Chapter One

Approaches to Accounting Theory

"Perhaps I am busied with pure numbers and the laws they symbolize: nothing of this sort is present in the world about me, this world of 'real fact.' And yet the world of numbers is also there for me, as the field of objects with which I am arithmetically busied; while I am thus occupied some numbers or constructions of a numerical kind will be at the focus of vision, girt by an arithmetical horizon partly defined, partly not; but obviously this being-there-for-me, like the being there at all, is something very different from this. *The arithmetical world is there for me only when and for so long as I occupy the arithmetical standpoint.*"

Edmund Husserl, Ideas p. 94 (italics original)

1.1. Historical Perspectives

Accounting is an ancient human activity. From the time when men and women first engaged in trade, whether for barter or money, it must have been necessary to keep some kind of record of incomings and out-goings, to which the origins of the double entry bookkeeping system can be traced. Already in the twelfth century of the Christian Era the Arabic writer Ibn Taymiyyah mentioned in his book Hisba (literally, "verification" or "calculation") accounting systems used by Muslims as early as the seventh century. A critical development in the history of accounting was the publication in Venice in 1494 of the book "Summa de Arithmetica, Geometria, Proportioni et Proportionalita" by the Franciscan monk and mathematician Luca Pacioli (1445-1517) – see Pacioli [1963]. This is the first known work to contain a detailed description of the practice of bookkeeping and the double entry system, "Particularis de Computis et Scripturis". Today it is widely regarded as the forerunner of modern bookkeeping practice. It was also Pacioli who introduced the symbols for plus and minus, which became standard notation in mathematics during the Renaissance. The first book on accounting in the English language appeared in London in 1543, authored by John Gouge. An important source for the early history of accounting is the writings of R. Mattessich ([1998], [2000], [2003], [2005b]).

While it seems clear that accounting was considered by Pacioli and his contemporaries to be part of arithmetic, its relationship with other parts of mathematics has had to wait much longer for recognition. The methods of statistics have long been used, almost since the importance of that branch of applied mathematics was first recognized in the seventeenth century. More recently probability theory and risk analysis have featured in economics. However, algebra has played little or no role, despite the precision of its language and its ability to describe complex situations concisely. The purpose of this monograph is to draw attention to the contribution that abstract algebra can make to accounting theory. Indeed it is the authors' contention that, at least in its deterministic form, accounting theory should be considered as a branch of applied algebra.

The book presents and develops a proof-based, algebraic approach to the study of accounting systems. The analysis provides a description of single firms in terms of abstract algebraic objects such as automata. It concentrates on the process of producing information from data provided by the environment through the double-entry system. This process, although considered by many to be the core of accounting, has often been ignored in accounting research. In attempting to address this issue, the book adds a level to the analysis of the information economists through the very act of exploring the production aspect of accounting information systems. The motivation is to expose the complexities and subtleties of information production in this field of research. The literature review which follows reflects the rather fragmented nature of the work which has been done up to this time in axiomatics, natural languages, formal grammars and information economics. The book shows how a basic accounting system can be represented as a formal algebraic language. The reduction of accounting systems to these types of languages will lead to a much stronger method of modeling information systems.

Although much discussion has occurred in the last fifty years concerning the treatment of accounting as a language and its justification as the language of business, surprisingly little progress has been made. This is perhaps due to the remarkable diversity of methods in linguistic research. In pure linguistic research, the various methods are divided into the natural language and formal language schools. The natural language schools study naturally occurring human languages as they have arisen from the historic acts of increasingly complex human communication. The formal language school arose from this tradition as methodologies were devised to study natural languages. These methodologies generally tried to reduce the complexity of natural language constructs to a finite system of grammatical rules. The formal language school became distinct from the natural language school when it was determined that certain domains of language, such as parts of mathematics and later computer science, could be completely specified by these finite systems.

Outside the area of pure linguistics, some applied fields such as speech communication and organizational behavior have adopted certain linguistic approaches and have developed other approaches independently. Semiotics has been used to determine what signs employees attend to in their everyday work relationships (Barley [1983]). Semiotics studies the meanings that people assign to language constructs in their search for understanding in their worlds. Recently, hermeneutics has been used to develop a criticism of the economics literature (McCloskey [1983]). This method employs the analysis of texts to identify repetition of linguistic constructs or changes in constructs over time and to study how the authors of the texts view their social realities.

With such a diversity of methods available, it is hardly surprising that the accounting profession has found little success in its search for a formalization behind the intrinsic meaning of the metaphor "accounting as the language of business". It is the contention of the present writers that the best way to proceed in the issue is to choose a potential methodological candidate, develop it and make a judgement based on its contribution to accounting research. The method chosen here is a formal, algebraic approach. In order to present this new approach to accounting in its contemporary setting, a detailed review of the language studies, both formal and natural, which have appeared in the accounting literature up to this point, is given in the sections which follow.

1.2. Algebraic and Proof-Based Approaches

As has been pointed out, the application of abstract algebra to accounting is something of a novelty. However, it would be wrong to suggest that nothing has been attempted in this direction. Already in 1894 the English algebraist Arthur Cayley wrote that "The principles of book-keeping by double entry constitute a theory which is mathematically by no means uninteresting; it is in fact, like Euclid's theory of ratios, an absolutely perfect one, and it is only its extreme simplicity which prevents it from being as interesting as it would otherwise be" (Cayley [1894]). Even before this time matrices had been introduced in the framework of accounting theory by Augustus De Morgan [1846], a route that was not followed by other writers until 100 years later. Indeed matrices reappeared as a topic of research interest in accounting only in the 1960s and 1970s, when a number of classic works in accounting theory were published, such as Edwards and Bell [1961], Chambers [1966], Ijiri [1967] and Mattessich [1964]. Here it should be understood that matrices were considered only as a tool to describe in a mathematical way the activity of accounting, and not as an attempt to formalize the concept of an accounting system. Paton [1922], one of the major personalities in accounting research in the United States in the 1920's, seems to have been the first author in formulate some accounting postulates. Nevertheless at that time fundamental research was not common in this area and the postulates never became part of a formal system.

