^n |W_n(f)(x_0) - W_n(f_m)(x_0)| \leq \frac{M}{n^{m+1}} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \rho(x_0)^k$$
$$\leq \frac{M}{n^{m+1}} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{1}{n^k}.$$

For $n > n_0 \geq 3$, by the ratio test we easily get that the series $\sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{1}{n^k}$ is convergent, which implies

$$\lim_{m\to\infty} W_n(f_m)(x_0) = W_n(f)(x_0).$$

But the above defined function $h(t)$ also is increasing on $[0, \infty)$, therefore for any $x \in [0, x_0]$ we get $\frac{x}{1+x} \leq \frac{x_0}{1+x_0}$ and $\rho(x) \leq \rho(x_0) < \frac{1}{n}$. Applying the above estimates for x instead of x_0 we get

$$\lim_{m\to\infty} W_n(f_m)(x) = W_n(f)(x),$$

which proves our assertion.

2) Simple examples of functions f satisfying the hypothesis in Theorem 1.9.1 are $f(z) = e^{-az}$, or $f(z) = sin(az)$ with $0 < a < 1$.

Now, let us recall that in the case of real Baskakov operators the following Voronovskaja-type formula is known.

Theorem 1.9.2. (Sikkema [163]) *If* $f : [0, +\infty) \to \mathbb{R}$ *is twice continuous differentiable on* $[0, +\infty)$*, then uniformly in any compact subinterval of* $[0, \infty)$ *we have*

$$\lim_{n\to\infty} n[W_n(f)(x) - f(x)] = \frac{x(1+x)}{2} f''(x).$$

In what follows we extend this result to the complex Baskakov operators obtaining, in addition, a quantitative estimate too.

We have :

Theorem 1.9.3. (Gal [90]) *Suppose that the hypothesis on the function* f *and on the constants* n_0, R, M, A *in the statement of Theorem 1.9.1 hold and let* $1 \leq r < \min\{\frac{n_0}{2}, \frac{1}{A}\}$ *be fixed. For all* $n > n_0, |z| \leq r$ *the following Voronovskaja-type result*

$$\left| W_n(f)(z) - f(z) - \frac{z(1+z)}{2n} f''(z) \right| \leq \frac{16M}{n^2} \sum_{k=3}^{\infty} (rA)^k (k-1)(k-2)^2,$$

holds, where for $rA < 1$ *we have* $\sum_{k=3}^{\infty} (rA)^k (k-1)(k-2) < \infty$.

Proof. Denote $e_k(z) = z^k$, $k = 0, 1, ...,$ and $T_{n,k}(z) = W_n(e_k)(z)$. By the proof of Theorem 1.9.1, (i), we can write $W_n(f)(z) = \sum_{k=0}^{\infty} c_k T_{n,k}(z)$, which implies

$$\left| W_n(f)(z) - f(z) - \frac{z(1+z)}{2n} f''(z) \right|$$
$$\leq \sum_{k=2}^{\infty} |c_k| \cdot \left| T_{n,k}(z) - e_k(z) - \frac{z^{k-1}(1+z)k(k-1)}{2n} \right|.$$

By the recurrence in the proof of Theorem 1.9.1, (i), satisfied by $T_{n,k}(z)$, denoting $E_{k,n}(z) = T_{n,k}(z) - e_k(z) - \frac{z^{k-1}(1+z)k(k-1)}{2n}$, we obtain the new recurrence

$$E_{k,n}(z) =$$

$$\frac{z(1+z)}{n}E'_{k-1,n}(z) + zE_{k-1,n}(z) + \frac{z^{k-2}(1+z)(k-1)(k-2)}{2n^2}[(k-2)+z(k-1)],$$

for all $k \geq 2$, $n \in \mathbb{N}$.

Therefore, for all $|z| \leq r$, $k \geq 2$, $n \in \mathbb{N}$ we obtain

$$\begin{aligned}|E_{k,n}(z)| \leq\ & \frac{r(1+r)}{n}|E'_{k-1,n}(z)| + r|E_{k-1,n}(z)| \\ & + \frac{(1+r)r^{k-2}(k-1)(k-2)}{2n^2}[(k-2)+r(k-1)].\end{aligned}$$

Since $E_{k-1,n}(z)$ is a polynomial of degree $\leq (k-1)$, by applying the Bernstein's inequality we get

$$\begin{aligned}|E'_{k-1,n}(z)| &\leq \frac{k-1}{r}\|E_{k-1,n}(z)\|_r \\ &\leq \frac{k-1}{r}\left[\|T_{n,k-1} - e_{k-1}\|_r + \frac{r^{k-2}(r+1)(k-1)(k-2)}{2n}\right] \\ &\leq \frac{k-1}{r}\left[\frac{6r^{k-1}k!(k-2)}{n} + \frac{r^{k-2}(r+1)(k-1)(k-2)}{2n}\right] \\ &\leq \frac{7r^{k-2}k!(k-1)(k-2)}{n}.\end{aligned}$$

Replacing above this inequality, for all $|z| \leq r$ we obtain

$$\begin{aligned}|E_{k,n}(z)| \leq\ & r|E_{k-1,n}(z)| + \frac{14r^k k!(k-1)(k-2)}{n^2} \\ & + \frac{(1+r)r^{k-2}(k-1)(k-2)}{2n^2}[(k-2)+r(k-1)] \\ \leq\ & r|E_{k-1,n}(z)| + \frac{16r^k k!(k-1)(k-2)}{n^2}.\end{aligned}$$

Since $E_{0,n}(z) = E_{1,n}(z) = E_{2,n}(z) = 0$, taking $k = 3, 4, \ldots,$ in the last inequality, step by step finally we arrive at

$$|E_{k,n}(z)| \leq \frac{16r^k}{n^2}\sum_{j=3}^{k} j!(j-1)(j-2) \leq \frac{16r^k k!(k-1)(k-2)^2}{n^2},$$

which implies

$$\begin{aligned}\left|W_n(f)(z) - f(z) - \frac{z(1+z)}{2n}f''(z)\right| &\leq \sum_{k=3}^{\infty}|c_k| \cdot |E_{k,n}(z)| \\ &\leq \frac{16M}{n^2}\sum_{k=3}^{\infty}(rA)^k(k-1)(k-2)^2.\end{aligned}$$

Here for $rA < 1$ we obviously have $\sum_{k=3}^{\infty}(rA)^k(k-1)(k-2) < \infty$, which proves the theorem. $\square$

Remark. Under the hypothesis in Theorem 1.9.3 we have the equivalence

$$\left\|W_n(f) - f(z) - \frac{e_1(1+e_1)}{2n}f''\right\|_r \sim \frac{1}{n^2}.$$

For the proof of this equivalence we follow the ideas in the proof of Theorem 1.8.2 (ii), with the only differences that we use the upper estimate for $|E_{k,n,1}(z)| := |E_{k,n}(z)|$ in Theorem 1.9.3 and the recurrence formula satisfied in this case by $A_{n,j}(z) := W_n[(\cdot - z)^j](z)$, that is

$$A_{n,j+1}(z) = z(1+z)[A'_{n,j}(z) + njA_{n,j-1}(z)],$$

(the proof of this recurrence formula uses the same Lemma 1.3 in Pop [156]). Finally, everything one reduces to prove that the differential equation

$$P_{2p+1}(z)f^{(2p+1)}(z) + P_{2p+2}(z)f^{(2p+2)}(z) = 0,\ z \in \mathbb{D}_r,$$

has the only solutions for f a polynomial of degree $\leq 2p$. Here by the above recurrence formula we get $P_{2p+1}(z) = c_p[z(1+z)]^p(1+2z)$ and $P_{2p+2}(z) = d_p[z(1+z)]^{p+1}$, with $c_p, d_p > 0$. The proof is easy by writing $y(z)$ in the form $y(z) = \sum_{k=0}^{\infty} a_k z^k$.

In what follows we obtain the exact degree in approximation by $W_n(f)(z)$.

The first main result is the following.

Theorem 1.9.4. (Gal [90]) *Suppose that the hypothesis on the function f and on the constants n_0, R, M, A in the statement of Theorem 1.9.1 hold and let $1 \leq r < \min\{\frac{n_0}{2}, \frac{1}{A}\}$. If f is not a polynomial of degree ≤ 1, then for all $n > n_0$ the lower estimate*

$$\|W_n(f) - f\|_r \geq \frac{C_r(f)}{n}$$

holds, where the constant $C_r(f)$ depends only on f (that is on A, M) and r.

Proof. For all $|z| \leq r$ and $n > n_0$ we get

$$W_n(f)(z) - f(z)$$

$$= \frac{1}{n}\left\{\frac{z(1+z)}{2}f''(z) + \frac{1}{n}\left[n^2\left(W_n(f)(z) - f(z) - \frac{z(1+z)}{2n}f''(z)\right)\right]\right\}.$$

We apply to this identity the following simple property :

$$\|F + G\|_r \geq |\, \|F\|_r - \|G\|_r \,| \geq \|F\|_r - \|G\|_r.$$

We get

$$\|W_n(f) - f\|_r$$

$$\geq \frac{1}{n}\left\{\left\|\frac{e_1(1+e_1)}{2}f''\right\|_r - \frac{1}{n}\left[n^2\left\|W_n(f) - f - \frac{e_1(1+e_1)}{2n}f''\right\|_r\right]\right\}.$$

Since f is not a polynomial of degree ≤ 1 in $\mathbb{D}_R$, we get $\left\|\frac{e_1(1+e_1)}{2}f''\right\|_r > 0$. Indeed, supposing the contrary it follows that $\frac{z(1+z)}{2}f''(z) = 0$ for all $z \in \overline{\mathbb{D}}_r$, which clearly implies $f''(z) = 0$ for all $z \in \overline{\mathbb{D}}_r \setminus \{0, -1\}$. Since f is analytic, from the identity theorem on analytic functions this implies that $f''(z) = 0$, for all $z \in \mathbb{D}_R$, that is f is a polynomial of degree ≤ 1, in contradiction with the hypothesis.

