Chapter 1

Preliminaries

1.1 Background Sketch

In this section, for the reader's convenience, we list some basic definitions, notation and well-known facts that will be needed in this book. For details, the reader is referred to standard books and recent books, in particular, to Anderson-Fuller [5], Faith [47], Harada [64], Lam [107], Mohamed-Mueller [115], Nicholson-Yousif [141], Dung-Huynh-Smith-Wisbauer [41], Clark-Lomp-Vanaja-Wisbauer [31] and Xue [184].

Throughout this book, all rings R considered are associative rings with identity and all R-modules are unital. Departing from convention, when we consider subrings of a ring, we do not assume that they have the same identity. All R-homomorphisms between R-modules are written on the opposite side of scalars. The notation M_R (resp. ${}_RM$) is used to stress that M is a right (resp. left) R-module. For a module M, we use $E(M)$ to denote its injective hull. An idempotent e of R is called *primitive* if eR_R is an indecomposable module or, equivalently, if ${}_RRe$ is an indecomposable module. A set $\{e_1, \ldots, e_n\}$ of orthogonal primitive idempotents of R with $1 = e_1 + \cdots + e_n$ is called a *complete set of orthogonal primitive idempotents* of R. We put $Pi(R) = \{e_1, \ldots . e_n\}$.

Let M be a right R-module and N a submodule of M. Then N is called an *essential* submodule of M (or M is an *essential* extension of N) if $N \cap X \neq 0$ for any non-zero submodule X of M and we denote this by $N \subseteq_e M$. We use $Z(M)$ to denote the *singular* submodule of M, i.e., $Z(M) = \{ m \in M \mid r_R(m) \subseteq_e R_R \}$, where $r_R(m)$ means the right annihilator of m in R. Dually, a submodule N of M is called a *small* submodule (or *superfluous* submodule) of M, abbreviated $N \ll M$, if, given a submodule $L \subseteq M$, $N + L = M$ implies $L = M$.

For an R-module M, the (*Jacobson*) *radical* $J(M)$ of M is defined as the intersection of all maximal submodules of M, i.e., $J(M) = \cap\{ K \leq M \mid K$ is a maximal submodule of $M \}$. If M has no maximal submodule, we define $J(M) = M$.

For a ring R, we say that $J(R_R)$ $(= J({}_RR))$ is the *Jacobson radical* of R and denote it by $J(R)$ or, simply, by J if no confusion arises.

For a module M_R, by transfinite induction, *the upper Loewy series*

$$J_0(M) \supseteq J_1(M) \supseteq J_2(M) \supseteq \cdots$$

of M is defined as follows:

$J_0(M) = M,\ J_1(M) = J(M)$,
$J_\alpha(M) = J(J_{\alpha-1}(M))$ if α is a successor ordinal, and
$J_\alpha(M) = \cap_{\beta<\alpha} J_\beta(M)$ if α is a limit ordinal.

Here $J_\alpha(M)$ is called the *α-th radical* of M.

We record some fundamental properties of $J(M)$ (see, for example, [5]).

Proposition 1.1.1. *Let M be a right R-module. Then*

$$J(M) = \sum\{ L \mid L \ll M \}.$$

The following basic result is due to Bass [21].

Theorem 1.1.2. *Let R be a ring and P a non-zero projective right R-module. Then the following hold:*

(1) $J(P) = PJ$.
(2) $J(P) \neq P$, *i.e., P has a maximal submodule.*

Theorem 1.1.3. *Let M be a finitely generated right R-module. Then the following hold:*

(1) $MJ \subseteq J(M)$.
(2) $J(M) \ll M$.

Remark 1.1.4. Theorem 1.1.3 (2) is well-known as *Nakayama's lemma.* We can easily prove this using Theorem 1.1.2 (1) and the fact that every submodule of M is contained in a maximal submodule of M. Nakayama's lemma is important for its use in various situations in the ring and module theory.

Throughout this book, $\mathbb{N}$ denotes the set of positive integers while $\mathbb{N}_0$ denotes the set of non-negative integers.

A ring R is called a *right noetherian* (*artinian*) ring if R satisfies the ascending (descending) chain condition (abbreviated *ACC* (*DCC*)) for right ideals. A *left noetherian* (*artinian*) ring is similarly defined. A left and right noetherian (artinian) ring is called a *noetherian* (*artinian*) ring. A ring R is called a *semiprimary* ring if R/J is semisimple and J is nilpotent, i.e., there exists $k \in \mathbb{N}$ with $J^k = 0$.

The following theorem records some fundamental facts of ring theory.

Theorem 1.1.5. *Let R be a right artinian ring. Then the following hold:*

(1) *R is a right noetherian ring.*
(2) *J is nilpotent.*
(3) *R/J is semisimple.*
(4) *There exists a complete set $\{e_i\}_{i=1}^n$ of orthogonal primitive idempotents of R (so that $1 = e_1 + \cdots + e_n$ and $R = e_1R \oplus \cdots \oplus e_nR$) and each e_iJ is a maximal submodule of e_iR.*
(5) *Every complete set of orthogonal primitive idempotents of $\overline{R} = R/J$ lifts to a complete set of orthogonal primitive idempotents of R.*

Given a right R-module M, the *socle* of M is defined as the sum of all the simple submodules of M_R and is denoted by $S(M)$. When M does not have simple submodules, we put $S(M) = 0$. By transfinite induction, *the lower Loewy series*

$$S_0(M) \subseteq S_1(M) \subseteq S_2(M) \subseteq \cdots$$

of M is defined as follows:

$S_0(M) = 0, \; S_1(M) = S(M)$,
$S_\alpha(M)/S_{\alpha-1}(M) = S(M/S_{\alpha-1}(M))$ if α is a successor ordinal, and
$S_\alpha(M) = \cup_{\beta<\alpha} S_\beta(M)$ if α is a limit ordinal.

Here $S_\alpha(M)$ is called the *α-th socle* of M.

As a dual to Proposition 1.1.1, we have the following proposition (see, for example, [5, 9.7. Proposition]).

Proposition 1.1.6. *Let M be a right R-module. Then*

$$S(M) = \bigcap \{ L \mid L \subseteq_e M \}.$$

Let R be a ring. We will denote the category of all right (left) R-modules by *Mod-R* (*R-Mod*), and use *FMod-R* (*R-FMod*) to denote the category of all finitely generated right (left) R-modules. A right R-module M is called a *generator* in *Mod-R* if, for any right R-module X, there exists a direct sum $\sum M$ of copies of M and an epimorphism from $\sum M$ to X, equivalently, R is isomorphic to a direct summand of a finite direct sum of copies of M. Dually, a right R-module M is called a *cogenerator* in *Mod-R* if any right R-module X can be embedded in a direct product of copies of M. It is well-known that a right R-module M is a cogenerator in *Mod-R* if and only if, for any simple right R-module S, M contains an injective module which is isomorphic to the injective hull of S.