Perhaps the most famous axiomatization of accounting was given by Mattesich [1957, 1964]. The first of these publications relies on a matrix formulation of accounting to provide structure to the axiomatic system. Three axioms are included in this schema: a plurality axiom, a double effect axiom and a period axiom. The first asserts that there exist at least two objects with a common measurable property. This provides a basis for the recording of transactions. The second axiom states the existence of an event which causes an increase of a property of one object and the corresponding decrease of the property of another. In effect this is an axiom of double entry. The last axiom requires that accounting systems are capable of being divided into time periods, thus providing a basis for the construction of financial statements. In addition to these axioms, the paper provides numerous definitions and "requirements", as well as several theorems.

The proofs of the theorems in Mattessich's first paper give insight into the formal relationship between the axioms and the theorems. The proofs consist of algebraic manipulations of matrices using the sigma, i.e., summation, notation. While in a sense this does serve to "demonstrate" the theorems from the definitions, the proofs do not consist of formal deductions from the axioms, as would be the case in a strict deductive system. Thus the axioms do not serve as a complete basis for the proofs of the theorems. In his second publication Mattessich shifts from a matrix to a set theoretical approach. In this work he relies on primitive terms, definitions stated using the set notation, and propositions. The theorems which are proved appeal to the definitions and propositions and are basically algebraic in nature. Perhaps the absence of axioms in this second work was due in part to the difficulty noted above, i.e., axioms which are not used in the proofs of the theorems. Some might argue that the propositions substitute for axioms in this formulation, but the propositions here are generally set theoretic definitions of such concepts as an accounting period or the chart of accounts. Although they may be invoked as a proof proceeds, the proofs do not begin with the propositions, nor are the theorems deduced from them. Again the beginnings of a formal proof-based system can be discerned here, but it is not coupled with a formal deductive scheme. This type of scheme may be provided by including the axioms of the mathematical system – in Mattessich's case an algebra – as part of the axiom scheme, thereby specifically allowing for mathematical inference within the axiomatized system, as will be seen below.

Ijiri's [1975] book on accounting measurement also includes three axioms, but again it lacks any derivation of theorems from the framework of these axioms. He does, however, derive his axioms from the theoretical structure of the accounting system which he provides in the book. Therefore it is likely that he sees these axioms more as general statements about accounting, rather than as a basis for any formal deductive system. Indeed he makes no attempt at all at proving the theorems. One of the contributions of the current book is that it provides not only an axiomatization of accounting systems, but also a deductive inference scheme which can operate on the axioms in a formal way to derive the theorems as consequences.

Tippett [1978] derived axioms of accounting measurement, and more recently Cooke and Tippett [2000] used a structural matrix to represent the restrictions imposed in a double-entry bookkeeping system, employing the information in the matrix to predict financial ratios. Willett [1987, 1988] demonstrated in two papers the derivation of axioms of accounting measurement, following Tippett's methods. His analysis extended to the stochastic space of accounting variables. Gibbons and Willett [1997], building on Willett's earlier work, demonstrated that accounting data produced from implemented information systems have a statistical nature due to the error generated by processing: that statistical nature is shown to be of value to decision makers under certain conditions. Nehmer and Robinson [1997] provided an initial description of the algebraic structure of accounting which is greatly expanded upon in this book. Nehmer [2010] encodes the algebraic structure in first order logic and derives consequences for the resulting structures.

Aukrust [1955, 1966] made an important contribution to the standard methodology for national and international accounts, completing a theoretical discussion of the underlying principles in accounting at the national level. He presented some problems of definition, classification and measurement of national accounts in an axiomatic way. After stating a set of twenty postulates, he showed that the structure of a simple system of national accounting can be derived from them. In this way it is possible to establish algebraic relations among national accounting concepts. Aukrust concludes: "The set of twenty postulates used above to derive a national accounting system is, of course, not the only one which could be conceived of. Others are equally feasible. Some would lead to national accounting systems different from the one described here, in much the same sense as non-Euclidean geometries are different from Euclidean geometry".

The problem of financial statements was dealt with by Arya et al. [2000], emphasizing the power of the double entry system to determine all consistent transaction vectors. They showed how a graphical representation of the accounting system can be used to obtain the characteristics of the vectors, solving in a simple way the problems of inverting and selecting the most likely transaction vector from the set of consistent transaction vectors. Arya et al. [2004] provided a systematic approach to reconciling diverse financial data. Again the key is the ability to represent the double entry system by a network of flows. Two specific uses are investigated: the reconciliation of audit evidence with management by means of prepared financial statements and the creation of transaction level financial ratios.

The first collaboration in the area between a philosopher of science and a theoretical accountant materialized in Balzer and Mattessich [1991, 2000]. They considered the reconstruction of yield to be a viable way of capturing the essence and basic structure of accounting as rigorously as possible. The proposed reconstruction showed that accounting has the same overall structure as other empirical theories by presenting nine axiomatic principles to establish the following concepts: economic objects, economic transactions, state-space for accounting, accounting data systems, accounts, double entry accounting systems, accounting morphisms and accounting systems (in general). By combining these definitions, they obtain the kernel of a model for accounting and they claim that all special methods and procedures used by accountants can be obtained from this core model with some appropriate specifications. All theorems are proved, but the authors indicate the need for further development of the axiomatic system presented in the paper and they present details of certain specifications to appear in future work.

According to Ellerman [1982, 1985, 1986], "Double-entry bookkeeping illustrates one of the most astonishing examples of intellectual insulation between disciplines, in this case, between accounting and mathematics". He described a mathematical basis for a treatment of double-entry bookkeeping in terms of the so-called "group of differences", sometimes called the *Pacioli group*: for details of this connection see 3.1 below. The possible use of the algebraic concept of a group in accounting theory is also considered in Brewer [1987] and Botafogo [2009], but with little progress beyond the formulation of some definitions.

There have been many other attempts to formalize accounting in a scientific way. Since the present work does not pretend to give an exhaustive history of accounting, only some of them have been mentioned. Details of other attempts can be found in Mattessich [1995, 1998, 2000, 2003, 2005a, 2005b].