By Theorem 1.9.3 we have

$$n^2 \left\|W_n(f) - f - \frac{e_1(1+e_1)}{2n}f''\right\|_r \leq 16M\sum_{k=3}^{\infty}(rA)^k(k-1)(k-2)^2.$$

Therefore, there exists $n_1 > n_0$ depending only on f and r, such that for any $n \geq n_1$ we have

$$\left\|\frac{e_1(1+e_1)}{2}f''\right\|_r - \frac{1}{n}\left[n^2\left\|W_n(f) - f - \frac{e_1(1+e_1)}{2n}f''\right\|_r\right] \geq \frac{1}{2}\left\|\frac{e_1(1+e_1)}{2}f''\right\|_r,$$

which implies

$$\|W_n(f) - f\|_r \geq \frac{1}{n}\cdot\frac{1}{2}\left\|\frac{e_1(1+e_1)}{2}f''\right\|_r, \forall n \geq n_1.$$

For $n \in \{n_0+1, ..., n_1\}$ we get $\|W_n(f) - f\|_r \geq \frac{M_{r,n}(f)}{n}$ with $M_{r,n}(f) = n \cdot \|W_n(f) - f\|_r > 0$, which finally implies $\|W_n(f) - f\|_r \geq \frac{C_r(f)}{n}$ for all $n > n_0$, with $C_r(f) = \min\left\{M_{r,n_0+1}(f), ..., M_{r,n_1-1}(f), \ \frac{1}{2}\left\|\frac{e_1(1+e_1)}{2}f''\right\|_r\right\}$. This proves the theorem. □

From Theorem 1.9.4 and Theorem 1.9.1, (i), we get the following consequence.

Corollary 1.9.5. (Gal [90]) *Suppose that the hypothesis on the function f and on the constants n_0, R, M, A in the statement of Theorem 1.9.1 hold and let $1 \leq r < \min\{\frac{n_0}{2}, \frac{1}{A}\}$ be arbitrary fixed. If f is not a polynomial of degree ≤ 1, then for all $n > n_0$ the estimate*

$$\|W_n(f) - f\|_r \sim \frac{1}{n},$$

holds, where the constants in the equivalence depend only on f (i.e. on A, M) and r.

Regarding the simultaneous approximation we have the following result.

Theorem 1.9.6. (Gal [90]) *Suppose that the hypothesis on the function f and on the constants n_0, R, M, A in the statement of Theorem 1.9.1 hold and let $1 \leq r < r_1 < \min\{\frac{n_0}{2}, \frac{1}{A}\}$, $p \in \mathbb{N}$. If f is not a polynomial of degree $\leq \max\{1, p-1\}$, then for all $n > n_0$ the estimate*

$$\|W_n^{(p)}(f) - f^{(p)}\|_r \sim \frac{1}{n},$$

holds, where the constants in the equivalence depend only on f (that is on A, M), r, r_1 and p.

Proof. By Theorem 1.9.1, (ii) we have the upper estimate for $\|W_n^{(p)}(f) - f^{(p)}\|_r$, therefore it remains to prove the lower estimate for $\|W_n^{(p)}(f) - f^{(p)}\|_r$. Denoting by Γ the circle of radius r_1 and center 0 (where $\min\{\frac{n_0}{2}, \frac{1}{A}\} > r_1 > r \geq 1$), we have the inequality $|v-z| \geq r_1 - r$ for all $|z| \leq r$ and $v \in \Gamma$. By the Cauchy's formula we obtain

$$W_n^{(p)}(f)(z) - f^{(p)}(z) = \frac{p!}{2\pi i}\int_\Gamma \frac{W_n(f)(v) - f(v)}{(v-z)^{p+1}}dv.$$

As in the proof of Theorem 1.9.1, (ii), for all $v \in \Gamma$ and $n > n_0$ we get

$$W_n(f)(v) - f(v)$$

$$= \frac{1}{n}\left\{\frac{v(1+v)}{2}f''(v) + \frac{1}{n}\left[n^2\left(W_n(f)(v) - f(v) - \frac{v(1+v)}{2n}f''(v)\right)\right]\right\}.$$

Replaced in the above Cauchy's formula implies

$$\begin{aligned} W_n^{(p)}(f)(z) - f^{(p)}(z) &= \frac{1}{n}\left\{\frac{p!}{2\pi i}\int_\Gamma \frac{v(1+v)f''(v)}{2(v-z)^{p+1}}dv\right.\\ &\quad \left. + \frac{1}{n}\cdot\frac{p!}{2\pi i}\int_\Gamma \frac{n^2\left(W_n(f)(v) - f(v) - \frac{v(1+v)}{2n}f''(v)\right)}{(v-z)^{p+1}}dv\right\}\\ &= \frac{1}{n}\left\{\left[\frac{z(1+z)}{2}f''(z)\right]^{(p)}\right.\\ &\quad \left. + \frac{1}{n}\cdot\frac{p!}{2\pi i}\int_\Gamma \frac{n^2\left(W_n(f)(v) - f(v) - \frac{v(1+v)}{2n}f''(v)\right)}{(v-z)^{p+1}}dv\right\} \end{aligned}$$

and passing to the norm $\|\cdot\|_r$, for all $n > n_0$ we obtain

$$\|W_n^{(p)}(f) - f^{(p)}\|_r \geq \frac{1}{n}\left\{\left\|\left[\frac{e_1(1+e_1)}{2}f''\right]^{(p)}\right\|_r\right.$$

$$\left. - \frac{1}{n}\left\|\frac{p!}{2\pi}\int_\Gamma \frac{n^2\left(W_n(f)(v) - f(v) - \frac{v(1+v)}{2n}f''(v)\right)}{(v-z)^{p+1}}dv\right\|_r\right\},$$

where by Theorem 1.9.3, for all $n > n_0$ we have

$$\begin{aligned} &\left\|\frac{p!}{2\pi}\int_\Gamma \frac{n^2\left(W_n(f)(v) - f(v) - \frac{v(1+v)}{2n}f''(v)\right)}{(v-z)^{p+1}}dv\right\|_r\\ &\leq \frac{p!}{2\pi}\cdot\frac{2\pi r_1 n^2}{(r_1-r)^{p+1}}\left\|W_n(f) - f - \frac{e_1(1+e_1)}{2n}f''\right\|_{r_1}\\ &\leq 16M\sum_{k=3}^{\infty}(r_1A)^k(k-1)(k-2)^2\cdot\frac{p!r_1}{(r_1-r)^{p+1}}. \end{aligned}$$

But by hypothesis on f, we have $\left\|\left[\frac{e_1(1+e_1)}{2}f''\right]^{(p)}\right\|_r > 0$. Indeed, supposing the contrary it follows that $\frac{z(1+z)}{2}f''(z)$ is a polynomial of degree $\leq p-1$. Now, if $p=1$ and $p=2$ then the analyticity of f implies that f is a polynomial of degree $\leq 1 = \max\{1, p-1\}$, in contradiction with the hypothesis. If $p > 2$ then the analyticity of f implies that f is a polynomial of degree $\leq p-1 = \max\{1, p-1\}$, which is again in contradiction with the hypothesis.

Finally, reasoning exactly as in the proof of Theorem 1.9.4 we immediately obtain the required conclusion. □

At the end of this section we deal with the approximation properties of $Z_n(f)(z)$ given by

$$Z_n(f)(z) = (1+z)^{-n}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\left(\frac{z}{1+z}\right)^k f(k/n),$$

mentioned and discussed at the beginning of this section. First we present some sufficient conditions on f for the analyticity of $Z_n(f)(z)$.

Theorem 1.9.7. (Gal [90]) *Suppose that* $f : [0,\infty) \to \mathbb{C}$ *is of exponential growth on* $[0,\infty)$, *that is there exists* $M > 0$ *and* $A \geq 0$ *such that* $|f(x)| \leq Me^{Ax}$ *for all* $x \in [0,\infty)$.

(i) For any $n \in \mathbb{N}$ *there exists a* $0 < \rho < 1$ *(depending on* n*) such that* $Z_n(f)(z)$ *is well defined and analytic in the compact disk* $\overline{\mathbb{D}}_{\rho/2}$.

(ii) Let $r \geq 1$ *be fixed and denote* $n_0 = \left[\frac{2A}{ln(1+1/r^2)}\right] + 2$. *For all* $n \geq n_0$, $Z_n(f)(z)$ *is analytic in the compact semi-disk* $\overline{\mathbb{D}}_r^+ = \{z \in \mathbb{C}; |z| \leq r, Re z \geq 0\}$.

Proof. (i) It is clear that if $0 < \rho < 1$ then for $z \in \overline{\mathbb{D}}_{\rho/2}$ it follows $1+z \neq 0$ and therefore $(1+z)^{-n}$ is analytic. We get

$$|Z_n(f)(z)| \leq M|1+z|^{-n}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\left(\frac{|z|}{|1+z|}\right)^k e^{ak/n}.$$

For $z = x+iy$ with $|x| \leq \sqrt{x^2+y^2} \leq \rho/2$ we obtain (since $h(t) = t/(1-\rho+t)$ is increasing for $t \geq 0$)

$$\left(\frac{|z|}{|1+z|}\right)^2 = \frac{x^2+y^2}{1+2x+(x^2+y^2)} = \frac{x^2+y^2}{1-\rho+(\rho+2x)+(x^2+y^2)}$$
$$\leq \frac{x^2+y^2}{1-\rho+(x^2+y^2)} \leq \frac{(\rho/2)^2}{1-\rho+(\rho/2)^2}.$$

Denoting $\eta := \sqrt{\frac{(\rho/2)^2}{1-\rho+(\rho/2)^2}} < 1$ we obtain

$$|Z_n(f)(z)| \leq M|1+z|^{-n}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\eta^k e^{ak/n} := M|1+z|^{-n}\sum_{k=0}^{\infty}a_k.$$

We apply the ratio test to the last series. We have $\frac{a_{k+1}}{a_k} = \eta e^{a/n}\cdot\frac{k+n}{k+1}$. Since for $\rho \to 0$ we obtain $\eta \to 0$, for fixed n choose ρ so small that $\beta := \eta e^{a/n}n < 1$. By

$\frac{k+n}{k+1} \le n$ for all $k \ge 0$ obviously that $\frac{a_{k+1}}{a_k} \le \beta < 1$, for all $k \ge 0$, which proves that the series $\sum_{k=0}^{\infty} a_k$ is convergent and therefore $Z_n(f)(z)$ is analytic in $\overline{\mathbb{D}}_{\rho/2}$ for ρ sufficiently small (depending on n) chosen as above.

