Chapter 1

Paradox

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A. The Liar

Epimenides the Cretan said that all Cretans lie; did he tell the truth, or not? Let us assume, for the sake of argument, that every Cretan, except possibly Epimenides himself, was in fact a liar; but what then of Epimenides?

In effect, he says he himself lies; but if he is lying, then he is telling the truth; and if he is telling the truth, then he is lying! Which then is it?

The same conundrum arises from the following sentence:

"This sentence is false".

That sentence is known as the "Liar Paradox", or "pseudomenon".

The pseudomenon obeys this equation:

$L = \text{not } L.$

It's true if false, and false if true. Which then is it?

That little jest is King of the Contradictions. They all seem to come back to that persistent riddle. If it is false then it is true, by its own definition; yet if it is true then it is false, for the exact same reason! So which is it, true or false? It seems to undermine dualistic reason itself. Dualists fear this paradox; they would banish it if they could.

Since it is, so to speak, the leader of the Opposition Party, it naturally bears a nasty name; the "Liar" paradox. Don't trust it, say the straight thinkers; and it agrees with them! They denigrate it, but it denigrates *itself*; it admits that it is a liar, and thus it is not *quite* a liar! It is straightforward in its deviation, accurate in its errors, and honest in its lies! Does that make sense to you, dear reader? I must admit that it has never quite made sense to me.

The name "Liar" paradox is nonetheless a gratuitous insult. The pseudomenon merely denies its truth, not its intentions. It may be false innocently, out of lack of ability or information. It may be contradicting itself, not bitterly, as the name "Liar" suggests, but in a milder tone.

Properly speaking, the Liar paradox goes:

"This statement is a lie".

"I am lying".

"I am a liar".

But consider these statements:

"This statement is wrong".

"I am mistaken".

"I am a fool".

This is the Paradox of the Fool; for the Fool is wise if and only if the Fool is foolish! The underlying logic is identical, and rightly so. For whom, after all, does the Liar fool best but the Liar? And whom else does the Fool deceive except the Fool? The Liar is nothing but a Fool, and vice versa!

Therefore I sometimes call the pseudomenon (or Paradox of Self-Denial) the "Fool Paradox", or "Fool's Paradox", or even "Fool's Gold". The mineral "fool's gold" is iron pyrite; a common ore. This fire-y and ironic little riddle is also a common'ore, with a thousand wry offspring.

For instance:

"I am not a Marxist". — *Karl Marx*

"Everything I say is self-serving". — *Richard Nixon*

Tell me, dear reader; would you believe either of these politicos?

Compare the Liar to the following quarrel:

Tweedledee: "Tweedledum is a liar".

Tweedledum: "Tweedledee is a liar".

— *two* calling *each other* liars rather than *one* calling *itself* a liar! This dispute, which I call "Tweedle's Quarrel", is also known as a "toggle".

Its equations are:

$$EE = \text{not } UM$$

$$UM = \text{not } EE$$

This system has two boolean solutions: (true, false) and (false, true). The brothers, though symmetrical, create a difference between them; a memory circuit! It seems that paradox, though chaotic, contains order within it.

Now consider this three-way quarrel:
Moe: "Larry and Curly are liars".
Larry: "Curly and Moe are liars".
Curly: "Moe and Larry are liars".

$$M = \text{not } L \text{ nor } K$$

$$L = \text{not } K \text{ nor } M$$

$$K = \text{not } M \text{ nor } L$$

This system has three solutions: (true, false, false), (false, true, false), and (false, false, true). *One* of the Stooges is honest; but which one?

B. The Anti-Diagonal

Here are two paradoxes of mathematical logic, generated by an "anti-diagonal" process:

Grelling's Paradox. Call an adjective *autological* if it applies to itself, *heterological* if it does not: "A" is heterological = "A" is not A.

Thus, *short* and *polysyllabic* are autological, but *long* and *monosyllabic* are heterological.

> Is *heterological* heterological?
> "Heterological" is heterological = "Heterological" is not heterological.
> It is to the extent that it isn't!

Quine's Paradox. Let *quining* be the action of preceding a sentence fragment by its own quotation. For instance, when you quine the fragment *is true when quined*, you get:

> "Is true when quined" is true when quined.

> — a sentence which declares itself true.

In general the sentence:

> "Has property P when quined" has property P when quined.

is equivalent to the sentence:

> "This sentence has property P".

