Preface

The *Bernstein* polynomials attached to $f : [0,1] \to \mathbb{R}$ and given by

$$B_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f(\frac{k}{n}), \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, x \in [0,1],$$

probably are the most famous algebraic polynomials in Approximation Theory and were introduced in 1912 by S.N. Bernstein [38] in order to give the first constructive (and simple) proof to the Weierstrass' approximation theorem. Many books and papers were dedicated to their study, probably the most known being the book of Lorentz [125]. The importance of Bernstein polynomials also consists in the fact that their form has suggested and still continues to suggest to mathematicians the construction of a great variety of other approximation operators, like the *Schurer* polynomials, *Kantorovich* polynomials, *Stancu* polynomials, *q-Bernstein* polynomials, *Durrmeyer* polynomials, *Favard-Szász-Mirakjan* operators, *Baskakov* operators, and the list can continue with very many others.

A natural question concerning the Bernstein polynomials of a real variable (and by analogy, concerning any Bernstein-type operator of a real variable x) is the following : if in the expression of $B_n(f)(x)$ one replaces $x \in [0,1]$ by z in some regions in $\mathbb{C}$ (containing $[0,1]$) where f is supposed to be analytic, (a process we call *complexification*), then what convergence properties have the complex Bernstein polynomials

$$B_n(f)(z) = \sum_{k=0}^{n} p_{n,k}(z) f(\frac{k}{n}), \quad p_{n,k}(z) = \binom{n}{k} z^k (1-z)^{n-k}, z \in \mathbb{C} \ ?$$

In other words, the problem is to study the *overconvergence phenomenon* (not in the sense known as Wash's overconvergence in the interpolation of functions !) for the Bernstein polynomials, that is to extend their convergence properties and orders of these convergencies to larger sets in the complex plane than the real interval $[0,1]$.

The first goal of the present book is to give some answers in Chapter 1 to the above question of *overconvergence*, for several classes of Bernstein-type operators. In essence, it will be shown that for all the Bernstein-type operators, the orders of approximation from the real axis are preserved in complex domains too.

We recall that concerning the complex Bernstein polynomials, Wright [199], Kantorovich [113], Bernstein [39; 40; 41], Lorentz [125] and Tonne [190] have given interesting answers to this question. It is worth noting that an entire Chapter 4 of 38 pages is dedicated to it in the book of Lorentz [125]. In that book interesting convergence properties of $B_n(f)(z)$ and of its so-called degenerate form, in various domains in $\mathbb{C}$, like compact disks, ellipses, loops, autonomous sets are presented. Notice that in the above mentioned papers no quantitative estimates of these convergence results were obtained. Also, convergence results without any quantitative estimate were obtained for the complex Favard-Szász-Mirakjan operators by Dressel-Gergen and Purcell [65] and for the complex Jakimovski-Leviatan operators by Wood [200]. The above qualitative results are theoretically based on the "bridge" made by the classical result of Vitali (see Theorem 1.0.1), between the (well-established) approximation results for the Bernstein-type operators of real variable and those for the Bernstein-type operators of complex variable.

It is worth noting that in the other books or long surveys dealing with complex approximation, like those of Sewell [162], Dzjadyk [69], Gaier [76], Suetin [186], Andrievskii-Belyi-Dzjadyk [26], [27], or the surveys of Dyn'kin [62] and Andrievskii [25], the topic of the present book is not considered. Also, in other books like Lorentz [125] (Chapter 4) and Gal [77] (Chapters 3 and 4), the topic of the present book is attended in a tangential (and somehow different) way only.

In the Preface of their important book [60] in 1993 concerning the Approximation Theory of functions of real variable, DeVore and Lorentz note that, I cite "the Approximation Theory of functions of complex variables would require new books". The present book seeks to be one among the answers to this requirement and can briefly be described as follows.

In Chapter 1 one deepens the study of the approximation properties for the complex Bernstein polynomials $B_n(f)(z)$ in compact disks and in some special compact subsets of $\mathbb{C}$. In addition, similar results for other Bernstein-type polynomials/operators including those mentioned above are presented.

In detail, Chapter 1 can be described as follows :

– Section 1.0 contains the main results and concepts in complex analysis required for the proofs of the results in this book. For example, we mention here the Vitali's theorem, Cauchy's formula, Bernstein's inequality, Faber polynomials associated to a domain in $\mathbb{C}$, Faber series, Faber coefficients, Faber mapping.