Let R and S be rings. Then *Mod-R* and *Mod-S* are called *equivalent* if there exist additive covariant functors F : *Mod-R* $\to$ *Mod-S* and G : *Mod-S* $\to$ *Mod-R* such that GF and FG are isomorphic to the identity functors of *Mod-R* and *Mod-S*, respectively. In this case, we also say that the rings R and S are *Morita equivalent*.

Given $n \in \mathbb{N}$, we let $(R)_n$ denote the ring of $n \times n$-matrices over the ring R. Given any R-module M, we let $\mathrm{End}(M)$ denote the ring of endomorphisms of M.

Theorem 1.1.7. (Morita equivalence) *For two rings R and S, the following are equivalent:*

(1) *Mod-R is equivalent to Mod-S.*
(2) *R-Mod is equivalent to S-Mod.*
(3) *There exists a finitely generated projective generator P_R such that* $\mathrm{End}(P_R) \cong S$.
(4) *There exist $n \in \mathbb{N}$ and an idempotent e of $T = (R)_n$ such that $TeT = T$ and $eTe \cong S$.*

Let P_R be a projective module as given in Theorem 1.1.7 (3) above. Then, setting $Q = \mathrm{Hom}_R({}_SP_R, R)$, the functors $\mathrm{Hom}_R({}_SP_R, -)$ and $\mathrm{Hom}_S({}_RQ_S, -)$ realize the equivalence between *Mod*-R and *Mod*-S.

A property $(\mathcal{P})$ in the class of all rings is called *Morita invariant* if, whenever R has the property $(\mathcal{P})$ and S is Morita equivalent to R, then S has the property $(\mathcal{P})$ as well.

Let R and S be rings. Let $\mathcal{C}$ and $\mathcal{D}$ be full subcategories of *Mod*-R and S-*Mod*, respectively. If there exist additive contravariant functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that GF and FG are isomorphic to the identity functors of $\mathcal{C}$ and $\mathcal{D}$, respectively, then we say that the pair (F, G) *is a duality* between $\mathcal{C}$ and $\mathcal{D}$. Further we say that a duality (F, G) between $\mathcal{C}$ and $\mathcal{D}$ is a *Morita duality* if $R_R \in \mathcal{C}$ and ${}_SS \in \mathcal{D}$ and $\mathcal{C}$ and $\mathcal{D}$ are closed under submodules and factor modules.

Consider a bimodule ${}_SE_R$. For a module $M = M_R$, we put $M^* = \mathrm{Hom}_R(M, E)$. Then M^* is a left S-module. For a module $N = {}_SN$, we also put $N^* = \mathrm{Hom}_S(N, E)$. Then N^* is a right R-module. For a right R-module M, the map $\sigma_M : M \to M^{**}$ given by

$$\sigma_M(m)(f) = f(m) \qquad (m \in M,\ f \in M^*)$$

is a homomorphism, which we call an *evaluation map*. When $M \cong M^{**}$ under σ_M, M is called an *E-reflexive* right R-module. Similarly, beginning with a left S-module N, we may define N^{**} and the evaluation map σ_N and, when $N \cong N^{**}$ under σ_N, N is called an *E-reflexive* left S-module.

We let $R[E]_R$ and ${}_SR[E]$ denote the family of all E-reflexive right R-modules and that of all E-reflexive left S-modules, respectively. Then the pair of the functors $F(-) = \mathrm{Hom}_R(-, E)$ and $G(-) = \mathrm{Hom}_S(-, E)$ induces a duality between $R[E]_R$ and ${}_SR[E]$.

For a given bimodule ${}_SE_R$, there exist canonical ring homomorphisms $S \to \mathrm{End}(E_R)$ and $R \to \mathrm{End}({}_SE)$. If these are isomorphisms, ${}_SE_R$ is called a *faithfully balanced* bimodule.

The next two theorems give the essence of Morita duality.

Theorem 1.1.8. (Morita duality I) *Let $\mathcal{C}$ and $\mathcal{D}$ be full subcategories of Mod-R and S-Mod, respectively, and let (F, G) be a Morita duality between $\mathcal{C}$ and $\mathcal{D}$. Then there exists a bimodule ${}_SE_R$ such that*

(1) ${}_SE \cong F(R_R)$ *and* $E_R \cong G({}_SS)$,
(2) *there exist natural isomorphisms* $F(-) \cong \mathrm{Hom}_R(-, E)$ *and* $G(-) \cong \mathrm{Hom}_S(-, E)$ *and*
(3) *all modules M_R in $\mathcal{C}$ and all modules ${}_SN$ in $\mathcal{D}$ are E-reflexive.*

As the above result shows, every Morita duality can be realized as functors $\mathrm{Hom}_R(-, E)$ and $\mathrm{Hom}_S(-, E)$ for a suitable bimodule ${}_SE_R$. We say that a bimodule ${}_SE_R$ *defines a Morita duality* in case the pair of the functors $\mathrm{Hom}_R(-, E)$ and $\mathrm{Hom}_S(-, E)$ induces a Morita duality.

Theorem 1.1.9. (Morita duality II) *The following are equivalent for a bimodule ${}_SE_R$:*

(1) ${}_SE_R$ *defines a Morita duality.*
(2) *Every factor module of R_R, ${}_SS$, E_R and ${}_SE$ are E-reflexive.*
(3) ${}_SE_R$ *is a faithfully balanced bimodule and E_R and ${}_SE$ are injective cogenerators.*

The following theorem is known as the Morita-Azumaya Theorem:

Theorem 1.1.10. (Azumaya [9], Morita [122]) *Let R be a right artinian ring and S a ring. For a bimodule ${}_SE_R$, the following are equivalent:*

(1) *${}_SE_R$ defines a Morita duality between S-FMod and FMod-R.*
(2) *E_R is a finitely generated injective cogenerator and the canonical ring homomorphism $S \to End\ (E_R)$ is an isomorphism.*

When this is so, the following hold:

(1) *S is left artinian, FMod-$R = R[E]_R$ and S-FMod $= {}_SR[E]$.*
(2) *Every indecomposable injective right R-module and every indecomposable injective left S-module are finitely generated.*

From this theorem we know that, for a right artinian ring R, if $E = E((R/J)_R)$ is finitely generated, then ${}_SE_R$ defines a Morita duality between S-*FMod* and *FMod*-R, where $S = \text{End}(E_R)$. We note that this important fact is also shown in Tachikawa [172].