Now consider the sentence:

> "Is false when quined" is false when quined.

That sentence declares itself false. Is it true or false?

C. Russell's Paradox

Let R be the set of all sets which do not contain themselves:

$$R = \{x \mid x \notin x\}$$

R is an anti-diagonal set. Is it an element of itself?

In general: $x \in R = x \notin x$

and therefore: $R \in R = R \notin R$.

Therefore R is paradoxical. Does R exist?

Here's a close relative of Russell's set; the "Short-Circuit Set":

$$S = \{x : S \notin S\}.$$

S is a constant-valued set, like the universal and null sets:

For all x, $(x \in S) = (S \notin S) = (S \in S)$.

All sets are paradox elements for S.

Bertrand Russell told a story about the barber of a Spanish village. Being the only barber in town, he boasted that he shaves all those — and only those — who do not shave themselves. Does the barber shave himself?

To this legend I add a political postscript. That very village is guarded by the watchmen, whose job is to watch all those, and only those, who do not watch themselves. But who shall watch the watchmen?

(Thus honesty in government is truly imaginary!)

The town is also guarded by the trusty watchdog, whose job is to watch all houses, and only those houses, that are not watched by their owners. Does the watchdog watch the doghouse?

Not too long ago that village sent its men off to fight the Great War, which was a war to end all wars, and only those wars, which do not end themselves. Did the Great War end itself?

> That village's priest often ponders this theological riddle: God is worshipped by all those, and only those, who do not worship themselves. Does God worship God?

D. Santa and the Grinch

Suppose that a young child were to proclaim:

"If I'm not mistaken, then Santa Claus exists".

If one assumes that Boolean logic applies to this sentence, then its mere existence would imply the existence of Santa Claus!

Why? Well, let the child's statement be symbolized by "R", and the statement "Santa exists" be symbolized by "S". Then we have the equation:

$$R = \text{if } R \text{ then } S = R \Rightarrow S = {\sim}R \vee S.$$

Then we have this line of argument:

$R = (R \Rightarrow S)$; assume that R is either true or false.
If R is false, then $R = (\text{false} \Rightarrow S) = (\text{true} \vee S) = \text{true}$.

$R = $ false implies that $R = $ true;
therefore (by contradiction) R must be true.

Since $R = (R \Rightarrow S)$, $(R \Rightarrow S)$ is also true.
R is true, $(R \Rightarrow S)$ is true; so S is true.
Therefore Santa Claus exists!

This proof uses proof by contradiction; an indirect method, suitable for avoiding overt mention of paradox. Here is another argument, one which confronts the paradox directly:

S is either true or false. If it's true, then so is R:

$R = (\sim R) \vee$ true $=$ true.

No problem. But if S is false, then R becomes a liar paradox:

$R = (\sim R) \vee$ false $= \sim R$.

If S is false, then R is non-boolean.
therefore: If R is boolean, then S is true.

Note that both arguments work equally well to prove any other statement besides S to be true; one need merely display the appropriate "santa sentence". Thus, for instance, if some skeptic were to declare:

"If I'm not mistaken, then Santa Claus does not exist".

— then by identical arguments we can prove that Santa Claus does *not* exist!

Given two opposite Santa sentences:

$$R_1 = (R_1 \Rightarrow S); \quad R_2 = (R_2 \Rightarrow \sim S)$$

then at least one of them must be paradoxical.

We can create Santa sentences by Grelling's method. Let us call an adjective "Santa-logical" when it applies to itself only if Santa Claus exists;
"A" is Santa-logical = If "A" is A, then Santa exists.
Is "Santa-logical" Santa-logical?
"Santa-logical" is Santa-logical =
If "Santa-logical" is Santa-logical, then Santa exists.

Here is a Santa sentence via quining:

"Implies that Santa Claus exists when quined" implies that Santa Claus exists when quined.

If that statement is boolean, then Santa Claus exists.

Here's the "Santa Set for sentence G":

$$S_G = \{x \mid (x \in x) \Rightarrow G\}$$

S_G is the set of all sets which contain themselves only if sentence G is true:

$$x \in S_G = ((x \in x) \Rightarrow G).$$

Then "S_G is an element of S_G" equals a Santa sentence for G:

$$S_G \in S_G = ((S_G \in S_G) \Rightarrow G).$$

"$S_G \in S_G$", if boolean, makes G equal true; another one of Santa's gifts. If G is false, then "$S_G \in S_G$" is paradoxical.