(ii) Clearly that for $z \in \overline{\mathbb{D}}_r^+$ we get $1+z \ne 0$ and therefore $(1+z)^{-n}$ is analytic. Reasoning exactly as at the above point (i) and denoting $\eta = \frac{r}{\sqrt{1+r^2}} < 1$, for all $z \in \overline{\mathbb{D}}_r^+$ we get

$$\begin{aligned}|Z_n(f)(z)| &\le M|1+z|^{-n}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\eta^k e^{Ak/n}\\ &= M|1+z|^{-n}\sum_{k=0}^{\infty}\binom{n+k-1}{k}[\eta e^{A/n}]^k.\end{aligned}$$

Denote $C = \frac{2A}{ln(1+1/r^2)}$, $n_1 = [C]+1 \ge C$. We get that for $n \ge n_0 > n_1$ we have

$$\eta e^{A/n} \le \eta e^{A/n_0} < \eta e^{A/n_1} \le \eta e^{A/C} = 1.$$

Denoting $\gamma = \eta e^{A/n_0} < 1$, for all $n \ge n_0$ we obtain

$$\begin{aligned}|Z_n(f)(z)| &\le M|1+z|^{-n}\sum_{k=0}^{\infty}\binom{n+k-1}{k}[\eta e^{A/n}]^k\\ &\le M|1+z|^{-n}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\gamma^k := M|1+z|^{-n}\sum_{k=0}^{\infty}a_k.\end{aligned}$$

Let $n \ge n_0$. By the ratio test applied to the last series we have $\frac{a_{k+1}}{a_k} = \gamma \cdot \frac{k+n}{k+1}$. Since there exists a k_0 with $\frac{k+n}{k+1} < 2-\gamma$ for all $k \ge k_0$, it follows that $\frac{a_{k+1}}{a_k} = \gamma \cdot \frac{k+n}{k+1} < \gamma(2-\gamma) < 1$, for all $k \ge k_0$. This implies the convergence of the series $\sum_{k=0}^{\infty} a_k$ and therefore we obtain the analyticity of $Z_n(f)(z)$. □

In what follows we show that $Z_n(f)(z)$ can be written under the form of divided differences.

Corollary 1.9.8. (Gal [90]) *(i) Suppose that f satisfies the hypothesis in Theorem 1.9.7. Then for any $n \in \mathbb{N}$ there exists a sufficiently small $0 < \rho < 1$ (depending on n) such that for all $z \in \overline{\mathbb{D}}_{\rho/2}$ we have*

$$Z_n(f)(z) = \sum_{j=0}^{\infty}\frac{n(n+1)...(n+j-1)}{n^j}[0, 1/n, ..., j/n; f]z^j.$$

(ii) Let $p \in \mathbb{N}\cup\{0\}$ be fixed. For any $n \in \mathbb{N}$ there exists a sufficiently small $0 < \rho < 1$ such that for all $z \in \overline{\mathbb{D}}_{\rho/2}$ we have

$$Z_n(e_p)(z) = \sum_{k=0}^{p}\frac{n(n+1)...(n+k-1)}{n^k}[0, 1/n, ..., k/n; e_p]z^k.$$

(iii) Let $p \in \mathbb{N}\cup\{0\}$ and $r \ge 1$ be fixed and denote $n_0 = \left[\frac{2}{ln(1+1/r^2)}\right]+2$. For all $n \ge n_0$ and $z \in \overline{\mathbb{D}}_r^+$ we have

$$Z_n(e_p)(z) = \sum_{k=0}^{p}\frac{n(n+1)...(n+k-1)}{n^k}[0, 1/n, ..., k/n; e_p]z^k.$$

Proof. (i) For fixed n let us define $g_{k,n}(z) = (1+z)^{-n-k}$, $k = 0, 1, 2, ...,$. Since $g_{k,n}$ is analytic in $\overline{\mathbb{D}}_{\rho/2}$, we get

$$g_{k,n}(z) = \sum_{p=0}^{\infty} \frac{g_{k,n}^{(p)}(0)}{p!} z^p = \sum_{p=0}^{\infty} (-1)^p \frac{(n+k-1+p)!}{(n+k-1)!p!} z^p,$$

which replaced in the expression of $Z_n(f)(z)$ implies

$$\begin{aligned} Z_n(f)(z) &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k f(k/n) \sum_{p=0}^{\infty} (-1)^p \frac{(n+k-1+p)!}{(n+k-1)!p!} z^p \\ &= \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} z^{k+p} \left[f(k/n)(-1)^p \frac{(n+k-1+p)!}{(n-1)!k!p!} \right] \\ &:= \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} z^{k+p} A_{k,p} = \sum_{j=0}^{\infty} z^j B_j. \end{aligned}$$

This implies the formulas

$$\begin{aligned} B_j &= \sum_{\nu=0}^{j} A_{\nu,j-\nu} = \sum_{\nu=0}^{j} f(\nu/n)(-1)^{j-\nu} \frac{(n+j-1)!}{(n-1)!\nu!(j-\nu)!} \\ &= \frac{(n+j-1)!}{(n-1)!n^j} \sum_{\nu=0}^{j} f(\nu/n)(-1)^{j-\nu} \frac{n^j}{\nu!(j-\nu)!} \\ &= \frac{(n+j-1)!}{(n-1)!n^j} [0, 1/n, ..., j/n; f], \end{aligned}$$

which proves the corollary.

(ii) It is immediate from the above point (i) since each e_p satisfies the hypothesis in Theorem 1.9.7 with $A = 1$.

(iii) By Theorem 1.9.7 (ii) we get that for all $n \geq n_0$, $Z_n(e_p)(z)$ is analytic in $\overline{\mathbb{D}}_r^+$. On the other hand, by the above point (ii) , $Z_n(e_p)(z)$ is a polynomial of degree $\leq p$ in a small disk $\overline{\mathbb{D}}_{\rho/2}$, that is its derivative of order $p+1$ is zero in $\overline{\mathbb{D}}_{\rho/2}$. By the identity theorem on analytic functions it is clear that the derivative of order $p+1$ of $Z_n(e_p)(z)$ also is zero in $\overline{\mathbb{D}}_r^+$, that is $Z_n(e_p)(z)$ is a polynomial of degree $\leq p$ in $\overline{\mathbb{D}}_r^+$. Since $Z_n(f)(z)$ analytically extends the values in $\overline{\mathbb{D}}_{\rho/2}$ to $\overline{\mathbb{D}}_r^+$, clearly that $Z_n(e_p)(z)$ must be of the same form

$$Z_n(e_p)(z) = \sum_{k=0}^{p} \frac{n(n+1)...(n+k-1)}{n^k} [0, 1/n, ..., k/n; e_p] z^k,$$

for all $z \in \overline{\mathbb{D}}_r^+$. □

We are in position to prove the following result.

Theorem 1.9.9. (Gal [90]) *For* $R > 1$ *suppose that* $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ *is analytic in* $\mathbb{D}_R$, *that is* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{D}_R$, *and that there exist*

$M > 0$ and $A \in (\frac{1}{R}, 1)$, with the property $|c_k| \le M\frac{A^k}{k!}$, for all $k = 0, 1, ...,$ (this implies $|f(z)| \le Me^{A|z|}$ for all $z \in \mathbb{D}_R^+$). In addition, let us suppose that f is of exponential growth on $[0, \infty)$ (for simplicity suppose that the exponent in the exponential growth also is A). Then the upper estimates in Theorems 1.9.1 (i) and 1.9.3 and the exact estimate in Corollary 1.9.5 hold for $Z_n(f)(z)$ in the compact semi-disk $\overline{\mathbb{D}}_r^+$, for any $1 \le r < R$ and $n \ge n_0$ with $n_0 = \left[\frac{2A}{ln(1+1/r^2)}\right] + 2$.

Proof. By Corollary 1.9.8 (iii), the recurrence for $Z_n(e_p)(z)$ is similar to that in the proof of Theorem 1.9.1, taking place for z in the compact semi-disk $\overline{\mathbb{D}}_r^+ = \{z; |z| \le r,\ Rez \ge 0\}$. The reasonings in the proofs of the corresponding results mentioned in the statement are similar. What remains to prove is the equality

$$Z_n(f)(z) = \sum_{k=0}^{\infty} c_k Z_n(e_k)(z),$$

for all $z \in \overline{\mathbb{D}}_r^+$ and $n \ge n_0$.

For this purpose, for any $m \in \mathbb{N}$ let us define

$$f_m(z) = \sum_{j=0}^{m} c_j z^j \text{ if } z \in \overline{\mathbb{D}}_r^+ \text{ and } f_m(x) = f(x) \text{ if } x \in (r, +\infty).$$

From the hypothesis on the coefficients of f it is clear that for any $m \in \mathbb{N}$ we have

$$|f_m(x)| \le \sum_{j=0}^{m} |c_j| x^j \le M \sum_{j=0}^{m} \frac{(Ax)^j}{j!} \le Me^{Ax}, \text{ for all } x \in [0, r],$$

which from the hypothesis on f on $[0, \infty)$, immediately implies that for any $m \in \mathbb{N}$ we have

$$|f_m(x)| \le Me^{Ax}, \text{ for all } x \in [0, \infty).$$

By Theorem 1.9.7 (ii) we get that $Z_n(f_m)(z)$, are well-defined and analytic in the compact semi-disk $\overline{\mathbb{D}}_r^+$, for all $m \in \mathbb{N}$ and $n \ge n_0$.

Denoting

$$f_{m,k}(z) = c_k e_k(z) \text{ if } z \in \overline{\mathbb{D}}_r^+ \text{ and } f_{m,k}(x) = \frac{f(x)}{m+1} \text{ if } x \in (r, \infty),$$

each $f_{m,k}$ is of exponential growth on $[0, \infty)$ (with the same exponent A) and $f_m(z) = \sum_{k=0}^{m} f_{m,k}(z)$. Since from the linearity of Z_n we have

$$Z_n(f_m)(z) = \sum_{k=0}^{m} c_k Z_n(e_k)(z), \text{ for all } z \in \overline{\mathbb{D}}_r^+,$$

it suffices to prove that $\lim_{m\to\infty} Z_n(f_m)(z) = Z_n(f)(z)$ for any fixed $n \ge n_0$ and $z \in \overline{\mathbb{D}}_r^+$. But this is immediate from $\lim_{m\to\infty} \|f_m - f\|_r = 0$, from $\|f_m - f\|_{B[0,+\infty)} \le$

$\|f_m - f\|_r$ and from the inequality

$$\begin{aligned}
&|Z_n(f_m)(z) - Z_n(f)(z)| \\
&\leq |1+z|^{-n} \sum_{j=0}^{\infty} \binom{n+j-1}{j} \left(\frac{|z|}{|1+z|} \right)^j \|f_m - f\|_{B[0,\infty)} \\
&\leq |1+z|^{-n} \sum_{j=0}^{\infty} \binom{n+j-1}{j} \left(\frac{r}{\sqrt{1+r^2}} \right)^j \|f_m - f\|_{B[0,\infty)},
\end{aligned}$$

valid for all $z \in \overline{\mathbb{D}}_r^+$. Here, by using the ratio test it follows that the series $\sum_{j=0}^{\infty} \binom{n+j-1}{j} \left(\frac{r}{\sqrt{1+r^2}} \right)^j$ is convergent by using the ratio test. Also, $\| \cdot \|_{B[0,+\infty)}$ denotes the uniform norm on $C[0,+\infty)$-which represents the space of all real-valued bounded functions on $[0,+\infty)$. □

1.10 Balázs-Szabados Operators

The goal of the present section is to obtain similar type of results for the rational complex Balázs-Szabados operators given by

$$R_n(f)(z) = \frac{1}{(1+a_n z)^n} \sum_{j=0}^{n} f(j/b_n) \binom{n}{j} (a_n z)^j,$$

where $a_n = n^{\beta-1}$, $b_n = n^{\beta}$, $0 < \beta \leq 2/3$, $n \in \mathbb{N}$ and $z \in \mathbb{C}, z \neq -\frac{1}{a_n}$.