Let R be a right or left noetherian ring. We say that R is *self-dual* or has a *self-duality* if there is a duality between $FMod$-R and R-$FMod$.

For an R-module M, the top $M/J(M)$ of M is denoted by $T(M)$. The right (resp. left) annihilator of a subset T of M_R (resp. ${}_RM$) in a subset S of R is denoted by $r_S(T)$ (resp. $l_S(T)$).

A ring R is called a *quasi-Frobenius* ring (abbreviated as *QF-ring*) if R is a left and right artinian and left and right self-injective ring. *QF*-rings were introduced by Nakayama [133], as artinian rings with the condition (5) in the theorem below (see Section 1.3).

The following theorem records well-known characterizations of *QF*-rings.

Theorem 1.1.11. *The following are equivalent for a ring R:*

(1) *R is QF.*

(2) *R is left or right artinian and left or right self-injective.*

(3) *R is left or right noetherian and left or right self-injective.*

(4) *R is a left or right noetherian ring and ${}_RR_R$ defines a self-duality between R-FMod and FMod-R.*

(5) *R is a left or right artinian ring and, for any primitive idempotent e of R, there exists a primitive idempotent f of R such that $(eR_R; {}_RRf)$ is an i-pair, i.e., $S(eR_R) \cong T(fR_R)$ and $S({}_RRf) \cong T({}_RRe)$.*

(6) *R is a left or right noetherian ring with double annihilator conditions, i.e.,*

$$l_R(r_R(I)) = I \text{ for any left ideal } I \text{ of } R \text{ and}$$
$$r_R(l_R(K)) = K \text{ for any right ideal } K \text{ of } R.$$

(7) *Every projective left (or right) R-module is injective.*

(8) *Every injective left (or right) R-module is projective.*

A right R-module M is said to be *uniserial* if the family of all submodules of M is linearly ordered by inclusion. A right (resp. left) artinian ring is called a *right* (resp. *left*) *Nakayama* ring if, for any primitive idempotent e of R, eR_R (resp. ${}_RRe$) is uniserial. The ring R is called a *Nakayama ring* if it is a right and left Nakayama ring.

Theorem 1.1.12. ([133], cf. [43]) *Let R be a Nakayama ring. Then every right R-module can be expressed as a direct sum of uniserial modules.*

For a given right R-module M, there exists an injective module $E(M)$ containing M as an essential submodule. The existence of $E(M)$ is known as the Eckmann-Schopf Theorem ([42]), and $E(M)$ is called an *injective hull* of M.

For a module M and an indexed set A, we let $M^{(A)}$ denote the direct sum of $\sharp A$ copies of M. An injective module M is said to be $\sum$-*injective* if $M^{(A)}$ is injective for every indexed set A.

The following fundamental theorem is due to Faith.

Theorem 1.1.13. ([47, 20.3 A and 20.6 A] or [5, Theorem 25.1]) *The following are equivalent for an injective module M:*

(1) *M is $\sum$-injective.*
(2) *ACC holds on $\{ r_R(X) \mid X$ is a subset of $M \}$.*
(3) *$M^{(\mathbb{N})}$ is injective.*

We say that a module M has *the exchange property* if, for every R-module L containing M and for submodules N and $\{L_i\}_{i\in I}$ of L, decompositions

$$L = M \oplus N = \oplus_{i\in I} L_i$$

imply the existence of a submodule K_i of L_i for every $i \in I$ satisfying

$$L = M \oplus (\oplus_{i\in I} K_i).$$

We note that every quasi-injective R-module has the exchange property (see, for instance, Fuchs [52], Warfield [180]). M is said to have the *finite exchange property* if this condition is satisfied whenever the above indexed set I is finite.

Let M be a right R-module with a decomposition $M = \oplus_{i\in I} A_i$. We say that the decomposition $M = \oplus_{i\in I} A_i$ is *exchangeable* (or *complements direct summands*) if, for any direct summand N of M, there exists a submodule A_i' of A_i for every $i \in I$ such that $M = N \oplus (\oplus_{i\in I} A_i')$.

Theorem 1.1.14. ([179]) *Let A be an indecomposable module and $M = A_1 \oplus A_2$ with $A_i \cong A$ for $i = 1, 2$. Then the following are equivalent:*

(1) *A satisfies the (finite) exchange property.*
(2) *A satisfies the exchange property for $A \oplus A$, i.e., for any direct summand N of M with $N \cong A$, either $M = N \oplus A_1$ or $M = N \oplus A_2$.*
(3) *$M = A_1 \oplus A_2$ is exchangeable.*
(4) *$\mathrm{End}(A)$ is a local ring.*

An infinite family $\{M_\alpha\}_{\alpha\in I}$ of modules is called *locally semi-transfinitely nilpotent* (abbreviated lsTn) if, for any subfamily $\{M_{\alpha_i}\}_{i\in\mathbb{N}}$ of $\{M_\alpha\}_{\alpha\in I}$ with distinct α_i, any family $\{f_i : M_{\alpha_i} \to M_{\alpha_{i+1}}\}_{i\in\mathbb{N}}$ of non-isomorphisms and any $x \in M_{\alpha_1}$, there exists $n \in \mathbb{N}$ (depending x) such that $f_n \cdots f_2 f_1(x) = 0$.

Theorem 1.1.15. ([64], cf. [191]) *Let $M_R = \oplus_{i\in I} M_i$ be an indecomposable decomposition, where* $\mathrm{End}(M_i)$ *is a local ring for all $i \in I$. Then the following are equivalent:*

(1) *M has the (finite) exchange property.*
(2) *$M_R = \oplus_{i\in I} M_i$ is exchangeable.*
(3) *(If I is an infinite set) the family $\{M_i\}_{i\in I}$ is lsTn.*

Given $M_R \supseteq B_R \supseteq A_R$, we may interpret the dual of "$A \subseteq_e B$" in M to mean "B/A is a small submodule of M/A" or, equivalently, "for any submodule X of M with $A \subseteq X$, $M = X + B$ implies $X = M$". If this dual condition holds, A is called a *co-essential* submodule of B (in M) and this is denoted by $A \subseteq_c B$ *in* M.

An R-module M is called *uniform* (resp. *hollow*) if any non-zero (resp. proper) submodule of M is an essential (resp. a small) submodule of M.