One could presumably tell Barber-like stories about Santa sets. For instance, in another Spanish village, the barber takes weekends off; so he shaves all those, and only those, who shave themselves only on the weekend:

B shaves M = If M shaves M, then it's the weekend.

One fine day someone asked: does the barber shave himself?

B shaves B = If B shaves B, then it's the weekend.

Has it been weekends there ever since?

That village is watched by the watchmen, who watch all those, and only those, who watch themselves only when fortune smiles:

W watches C = if C watches C, then fortune smiles.

One fine day someone asked: who watches the watchmen?

W watches W = if W watches W, then fortune smiles.

Does fortune smile on that village?

Recently, that village saw the end of the Cold War, which ended all wars, and only those wars, which end themselves only if money talks:

CW ends W = if W ends W, then money talks.

Did the Cold War end itself?

CW ends CW = if CW ends CW, then money talks.

Does money talk?

That village's priest proclaimed this theological doctrine:

God blesses all those, and only those, who bless themselves only when there is peace:

G blesses S = If S blesses S, then there is peace.

One fine day someone asked the priest: Does God bless God?

G blesses G = If G blesses G, then there is peace.

Is there peace?

Consider the case of Promenides the Cretan, who always disagrees with Epimenides. Recall that Epimenides the Cretan accused all Cretans of being liars, including himself. If we let $E =$ Epimenides, $P =$ Promenides, and $H =$ "honest Cretans exist", then:

$$E = (\sim E) \wedge (\sim H)$$

$$P = \sim E = \sim((\sim E) \wedge (\sim H))$$

$$= E \vee H = (\sim P) \vee H = (P \Rightarrow H)$$

Thus we get this dialog:

Epimenides: All Cretans are liars.
Promenides: You're a liar.
Epimenides: All Cretans are liars, and I am a liar.
Promenides: Either some Cretan is honest, or you're honest.
Epimenides: You're a liar.
Promenides: Either some Cretan is honest, or I'm a liar.
Epimenides: All Cretans are liars, including myself.
Promenides: If I am honest, then some Cretan is honest.

Promenides is the Santa Claus of Crete; for if his statement is boolean, then some honest Cretan exists.

Now suppose that some sarcastic Grinch were to proclaim: "Santa Claus exists, and I am a liar".

$G = (S \wedge \sim G)$

If Boolean logic applies to this "Grinch Sentence", then it refutes both itself and Santa Claus! For consider this line of argument:

$G = (S \wedge \sim G)$; assume that G is either true or false.
If G is true, then $G = (S \wedge \sim \mathrm{T}) = \mathrm{F}$.
$G =$ true implies $G =$ false;
therefore (by contradiction) G must be false.
False $= G = (S \wedge \sim G) = (S \wedge \sim \mathrm{F}) = S$.

Therefore S is false. Therefore Santa Claus does not exist!

This proof uses proof by contradiction; an indirect method, suitable for avoiding overt mention of paradox. Here is another argument, one which confronts the paradox directly:

S is either true or false. If it's false, then so is G:

$G = (F \wedge \sim G) =$ false.

No problem. But if S is true, then G becomes a liar paradox:

$G = (T \wedge \sim G) = \sim G$.

If S is true, then G is non-boolean.
Therefore; if G is boolean, then S is false. QED

Call an adjective *Grinchian* if and only if it does not apply to itself, and Santa Claus exists:

"*A*" is Grinchian = Santa exists, and "*A*" is not *A*.

Is "Grinchian" Grinchian?

"*G*" is *G* = Santa exists, and "*G*" is not *G*.

The "Grinch Set for sentence *H*" is:

$$G_H = \{x \mid H \wedge (x \notin x)\}$$

$$(G_H \in G_H) = (H \wedge (G_H \notin G_H))$$

In paradox logic, the threatened paradox need not affect any other truth value. If Santa Claus *does* exist after all, then the Grinch is exposed as a Liar!

The Grinch sets suggest Grinch stories. Consider the Weekend Barber, who only shaves on the weekends, and only those who do not shave themselves:

WB shaves *M* = It's the weekend, and *M* does not shave *M*.

Does the Weekend Barber shave himself?

WB shaves *WB* = It's the weekend, and *WB* does not shave *WB*.

Note that Epimenides's statement:

"All Cretans are liars, including myself".

— makes him the Grinch of Crete!