– in Section 1.1 the exact orders in simultaneous approximation by $B_n(f)(z)$ and its derivatives, Voronovskaja's result with quantitative upper estimate and shape preserving properties of $B_n(f)(z)$ are obtained ; Subsection 1.1.1 contains the results on compact disks centered at origin while Subsection 1.1.2 contains some approximation results on compact sets in $\mathbb{C}$ for the so-called Bernstein-Faber polynomials ;

– Section 1.2 contains convergence results with quantitative estimates of the iterates of $B_n(f)(z)$, connected with the theory of the semigroups of operators and the shape preserving properties of these iterates, in the sense that beginning with an index they preserve some properties of f in Geometric Function Theory, like the starlikeness, convexity and spirallikeness ;

– in Section 1.3 the exact order in the generalized Voronovskaja's theorem for $B_n(f)(z)$ is obtained ;

– Section 1.4 presents the exact orders of approximation by Butzer's linear combinations of complex Bernstein polynomials and of Bernstein-Faber polynomials in compact disks and in compact Faber sets, respectively ;

– in Sections 1.5, 1.6, 1.7, 1.8, 1.9 and 1.10 we prove some similar properties for the complex q-Bernstein polynomials, Bernstein-Stancu polynomials, Bernstein-Kantorovich polynomials, Favard-Szász-Mirakjan operators, Baskakov operators and Balázs-Szabados operators, respectively. Besides the approximation results in compact disks for all these complex Bernstein-type operators, it is worth mentioning here the approximation results for the Bernstein-Stancu-Faber polynomials in compact sets in $\mathbb{C}$, a weighted-kind approximation result for the Favard-Szász-Mirakjan operator in strips and the study of two kinds of complex Baskakov operators generated by the real one ;

– Section 1.11 contains bibliographical notes and some open problems. The open problems mainly consist in proposals of similar studies for other types of complex Bernstein-type operators too.

In Chapter 2 we extend some of the results in Chapter 1 to the case of several complex variables. In Section 2.1 the concepts and results in the complex analysis of functions of several complex variables that we need for this chapter are presented. Section 2.2 deals with the approximation by two kinds of bivariate complex Bernstein polynomials, while Sections 2.3 and 2.4 treat the bivariate case of complex Favard-Szász-Mirakjan and Baskakov operators, respectively. All the approximation results are obtained in compact polydisks. Section 2.5 contains some bibliographical notes and open problems.

Chapter 3 deals with the approximation and geometric properties of several types of complex convolutions. Section 3.1 contains the approximation properties of some complex linear convolution polynomials : of de la Vallée Poussin, Fejér, Riesz-Zygmund, Jackson and Rogosinski kinds. More exactly, for these complex linear convolutions Voronovskaja-type results and the exact orders of approximation in compact disks are proved. Section 3.2 studies several kinds of linear non-polynomial convolutions. Thus, in Subsection 3.2.1 one studies the approximation properties (including the exact orders of approximation) in compact disks and compact sets in $\mathbb{C}$ of the non-polynomial complex convolutions of Picard, Poisson-Cauchy, and Gauss-Weierstrass. Also, their geometric properties are studied and applications to PDE in complex setting (i.e to heat and Laplace equations of complex spatial variable) are presented. In the Subsections 3.2.2, 3.2.3, 3.2.4 and 3.2.5 the approximation and geometric properties in compact disks of the complex q-Picard and q-Gauss-Weierstrass convolutions, Post-Widder complex convolution, rotationinvariant complex convolution and Sikkema complex convolution, respectively are presented. Section 3.3 contains the approximation and geometric properties of a nonlinear-type complex convolution in compact disks.

Finally, in Chapter 4 one presents several related topics : approximation by Bernstein polynomials of quaternion variable in Section 4.1, approximation of vector-valued functions of real and complex variables by operators of the type introduced in the previous chapters in Section 4.2 and strong approximation by Taylor series in the unit disk in Section 4.3.

Let us mention that most of the results in this book have been obtained by the author of this monograph : in a series of papers, single or jointly written with other researchers (as can be seen in the bibliography) and as new results that appear for the first time here.

It is important to note that the present book suggests for further research similar studies for other complex linear and nonlinear convolutions and for the complex forms of other Bernstein-type operators in approximation theory, like those of Durrmeyer-type, Meyer-König-Zeller-type, Jakimovski-Leviatan-type, Bleimann-Butzer-Hahn-type, Gamma-type, beta-type, to mention only a few.

The book mainly is addressed to researchers in the fields of complex approximation of functions and its applications, mathematical analysis and numerical analysis.

Also, since most of the proofs use elementary complex analysis, it is accessible to graduate students and suitable for graduate courses in the above domains.

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