Let A be a submodule of M_R. We say that M satisfies the *extending property for* A if A can be essentially extended to a direct summand of M. In case any uniform submodule of M can be essentially extended to a direct summand of M, we say that M satisfies the *extending property for uniform submodules.* We say that M is an *extending* module or a *CS*-module if M satisfies the extending property for all of its submodules.

Note that, for an R-module M_R and a submodule A of M, we can find a submodule B of M such that $A \subseteq_e B$ and there exists no proper essential extension of B in M. A submodule B which has no proper essential extension in M is called a *closed* submodule of M. So we may say that M_R is an extending module if any closed submodule of M is a direct summand. (The alternative CS name comes from this "closed submodule is a summand" property.)

Dually, given an R-module M and a submodule A of M, we say that M has the *lifting property for* M/A or A is *co-essentially lifted* to a direct summand of M if there exists a decomposition $M = A^* \oplus A^{**}$ satisfying $A^* \subseteq A$ and $A/A^* \ll M/A^*$, i.e., $A \cap A^{**} \ll A^{**}$.

This leads to the following dual notions of "extending property for uniform modules" and "extending module". If M satisfies the lifting property for all of its hollow factor modules, we say that M satisfies the *lifting property for hollow factor modules.* And if it satisfies the lifting property for all its factor modules, we say that M is a *lifting* module. Furthermore we say that M satisfies the *lifting property for simple factor modules* if it satisfies the lifting property for all of its simple factor modules.

The following result for the extending case follows from the fact that, for any $A \subseteq B \subseteq C$ such that A is a closed submodule of B and B is a closed submodule of C, A is a closed submodule of C (see Gooderal [55], or Oshiro [145]). The result for the lifting case follows quickly from the definition.

Proposition 1.1.16. *Any direct summand of an extending* (*resp. a lifting*) *module is an extending* (*resp. a lifting*) *module.*

Let M and N be R-modules. Then M is said to be N*-injective* if, for any submodule X of N and any homomorphism $\varphi : X \to M$, there exists $\tilde{\varphi} \in \mathrm{Hom}_R(N, M)$ such that the restriction map $\tilde{\varphi}|_X$ coincides with φ. If in this definition we only consider homomorphisms φ with simple images, M is then said to be N*-simple-injective.*

Dually, M is said to be N*-projective* if, for any submodule X of N and any homomorphism $\varphi : M \to N/X$, there exists $\tilde{\varphi} \in \mathrm{Hom}_R(M, N)$ such that $\pi_X \tilde{\varphi} = \varphi$, where $\pi_X : N \to N/X$ is the natural epimorphism.

A module M is then called *quasi-injective* (resp. *quasi-projective*) if M is M-injective (resp. M-projective).

Let A and B be right R-modules. A is said to be *generalized* B*-injective* (or *B-ojective*) if, for any submodule X of B and any homomorphism $f : X \to A$, there exist decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, a homomorphism $h_1 : B_1 \to A_1$, and a monomorphism $h_2 : A_2 \to B_2$ satisfying

(a) $X \subseteq B_1 \oplus h_2(A_2)$, and
(b) for any $x \in X$ with $x = x_1 + x_2$, where $x_i \in B_i$ for $i = 1, 2$, we have $f(x) = h_1(x_1) + h_2^{-1}(x_2)$.

The following diagram illustrates this definition.

$$\begin{array}{ccc} 0 \longrightarrow X \xrightarrow{i} B \\ f\downarrow \\ A \end{array} \qquad \begin{array}{cccc} 0 \longrightarrow X \xrightarrow{i} & B = & B_1 \oplus & B_2 \\ f\downarrow & & h_1\downarrow & \uparrow h_2 \\ A & = & A_1 \oplus & A_2 \\ & & & \uparrow \\ & & & 0 \end{array}$$

A right R-module A is said to be *essentially B-injective* if, for any submodule X of B, any homomorphism $f : X \to A$ with $\operatorname{Ker} f \subseteq_e X$ can be extended to a homomorphism $B \to A$.

Remark 1.1.17. It is easy to see that if A is generalized B-injective, then A is essentially B-injective (cf. [57]). From this, it follows that, for uniform right R-modules A and B, the following are equivalent:

(1) A is generalized B-injective.
(2) For any submodule X of B and any homomorphism $f : X \to A$, if f is not monomorphic, then f extends to $B \to A$, and if f is monomorphic, then either f extends to $B \to A$ or f^{-1} extends to $A \to B$.

The following theorem is one of the fundamental results on extending modules.

Theorem 1.1.18. ([57], [116]) *Let $A_1, \ldots, A_n$ be extending right R-*

modules and set $M = A_1 \oplus \cdots \oplus A_n$*. Then the following are equivalent:*

(1) M *is an extending module and the decomposition* $M = A_1 \oplus \cdots \oplus A_n$ *is exchangeable.*

(2) A_i *is generalized* $\oplus_{j \neq i} A_j$*-injective for any* $i \in \{1, \cdots, n\}$.

(3) $\oplus_{j \neq i} A_j$ *is generalized* A_i*-injective for any* $i \in \{1, \cdots, n\}$.

In particular, when each A_i *is uniform, the following are also equivalent to these conditions:*

(4) *The direct sum* $A_i \oplus A_j$ *is extending and exchangeable for any* $i \neq j$.

(5) A_i *and* A_j *are matually relative generalized injective modules for any* $i \neq j$.

For $M = A_1 \oplus \cdots \oplus A_n$ in this theorem, we note that, if A_i and A_j are mutually relative injective for any $i \neq j$, then M is an extending module (cf. [115], [75]).

We now define the dual notion of generalized B-injectivity. Given modules A and B, we say that A is *generalized* B*-projective* (or *dual* B*-ojective*) if, for any module X, any homomorphism $f : A \to X$ and any epimorphism $g : B \to X$, there exist decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, a homomorphism $h_1 : A_1 \to B_1$, and an epimorphism $h_2 : B_2 \to A_2$ such that $gh_1 = f|_{A_1}$ and $fh_2 = g|_{B_2}$. The following diagram shows the decompositions and maps involved.

$$
\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \uparrow \\
 & & A & = A_1 & \oplus A_2 \\
 & & f \downarrow & h_1 \downarrow & \uparrow h_2 \\
0 \longleftarrow & & X \overset{g}{\longleftarrow} & B = B_1 & \oplus B_2
\end{array}
$$

A right R-module A is said to be *small* B*-projective* if, for any module X, any homomorphism $f : A \to X$ with $\operatorname{Im} f \ll X$ and any epimorphism $g : B \to X$, there exists a homomorphism $h : A \to B$ satisfying $gh = f$.