E. Antistrephon

That is, “The Retort”. This is a tale of the law-courts, dating back to Ancient Greece. Protagoras agreed to train Euathius to be a lawyer, on the condition that his fee be paid, or not paid, according as Euathius win, or lose, his first case in court. (That way Protagoras had an incentive to train his pupil well; but it seems that he trained him too well!) Euathius delayed starting his practice so long that Protagoras lost patience and brought him to court, suing him for the fee. Euathius chose to be his own lawyer; this was his first case.

Protagoras said, “If I win this case, then according to the judgement of the court, Euathius must pay me; if I lose this case, then according to our contract he must pay me. In either case he must pay me”.

Euathius retorted, “If Protagoras loses this case, then according to the judgement of the court I need not pay him; if he wins, then according to our contract I need not pay him. In either case I need not pay”.

How should the judge rule?

Here’s another way to present this paradox:

According to the contract, Euathius will avoid paying the fee — that is, win this lawsuit — exactly if he loses his first case; and Protagoras will get the fee — that is, win this lawsuit — exactly if Euathius wins his first case. But this lawsuit *is* Euathius’s first case, and he will win it exactly if Protagoras loses. Therefore Euathius wins the suit if and only if he loses it; ditto for Protagoras.

F. Parity of Infinity

What is the *parity* of infinity? Is infinity odd or even? In the standard Cantorian theory of infinity, ∞ equals its own successor:

$$\infty = \infty + 1$$

But therefore ∞ is even if and only if it is odd! Since ∞ is a *counting* number, it is presumably an integer; but any integer is even *or else* odd!

Infinity has paradoxical parity. We encounter this paradox when we try to define the limit of an infinite oscillation. Consider the sequence $\{x_0, x_1, x_2, \ldots\}$:

$$x_0 = \text{True};$$

$$x_{n+1} = {\sim}x_n, \quad \text{for all } n.$$

Therefore $x_{\text{even}} = \text{True}$, and $x_{\text{odd}} = \text{False}$.

The x's make an oscillation: {T, F, T, F, ...} Now, can we define a *limit* of this sequence? $\text{Lim}(x_n)$ equals what? If we cannot define this limit, in what sense does infinity have a parity at all? And if no parity, why other arithmetical properties?

We can illuminate the Parity of Infinity paradox with a fictional lamp; the Thompson Lamp, capable of infinitely making many power-toggles in a finite time. The Thompson Lamp clicks on for one minute, then off for a half-minute, then back on for a quarter-minute; then off for an eighth-minute; and so on, in geometrically decreasing intervals until the limit at two minutes, at which point the Lamp stops clicking. After the second minute, is the Lamp on or off?

G. The Heap

Surely one sand grain does not make a heap of sand. Surely adding another grain will not make it a heap. Nor will adding another, or another, or another. In fact, it seems absurd to say that adding one single grain of sand will turn a non-heap into a heap. By adding enough ones, we can reach any finite number; therefore no finite number of grains of sand will form a sand heap. Yet sand heaps exist; and they contain a finite number of grains of sand!

Let's take it in the opposite direction. Let us grant that a finite sand heap exists. Surely removing one grain of sand will not make it a non-heap. Nor will removing another, nor another, nor another. By subtracting enough ones, we can reduce any finite number to one. Therefore one grain of sand makes a heap!

What went wrong?

Let's try a third time. Grant that one grain of sand forms no heap; but that some finite number of grains do form a heap. If we move a single grain at a time from the heap to the non-heap, then they will eventually become indistinguishable in size. Which then will be the heap, and which the nonheap?

The First Boring Number. This is closely related to the paradox of the Heap. For let us ask the question: are there any boring (that is, uninteresting) numbers? If there are, then surely that collection has a *smallest* element; the *first* uninteresting number. How interesting!

Thus we find a contradiction; and this seems to imply that there are no uninteresting numbers!

But in practice, most persons will agree that most numbers are stiflingly boring, with no interesting features whatsoever! What then becomes of the above argument?

Simply this; that the *smallest* boring number is inherently paradoxical. If being the first boring number were a number's only claim to our interest, then we would find it interesting if and only if we do *not* find it interesting.

Which then is it?

Berry's Paradox. What is "the smallest number that cannot be defined in less than 20 syllables"? If this defines a number, then we have done so in 19 syllables! So this defines a number if and only if it does not.