On a final note, recently Demski [2007] has tried to answer to the question "Is accounting an academic discipline?" After analyzing the meaning of "discipline" and "academic", his immediate conclusion was negative. However, Demski was not pleased with this answer and therefore he preferred to analyze the ten indicators of the accounting as an academic discipline, ending with "... accounting is not today an academic discipline; it is an ever-narrowing insular vocational enterprise. But it could and should, in my opinion, be an academic discipline. Even if you disagree with my assessment, you should consider whether the state of academic accounting is, in your view, what it could and should be. The stakes in this game are enormous and serious".

1.3. Natural Language Approaches

Research in accounting as a natural language, as opposed to an proof-based system, has fallen into three broad categories: connotative and denotative meanings, readability of reports and linguistic relativity (McClure [1983]). The connotative and denotative meanings of language refer to its subjective and objective meanings respectively. The research in this category has emphasized the interpretation of accounting concepts by different groups including certified public accountants (CPA's), users, students and academics. The results have generally indicated agreement on the connotative meaning between groups, but there is some evidence of disagreement over denotative meanings (Belkaoui [1980b]). Research into the readability of financial reports has stressed the ability of the reports to communicate information on several levels. Levels of reading ability needed to comprehend the reports have been tested, but the tests were found to be inappropriate for the analysis of materials in a report format. Lebar [1982] tested several different types of financial report along an extentional - intentional axis. Extentional language is more descriptive and objective, whereas intentional language is more general and unqualified. She found that 10-K reports (a specific type of filing that a company makes to the Security and Exchange Commission) scored well on the extentional components as compared to the annual reports.

The third category of linguistic research in accounting is based on linguistic relativity (the Sapir-Whorf hypothesis). The two basic concepts of the hypothesis are that language determines thought and that consequently individuals with different linguistic backgrounds have different world views. Belkaoui [1978, 1980a] used this hypothesis to study disclosure issues in the area of pollution control costs, with results generally supporting the hypothesis. All three categories of research in accounting as a language have viewed it as a natural language and applied natural language techniques to its study.

Some more recent studies of business communication include Tyrvainen et al. [2005], who examine the internal and external communication of three business units, looking at digital, paper-based and oral communication. In a series of articles in the accounting area, Fisher [2004], Fisher and Garnsey [2006] and Garnsey and Fisher [2008] codify the professional accounting literature. This codification is then used to critique the adequacy of the literature (Fisher [2004]) and to examine amendments to the literature (Fisher and Garnsey [2006]). Garnsey and Fisher [2008] implement a software retrieval solution to the professional accounting literature.

An alternative approach is to view accounting as a formal language built up from a detailed specification of its grammar by exact rules of composition known as production rules. Formal grammars and languages were originally developed for natural language research and are still used there, especially in computational linguistics research. They have been largely absorbed into computer science because they are an alternative representation of finite state automata. Such automata are used in computer science for the general representation of computer languages. The concept is easy to relate to for anyone who have ever tried to learn a computer language with its peculiar sentence structure and rules. A good example of research using the automata approach is Cruz Rambaud and García Pérez [2005].

Demski et al. [2006], looking for a new language for the treatment of accounting information, examined the nature of quantum information in order to search for promising conceptual applications to accounting. They present some important features of quantum information such as quantum superposition, randomness, entanglement and unbreakable cryptography, and they begin to explore the possible link between quantum information and double-entry information which lies in the core of accounting information. The starting point is the work of Cayley [1894] on the parallel between the Euclid's theory of ratio and the double entry theory. As a consequence, it is intended to explore the possibility of a hybrid between accounting information and quantum information, "quantum double-entry information". In a second article, Demski et al. [2009] studied the applications of conceptual topology to quantum information and accounting information. The use of topology allows one to emphasize the qualitative characteristics of accounting information and to maintain the quantitative ones.

A reasonable and effective mathematization and axiomatization of the economy, and in particular of accounting, necessarily implies Diophantine formalisms (Velupillai [2005]), which raises issues of undecidability and non-computability. In the future there should be greater freedom for experimental research supported by alternative mathematical structures. In conclusion Velupillai speaks of "the notion of a Universal Accounting System, implied by and implying Universal Turing Machines and universality in cellular automata".

1.4. A Formal Grammar Approach

One exception to the exclusive use of natural language research methods in accounting is Stephens, Dillard, and Dennis [1985], hereafter Stephens et al. The article is entitled "Implications of Formal Grammars for Accounting Policy Development" and it presents a classification scheme for proposed and existing Financial Accounting Standards Board (FASB) statements. The examples provided in the article are partial formal grammars, reflecting the accounting rules promulgated by a specific standard. The level of analysis is macro in the sense that it considers the standard for all firms to which they apply. As such the analysis focuses on establishing criteria with which to evaluate standards through formal grammars. The three criteria used are possibility, consistency and resolution.

Possibility refers to the ability to reduce the statement to a formal grammar. One potential problem here, albeit one which is discussed in a different section of the article, is the difficulty in determining the primitives of the grammar. In the article the example of leases is cited. The determination of whether a certain economic event should be classified as a rental arrangement or a purchase has become increasingly problematic in accounting. Unless a clear demarcation is allowed or imposed on the "correct" interpretation of such an event under every circumstance, the formal grammar will not be capable of operating in these types of situation.

The second criterion, consistency, refers to the cross-statement compatibility of the grammars. This compatibility can perhaps best be addressed in terms of first order logic, rather than the formal grammar approach used in the article. The two systems are equivalent, so the change in approach is warranted. In first order logic consistency is defined in terms of the sentences which can be proved from the axioms. If both a sentence and its negation are provable from the axioms, then the system is inconsistent and in fact any sentence is then provable from the axiom system. In terms of the article, in order for a formal grammatical analysis to succeed, a single formal grammar containing all accounting standards must be demonstrated. Then any proposed new standards could be appraised in terms of their consistency with the current formal grammar.

The third criterion, resolution, is an attempt to deal with problems of inconsistency arising from the different rules specified in the single formal grammar mentioned above. The article proposes that uniform ranking rules be included in the grammar in order to remove such inconsistencies. It points out that the FASB does provide such rules in certain situations, but that the rankings so provided have not been uniform in the past. Stephens et al. classified the resulting inconsistencies as being due to one of three situations: arbitrary selection among possible standards, stipulation of standards without theory and the inability to write a definitive grammar.