Remark 1.1.19. (cf. [106]) We can easily see that if A is a generalized B-projective module and B is lifting, then A is small B-projective. From this fact, we note that, when A and B are hollow modules, the following are equivalent:

(1) A is generalized B-projective.

(2) For any X, any homomorphism $f : A \to X$ and any epimorphism $g : B \to X$, if f is not epimorphic, there exists a homomorphism $h : A \to B$ satisfying $gh = f$, and if f is epimorphic, then there exists an epimorphism $h : A \to B$ or $B \to A$ satisfying $gh = f$ or $fh = g$, respectively.

Theorem 1.1.20. ([106], [117], cf. [31]) *Let $A_1, \ldots, A_n$ be lifting right R-modules and set $M = A_1 \oplus \cdots \oplus A_n$. Then the following are equivalent:*

(1) *M is a lifting module and the decomposition $M = A_1 \oplus \cdots \oplus A_n$ is exchangeable.*

(2) *A'_i and T are mutually generalized projective for any summand A'_i of A_i and any summand T of $\oplus_{j\neq i} A_j$ for any $i \in \{1, \cdots, n\}$.*
In particular, when each A_i is a hollow module, the following are equvalent to these conditions:

(3) *The direct sum $A_i \oplus A_j$ is an exchangeable and lifting module for any $i \neq j$.*

(4) *A_i and A_j are mutually relative generalized projective modules for any $i \neq j$.*

For $M = A_1 \oplus \cdots \oplus A_n$ in this theorem, we note that, if A_i and A_j are mutually relative projective for any $i \neq j$, then M is a lifting module and the direct sum $A_i \oplus A_j$ is exchangeable ([106], cf. [92]).

For a right R-module M, we consider the following conditions:

(C_1) M is an extending module.

(C_2) If a submodule X of M is isomorphic to a direct summand of M, then X is a direct summand.

(C_3) If M_1 and M_2 are direct summands of M with $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M.

(D_1) M is a lifting module.

(D_2) If X is a submodule of M such that M/X is isomorphic to a direct summand of M, then X is a direct summand of M.

(D_3) If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of M.

M is called a *continuous* module if (C_1) and (C_2) hold, and is called a *quasi-continuous module* if (C_1) and (C_3) hold. Dually M is called a *discrete* module (or *semiperfect* module) if (D_1) and (D_2) hold, and is called a *quasi-discrete* (or *quasi-semiperfect*) module if (D_1) and (D_3) hold.

Remark 1.1.21. We have the following implications for these module properties:

(1) injective $\Rightarrow$ quasi-injective $\Rightarrow$ continuous $\Rightarrow$ quasi-continuous $\Rightarrow$ extending

(2) projective $\Rightarrow$ quasi-projective $\not\Rightarrow$ discrete $\Rightarrow$ quasi-discrete $\Rightarrow$ lifting

(3) (quasi-)projective lifting $\Rightarrow$ discrete.

It follows from the last of these and Theorem 1.1.20 that any finite direct sum of projective lifting modules is lifting.

The following results give fundamental facts on (quasi-)discrete modules.

Theorem 1.1.22. ([146], cf. [115]) *If M is a quasi-discrete module, then M can be expressed as a direct sum $\oplus_{i\in I} M_i$ of hollow modules M_i. In particular, if M is discrete, then each* End(M_i) *is a local ring and $M = \oplus_{i\in I} M_i$ is exchangeble, and hence the family $\{M_i\}_{i\in I}$ is lsTn. Consequently, M has* *the exchange property by Theorem 1.1.15.*

Theorem 1.1.23. ([146], cf. [115]) *Let M be a discrete right R-module.*

(1) *If $M = \sum_{i \in I} M_i$, M_i is a summand of M and $M_i \cap (\sum_{j \in I - \{i\}} M_j) \ll M$ for any $i \in I$, then $M = \sum_{i \in I} M_i$ is a direct sum.*

(2) *If $M = \sum_{i \in I} M_i$ is an irredundant sum of indecomposable submodules M_i, then the sum $M = \sum_{i \in I} M_i$ is a direct sum.*

Let M be a right R-module and N a submodule of M. We say that N is a *fully invariant* submodule of M if N is a right R- left $\mathrm{End}(M_R)$-subbimodule of M. Let M and P be right R-modules. An epimorphism $\varphi : P \to M$ is called *small* (or *superfluous*) if $\mathrm{Ker}\, \varphi \ll P$. A pair (P, φ) is called a *projective cover* of M if P is projective and there exists a small epimorphism $\varphi : P \to M$. In this case we usually just say that $\varphi : P \to M$ is a projective cover or, more simply, that P is a projective cover of M.

To finish this section, we record several properties of quasi-injective modules and quasi-projective modules.

Theorem 1.1.24. *Let M be a right R-module.*

(1) *If M is a quasi-injective module, then it is a fully invariant submodule of $E(M)$.*

(2) *If M is a quasi-injective module with injective hull $E(M)$, then any direct sum decomposition $E(M) = E_1 \oplus \cdots \oplus E_n$ induces $M = (M \cap E_1) \oplus \cdots \oplus (M \cap E_n)$.*

(3) *If M is a quasi-projective module with projective cover $\varphi : P \to M$, $\mathrm{Ker}\, \varphi$ is a fully invariant submodule of P; consequently, any endomorphism of P induces an endomorphism of M.*

(4) *If M is a quasi-projective module with projective cover $\varphi : P \to M$, then any decomposition $P = P_1 \oplus \cdots \oplus P_n$ induces a decomposition $M = \varphi(P_1) \oplus \cdots \oplus \varphi(P_n)$.*

(5) *If* $\varphi : P \to M$ *is an epimorphism with* P *projective and* M *has a projective cover, then there exists a decomposition* $P = P_1 \oplus P_2$ *such that* $\varphi(P_2) = 0$ *and the restriction map* $\varphi|_{P_1} : P_1 \to M$ *is a projective cover.*

We note that (1), (3), (5) are easily verified, and (2), (4) can be proved using (1), (3), respectively.

1.2 Semiperfect Rings and Perfect Rings

Semiperfect rings and perfect rings were introduced by Bass [21] in 1960. Since our book is based on these rings, for the sake of the reader's convenience, we shall recall fundamental facts on these rings.

In [21], as a dual to the notion of an injective hull, Bass introduced a projective cover of a module. In general projective covers need not exist. A ring R is called *semiperfect* (resp. *right perfect*) if every finitely generated right module (resp. every right R-module) has a projective cover.