The above complex form is obtained simply replacing x by z in the real form of rational operators introduced and studied in Balázs [33] and Balázs-Szabados [34]. Further studies on these operators in the case of real variable can be found in e.g. the paper Abel-Della Vecchia [1].

Remarks. 1) The complex operators $R_n(f)(z)$ are well-defined and analytic for all $n \geq n_0$ and $|z| \leq r < n_0^{1-\beta}$. Indeed, in this case we easily obtain that $z \neq -\frac{1}{a_n}$, for all $|z| \leq r < n_0^{1-\beta}$ and $n \geq n_0$, which implies that $\frac{1}{(1+a_n z)^n}$ is analytic.

2) There exists a close connection between $R_n(f)(z)$ and the classical complex Bernstein polynomials given by $B_n(f)(z) = \sum_{j=0}^{n} f(j/n) \binom{n}{j} z^j (1-z)^j$. Indeed, denoting $e_k(z) = z^k$, we easily get

$$R_n(e_k)(z) = n^{k(1-\beta)} B_n(e_k) \left(\frac{a_n z}{1+a_n z} \right),$$

valid for all $n \geq n_0$, $k \in \mathbb{N}$ and $|z| \leq r < n_0^{1-\beta}$. This connection will be essential in our reasonings.

First we will find some classes of analytic functions for which the uniform convergence of $R_n(f)(z)$ to $f(z)$ holds in some compact disks. As in the case of complex Favard-Szász-Mirakjan and complex Baskakov operators, for the complex Balázs-Szabados operators too let us note that in our results, the domain of definition of the approximated function $f : \mathbb{D}_R \bigcup [R, \infty) \to \mathbb{C}$ seem to be rather strange. However, the analyticity of f on $\mathbb{R}$ on $\mathbb{D}_R$ assures the representation $f(z) = \sum_{k=0}^{\infty} c_k z^k$, which is essential in the proof of quantitative estimates in any $\overline{\mathbb{D}}_r$ with $1 \leq r < R$ (while on $[0, \infty)$ the well known estimates in the case of real variable can be used).

Probably a more natural domain of definition for f would be a strip around the OX-axis, but in this case the representation $f(z) = \sum_{k=0}^{\infty} c_k z^k$ fails, fact which produces the failure of the proofs in this case.

Because of the Remark 2 from the beginning of this section first we need to deal with the estimate of $|B_n(e_k)(z)|$ for $|z| \leq r$ with $0 < r < 1$. Note that in the case when $r \geq 1$ the estimate of $|B_n(e_k)(z)|$ is completely different and it was found in the proof of Theorem 1.1.2 (i).

Lemma 1.10.1. *Denote* $\pi_{k,n}(z) = B_n(e_k)(z)$, $\|f\|_r = \sup_{|z| \leq r}\{|f(z)|\}$ *and consider* $0 < r < 1$.

(i) For all $|z| \leq r$, $k \in \mathbb{N} \bigcup \{0\}$ *and* $n \geq 1 + \frac{1}{r}$ *we have*

$$\|\pi_{k,n}\|_r \leq k!(1+r)r^k.$$

(ii) For all $|z| \leq r < 1$, $k = 0, 1, 2, ...$ *and* $n \in \mathbb{N}$ *we have* $\|\pi_{k,n}\|_r \leq r^k + \frac{(1+r)k(k-1)}{2n}$.

Proof. (i) We consider the following recurrence formula for Bernstein polynomials (see the proof of Theorem 1.1.2 (i))

$$\pi_{k+1,n}(z) = \frac{z(1-z)}{n}\pi'_{k,n}(z) + z\pi_{k,n}(z),$$

$z \in \mathbb{C}$, $k = 0, 1, 2, ...,$, $n \in \mathbb{N}$.

We will use the mathematical induction. For $k = 0$ the inequality in statement is obvious. Suppose that it is true for k. By the above recurrence, by the Bernstein's inequality (since $\pi_{k,n}(z)$ is a polynomial of degree $\leq k$) and since $\frac{k(1+r)}{n} \leq kr$ we obtain

$$\begin{aligned}\|\pi_{k+1,n}\|_r &\leq \frac{r(1+r)}{n} \cdot \frac{k}{r} \cdot \|\pi_{k,n}\|_r + r\|\pi_{k,n}\|_r \\ &\leq k!(1+r)r^k \left[k \cdot \frac{1+r}{n} + r\right] \\ &\leq k!(1+r)r^k[(k+1)r] = (k+1)!(1+r)r^{k+1}.\end{aligned}$$

(ii) Applying the Bernstein's inequality to the above recurrence formula , for all $|z| \leq r$ we get

$$\begin{aligned}|\pi_{k+1,n}(z)| \leq\ & \frac{r(1+r)}{n}\frac{k}{r} \cdot \|\pi_{k,n}\|_r + r|\pi_{k,n}(z)| \leq r|\pi_{k,n}(z)| \\ &+ \frac{(1+r)}{n} \cdot k\|\pi_{k,n}\|_r \leq r|\pi_{k,n}(z)| + \frac{(1+r)k}{n},\end{aligned}$$

since by the proof of Theorem 1.1.2 (i) we have $\|\pi_{k,n}\|_1 \leq 1$, for all $k, n \in \mathbb{N}$.

Now, taking $k = 1, 2, 3, ...$ in the above inequality, by recurrence we easily obtain for all $|z| \leq r$

$$|\pi_{k,n}(z)| \leq r^k + \frac{1+r}{n}[1+2+...+(k-1)] = r^k + \frac{(1+r)k(k-1)}{2n},$$

which proves (ii) and the lemma. □

Also we need the following.

Lemma 1.10.2. *Let* $n_0 \geq 2$*,* $0 < \beta \leq 2/3$ *and* $\frac{1}{2} < r < \frac{n_0^{1-\beta}}{2}$*. If we denote* $r_{k,n}(z) = R_n(e_k)(z)$ *then for all* $n \geq \max\{n_0, \frac{1}{r^{1/\beta}}\}$*,* $|z| \leq r$ *and* $k = 0, 1, 2, ...,$ *we have*

$$|r_{k,n}(z)| \leq (k!) \cdot (2r)^k.$$

Proof. By Remark 2 from the beginning of this section it follows

$$R_n(e_k)(z) = n^{k(1-\beta)} B_n(e_k)\left(\frac{a_n z}{1+a_n z}\right).$$

But for all $n \geq n_0 \geq 2$ and $|z| \leq r < \frac{n_0^{1-\beta}}{2}$ it is easy to see that

$$\left|\frac{a_n z}{1+a_n z}\right| \leq \frac{a_n r}{1-a_n r} < 1.$$

Therefore, applying Lemma 1.10.1 (i) with $\frac{a_n r}{1-a_n r}$ instead of r, for all $|z| \leq \frac{a_n r}{1-a_n r}$, $k \in \mathbb{N}\bigcup\{0\}$ and $n \geq 1 + \frac{1-a_n r}{a_n r} = \frac{1}{a_n r}$ we get

$$|r_{k,n}(z)| = n^{k(1-\beta)}\left|B_n(e_k)\left(\frac{a_n z}{1+a_n z}\right)\right|$$
$$\leq n^{k(1-\beta)} k! \left(1 + \frac{a_n r}{1-a_n r}\right)\left(\frac{a_n r}{1-a_n r}\right)^k.$$

But it is easy to see that the condition $n \geq \frac{1}{a_n r}$ is equivalent with $n \geq \frac{1}{r^{1/\beta}}$. Also, the conditions $n \geq \{n_0, \frac{1}{r^{1/\beta}}\}$, $|z| \leq r < \frac{n_0^{1-\beta}}{2}$ where $\frac{1}{2} < r$ implies that $\frac{n_0^{1-\beta}}{2} \leq \frac{a_n r}{1-a_n r}$.

Indeed, simple calculation shows that $\frac{n_0^{1-\beta}}{2} \leq \frac{a_n r}{1-a_n r}$ is equivalent to $n_0^{1-\beta} \leq r n^{\beta-1}(2 + n_0^{1-\beta})$. But $\frac{1}{2+n_0^{1-\beta}} < \frac{1}{2} \leq r$, which implies $\left(\frac{n_0}{n}\right)^{1-\beta} \leq 1 \leq r(2 + n_0^{1-\beta})$, that is exactly $n_0^{1-\beta} \leq r n^{\beta-1}(2 + n_0^{1-\beta})$.

In conclusion, for all $n \geq \max\{n_0, \frac{1}{r^{1/\beta}}\}$, $|z| \leq r < \frac{n_0^{1-\beta}}{2}$ and $k = 0, 1, 2, ...,$ we obtain

$$|r_{k,n}(z)| \leq n^{k(1-\beta)} k! \frac{n^{k(\beta-1)} r^k}{(1-a_n r)^{k+1}} \leq 2(k!)(2r)^k,$$

if we can prove that $\frac{1}{1-a_n r} < 2$ for $n \geq n_0 \geq 2$ and $r < \frac{n_0^{1-\beta}}{2}$. But $r < \frac{n_0^{1-\beta}}{2} < \frac{n^{1-\beta}}{2}$ implies $r < \frac{1}{2a_n}$ for all $n \geq n_0$, which is equivalent to $\frac{1}{1-a_n r} < 2$. The lemma is proved. □

Now we are in position to prove the following convergence result.