In the first situation a choice is made and a particular standard must be selected, when alternative standards have possible correct economic interpretations and their own supporters. Stephens et al. contend that this and the next situation result primarily from lobbying by factions of the accounting community. The next situation occurs when a standard is stipulated which is lacking in theoretical support; this seems to mean lacking in terms of a justifiable economic interpretation. The interpretation is usually only provided *a posteriori* and may be thought of as imposing a new economic reality based on the standard. The last situation is the problem of specifying the primitives of the grammar, which was discussed under "possibility" above.

Stephens et al. divide the economic realm into three parts, the environment, accounting and decision. The effects of economic events in the environment are actions which play the role of primitives subject to the grammatical rules of accounting. The rules produce accounting results which are used by decision makers to produce decisions. Stephens et al. restrict their analysis to the accounting component only, so that the evidence of a transaction occurring is taken as a given and the use of the output is not analyzed. The same position is adopted throughout this book.

However, there are several differences between the article by Stephens et al. and this book, perhaps the most important being the level of analysis. The analysis presented here is at a micro level, as opposed to the macro level of the article. Specifically the analysis here pertains to the accounting system of a single firm. Secondly, a complete axiom system is developed for the firm, based on the double-entry components of the system only. The necessity of developing such a restricted system is based on the requirement of demonstrating the existence of such representations of accounting systems before proceeding with higher level analysis, as is recognized by the authors of the article.

A contribution of this research is to provide a basic method for constructing formal proofs in accounting. It interfaces with the axiomatization and formal inference scheme to yield a formal abstract specification; this leads directly to axioms for an accounting system, as well as to a system of inference which can be used to derive consequences of those axioms. In fact, the analysis of the paper includes the consideration of information systems as finite state grammars (FSG's) and automata. This representation is the basis of the computer languages which form the structure of any computerized system. Therefore finite state grammars can be used as a general representation of the process involved in converting states into signals. Such FSG's include relation as well as function operators, thereby providing a more powerful means of analysis in exploring the possibilities and limitations of the signal/output generation process of information systems.

The representation of information systems as FSG's serves two purposes in this analysis. First it addresses some problems noted below with information economics methodology, i.e., it provides a specific formulation of the internal production of information and assigns a specific interpretation to the states recognized by the system as well as its outputs. Furthermore, it allows for the production of multiple derivations from the capture of an additional piece of data. The second use of FSG's is to provide a convenient bridge between the representation of accounting systems as FSG's and their representation as proof-based systems. This is accomplished through the conversion of the production rules of the FSG into axioms of a first order logical system.

1.5. Information Systems in Information Economics

This book addresses some of the issues in the comparison of information systems which occur in the information economics literature, this being the current standard of comparison of systems in accountancy. A large body of work has been done in the area using utility analysis and relying on the results of Blackwell's "Comparison of Experiments" (Blackwell [1951]). As the title indicates, Blackwell's procedure shows that if an experiment A is a sufficient procedure for a different experiment B, then A is more informative than B, i.e., it provides at least as many statistical measures. Authors such as Gjesdal [1981] have used the matrix form of Blackwell's results to analyze different information systems. Demski [1980] and Demski, Patell and Wolfson [1984] have used the basic matrix framework of states crossed with signals and in the latter paper relied on Gjesdal's information systems comparison result. All of these comparisons of information systems are founded on the partitioning of the states of nature, the idea being that different information systems will be able to "recognize" different states at various levels of fineness. That is, a certain information system may produce signal Y_1 when it recognizes S_1 and signal Y_2 when it recognizes S_2, whereas another information system may not be able to distinguish S_1 from S_2, and produce the same signal for either realization.

The implication of the states to signals model of information systems is that there is a set of functions corresponding to the set of information systems under comparison. Mathematically the conversion of the states to signals is a mapping from the set of possible states to the set of possible signals. Over the entire state and signal spaces the function family is neither injective nor surjective. In the first place a particular information system function may map two or more states to the same signal, so the function is not injective. Secondly, an information system function may not be able to generate certain signals in the codomain at all, so it is not surjective. Indeed in Demski's 1980 examples, it is only in the perfect information case that the mapping can be bijective, i.e., both injective and surjective. It is this lack of uniformity in the construction of the state to signal functions (or information systems) regarding their relevant domains and codomains which partially explains the failure of Blackwell's comparison technique in proof-based systems.

Several topics are important to the present analysis. Firstly, Blackwell's result lies in the domain of experimental procedures, whereas an information system is, in a practical sense, an extant structure generating outputs from inputs by a formalized system of rules. As such there are several differences in the level of analysis which are apparent. Most importantly the information economics analysis considers the external or environmental states of nature only, without considering the internal states of the information system itself. It therefore ignores the interaction of the internal components of the system in the production of its outputs.

The unspecified nature of the internal components prevents the methodology from addressing questions relating to how changes in the configuration of the system will alter the signal set generated. Of course, information economists use the term "information system" in a different sense than is used here. But it is the difference in representation of the system which allows this additional analysis to occur. These are important questions for the accounting profession since they involve the production of information for decision makers in an organization from the design of the accounting system.

This lack of concern for the internal state of the system also forces the information economics methodology to ignore explicitly the problem of data capture versus information production. That is, a state may occur in the external environment which is captured or recognized by the system but is not processed in a timely manner. While the techniques of information economics do implicitly take this into consideration by collecting states into sets based on the concept of fineness, this does not help in determining why a particular output is not being generated, i.e., whether the data are being processed too slowly or are not available at all.

A second deficiency in the statistical analysis of information systems is its inability to recognize that a particular state may generate more than one signal. The information economics approach provides, at best under perfect information, a single state or a single signal mapping for output production from information systems. Under imperfect information several different states may produce the same signal, but the reverse situation, of a single state being mapped onto multiple signals, is not considered. For instance, a decline in interest rates may cause changes in pension funding requirements, a decline in the mortgage interest rates being paid by an organization and declines in the dividend rate expected from an investment in mutual funds. Further, it is possible within an information systems methodology to develop single states to multiple signals if a recognition of the interrelationships between states and signals is provided.