Let $M = M_1 \oplus M_2$ and let $\varphi : M_1 \to M_2$ be an R-homomorphism. Let $\langle M_1 \xrightarrow{\varphi} M_2 \rangle$ denote the submodule of M given by $\{ m_1 - \varphi(m_1) \mid m_1 \in M_1 \}$. This is called the *graph* of φ. Note that $M = M_1 \oplus M_2 = \langle M_1 \xrightarrow{\varphi} M_2 \rangle \oplus M_2$.

Proposition 1.2.1. *Let* R *be a ring such that every maximal right ideal is a direct summand of* R_R. *Then* R *is a semisimple ring.*

Proof. Assume that $S(R_R) \subsetneq R_R$. Then there exists a maximal right ideal I_R of R such that $S(R_R) \subseteq I_R$. By hypothesis, there exists a decomposition $R_R = I \oplus X$. Then X is a simple submodule of R_R and hence $X \subseteq S(R_R) \subseteq I$, a contradiction. Hence $R = S(R_R)$.

□

Proposition 1.2.2. *Let* e *be an idempotent of a ring* R. *For any* $s \in eRe$, $s \in J(eRe) = eJe$ *if and only if* $sR \ll eR$.

Proof. ($\Rightarrow$). Let $s \in J(eRe) = eJe$. Then $sR \subseteq eJ$, so $sR \ll R$. Hence $sR \ll eR$.

($\Leftarrow$). Assume $sR \ll eR$. Then $sR \ll R$ and hence $sR \subseteq J$. Hence $esRe \subseteq eJe$. Thus $s = ese \in eJe$. □

For a subset S of a ring R and $a \in R$, the left multiplication map $: S \to aS$ defined by $s \mapsto as$ is denoted by $(a)_L$. (Similarly, the right multiplication map $: S \to Sa$ defined by $s \mapsto sa$ is denoted by $(a)_R$.)

Proposition 1.2.3. *For an idempotent e of a ring R, the following are equivalent:*

(1) *eRe is a local ring.*
(2) *eJ is the unique maximal submodule of eR_R.*
(3) *Je is the unique maximal submodule of ${}_RRe$.*

Proof. It suffices to show (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (2). Let K be a proper submodule of eR_R and let $eR = K + L$. Then $eR/K \cong L/(L \cap K)$. Consider the canonical epimorphism $f : L \to L/(L \cap K)$. Since eR_R is projective, there exists a homomorphism $\rho : eR \to L$ such that $\operatorname{Im}\rho + \operatorname{Ker} f = L$. Since $\rho \in \operatorname{End}(eR_R)$, ρ is realized by a left multiplication $(s)_L$ for some $s \in eRe$. Since $\operatorname{Ker} f \neq L$, $\operatorname{Im}\rho$ is not small in L. By Proposition 1.2.2, $s \notin J(eRe) = eJe$. Then, since eRe is a local ring, s is unit and hence $\operatorname{Im}\rho = L = eR$. Hence $K \ll eR$, and $K \subseteq eJ$.

(2) $\Rightarrow$ (1). Since eJ is the unique maximal submodule of eR_R, we see that the set of all non-unit elements is a two-sided ideal of $\operatorname{Hom}_R(eR, eR) \cong eRe$, whence eRe is a local ring. □

An idempotent e of R is called a *local* idempotent if eRe is a local ring.

Proposition 1.2.4. *Let R be a ring such that R_R is a lifting module. Then the following hold:*

(1) *$\overline{R} = R/J$ is a semisimple ring.*

(2) *If e is a primitive idempotent of R, then eJ is the unique maximal submodule of eR_R, i.e., eRe is a local ring.*

(3) *Every complete set of orthogonal (primitive) idempotents of $\overline{R} = R/J$ lifts to a complete set of orthogonal (primitive) idempotents of R.*

Proof. (1). Though this follows from Theorem 1.1.22, we give a direct proof. Let A be a submodule of R_R with $A \supseteq J$. We put $\overline{A} = A/J$ and $\overline{R} = R/J$. We may show that $\overline{A} \lessdot\!\oplus\, \overline{R}$. Since R_R is lifting, there exists a decomposition $R_R = A^* \oplus A^{**}$ such that $A^* \subseteq A$ and $A \cap A^{**} \ll R$. Hence $\overline{A} = \overline{A^*} \lessdot\!\oplus\, \overline{R}$, as required.

(2). Consider a proper submodule $K_R \subsetneq eR$. Since eR_R is an indecomposable lifting module, we have $K \ll eR$. Hence $K \subseteq eJ$. Therefore eJ is the unique maximal submodule of eR_R.

(3). Let $\overline{R} = \overline{g_1 R} \oplus \cdots \oplus \overline{g_n R}$, where $\{\overline{g_1}, \ldots, \overline{g_n}\}$ is a complete set of orthogonal idempotents of $\overline{R}$. Let $\varphi : R \to \overline{R}$ be the canonical epimorphism. Since R_R is lifting, there exists a decomposition $R_R = A_i \oplus A_i^*$ such that $A_i \subseteq \varphi^{-1}(\overline{g_i R})$ and $\varphi^{-1}(\overline{g_i R}) \cap A_i^* \ll A_i^*$ for $i = 1, 2, \ldots, n$. Then $R = A_1 + \cdots + A_n + \operatorname{Ker}\varphi$. Since $\operatorname{Ker}\varphi \ll R_R$, this implies that $R = A_1 + \cdots + A_n$. Moreover, since $A_i \cap (\sum_{j \neq i} A_j) \ll R_R$, we see that $R = A_1 \oplus \cdots \oplus A_n$ by Theorem 1.1.22. Then $\overline{A_i} = \overline{g_i R}$ for $i = 1, \ldots, n$. Now take a complete set $\{e_1, \ldots, e_n\}$ of orthogonal idempotents of R such that $A_i = e_i R$ for $i = 1, 2, \ldots, n$. Since $\varphi(1) = \overline{1} = \overline{e_1} + \cdots + \overline{e_n} = \overline{g_1} + \cdots + \overline{g_n}$, we see that $\overline{e_i} = \overline{g_i}$ for $i = 1, \ldots, n$. □

Proposition 1.2.5. *If M_R has a projective cover, then, for any epimorphism $\varphi : P \to M$, where P_R is a projective module, there exists a decomposition $P = P_1 \oplus P_2$ such that $P_1 \subseteq \operatorname{Ker}\varphi$ and $\varphi|_{P_2} : P_2 \to M$ is a projective cover of M.*