Theorem 1.10.3. *Let* $n_0 \geq 2$, $0 < \beta \leq 2/3$ *and* $\frac{1}{2} < r < R \leq \frac{n_0^{1-\beta}}{2}$. *Suppose that* $f : \mathbb{D}_R \bigcup [R, +\infty) \to \mathbb{C}$ *is uniformly continuous and bounded on* $[0, +\infty)$, *is analytic in* $\mathbb{D}_R$, *that is* $f(z) = \sum_{j=0}^{\infty} c_k z^k$ *for all* $z \in \mathbb{D}_R$ *and there exist* $M > 0$, $0 < A < \frac{1}{2r}$ *with* $|c_k| \leq M \frac{A^k}{k!}$ *for all* $k = 0, 1, 2, ...$, *(which implies* $|f(z)| \leq M e^{A|z|}$, *for all* $z \in \mathbb{D}_R$*). Then the sequence* $(R_n(f)(z))_{n \geq n_0}$ *is uniformly convergent to* f *in* $\mathbb{D}_r$.

Proof. First we prove that $R_n(f)(z) = \sum_{j=0}^{\infty} R_n(e_j)(z)$ for all $z \in \mathbb{D}_r$, where $e_j(z) = z^j$. In this sense, for any $m \in \mathbb{N}$ define

$$f_m(z) = \sum_{j=0}^{m} c_j z^j \text{ if } |z| \leq r \text{ and } f_m(x) = f(x) \text{ if } x \in (r, +\infty).$$

From the hypothesis on f it is clear that each f_m is bounded on $[0, +\infty)$, which implies that

$$|R_n(f_m)(z)| \leq \frac{1}{|1 + a_n z|^n} \sum_{j=0}^{n} (a_n |z|)^j \binom{n}{j} M(f_m) < \infty,$$

that is all $R_n(f_m)(z)$ with $n \geq n_0$, $r < \frac{n_0^{1-\beta}}{2}$, $m \in \mathbb{N}$ are well-defined for $z \in \mathbb{D}_r$.

Denoting

$$f_{m,k}(z) = c_k e_k(z) \text{ if } |z| \leq r \text{ and } f_{m,k}(x) = \frac{f(x)}{m+1} \text{ if } x \in (r, \infty),$$

it is clear that each $f_{m,k}$ is bounded on $[0, \infty)$ and that $f_m(z) = \sum_{k=0}^{m} f_{m,k}(z)$. Since from the linearity of S_n we have

$$R_n(f_m)(z) = \sum_{k=0}^{m} c_k R_n(e_k)(z), \text{ for all } |z| \leq r,$$

it suffices to prove that $\lim_{m \to \infty} R_n(f_m)(z) = R_n(f)(z)$ for any fixed $n \in \mathbb{N}$, $n \geq n_0$ and $|z| \leq r$. But this is immediate from $\lim_{m \to \infty} \|f_m - f\|_r = 0$, from $\|f_m - f\|_{B[0,+\infty)} \leq \|f_m - f\|_r$ and from the inequality

$$|R_n(f_m)(z) - R_n(f)(z)| \leq M_{r,n} \|f_m - f\|_{B[0,\infty)} \leq M_{r,n} \|f_m - f\|_r,$$

valid for all $|z| \leq r$. Here $\| \cdot \|_{B[0,+\infty)}$ denotes the uniform norm on $C[0, +\infty)$-the space of all real-valued bounded functions on $[0, +\infty)$.

Therefore for all $|z| \leq r$, $n \geq n_0$ we obtain

$$|R_n(f)(z)| \leq \sum_{k=0}^{\infty} |c_k| \cdot |R_n(e_k)(z)| \leq 2 \sum_{k=0}^{\infty} |c_k| \cdot k!(2r)^k \leq 2M \sum_{k=0}^{\infty} (2rA)^k < \infty.$$

Now, since we have $\lim_{n \to \infty} R_n(f)(x) = f(x)$ for all $x \in [0, r)$ (by Theorem 1 in Balázs-Szabados [34]), by the classical Vitali's theorem it follows that $R_n(f)(z)$ uniformly converges to $f(z)$ in $\mathbb{D}_r$. □

As in the case of complex Bernstein polynomials, for an upper estimate in approximation by $R_n(f)(z)$ we would need a recurrence formula. For this purpose, formally differentiating $R_n(f)(z)$ we obtain

$$R_n'(f)(z) = \frac{1}{(1+a_nz)^k}\sum_{j=1}^{n} f(j/b_n)\binom{n}{j} ja_n(a_nz)^{j-1} - \frac{na_n}{1+a_nz}R_n(f)(z).$$

Taking here $f(z) = e_k(z)$ by simple calculation we obtain $R_n'(e_k)(z) = \frac{b_n}{z}R_n(e_{k+1})(z) - \frac{na_n}{1+a_nz}R_n(e_k)(z)$, or denoting $r_{k,n}(z) = R_n(e_k)(z)$ finally we easily arrive to the recurrence formula

$$r_{k+1,n}(z) = \frac{z}{b_n}r_{k,n}'(z) + \frac{z}{1+a_nz}r_{k,n}(z),$$

valid for all $k = 0, 1, 2, ..., |z| \le r < \frac{n_0^{1-\beta}}{2}$ and $n \ge n_0$.

Note that from this recurrence formula we easily get that $r_{k,n}(z)$ is a rational function of the form $r_{k,n}(z) = \frac{P_{k,n}(z)}{(1+a_nz)^k}$ with $P_{k,n}(z)$ a polynomial of degree $\le k$ in z.

Also, simple calculation leads us to the recurrence

$$\begin{aligned} r_{k+1,n}(z) - z^{k+1} &= \frac{z}{b_n}[r_{k,n}(z) - z^k]' + \frac{z}{1+a_nz}[r_{k,n}(z) - z^k] \\ &\quad + z^k\left[\frac{k}{b_n} - \frac{a_nz^2}{1+a_nz}\right]. \end{aligned}$$

In order to use the kinds of reasonings in the case of complex Bernstein polynomials in Section 1.1, we need a Bernstein type inequality for rational functions. The following direct consequence of the Bernstein's inequality in closed unit disk for rational functions in Borwein-Erdélyi [47], Corollary 6, will be useful.

Corollary 1.10.4. *Let* $f(z) = \frac{p_k(z)}{\Pi_{j=1}^k(z-a_j)}$, *where* $p_k(z)$ *is a polynomial of degree* $\le k$ *and we suppose that* $|a_j| \ge R > 1$, *for all* $j = 1, ..., k$. *If* $1 \le r < R$ *then for all* $|z| \le r$ *we have*

$$|f'(z)| \le \frac{R+r}{R-r}\cdot\frac{k}{r}\|f\|_r.$$

Proof. Denote $g(u) = f(ru)$, $|u| \le 1$. We get

$$g(u) = \frac{p_k(ru)}{\Pi_{j=1}^k(ru-a_j)} = \frac{p_k(ru)/r^k}{\Pi_{j=1}^k(u-a_j/r)}, |u| \le 1.$$

Since $\frac{|a_j|}{r} \ge \frac{R}{r} > 1$, we can apply Corollary 6 in Borwein-Erdélyi [47] so that we obtain

$$|g'(u)| \le \frac{R/r+1}{R/r-1}\cdot k\|g\|_1 = \frac{R+r}{R-r}\cdot k\|g\|_1 = \frac{R+r}{R-r}\cdot k\|f\|_r.$$

But $g'(u) = rf'(ru)$, which proves the corollary. □

Now we are in position to prove the following upper estimate in approximation by $R_n(f)(z)$.

Theorem 1.10.5. *Let* $n_0 \geq 2$, $0 < \beta \leq 2/3$ *and* $\frac{1}{2} < r < R \leq \frac{n_0^{1-\beta}}{2}$. *Suppose that* $f : \mathbb{D}_R \bigcup [R, +\infty) \to \mathbb{C}$ *is uniformly continuous and bounded on* $[0, +\infty)$, *is analytic in* $\mathbb{D}_R$, *that is* $f(z) = \sum_{j=0}^{\infty} c_k z^k$ *for all* $z \in \mathbb{D}_R$ *and there exist* $M > 0$, $0 < A < \frac{1}{2r}$ *with* $|c_k| \leq M \frac{A^k}{k!}$ *for all* $k = 0, 1, 2, \ldots.$, *(which implies* $|f(z)| \leq M e^{A|z|}$, *for all* $z \in \mathbb{D}_R$*). Then for all* $n \geq \max\{n_0, \frac{1}{r^{1/\beta}}\}$ *and* $|z| \leq r$ *we have the upper estimate*

$$|R_n(f)(z) - f(z)| \leq \frac{1}{n^{1-\beta}} \cdot 2Mre^{2rA} + \frac{1}{n^{\beta}} \cdot \frac{8M}{r} \sum_{k=0}^{\infty} k(2rA)^k.$$

Proof. Let $|z| \leq r < R \leq \frac{n_0^{1-\beta}}{2}$, $n \geq n_0$ and $k \in \mathbb{N} \bigcup \{0\}$. From the last proved recurrence, Corollary 1.10.4 and Lemma 1.10.2 we obtain

$$\begin{aligned}
&|r_{k+1,n}(z) - z^{k+1}| \\
&\leq \frac{r}{b_n}[|r'_{k,n}(z)| + kr^{k-1}] + \frac{r}{1 - a_n r}|r_{k,n}(z) - z^k| + r^k \left[\frac{k}{b_n} + \frac{a_n r^2}{1 - a_n r}\right] \\
&\leq \frac{r}{b_n}\left[\frac{k}{r} \cdot \frac{n_0^{1-\beta} + r}{n_0^{1-\beta} - r} \cdot \|r_{k,n}\|_r\right] + \frac{kr^k}{b_n} + \frac{r}{1 - a_n r}|r_{k,n}(z) - z^k| + \frac{kr^k}{b_n} \\
&\quad + \frac{a_n r^{k+2}}{1 - a_n r} \\
&\leq \frac{r}{1 - a_n r}|r_{k,n}(z) - z^k| + \frac{2k(k!)}{b_n} \cdot (2r)^k \cdot \frac{n_0^{1-\beta} + r}{n_0^{1-\beta} - r} + \frac{2kr^k}{b_n} + \frac{a_n r^{k+2}}{1 - a_n r}.
\end{aligned}$$

But from the end of the proof of Lemma 1.10.2 we have $\frac{1}{1 - a_n r} < 2$ while the condition $r < \frac{n_0^{1-\beta}}{2}$ is equivalent to $\frac{n_0^{1-\beta} + r}{n_0^{1-\beta} - r} < 3$, which immediately implies

$$|r_{k+1,n}(z) - z^{k+1}| \leq 2r|r_{k,n}(z) - z^k| + \frac{6k(k!)}{b_n}(2r)^k + \frac{2kr^k}{b_n} + 2r^{k+2}a_n,$$

that is

$$|r_{k+1,n}(z) - z^{k+1}| \leq 2r|r_{k,n}(z) - z^k| + \frac{8k(k!)}{b_n}(2r)^k + 2r^{k+2}a_n,$$

for all $k = 0, 1, 2, \ldots.$ Taking step by step $k = 0, 1, 2, \ldots$ we easily obtain

$$\begin{aligned}
|r_{k,n}(z) - z^k| &\leq r^{k+1}a_n \sum_{j=1}^{k} 2^j + \frac{8(2r)^{k-1}}{b_n} \cdot \sum_{j=1}^{k-1} j(j!) \leq 2r^{k+1}a_n(2^{k-1} - 1) \\
&\quad + \frac{8(2r)^{k-1}}{b_n}k(k!) \leq (2ra_n)(2r)^k + \frac{8(2r)^{k-1}}{b_n}k(k!),
\end{aligned}$$

taking into account that $\sum_{j=1}^{k-1} j(j!) \leq k(k!)$.