A final problem with the current method of analysis is that it does not provide a convenient way to interpret the states and signals. As an example consider Gjesdal's [1981] description of an information system. Here he reduces the system to merely the specification of the signal's functional form and proceeds to assert that "the nature of the signals is of no concern" (p. 212). One can only assume that the nature of the information system is of no concern as well, yet it is difficult to comprehend the purpose of comparing objects whose nature is not the object of comparison.

Generally the matrix representation of information systems and especially the concept of state (and hence signal) partitioning does not address the problem of how the signals are generated. This leaves open the question as to whether and to what extent in the context of axiomatic information systems, such a partitioning is possible. This problem is addressed in Chapters 2 and 3 where the algebraic core of the model is constructed.

Demski's ([1980]) analysis of information systems differs from an axiomatic approach in that his complete model consists of a set of acts, states, state probability functions and utility functions, with states and acts as parameters. This model is conditioned on the decision makers' experience and assumes that the four factors mentioned above are correctly specified. He presents two cases, the perfect and the imperfect information situations. Under perfect information, the decision maker can directly observe the realization of the states of nature. Therefore there is no need for the information system to produce signals relating to the acts of an agent. Since the state is known with certainty before an act is chosen, this situation will match well the derivations of an axiomatized information system. If the state is known for certain prior to the act, there must be some decision procedure which would indicate which state will occur and such a procedure is axiomatizable. This is the case because under state certainty the state must already be a fact, in which case its truth value is known or must be determinable under some formulation which perfectly correlates its own predictions with the actualization of those predictions. In the latter case, the decision procedure will be reducible to a first order formula, barring the serious consideration of some form of crystal gazing as providing perfect information. Of course Demski would agree that this is an unlikely situation in any complex decision making problem. In fact, testing which state will occur is likely to involve a great deal of computational complexity in a complex, decision making environment: the results, if they can be determined with certainty, may not be produced in a timely manner.

In the imperfect information situation the information system cannot necessarily distinguish each state uniquely, so the same signal or output from the system may occur after the realization of different states of nature. When the state is not known with certainty prior to the act, Demski posits the information system as producing a set of signals which may, but usually do not, indicate the state which was achieved or which has transpired. The signal is a function of the information system with the states as the input and the signals as the output.

One and only one signal is associated with each state occurrence, although the same signal may be produced by different states. The state space is partitioned into different subsets of the power set of the set of states, with the information system as the partitioning agent, i.e., different information systems produce different partitions. Of course Demski's book has a wealth of ideas and constructs which cannot be explored here, but with this basic framework in mind, we note that results for algebraic systems are obtainable which differ from Demski's conclusions.

In effect the analysis presented here adds a level to the work of the information economists, exploring the possible derivations of, and treating the information systems used in, their formulations as information systems: these are representable as systems of first order formulas and consequently are amenable to analysis as structures in model theory. Whereas the information economics approach formalizes information systems as collections of functions from the states to the signals, our approach imposes additional constraints on the production of the signals themselves by explicitly considering the language used to express the functional formulation of the information systems. These additional constraints will be of consequence when the information system is represented as a proof-based system.

1.6. Location of the Research Justified

Returning to the article of Stephens et al. [1985] which was discussed in 1.4, we note their description of accounting interfaces. If this description is reduced to an individual firm, then the accounting system of the firm can be seen as a filter which captures certain data from the environment, to be processed and presented to decision makers. It is this filter, the specification of which data are captured, how they are processed and in what general form they are presented, which locate this book within the accounting process. In this location an accounting system is constructed as a machine which follows strict rules, namely the axioms, in converting inputs to outputs. This procedure is also strictly defined as an inference scheme, determining how occurrences of inputs combine within the rules to produce outputs and other secondary rules. These outputs are the derivates of the accounting system and may also arise from combining rules only from within the inference scheme. Thus we are dealing with the construction of a deterministic system.

In addition the book considers how to control accounting systems which operate under different rules. This requires building on the derivations of rule-based accounting systems. The control is achieved as follows. The derivations of an axiom system can be thought of as formal deductions from given premises. In this case, the formal deductions arise from the inference scheme and the premises are the axioms and derivations already deduced. The control of accounting systems under this methodology would then look at the differences in the sets of consequence of the separate proof-based accounting systems.

As mentioned previously, the Stephens et al. article describes accounting as an environment to accommodate accounting information systems (AIS) and decision maker flow. The link between the environment and the AIS and between the AIS and the decision maker are both areas of considerable research in accounting. The first link contains problem areas involving the recognition of economic events as transactions. Research has been concerned with when and whether an economic event such as a contingency should be captured by the system and thereafter reported to the decision maker. The crux of this problem is when an economic event should be interpreted as being probable. On the other hand, the link between the AIS and the decision maker involves the interpretation of whether and under what circumstances data presented by the AIS change decisions and thereby become information of some value to decision makers. Thus there are two general types of interpretation which occur between the AIS and its environment and the AIS and the decision makers. However, in the context developed in this book a third and more formal approach to interpretation is employed. This third type occurs entirely within the AIS and is specifically related to the axiomatization of the system. Within axiomatized systems there is a formal logical interpretation, indeed an interpretation function, between the syntactic components of the system and their semantic interpretations.

1.7. Accounting and Formal Languages

The axioms and derivations of a formal system are strings of symbols called sentences or formulas. At the syntactic level these strings are manipulated by the inference scheme in a purely formal way, without regard to the meanings which may be attached to the original or deduced sentences. The syntactic level therefore is merely concerned with which sentences can be produced by following the inference scheme. So the only way that a sentence is in essence "meaningless" in syntax is if it is not derivable from the axioms via a sequence of inferences. A logical interpretation is a formal map from the syntactic level to the semantic level which provides meaning or a translation of the combination of symbols in the sentences.

As an example, consider the standard rule of inference *modus ponens*. According to this rule, if there are two sentences $x \rightarrow P(x)$ and x, then $P(x)$ is derivable. Notice that no meaning is attached to the symbols x, $\rightarrow$ or $P(x)$, so that modus ponens is a strictly syntactic construction. Now suppose the interpretation function maps x to "cash", $\rightarrow$ to "implies" and $P(x)$ to "x is a current asset". Then at the semantic level the interpretation of this instance of modus ponens is that "cash implies cash is a current asset", so that "cash is a current asset" is derivable.