Proof. Let $f : Q \to M$ be a projective cover. Then we have a homomorphism $h : P \to Q$ satisfying $fh = \varphi$. Noting that $\operatorname{Ker} f \ll Q$, we see that h is epimorphic. Since Q is projective, h splits, i.e., there exists an R-homomorphism $g : Q \to P$ such that $hg = 1_Q$, and hence $P = \operatorname{Im} g \oplus \operatorname{Ker} h$. Put $P_2 = \operatorname{Im} g$ and $P_1 = \operatorname{Ker} h$. Then $P_1 \subseteq \operatorname{Ker} \varphi$ since $\operatorname{Ker} h \subseteq \operatorname{Ker} fh = \operatorname{Ker} \varphi$. Since $P_2 \cong Q$ by $h|_{P_2}$ and $fh|_{P_2} = \varphi|_{P_2}$, we see that $\varphi|_{P_2} : P_2 \to M$ is a projective cover of M. □

Proposition 1.2.6. *Let P be a projective module. Then the following are equivalent:*

(1) *Every factor module of P has a projective cover.*
(2) *P is a lifting module.*

Proof. (1) ⇒ (2). Let A be a submodule of P and let $\varphi : P \to P/A$ be the canonical epimorphism. Then, by Proposition 1.2.5, there exists a decomposition $P = P_1 \oplus P_2$ such that $P_1 \subseteq A$ and $\varphi|_{P_2} : P_2 \to P/A$ is a projective cover. Since $A \cap P_2 \subseteq \operatorname{Ker} \varphi|_{P_2} \ll P_2$, P satisfies the lifting property for A.

(2) ⇒ (1). We must show that, for any submodule A of P, P/A has a projective cover. Since P is a lifting module, we have a decomposition $P = P_1 \oplus P_2$ such that $P_1 \subseteq A$ and $A \cap P_2 \ll P_2$. Then $\varphi|_{P_2} : P_2 \to P/A$ is epimorphic and $\operatorname{Ker} \varphi|_{P_2} = A \cap P_2$. Thus $\varphi|_{P_2} : P_2 \to P/A$ is a projective cover of P/A. □

Using similar arguments, we can show the following two propositions.

Proposition 1.2.7. *Let R be a ring. The following are equivalent:*

(1) *Every cyclic right R-module has a projective cover.*
(2) *R_R is a lifting module.*

Proposition 1.2.8. *Let R be a ring. The following are equivalent:*

(1) *Every simple right R-module has a projective cover.*
(2) *R_R satisfies the lifting property for simple factor modules.*

Proposition 1.2.9. *If $P_1, \ldots, P_n$ are projective lifting R-modules, then $P = P_1 \oplus \cdots \oplus P_n$ is a lifting module.*

Proof. Since any projective lifting module is a discrete module, the statement follows from Theorem 1.1.20. □

Proposition 1.2.10. *Let R be a ring such that R/J is semisimple and every idempotent of $\overline{R} = R/J$ lift modulo J. Then R satisfies the lifting property for simple factor modules.*

Proof. Let M be a maximal right ideal. By the assumption, we can take an idempotent e of R such that $\overline{eR} = \overline{M}$. Then $\overline{(1-e)R}$ is simple and $eR + J = M + J = M$. Hence $M = eR \oplus (M \cap (1-e)R) \subseteq eR \oplus (1-e)J$. Since $\overline{(1-e)R}$ is simple, $(1-e)J$ is the unique maximal submodule of $(1-e)R_R$. Hence $M \cap (1-e)R \ll (1-e)R$. □

Proposition 1.2.11. *Let R be a ring and A a right ideal of R. If $R/(A+J)_R$ has a projective cover, then so does R/A_R.*

Proof. Consider the canonical epimorphisms: $R \xrightarrow{f} R/A \xrightarrow{g} R/(A+J)$. Then, by Proposition 1.2.5, we can take an idempotent $e \in R$ for which $gf\,|_{eR} : eR \to R/(A+J)$ is a projective cover. Then $\mathrm{Ker}(gf\,|_{eR}) \ll eR$. Since $R = eR + A + J$, we have $R = eR + A$. Hence $f\,|_{eR} : eR \to R/A$ is epimorphic. Since $\mathrm{Ker}(f\,|_{eR}) \subseteq \mathrm{Ker}(gf\,|_{eR}) \ll eR$, $f\,|_{eR} : eR \to R/A$ is a projective cover. □

Proposition 1.2.12. *Let R be a ring such that R_R satisfies the lifting property for simple factor modules. Then R_R is a lifting module. In other words, if every simple R-module has a projective cover, then every cyclic R-module has a projective cover.* (*Hence R is a lifting module by Proposition 1.2.7.*)

Proof. Let M be a maximal right ideal of R. By assumption, we have a decomposition $R = X \oplus Y$ such that $X \subseteq M$ and $M \cap Y \ll Y$. Hence $(M + J)/J = (X + J)/J$ and $R/J = (X + J)/J \oplus (Y + J)/J$. Thus every maximal right ideal of R/J is a direct summand, and hence R/J is semisimple by Proposition 1.2.1. Let $A_R \subseteq R_R$. We show that R/A has a projective cover. By Proposition 1.2.11, we may assume that $J \subseteq A$. Noting that $(R/J)/(A/J) \cong R/A$, we can show that R/A is expressible as a direct sum of simple submodules. Since any simple module has a projective cover, R/A has a projective cover. □

Theorem 1.2.13. *Let R be a ring. Then the following are equivalent:*

(1) *R is semiperfect.*

(2) *R/J is semisimple and idempotents of R/J lift modulo J.*

(3) *R/J is semisimple and every complete set of orthogonal (primitive) idempotents of R/J lifts to a complete set of orthogonal (primitive) idempotents of R.*

(4) *R is expressed as $R = e_1R \oplus \cdots \oplus e_nR$, where $\{e_i\}_{i=1}^n$ is a complete set of orthogonal primitive idempotents of R and each e_i is a local idempotent.*

(5) *R is expressed as $R = e_1R \oplus \cdots \oplus e_nR$, where $\{e_i\}_{i=1}^n$ is a complete set of orthogonal primitive idempotents of R and each e_iJ is the unique maximal submodule of e_iR_R.*

(6) *Every cyclic right R-module has a projective cover.*

(7) *Every simple right R-module has a projective cover.*

(8) *Every finitely generated projective right R-module is a lifting module.*

(9) *R_R is a lifting module.*

(10) *R_R satisfies the lifting property for simple factor modules.*

Proof. (1) ⇒ (6) ⇒ (7) are obvious. (3) ⇒ (4) is obvious. (1) ⇔ (8) follows from Proposition 1.2.6. Proposition 1.2.7 gives (6) ⇔ (9). (7)

$\Leftrightarrow$ (10) follows from Proposition 1.2.8. (8) $\Rightarrow$ (9) $\Rightarrow$ (10) are obvious. Proposition 1.2.12 gives (10) $\Rightarrow$ (9). (9) $\Rightarrow$ (8) follows from Proposition 1.2.9. Proposition 1.2.4 gives (9) $\Rightarrow$ (3), (4). (2) $\Rightarrow$ (10) follows from Proposition 1.2.10. (3) $\Rightarrow$ (2) is obvious. (4) $\Leftrightarrow$ (5) follows from Proposition 1.2.3.