Taking into account the hypothesis in the coefficients of f we therefore obtain

$$\begin{aligned}|R_n(f)(z)-f(z)| &\le \sum_{k=0}|c_k|\cdot|r_{k,n}(z)-z^k|\\ &\le 2Mra_n\sum_{k=0}^{\infty}\frac{(2rA)^k}{k!}+\frac{1}{b_n}\cdot\frac{8M}{r}\sum_{k=0}^{\infty}k(2rA)^k\\ &= \frac{2Mr}{n^{1-\beta}}\cdot e^{2rA}+\frac{1}{b_n}\cdot\frac{8M}{r}\sum_{k=0}^{\infty}k(2rA)^k,\end{aligned}$$

where $\sum_{k=0}^{\infty}k(2rA)^k<\infty$ for $2rA<1$. The theorem is proved. □

Remark. The upper estimate in Theorem 1.10.5 can obviously be written in the form

$$|R_n(f)(z)-f(z)|=O\left(a_n+\frac{1}{b_n}\right).$$

The Voronovskaja-type result for $R_n(f)(x)$, $x\in[0,\infty)$ was obtained by Balázs [33]. In the case of complex variable, the Voronovskaja-type result for $R_n(f)(z)$ can be stated as follows.

Theorem 1.10.6. *Let $n_0\ge 2$, $0<\beta\le 2/3$ and $\frac{1}{2}<r<R\le\frac{n_0^{1-\beta}}{2}$. Suppose that $f:\mathbb{D}_R\bigcup[R,+\infty)\to\mathbb{C}$ is uniformly continuous and bounded on $[0,+\infty)$, is analytic in $\mathbb{D}_R$, that is $f(z)=\sum_{j=0}^{\infty}c_kz^k$ for all $z\in\mathbb{D}_R$ and there exist $M>0$, $0<A<\frac{1}{2r}$ with $|c_k|\le M\frac{A^k}{k!}$ for all $k=0,1,2,....$, (which implies $|f(z)|\le Me^{A|z|}$, for all $z\in\mathbb{D}_R$). Then for all $n\ge\max\{n_0,\frac{1}{r^{1/\beta}}\}$ and $|z|\le r$ we have*

$$\left|R_n(f)(z)-f(z)+\frac{a_nz^2}{1+a_nz}f'(z)-\frac{a_n^2b_nz^4+z}{2b_n(1+a_nz)^2}f''(z)\right|$$

$$\le C_r(f)\left(a_n+\frac{1}{b_n}\right)^2.$$

Proof. We clearly have

$$\left|R_n(f)(z)-f(z)+\frac{a_nz^2}{1+a_nz}f'(z)-\frac{a_n^2b_nz^4+z}{2b_n(1+a_nz)^2}f''(z)\right|\le\sum_{k=0}^{\infty}|c_k|\cdot|E_{k,n}(z)|,$$

where

$$E_{k,n}(z)=r_{k,n}(z)-e_k(z)+\frac{a_nkz^{k+1}}{1+a_nz}-\frac{a_n^2b_nz^4+z}{2b_n(1+a_nz)^2}k(k-1)z^{k-2}.$$

Here we easily obtain $E_{0,n}(z)=E_{1,n}(z)=0$ for all z.

Now, from the recurrence formula for $r_{k,n}(z)$ obtained after the proof of Theorem 1.10.3, by long but simple calculation we obtain the recurrence for $E_{k,n}(z)$ given by

$$E_{k+1,n}(z)=\frac{z}{b_n}E'_{k,n}(z)+\frac{z}{1+a_nz}E_{k,n}(z)+A_{k,n}(z),$$

where

$$A_{k,n}(z)=\frac{ka_n^2z^{k+3}}{(1+a_nz)^2}+kz^{k+1}\cdot\frac{a_n}{b_n}\cdot\frac{6a_nz+2a_n^2z^2+5-k}{2(1+a_nz)^3}.$$

Under the hypothesis in the statement of Theorem 1.10.6, by the proof of Lemma 1.10.2 we have

$$\left|\frac{1}{1+a_nz}\right|\leq\frac{1}{1-a_nr}<2,$$

which immediately implies

$$\begin{aligned}|A_{k,n}(z)|&\leq C_rk(k+1)(k+2)(\max\{r^{k+1},r^{k+3}\})\left(a_n^2+\frac{a_n}{b_n}\right)\\&\leq C_rk(k+1)(k+2)(2r)^k\left(a_n+\frac{1}{b_n}\right)^2.\end{aligned}$$

Therefore, from the recurrence we obtain

$$|E_{k,n}(z)|\leq\frac{r}{b_n}|E'_{k-1,n}(z)|+2r|E_{k-1,n}(z)|+|A_{k,n}(z)|.$$

First we will estimate the quantity $|E'_{k-1,n}(z)|$. Let $r<r_1<R<\frac{n_0^{1-\beta}}{2}$ be with $r_1<\frac{1}{2A}$ and denote by Γ the circle of center 0 and radius r_1. By the Cauchy's theorem and taking into account the estimate for $\|r_{k-1,n}-e_{k-1}\|_{r_1}$ in the proof of Theorem 1.10.5, for all $|z|\leq r$ we obtain

$$\begin{aligned}|E'_{k-1,n}(z)|&\leq\frac{1}{2\pi}\left|\int_\Gamma\frac{E_{k-1,n}(u)du}{u-z}\right|\leq\frac{r}{r_1-r}\|E_{k-1,n}\|_{r_1}\\&\leq\frac{r}{r_1-r}\left[\|r_{k-1,n}-e_{k-1}\|_{r_1}+C_{r_1}k(k-1)(2r_1)^k\left(a_n+\frac{1}{b_n}\right)\right]\\&\leq C_{r,r_1}k(k-1)k!(2r_1)^k\left(a_n+\frac{1}{b_n}\right).\end{aligned}$$

By the inequality $\frac{1}{b_n}\left(a_n+\frac{1}{b_n}\right)\leq\left(a_n+\frac{1}{b_n}\right)^2$, this immediately implies

$$\begin{aligned}|E_{k,n}(z)|&\leq 2r|E_{k-1,n}(z)|+\frac{r}{b_n}C_{r,r_1}k(k-1)k!(2r_1)^k\left(a_n+\frac{1}{b_n}\right)+|A_{k,n}(z)|\\&\leq 2r|E_{k-1,n}(z)|+C_{r,r_1}(k-1)k(k+1)k!(2r_1)^k\left(a_n+\frac{1}{b_n}\right)^2.\end{aligned}$$

Now taking in this inequality $k=1,2,...$, by mathematical induction we easily arrive at

$$\begin{aligned}|E_{k,n}(z)|&\leq C_{r,r_1}\left(a_n+\frac{1}{b_n}\right)^2\left[\sum_{j=2}^{k}(j-1)j(j+1)j!(2r_1)^j\right]\\&\leq C_{r,r_1}\left(a_n+\frac{1}{b_n}\right)^2(k-1)k(k+1)(k+2)k!(2r_1)^k.\end{aligned}$$

Here the constants C_{r,r_1} can be different at each occurrence. Finally, taking into account the hypothesis too we obtain

$$\begin{aligned}&\left|R_n(f)(z) - f(z) + \frac{a_n z^2}{1+a_n z} f'(z) - \frac{a_n^2 b_n z^4 + z}{2b_n(1+a_n z)^2} f''(z)\right| \\ &\leq \sum_{k=0}^{\infty} |c_k| \cdot |E_{k,n}(z)| \\ &\leq C_{r,r_1}\left(a_n + \frac{1}{b_n}\right)^2 \cdot \sum_{k=2}^{\infty} |c_k| k!(k-1)k(k+1)(k+2)(2r_1)^k \\ &\leq C_{r,r_1}\left(a_n + \frac{1}{b_n}\right)^2 \sum_{k=2}^{\infty}(k-1)k(k+1)(k+2)(2r_1 A)^k,\end{aligned}$$

which proves the theorem since $2r_1 A < 1$. □

Now we are in position to obtain the exact degree of approximation for $R_n(f)(z)$.

Theorem 1.10.7. *Let* $n_0 \geq 2$, $0 < \beta \leq 2/3$, $\beta \neq 1/2$ *and* $\frac{1}{2} < r < R \leq \frac{n_0^{1-\beta}}{2}$. *Suppose that* $f : \mathbb{D}_R \bigcup [R, +\infty) \to \mathbb{C}$ *is uniformly continuous and bounded on* $[0, +\infty)$, *is analytic in* $\mathbb{D}_R$, *that is* $f(z) = \sum_{j=0}^{\infty} c_k z^k$ *for all* $z \in \mathbb{D}_R$ *and there exist* $M > 0$, $0 < A < \frac{1}{2r}$ *with* $|c_k| \leq M\frac{A^k}{k!}$ *for all* $k = 0, 1, 2, ...$, *(which implies* $|f(z)| \leq Me^{A|z|}$, *for all* $z \in \mathbb{D}_R$*). If* f *is not a polynomial of degree* ≤ 1 *then for all* $n \geq \max\{n_0, \frac{1}{r^{1/\beta}}\}$ *we have*

$$\|R_n(f) - f\|_r \sim \left(a_n + \frac{1}{b_n}\right),$$

where the constants in the equivalence are independent of n.