The syntactic rules are akin to the grammatical rules of a natural language. In natural languages the meaning of a sentence is based on an interpretation of its form. This form is regulated by distinguishing which sentences are grammatical. However, the distinction between syntax and semantics in natural language is often a hazy one because the grammatical rules are not specified in advance, but have been deduced from the structure of the language by linguists. Therefore it may be impossible to describe accurately the syntax of a language by a finite number of rules. For example, a basic sentence form in English is subject-verb-object. This rule works well for "sensible" sentences such as "The computer ran the program." Unfortunately, without further rules of grammatical construction, a naive foreign speaker might deduce the following sentence from the rule: "The computer walked the program." The purpose of indicating this type of problem in a natural language is to point out the close relationship of both syntax and semantics to the interpretation of meaning in these languages. It appears as if the human mind attends to both syntax and semantics simultaneously through learned patterns when constructing the meaning of natural language sentences.

Another type of interpretation known as hermeneutics has been developed in the naturalistic research methodology. By using this methodology the researcher attempts to interpret the world as a text in order to understand the meanings which the actors in the study attach to objects, to themselves and others, and to actions. Here the objective is similar to reducing the semantic context of the world to a somewhat less complex and perhaps hidden syntactic component. The syntactic component in hermeneutics is seen to be dynamic, with the actors and their environment constantly interacting to reconstitute meaning and form. This technique is essentially a meta-analysis of sentences which not only looks at sentences in their own contexts, but also across the contexts of different actors and environments.

In order to relate accounting to the concepts of syntax and semantics, it should be remembered that these concepts are used in different ways in various types of analysis. In the case of natural language, the syntactic component of accounting is the systems of rules, such as the mechanics of double entry bookkeeping, statements of auditing standards, Financial Accounting Standards Board (FASB) and International Accounting Standards Board publications, and Security and Exchange Commission rulings which affect transactions and manipulations of transactions, including disclosure. As with all natural languages, the syntactic and semantic components lie very close to one another when accounting is viewed as a natural language. For example, take a common occurrence when beginning students are introduced to accounting for merchandizing firms. A typical error is for the student to debit inventory and credit accounts payable when merchandize is purchased on account, instead of debiting purchases. This may happen because the student is confused about the semantic meaning of the problem of costs of goods sold, as against the meaning of accounting for inventories.

A further phenomenon which occurs when accounting is viewed as a natural language is that the interpretational component becomes closely intertwined with both the syntactic and the semantic components. The conceptual framework and the FASB statements which refine previous interpretations in order to standardize interpretation of economic events indicate the closeness of this relationship. For example, FASB statement number 1 is an attempt to standardize the interpretation of what constitutes an operating lease, as opposed to a capital lease for both the lessee and the lessor. This is similar in form to the hermeneutic concept of interpretation acting as the meta-rule intermediating between the actors and their environment, in this case certified public accountants, their clients, the FASB members and the accounting environment.

The explanation of the interrelationships between form, meaning and interpretation was the original inspiration to the formulation of formal logics and proof-based systems. The ancient Greeks were concerned with problems of valid arguments, proceeding from the development of schools of rhetoric. At the time work was concentrated on developing techniques for identifying correct inferences and exposing fallacious ones. One of the arguments which arose was between Diodorus Cronus and his pupil Philo of Megara. The argument revolved around the correct interpretation of the rule of inference modus ponens, which is also known as a conditional statement. If the conditional statement is formulated as "$a \to b$", then a is termed the antecedent and b the consequent. Diodorus and Philo differed as to what would be the conclusion if the antecedent were false. Diodorus took the position that a false antecedent negated the conditional, so that the statement is false. Thus the statement "If the FASB is a governmental agency, then this book is deposited" is false in Diodorus' system, since the FASB is not a governmental agency. Philo took the opposite view, arguing that the only case where the conditional is false is when the antecedent is true and the consequent is false.

Philo's reasoning is important because his position became the standard one in formal logic. Under his interpretation, $a \to b$, which semantically might be read as "a implies b" or "if a, then b", is logically equivalent to "not a or b". In this case, the "or" is interpreted as inclusive, meaning that "not a is true" or "b is true" or "both are true". (In the case of an exclusive or, the last case is disallowed.) Under this interpretation, treating the sentence "If the FASB is a governmental agency, then this book is deposited" is equivalent to "either the FASB is not a governmental agency or this book is deposited", which is true since the FASB is not currently a governmental agency. Notice that the second clause "this book is deposited" can be either true or false and the entire statement remains true as long as the FASB remains independent. As such, the Philonian interpretation of false antecedents is often referred to as the case of trivial truth of the conditional.

Whatever justification there may be for the specific interpretations that have been given to inference schemes, and there are many equivalences between rules of inference schemes as well, the point is that the construction of formal systems requires the specification of exact syntactic rules and specific interpretive mappings to semantic meaning. In addition, even after the specification of the formal inference scheme, it may be possible to reduce the number of allowed inferences by eliminating inferences which are logically equivalent to one another. For example, many of the inferences allowed in formal logics currently used in philosophical and linguistic texts on the subject were developed in the Middle Ages by the scholastics in order to match natural language inferences used in disputation and rhetoric. One such rule of inference is *modus tollens*, a type of negated modus ponens. With modus tollens the conclusion "not a" is deduced from "$a \to b$" and "not b".

This rule is equivalent to modus ponens, as can be seen when the conditional is translated into the form "not a" or "b"'. In the case of modus ponens, given "not b" along with the translated conditional, the only case where "not a or b" is true is when a is false, since then "not a" is true. This characterizes an important fact about formal inferences, they preserve truth. This means that if the premises of the inference, here "$a \to b$" and "not b", are true, then the conclusion must be true as well for the inference to be valid. The disadvantage of eliminating equivalent inferences is that it moves the logical system, as represented by the sentences, further away from natural language. This is true because the natural language inferences are translated into a reduced set of inferences, which removes some of the variety from the corresponding formal language. The variety lost does not entail a loss of content however, since the reduced system is logically equivalent to the system with the larger set of inferences. The reduced system does possess syntactic advantages however, since the number of rules has been reduced. This allows for simpler analysis at the syntactic level. In fact, many mathematical logic systems only include modus ponens in their inference schemes and these are almost always equivalent to systems which allow a greater number of inference types.