Presumably Berry's number equals the first boring number, if your boredom threshold is 20 syllables.

These paradoxes connect to the paradox of the Heap by simple psychology. If, for some mad reason, you actually *did* try to count the number of grains in a sand heap, then you will eventually get bored with such an absurd task. Your attention would wander; you would lose track of all those sand grains; errors would accumulate, and the number would become indefinite.

The Heap arises at the onset of uncertainty. In practice, the Heap contains a boring number of sand grains; and the smallest Heap contains the smallest boring number of sand grains!

H. Finitude

Finite is the opposite of infinite; but in paradox-land, that's no excuse! In fact the concept of finiteness is highly paradoxical; for though finite numbers are finite individually and in finite groups, yet they form an infinity.

Let us attempt to *evaluate* finiteness. Let F = 'Finitude', or 'Finity'; the *generic* finite expression. You may replace it with any finite expression.

Is Finity finite?

If F is finite, then you can replace it by $F+1$, and thus by $F+2$, $F+3$, etc. But such a substitution, indefinitely prolonged, yields an infinity.

If F is not finite, then you may not replace F by F, nor by any expression involving F; you must replace F by a well-founded finite expression, which will then be limited.

Therefore F is finite if and only if it is not finite.

Finitude is *just short* of infinity! It is infinity seen from underneath. You may think of it as that mysterious 'large finite number' N, bigger than any number you care to mention; that is, bigger than any *interesting* number.

Call a number "large" if it is bigger than any number you care to mention; that is, bigger than any interesting number. Call a number "medium" if it is bigger than some boring number but less than some interesting number. Call a number "small" if it is less than any boring number.

Then Finitude is the smallest large number; that is, the smallest number bigger than any interesting number. How interesting!

Finitude is dual to the Heap, which is the largest number less than any uninteresting number. The Heap is the lower limit of boredom; Finitude is the upper limit of interest.

We have these inequalities:

small interesting numbers

$<$ The Heap $=$ first boring number $=$ last small number

$<$ medium numbers

$<$ Finitude $=$ last interesting number $=$ first large number

$<$ large boring numbers

Consider this Berry-like definition:

"One plus the largest number defineable in less than 20 syllables".

If this defines a number, then it has done so in only 19 syllables, and therefore is its own successor. If your boredom threshold is 20 syllables, then this number $=$ "one plus the last interesting number" $=$ Finitude.

Now consider "well-foundedness". Set theorists were so disturbed by Russell's paradox that they decided to acknowledge only "well-founded" sets. A set is well-founded if and only if it has no "infinite descending element chains"; that is, there is no infinite sequence of sets $X_1, X_2, X_3, \ldots$ such that

$$\cdots X_4 \in X_3 \in X_2 \in X_1 \in X_0.$$

Well-founded sets include { }; {{ }}; {{{}}, {{},{{}}}}; and even infinite sets such as {{},{{}}, {{{}}}, {{{{}}}},…}; for well-founded sets can be infinitely "wide", so long as they are finitely "deep" along each "branch".

On the other hand, well-foundedness excludes sets such as

$$A = \{A\} = \{\{\{\{\{\{\ldots\}\}\}\}\}\}$$

for it has the infinite descending element chain $\cdots \in A \in A \in A$.

Let *WF* be the set containing all well-founded sets. Is *WF* well-founded?

If *WF is* well-founded, then *WF* is in *WF*; but this yields the infinite descending element chain $\cdots \in WF \in WF \in WF$.

On the other hand, if *WF* is *not* well-founded, then any element of *WF* is well-founded, and element chains deriving from those will be finite. Thus all element chains from *WF* will be finite; and therefore *WF* would be well-founded.

And so we see that the concept of "well-foundedness" leads us to the paradox of Finitude.

Related to "well-foundedness" is "clear-foundedness". Call a set "clear-founded" if none of its descending chains are more than a Heap of sets deep. A set is clear-founded if it has only interesting depth. Is the class of clear-founded sets clear-founded?

And just as infinity has paradoxical parity, so do Finitude and the Heap. Is the first boring number odd or even? Is the last interesting number?

I. Game Paradoxes

Hypergame and the Mortal

Let “Hypergame” be the game whose initial position is the set of all “short” games — that is, all games that end in a finite number of moves. For one’s first move in Hypergame, one may move to the initial position of any short game.

Is Hypergame short?