Proof. We can write

$$\begin{aligned}&R_n(f)(z) - f(z) \\ &= \left(a_n + \frac{1}{b_n}\right)\left\{\frac{1}{a_n + 1/b_n} \cdot \frac{-a_n z^2 f'(z)}{1 + a_n z} + \frac{1}{a_n + 1/b_n} \cdot \frac{(a_n^2 b_n z^4 + z) f''(z)}{2b_n(1+a_n z)^2}\right. \\ &\quad + \left(a_n + \frac{1}{b_n}\right) \\ &\quad \left.\cdot \left[\frac{1}{(a_n + 1/b_n)^2}\left(R_n(f)(z) - f(z) + \frac{a_n z^2 f'(z)}{1+a_n z} - \frac{(a_n^2 b_n z^4 + z) f''(z)}{2b_n(1+a_n z)^2}\right)\right]\right\}.\end{aligned}$$

Since $a_n + \frac{1}{b_n} \to 0$ as $n \to \infty$, taking into account the estimate in Theorem 1.10.6 and the reasonings in the cases of the previous complex Bernstein-type operators, it remains to show that for sufficiently large n and for all $|z| \leq r$ we have $|T(z)| > \rho > 0$, where ρ is independent of n and

$$T(z) := \frac{1}{a_n + 1/b_n} \cdot \frac{-a_n z^2 f'(z)}{1 + a_n z} + \frac{1}{a_n + 1/b_n} \cdot \frac{(a_n^2 b_n z^4 + z) f''(z)}{2b_n(1 + a_n z)^2}.$$

But since $a_n = n^{\beta-1} \to 0$ as $n \to \infty$ and $b_n = n^{\beta}$, we obtain

$$\left|\frac{1}{a_n + 1/b_n} \cdot \frac{-a_n z^2 f'(z)}{1 + a_n z}\right| \geq \frac{n^{2\beta-1}}{1 + n^{2\beta-1}} \cdot \frac{1}{1 + a_1 r}|z^2 f'(z)|$$
$$= \frac{n^{2\beta-1}}{1 + n^{2\beta-1}} \cdot \frac{|z^2 f'(z)|}{1 + a_1 r}.$$

If $2\beta - 1 < 0$ then $\frac{n^{2\beta-1}}{1+n^{2\beta-1}} \to 0$ and therefore the term $\frac{1}{a_n+1/b_n} \cdot \frac{-a_n z^2 f'(z)}{1+a_n z} \to 0$ as $n \to \infty$, uniformly with respect to $|z| \leq r$. In this case the term does not count for our estimate.

If $2\beta - 1 > 0$ then $\frac{n^{2\beta-1}}{1+n^{2\beta-1}} \to 1$ and therefore for sufficiently large n we have

$$\left|\frac{1}{a_n + 1/b_n} \cdot \frac{-a_n z^2 f'(z)}{1 + a_n z}\right| \geq \frac{1}{2} \cdot \frac{|z^2 f'(z)|}{1 + a_1 r}$$

For the other term it follows

$$\frac{1}{a_n + 1/b_n} \cdot \frac{(a_n^2 b_n z^4 + z) f''(z)}{2b_n(1 + a_n z)^2} = \frac{n^{3\beta-2}}{1 + n^{2\beta-1}} \cdot \frac{z^4 f''(z)}{2(1 + a_n z)^2}$$
$$+ \frac{z f''(z)}{2(1 + n^{2\beta-1})(1 + a_n z)^2}.$$

Here it is clear that if $2\beta - 1 > 0$ then this term converges to 0 (uniformly with respect to $|z| \leq r$) and therefore does not count for our estimate, while if $2\beta - 1 < 0$, then $n^{2\beta-1} \to 0$ (as $n \to \infty$) and here only the term $\frac{z f''(z)}{2(1+n^{2\beta-1})(1+a_n z)^2}$ counts for the estimate.

Therefore, concluding all the above reasonings, there exists an index n_1 depending on β, such that if $2\beta - 1 > 0$ then for all $n \geq n_1$ and all $|z| \leq r$ we have $|T(z)| \geq \frac{1}{2} \cdot \frac{|z^2 f'(z)|}{1+r}$, while if $2\beta - 1 < 0$ then for all $n \geq n_1$ and all $|z| \leq r$ it follows

$$|T(z)| \geq \frac{1}{2} \cdot \left|\frac{z f''(z)}{2(1 + a_n z)^2}\right| \geq \frac{|z f''(z)|}{4(1 + r)^2}.$$

Since f is not a polynomial of degree ≤ 1, it easily follows that $\|e_2 f'\|_r > 0$ and $\|e_1 f''\|_r > 0$, which implies that in both cases $2\beta - 1 > 0$ and $2\beta - 1 < 0$ we have $\|T\|_r > \rho > 0$, with ρ independent of n.

In the case of $2\beta - 1 = 0$, that is $\beta = 1/2$, we obtain

$$\frac{n^{3\beta-2}}{1 + n^{2\beta-1}} \cdot \frac{z^4 f''(z)}{2(1 + a_n z)^2} = \frac{n^{1/2} z^4 f''(z)}{4(1 + a_n z)^2} \to \infty,$$

as $n \to \infty$, so that the case $\beta = 1/2$ remains unsettled.

In conclusion,

$$\|R_n(f) - f\|_r \geq \left(a_n + \frac{1}{b_n}\right)[\|T\|_r - \|G_n\|_r] \geq \left(a_n + \frac{1}{b_n}\right)\frac{1}{2} \cdot \|T\|_r,$$

for all $n \geq n_1$. (Here $(G_n(z))_{n\in\mathbb{N}}$ is a sequence of analytic functions uniformly convergent to zero with respect to $|z| \leq r$).

In the case when $n = 1, 2, ..., n_1 - 1$ the lower estimate is trivial.

Finally, taking into account the upper estimate in Theorem 1.10.5 (in fact see the Remark after the proof of Theorem 1.10.5), our theorem is proved. □

1.11 Bibliographical Notes and Open Problems

Theorems 1.1.8, 1.1.9, 1.4.2, Lemmas 1.4.3, 1.4.4, Theorems 1.4.5, 1.5.5, 1.6.14, 1.6.15, 1.7.6, Lemma 1.8.9, Corollary 1.8.10, Lemma 1.10.1, 1.10.2, Theorem 1.10.3, Corollary 1.10.4, Theorems 1.10.5, 1.10.6, 1.10.7 are new and appear for the first time here.

Note 1.11.1. From a very long list, some references concerning the Bernstein-type operators of one real variable, different from those considered in this Chapter 1, for which would be possible to develop similar results are, for example : Altomare [8], Altomare-Campiti [9], Altomare-Mangino [10], Altomare-Raşa [11], Bleimann-Butzer-Hahn [46], Cimoca-Lupaş [55], Derriennic [56; 57; 59], Durrmeyer [67], Lupaş [129; 130; 131], Lupaş-Müller [132], Meyer-König and Zeller [136], Moldovan G. [140], Raşa [159], Soardi [168], Stancu [176].

Open Problem 1.11.2. A Voronovskaja-type formula with a quantitative estimate and the exact orders of approximation in Theorem 1.6.9 for the complex polynomials $S_n^{<\gamma>}(f)(z)$ remain as open questions.

Open Problem 1.11.3. It is well-known that if $f : [a, b] \to \mathbb{R}$, where $a, b \in \mathbb{R}$, $a < b$, the Bernstein polynomials attached to f are given by the formula

$$B_n(f; [a, b])(x) = \frac{1}{(b-a)^n} \sum_{k=0}^{n} f(a_k) \binom{n}{k} (x-a)^k (b-x)^{n-k},$$

$a_k = a + k\frac{b-a}{n}$. Now, if we consider the complexified form $B_n(f; [a, b])(z)$ for z belonging to a disk containing the real interval $[a, b]$ and we suppose that f is analytic in that disk, then similar results with those for $B_n(f; [0, 1])(z)$ in Sections 1.1, 1.2, 1.3 and 1.4 can be obtained. But more interesting seems to be the case when $[a, b]$ is a "complex" interval in $\mathbb{C}$, that is $a, b \in \mathbb{C}$ and $[a, b] = \{z = \lambda a + (1-\lambda)b; \lambda \in [0, 1]\}$. In this case, for the complexified form $B_n(f; [a, b])(z)$ as above, it would be of interest to study its approximation properties in a disk containing the "complex" interval $[a, b]$, when f is supposed to be analytic in that disk.

Open Problem 1.11.4. Let $L_n : C[a, b] \to C[a, b]$, $n \in \mathbb{N}$, $a, b \in \mathbb{R}$, $a < b$, be a sequence of positive and linear operators (of one real variable) attached to $f \in C[a, b]$, satisfying for example the conditions in the classical Korovkin theorem, that is $\lim_{n\to\infty} L_n(e_k)(x) = e_k(x)$, uniformly for $x \in [a, b]$, for $k = 0, 1, 2$ (here we recall that $e_k(x) = x^k$). Taking into account the results in this chapter, it is natural to ask if the convergence properties of the sequence $(L_n(f))_{n\in\mathbb{N}}$ remain valid if we complexify $L_n(f)(x)$ (that is if we replace $x \in [a, b]$ by $z \in \mathbb{D}_R$), supposing in addition that f can be extended to an analytic function in the disk $\mathbb{D}_R$ containing the real segment $[a, b]$.

In general this does not happen. For example, if one considers as $L_n(f)(x)$ the Hermite-Fejér interpolation polynomials on an infinite interpolatory matrix in $[-1, 1]$ consisting of the roots of orthogonal polynomials, or more general on an

APPROXIMATION BY COMPLEX BERNSTEIN AND CONVOLUTION TYPE OPERATORS

arbitrary infinite triangular matrix in $[-1,1]$, then for $f(z)=z$ the polynomials $L_{2n-1}(f)(z)$ diverges for all $z\in\mathbb{C}\setminus[-1,1]$, see Brutman-Gopengauz [49] and Brutman-Gopengauz-Vértesi [50].

However, for other sequences of positive and linear operators (possibly of non-interpolatory kind) the question remains open.

More exactly, it would be interesting to see if the Korovkin theory for the complexified sequence of a sequence of positive and linear operators (possibly under some additional hypothesis) still holds, that is if under some possible additional hypothesis on $L_n(e_0)$, $L_n(e_1)$ and $L_n(e_2)$, the conditions

$$\lim_{n\to\infty} L_n(f)(z)=e_i(z), i=0,1,2, \text{ uniformly in } \mathbb{D}_r,$$

would imply $L_n(f)(z)\to f(z)$ (as $n\to\infty$), uniformly in $\mathbb{D}_r$, for any analytic function in $\mathbb{D}_R$ and $r<R$.

Remark. From Vitali's theorem it is immediate that if $(L_n)_{n\in\mathbb{N}}$ is a sequence of positive and linear operators satisfying the Korovkin theorem and if in addition the complexified sequence satisfies

$$|L_n(e_k)(z)|\le M_r r^k, \text{ for all } |z|\le r,\ k\in\mathbb{N}\cup\{0\}, n\in\mathbb{N},$$

then for any analytic function f we have $\lim_{n\to\infty} L_n(f)(z)=f(z)$, uniformly in $\mathbb{D}_r$. The problem is how to reduce the additional conditions on the set $\{L_n(e_0)$, $L_n(e_1)$, $L_n(e_2)\}$ only. In the case of Bernstein-type operators this seems to be possible because of a recurrence formula with respect to k satisfied by $L_n(e_k)$.