In this work we follow the formal systems of the mathematicians, rather than the philosophers and linguists, because the reduction in the number of inference rules reduces the complexity of specifying the consequences for computation. In order to prepare for the formal analysis which follows, some of the major concepts of the syntax and semantics of formal proof-based systems will now be introduced.

1.8. Proof-Based Systems

In order to formalize a language, there must be a specification of the signs and symbols of the formal language, as well as a specification of the permissible manipulations of the symbols. First an alphabet for the formal language is needed. The alphabet is divided into six disjoint subsets, the first of which are constants. Constants are symbols which have a single value such as 0 or 1. The second subset of the alphabet consists of variables, which can take on a range of values. Constants and variables are called atomic terms. The third subset consists of operations or functions. Each function has a specified degree $1, 2, 3, \ldots$; their values are called terms. Multiplication is an example of a function of degree 2 since it has two arguments. Functions map elements in their domains to elements in their codomains. They must be well-defined, meaning that each element in their domain is mapped to a unique element in the codomain. If f is a function of degree i and $t_1, t_2, ..., t_i$ are terms, then $f(t_1, t_2, \ldots, t_i)$ is also a term, although not an atomic term.

The next subset of the alphabet consists of predicates, which also have a specified degree. In effect a predicate makes a statement about its arguments. It does this because it is a defined subset of the domain of discourse or universe of the formal language. The universe contains all of the object-meanings which are allowed in the language. For example, if the universe consists of all of the accounts in an accounting system and a predicate $P(a)$ of degree 1 is defined to be "a is an asset", then $P(a)$ will be true only if a represents an asset account. This defines a mapping in which $P(a)$ is sent to "true" (or 1) in only those cases where a is in the subset P; otherwise $P(a)$ is mapped to "false" (or 0). This mapping is called the characteristic function of the predicate. If P is a predicate of degree i and $t_1, t_2, \ldots, t_i$ are terms, then $P(t_1, t_2, \ldots, t_i)$ is an atomic formula. Notice that it is possible to represent functions of degree i by predicates of degree $i + 1$ by merely adding the codomain of the function as the $(i+1)$th object of the subset defined by the predicate. For example, the binary degree function of addition translates into a tertiary predicate in which $\langle 2, 5, 7\rangle$ and $\langle 3, 8, 11\rangle$ would be included in the subset defined by the addition predicate. In general this predicate would consist of the ordered triplets $\langle x, y, z\rangle$ such that $x + y = z$.

The fifth subset of the alphabet consists of logical symbols, which are divided into connectives and quantifiers. The connectives are $\rightarrow$ (implication), $\vee$ ("or" = disjunction), $\wedge$ ("and" = conjunction), $\neg$ ("not" = negation) and $\leftrightarrow$ (if and only if or logical equivalence). The quantifiers are $\exists$ (there exists) and $\forall$ (for all). The two quantifiers are also called the existential and the universal quantifiers respectively. If F and G are formulas, then the following are also formulas:

$$(F) \rightarrow (G),\ (F) \vee (G),\ (F) \wedge (G),\ \neg(F),\ (F) \leftrightarrow (G)$$

and

$$\exists(x)(F),\quad \forall(x)(F),$$

where in the last two formulas x is a variable. The final subset contains punctuation marks, of which only left and right parentheses and occasional commas are used here.

The rules for forming terms and formulas provide the ability to recognize well-formed formulas in the language. A formula is well-formed if and only if it is built up from constants and variables by repeated application of the rules for forming terms and atomic formulas. In addition a formula in which all the variables are bound to quantifiers is called a sentence. A variable is bound if it occurs in a formula F and in the quantification of that formula, i.e., x is bound in F by the quantifications $\exists(x)(F)$ or $\forall(x)(F)$. A variable which is not bound is considered free.

Next the syntax and semantics of a formal language are constructed as follows. Both concepts are founded on the idea of the truth of formulas, sentences and inferences. Each logical symbol in the alphabet has a corresponding truth table associated with it. In the case of implication, the formula is false only when its antecedent is true and its consequent is false. Likewise, in the case of the inclusive or, the formula is false only when both arguments are false. For a conjunction, its truth value is true only when both arguments are true; in all other cases it is false. Negation takes only one argument and is true if its argument is false and false if its argument is true. Logical equivalence is true if and only if either both arguments are true or both are false.

The quantifiers "for all" and "there exists" are true in the following cases. "For all $x, F(x)$" is true only when every symbol of the alphabet which can be substituted for x in the formula leads to the formula being true. For "there exists $x,\ F(x)$", the formula is true if at least one symbol can be substituted for x leading to a true formula. The assignment of truth values for a complicated formula begins at the lowest level of atomic terms and atomic formulas and proceeds to higher levels, in the same manner as the term or formula was created in its definition.

It was mentioned earlier that in order for an inference to be valid, it must preserve truth. This means that it is not valid to deduce a false conclusion from true premises. The notion of validity is a syntactic one because it involves the construction of formulas through the application of the rules of inference. Given a set of axiom formulas, the formulas which can be validly constructed from the axioms by repeated inferences are called the consequences of the formulas and these are said to be deducible or derivable from the axiom formulas.

In terms of the semantic component of a formal language, all formulas that are true in the language are said to be provable in the language if the language is complete. Completeness is a semantic concept because it requires that if the meaning of some formula is true in the sense of the universe of the language, then that formula must be provable. The specific derivation of the formula does not have to be given however. Another general concept of formal languages is consistency. Consistency means that if a formula is derivable in the language, then its negation is not derivable. This is an important technical detail since, if both a formula and its negation can be proved in the language, then any formula in the language can be proved as well, a situation which certainly adds nothing to the sum of human knowledge.