If Hypergame is short, then the first move in Hypergame can be too — Hypergame! But this implies an endless loop, thus making Hypergame no longer a short game!

But if Hypergame is *not* short, then its first move must be into a short game; thus play is bound to be finite, and Hypergame a short game.

The Hypergame paradox resembles the paradox of Finitude. Presumably Hypergame lasts Finitude moves; one plus the largest number definable in less than twenty syllables.

Dear reader, allow me to dramatize this paradox by means of a fictional story about a mythical being. This entity I shall dub “the Mortal”; an unborn spirit who must now make this fatal choice; to choose some mortal form to incarnate as, and thus be be doomed to certain death.

The Mortal has a choice of dooms. Is the Mortal doomed?

Normalcy and the Rebels

Define a game as "normal" if and only if it does not offer the option of moving to its own starting position:

G is normal = the move $G \Rightarrow G$ is not legal.

Let "Normalcy" be the game of all normal games. In it one can move to the initial position of any normal game:

The move $N \Rightarrow G$ is legal = the move $G \Rightarrow G$ is not legal.

Is Normalcy normal? Let $G = N$:

The move $N \Rightarrow N$ is legal = the move $N \Rightarrow N$ is not legal.

Normalcy is normal if and only if it is *ab*normal!

That was Russell's paradox for game theory. Now consider this:

The Rebel is a being who must become one who changes. The Rebel may become all those, and only those, who do not remain themselves:

R may become B = B may not become B.

Can the Rebel remain a Rebel?

R may become R = R may not become R.

A Santa Rebel may become all those, and only those, who remain themselves only if Santa Claus exists:

SR may become B = ((B may become B) $\Rightarrow$ Santa exists)

Therefore:

SR may become SR = ((SR may become SR) $\Rightarrow$ Santa exists)

If the pivot bit is boolean, then Santa Claus exists!

J. Cantor's Paradox

Cantor's proof of the "uncountability" of the continuum relies on an "anti-diagonalization" process. Suppose we had a countable list of the real numbers between 0 and 1:

$$R_1 = 0.D_{11}\,D_{12}\,D_{13}\,D_{14}\ldots$$
$$R_2 = 0.D_{21}\,D_{22}\,D_{23}\,D_{24}\ldots$$
$$R_3 = 0.D_{31}\,D_{32}\,D_{33}\,D_{34}\ldots$$
$$R_4 = 0.D_{41}\,D_{42}\,D_{43}\,D_{44}\ldots$$
$$\vdots$$

where D_{NM} is the Mth binary digit of the Nth number.
Then we define Cantor's "anti-diagonal" number:

$$C = 0.{\sim}D_{11}, {\sim}D_{22}, {\sim}D_{33}, {\sim}D_{44}, \ldots$$

If $C = R_N$ for any N, then $D_{NX} = {\sim}D_{XX}$;
Therefore $D_{NN} = {\sim}D_{NN}$; the pivot bit buzzes.

From this paradox, Cantor deduced that the continuum has too many points to be counted, and thus is of a "higher order" of infinity. Thus a single buzzing bit implies infinities beyond infinities! Was more ever made from less?

I say, why seek "transfinite cardinals", whatever those are? Why not ask for Santa Claus? In this spirit, I introduce the *Santa*-diagonal number:

$$S = 0.(D_{11} \Rightarrow \text{Santa}), (D_{22} \Rightarrow \text{Santa}), (D_{33} \Rightarrow \text{Santa}) \cdots$$

If $S = R_M$ for any M, then $D_{MX} = (D_{XX} \Rightarrow \text{Santa exists})$;
Therefore $D_{MM} = (D_{MM} \Rightarrow \text{Santa exists})$.
If the pivot bit is boolean, then Santa Claus exists!

K. Paradox of the Boundary

The continuum is paradoxical because it is *continuous*, and boolean logic is discontinuous. This topological difference yields a logical riddle which I call the Paradox of the Boundary.

The paradox of the boundary has many formulations, such as:

What day is midnight?
Is noon A.M. or P.M.?
Is dawn day or night? Is dusk?
Which hemisphere is the equator on?
Which longitude are the poles at?
Which country owns the border?
Is zero plus or minus?

If a statement is true at point *A* and false at point *B*, then somewhere in-between lies a boundary. At any point on the boundary, is the statement true, or is it false?

(If line segment *AB* spanned the island of Crete, then somewhere in the middle we should, of course, find Epimenides!)