Open Problem 1.11.5. It is left to the reader to prove that the upper estimates in the Voronovskaja-type results for q-Bernstein polynomials in Theorem 1.5.2 (ii) in fact hold with equivalence of order $\frac{1}{[n]_q^2}$.

More general, let us consider the exponential-type operators introduced in May [134] by the general formula $M_n(f)(x)=\int_a^b W_n(x,t)f(t)dt$, where the kernel $W_n: I(a,b)\times I(a,b)\to\mathbb{R}$ has the following properties : 1) $W_n(x,t)\ge 0$ for all $(x,t)\in I(a,b)\times I(a,b)$; 2) $\int_a^b W_n(x,t)dt=1$, for all $x\in I(a,b)$; 3) $\frac{\partial}{\partial x}W_n(x,t)=\frac{n(t-x)}{p(x)}W_n(x,t)$, where $p(x)$ is a strictly positive polynomial for $x\in I(a,b)$. Here $I(a,b)$ can be of the form $[a,b]$, $(-\infty,b]$, $[a,+\infty)$ or $(-\infty,+\infty)$. Also, denote

$$A_{n,m}(x)=n^m\int_a^b W_n(x,t)(t-x)^m dt.$$

Note that for $a=0$, $b=1$ and $p(x)=x(1-x)$ we obtain the Bernstein polynomials, for $a=0$, $b=\infty$ and $p(x)=x$ we obtain the Favard-Szász-Mirakjan operators, for $a=0$, $b=\infty$ and $p(x)=x(1+x)$ we obtain the Baskakov operators, for $a=0$, $b=\infty$ and $p(x)=x^2$ we obtain the Post-Widder operators, for $a=-\infty$, $b=\infty$ and $p(x)=1$ we obtain the Gauss-Weierstrass operators.

In Pop [156] the following generalized Voronovskaja's theorem was proved : if f is $2p$ continuous differentiable in $I(a,b)$ then for all $x \in I(a,b)$ we have

$$|M_n(f)(x) - \sum_{i=0}^{2p} \frac{1}{n^i i!} A_{n,i}(x) f^{(i)}(x)| = o(1/n^p).$$

Our conjecture is that for the complex operators

$$M_n(f)(z) = \int_a^b W_n(z,t) f(t) dt,$$

where f is supposed to be analytic in a suitable disk $\overline{\mathbb{D}}_r$ depending on the operator (see the theorems mentioned at the beginning of this Open Problem), we have the equivalence

$$\|M_n(f) - \sum_{i=0}^{2p} \frac{1}{n^i i!} A_{n,i} f^{(i)}\|_r \sim \frac{1}{n^{p+1}}.$$

For general $p \in \mathbb{N}$ and general operators $M_n(f)$, it could be useful the ideas in the proof for the complex Bernstein polynomials (that is the proof of Corollary 1.3.4), see also the proof of Theorem 1.8.2 (ii) for the case of complex Favard-Szász-Mirakjan operators.

Open Problem 1.11.6. The complex Bernstein-Stancu polynomials depending on the parameter $\gamma \geq 0$ defined for disks of center in origin are given by the formula

$$S_n^{<\gamma>}(f)(z) = \sum_{p=0}^{n} \binom{n}{p} \frac{z(z+\gamma)...(z+(p-1)\gamma)}{(1+\gamma)...(1+(p-1)\gamma)} \Delta_{1/n}^p f(0), |z| \leq r.$$

Writing now

$$\frac{z(z+\gamma)...(z+(p-1)\gamma)}{(1+\gamma)...(1+(p-1)\gamma)} = \sum_{j=0}^{p} A_{p,j} z^j,$$

where by identification of the coefficients we evidently can explicitly find each $A_{p,j} \in \mathbb{R}$, it follows

$$S_n^{<\gamma>}(f)(z) = \sum_{p=0}^{n} \binom{n}{p} \left[\sum_{j=0}^{p} A_{p,j} z^j \right] \Delta_{1/n}^p f(0), |z| \leq r.$$

Then, for $G \subset \mathbb{C}$ a compact set such that $\tilde{\mathbb{C}} \setminus G$ is connected and by using the Faber polynomials $F_p(z)$ attached to G (see Definition 1.0.10), for $f \in A(\overline{G})$ we can introduce the Bernstein-Stancu-Faber polynomials depending on the parameter $\gamma \geq 0$, given by the formula

$$\mathcal{S}_n^{<\gamma>}(f;\overline{G})(z) = \sum_{p=0}^{n} \binom{n}{p} \left[\sum_{j=0}^{p} A_{p,j} F_j(z) \right] \Delta_{1/n}^p F(0), z \in G,\ n \in \mathbb{N},$$

where Ψ is the unique conformal mapping Ψ of $\tilde{\mathbb{C}}\setminus\overline{\mathbb{D}}_1$ onto $\tilde{\mathbb{C}}\setminus G$ satisfying $\Psi(\infty)=\infty$ and $\Psi'(\infty)>0$ and $F(w)=\frac{1}{2\pi i}\int_{|u|=1}\frac{f(\Psi(u))}{u-w}du$, $w\in\mathbb{D}_1$. Here, since $F(1)$ is involved in $\Delta^n_{1/n}F(0)$ and therefore in the definition of $\mathcal{S}_n^{<\gamma>}(f;G)(z)$ too, in addition we will suppose that F can be extended by continuity on the boundary $\partial\mathbb{D}_1$.

It is an open problem to extend the results in Section 1.6 (corresponding there to $S_n^{<\gamma>}(f)(z)$, $|z|\le r$) to the Bernstein-Stancu-Faber polynomials $\mathcal{S}_n^{<\gamma>}(f;\overline{G})(z)$, $z\in G$ (exactly as we did for the complex Bernstein polynomials and Bernstein-Stancu polynomials depending on two parameters $0\le\alpha\le\beta$).

Open Problem 1.11.7. It is known that the Kantorovich variant of the complex Bernstein polynomial for compact disks is given by

$$K_n(f)(z)=B'_{n+1}(g)(z),\ g(z)=\int_0^z f(u)du,\ |z|\le r,$$

where $B_{n+1}(g)(z)=\sum_{k=0}^{n+1}\binom{n+1}{k}\Delta^k_{1/(n+1)}g(0)z^k$ denotes the complex classical Bernstein polynomial of degree $n+1$. This immediately implies the representation

$$K_n(f)(z)=\sum_{j=0}^{n}\binom{n+1}{j+1}(j+1)\Delta^{j+1}_{1/(n+1)}g(0)z^j,\ |z|\le r,$$

and suggests the following expression for the Kantorovich-Faber polynomials attached to a set $G\subset\mathbb{C}$

$$\mathcal{K}_n(f;\overline{G})(z)=\sum_{j=0}^{n}\binom{n+1}{j+1}(j+1)\Delta^{j+1}_{1/(n+1)}F(0)F_j(z),\ z\in\overline{G},n\in\mathbb{N},$$

with F defined as in the above Open Problem 1.11.6 (with g instead of f).

Following the same procedure and taking into account that the complex Stancu-Kantorovich polynomials depending on two parameters $0\le\alpha\le\beta$ are given by

$$K_n^{(\alpha,\beta)}(f)(z)=\frac{n+1+\beta}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}\Delta^{k+1}_{1/(n+\beta)}g[\alpha/(n+\beta)]z^j,|z|\le r,$$

the expression of Stancu-Kantorovich-Faber polynomials attached to a set $G\subset\mathbb{C}$, can be given by

$$\mathcal{K}_n^{(\alpha,\beta)}(f;\overline{G})(z)=\frac{n+1+\beta}{n+1}\sum_{j=0}^{n}\binom{n+1}{j+1}(j+1)\Delta^{j+1}_{1/(n+\beta)}F[\alpha/(n+\beta)]F_j(z),$$

$z\in\overline{G}$, where F is defined as above.

Remain as open questions the approximation properties of the polynomials $\mathcal{K}_n(f;\overline{G})(z)$ and $\mathcal{K}_n^{(\alpha,\beta)}(f;\overline{G})(z)$.

Open Problem 1.11.8. In a similar manner, taking into account that the complex Favard-Szász-Mirakjan operators for compact disks can be written in the form

$$S_n(f)(z)=\sum_{j=0}^{\infty}[0,1/n,...,j/n;f]z^j,|z|\le r,$$

for a set $G \subset \mathbb{C}$ we can formally define the Favard-Szász-Mirakjan-Faber operators by

$$\mathcal{S}_n(f;\overline{G})(z) = \sum_{j=0}^{\infty}[0, 1/n, ..., j/n; f]F_j(z), z \in \overline{G},$$

where $F(w) = \frac{1}{2\pi i}\int_{|u|=1}\frac{f(\Psi(u))}{u-w}du,\ w \in \mathbb{D}_1$ and we suppose that F can be extended by continuity on the boundary $\partial\mathbb{D}_1$. Let us observe that for any a with $|a| > 1$, $F(a)$ is well-defined.

It is an open question to find the approximation properties of the Favard-Szász-Mirakjan-Faber operators $\mathcal{S}_n(f;\overline{G})(z)$.

Open Problem 1.11.9. Prove Theorem 1.10.7 for the case $\beta = 1/2$.

Open Problem 1.11.10. As in the case of Favard-Szász-Mirakjan operators, it is clear that in the case of Baskakov operators too, the domain of definition $[R, +\infty) \bigcup \overline{\mathbb{D}}_R$ is rather unusual. Taking into account the form of complex Baskakov operators, more natural seems to consider and approximate the analytic function f on an annulus $A_{r,\infty} = \{z \in \mathbb{C}; r \leq |z+1| < \infty\}$, with $r > 0$, where f can be represented as a Laurent series. The study of this problem is left as an open question.

Open Problem 1.11.11. It is well known that in the case of Bernstein-type operators of real variables, starting with the paper of Altomare [7] a theory of strongly continuous contraction semigroups on Banach spaces obtained as a limit of the iterations of these operators is much developed. Then it would be interesting to consider the Altomare's idea in the case of Bernstein-type operators of complex variables. Note that taking into account, for example, the considerations from the beginning of Section 1.2, in the complex case would correspond a Trotter's theorem and a theory of limit semigroups on a Fréchet space (i.e. a metrizable complete locally convex space) with respect to the topology of uniform convergence on compacts subsets of the open disk $\mathbb{D}_R$.