1.9. The Scope of the Present Work

After this extended discussion of methodologies in accounting, the final section describes the scope of this book and what the authors believe is accomplished therein. The purpose of the book is to demonstrate how and under what conditions a basic accounting system can be reduced to a formal proof-based language. When this is accomplished, a method for controlling such systems through their derivations is established which is significantly stronger than methodologies used currently in accounting. The exposition in Chapters 2 through 9 employs definitions, propositions and proofs to formalize the system. The definitions are intended to represent terms, concepts or constructions currently in use and are carefully stated in order to avoid confusion as to the precise meaning assigned to them in the book. Propositions are used to state results which follow logically from the definitions and are in all cases accompanied by complete proofs. These proofs are meant to demonstrate the correctness of the propositions and to illustrate the techniques used in the algebraic and logical analyses.

Since the location of the research is the accounting system after an economic event has occurred and been quantified, but before the output of the system has been used by decision makers, the method concentrates on the manipulation and processing of inputs to outputs. These procedures are reduced to a purely algebraic system which is capable of receiving transaction data, processing the data and generating information in the form of summaries of various types.

What happens when the accounting system is reduced to an algebraic system is that the entire range of speech is circumscribed. This means that all sentences or ideas are known to be true, false or outside the particular system. Accounting systems can be thought of as possessing different dialects, some quite similar, others nearly distinct. The control of accounting systems then takes on the quality of distinguishing very precisely how the systems differ, i.e., which have larger vocabularies and which are richer in expressiveness. From a practical viewpoint this allows accountants as designers to match the expressive power of particular systems to user needs for more or less expressive languages. In addition the methodology can provide a means by which to identify situations of data or information asymmetry and can therefore act as an indirect guide to action.

It cannot be claimed that this reduction is unique, for there are many different opinions about what constitutes an accounting system and consequently many ways to construct a formal system. The intention here is to provide a method which mirrors a specified basic accounting system and which is reasonably comprehensible. It is not the intention of the book to provide a blueprint for an accounting system which could be programmed and used in practice. Rather the concern is to allow the system to recognize and act on the transaction data itself. The base level justification is to develop a full formal language for a particular aspect of accounting instead of assuming that such a grammar could exist and proceeding with partial constructions or a higher level analysis. The success of this basic stage of proof-based research in accounting will furnish researchers with a secure, well established base for future investigations.

Algebraic concepts employed

It is time to be specific about the algebraic concepts that have proved useful in the analysis. There are four principal structures which are used repeatedly and which appear well suited to application in accounting, namely:

- balance vector;
- directed graph (or digraph);
- automaton;
- monoid.

These structures will be familiar to most algebraists. A few words will be given to elucidate their meaning and to justify the claim of utility in accounting theory.

A *balance vector* is a column vector or column matrix the sum of whose entries equals zero. In this case the relevance to accounting will be obvious: the zero sum reflects the fundamental property of any accounting system that it must always be in balance. Mathematicians will immediately recognize that balance vectors form a structure with known algebraic properties; they form a submodule or hyperplane. Balance vectors are able to represent the state of an accounting system at any instant. They are also capable of encoding the transactions that are applied to the system. There is an important comment to be made regarding signs: for the entries of a balance vector can be positive or negative. The great advantage of using positive and negative signs is that the signs take care of questions of credit or debit automatically; for example, a credit balance has a positive sign and a debit balance a negative one. The theory of balance vectors is developed in Chapters 2 and 3, where their application to accounting is clearly laid out.

The second useful algebraic notion is that of a *directed graph*. This is best thought of geometrically, although its definition is entirely algebraic. The digraph consists of *vertices*. i.e., points in the plane, and *edges*, or lines with a direction, joining certain vertices.

The vertices represent accounts and the edges indicate where there are flows of value within the system. Thus a digraph gives a picture of how value can flow around an accounting system. While in general different accounting systems might have the same digraph, for certain special types the digraph determines the system up to equivalence.

The third concept, that of an *automaton*, is frequently used in information science as a theoretical model of a computer. The automaton is at any instant in a certain state; it reads a symbol on an input tape, goes to a another state and then writes a symbol on an output tape. The applicability to accounting is clear: the states of the accounting system are the balance vectors, the inputs are the transactions and the outputs are the new balance vectors. This simple picture can be made more complex in order to represent further actions of an accounting system, as is expounded in detail in Chapters 6 and 9.

The final concept of a *monoid* is the most abstract. Every automaton has an associated monoid, which is an algebraic structure with a means of combining its elements subject to suitable rules. An input to the automaton produces a change in the state of the automaton and thus determines a function from states to states. The functions on the set of states form a monoid for which the operation is functional composition; the associated functions generate a submonoid of this monoid. Despite their abstraction, monoids provide useful ways of characterizing accounting systems with special properties, as is shown in Chapter 7.

With the aid of the concept of a balance vector, the definition of an abstract accounting system is laid out in Chapter 4 and its properties are expounded, with numerous accompanying examples. Relations between different accounting systems are considered in Chapter 5 by using standard constructions from algebra, namely quotient systems and homomorphisms. The latter are functions between different accounting systems that relate their structures.

An important topic in algebra is the possible existence of algorithms to perform certain computations or to make decisions: what is at stake here is the question of what can and cannot be computed, in principle at least. For example, is it possible to write a program which is able to test the final balance vector of an accounting system and decide if there have been any irregularities during the accounting period? The importance of the question is evident. Chapter 8 contains a full discussion of what one can expect to be able to decide or compute in an accounting system.

In Chapter 9 all the strands come together to form our final model of an accounting system. In this there are ten parameters, so the model is referred to as the *10-tuple model*. It has the capability to scan and process incoming transactions, keep track of balances, generate reports on the system, control access by individuals to the system, and keep track of frequency of application of transactions. It is also able to test final balances. Our main conclusion in this book is that the 10-tuple model goes a long way towards representing what is actually going on during the operation of an accounting system.

The final Chapter 10 is intended as a corrective after the many mathematical considerations of this work. It presents a detailed example of a small company engaged in trade and it exhibits the accounting system in the form of a 10-tuple model. The aim of the example is, of course, to help make the case for the relevance of the model to accounting practice and to justify the claim that all connection with reality has not been been eroded through the process of abstraction.