We finish by showing (5) $\Rightarrow$ (9). Since e_iJ is the unique maximal submodule of e_iR_R, any proper submodule of e_iR_R is small. Hence each e_iR_R is a lifting module and hence R is a lifting module by Proposition 1.2.9.

□

We note that the notion of a semiperfect ring is left-right symmetric by (2) in Theorem 1.2.13.

Let R be a semiperfect ring. By Theorem 1.2.13, R has a complete set of orthogonal primitive idempotents. Henceforth, we let $Pi(R)$ denote a complete set of orthogonal primitive idempotents of R. Let $Pi(R) = \{e_1, \ldots, e_n\}$. In case $e_iR_R \not\cong e_jR_R$ for any distinct $i, j \in \{1, \ldots, n\}$, R is called a *basic* semiperfect ring. If R is not basic, we may partition $E = E_1 \cup \cdots \cup E_k$ such that, for any $e \in E_i$ and any $f \in E_j$, $eR_R \cong fR_R$ if and only if $i = j$. If we choose one e_i in E_i for each $i = 1, \ldots, k$, and set $e = e_1 + \cdots + e_k$, then eR is a finitely generated projective generator in *Mod*-R, and hence R is Morita equivalent to eRe. (eRe is called a *basic* subring of R.) For this reason, we usually restrict our attention to basic semiperfect rings.

Let R be a basic semiperfect ring with $Pi(R) = \{e_1, \ldots, e_n\}$, and represent R as

$$R = \begin{pmatrix} e_1Re_1 & e_1Re_2 & \cdots & \cdots & e_1Re_n \\ e_2Re_1 & e_2Re_2 & \cdots & \cdots & e_2Re_n \\ & \cdots & \cdots & \cdots & \\ e_nRe_1 & e_nRe_2 & \cdots & \cdots & e_nRe_n \end{pmatrix}.$$

Then we have

$$J = \begin{pmatrix} e_1Je_1 & e_1Re_2 & \cdots & \cdots & e_1Re_n \\ e_2Re_1 & e_2Je_2 & \cdots & \cdots & e_2Re_n \\ & \cdots & \cdots & \cdots & \\ e_nRe_1 & e_nRe_2 & \cdots & \cdots & e_nJe_n \end{pmatrix}.$$

Using this representation, we can see that, for any subset $\{f_1, \ldots, f_k\} \subseteq Pi(R)$, fRf is also a basic semiperfect ring, where $f = f_1 + \cdots + f_k$.

Theorem 1.2.14. *Let R be a semiperfect ring. For two complete sets $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^m$ of orthogonal primitive idempotents of R, the following hold:*

(1) *$n = m$.*

(2) *There exist a permutation ρ of $\{1, \ldots, n\}$ and isomorphisms $g_i : e_iR_R \cong f_{\rho(i)}R_R$ for any i.*

(3) *There exists an inner automorphism τ of R satisfying $\tau(e_i) = f_{\rho(i)}$ for any i.*

Proof. (1), (2). These follow from the Krull-Remak-Schmidt-Azumaya Theorem.

(3). We take $\{g_i\}_{i=1}^n$ from (2). The direct sum map $g = \oplus_{i=1}^n g_i$ is an automorphism of R_R such that $g(e_iR) = f_{\rho(i)}R$. If $x = g(1)$, then $g(r) = g(1)r = xr$ for any $r \in R$. Since g is an automorphism, $1 = g(a) = xa$ for some $a \in R$. Since $1 = xa$, we see that $aR_R \cong R_R$ and $R = aR \oplus (1 - ax)R$. Then using (1), we see that $(1 - ax)R = 0$, so $1 = ax$ and hence x is a unit of R. Hence $xe_iRx^{-1} = xe_iR = g(e_iR) = f_{\rho(i)}R$ for any i. Since $1 = \sum_{i=1}^n e_i = x1x^{-1} = \sum_{i=1}^n xe_ix^{-1} = \sum_{i=1}^n f_{\rho(i)}$ and $xe_ix^{-1} \in f_{\rho(i)}R$, we see that $x^{-1}e_ix = f_{\rho(i)}$ for any i. □

A subset I of a ring R is called *right T-nilpotent* if, for every sequence $a_1, a_2, \ldots$ in I, there exists $n \in \mathbb{N}$ with $a_na_{n-1} \cdots a_1 = 0$. (Similarly, I is called *left T-nilpotent* if, for any sequence $a_1, a_2, \ldots$ in I, we have $a_1a_2 \cdots a_n = 0$ for some n.) We note that, if I is left or right T-nilpotent, then it is nil because $a, a, \ldots$ is a sequence in I whenever $a \in I$.

Theorem 1.2.15. (cf. [5]) *Let I be a right ideal of a ring R. Then the following are equivalent:*

(1) *I is right T-nilpotent.*
(2) *$MI \neq M$ for every non-zero right R-module M.*
(3) *$MI \ll M$ for every non-zero right R-module M.*
(4) *$FI \ll F$ for the countably generated free module $F = R^{(\mathbb{N})}$.*

The following theorem due to Bass is one of the fundamental results in ring theory. (For its proof, refer to Anderson-Fuller [5].)

Theorem 1.2.16. ([21]) *The following are equivalent for a ring R:*

(1) *R is right perfect.*
(2) *R/J is semisimple and J is right T-nilpotent.*
(3) *R/J is semisimple and every non-zero right R-module contains a maximal submodule.*
(4) *Every right flat R-module is projective.*
(5) *R satisfies DCC on principal left ideals.*
(6) *R contains no infinite set of orthogonal idempotents and every non-zero left R-module contains a minimal submodule.*
(7) *Every countably generated free right R-module is lifting.*

We now give a further characterization perfect rings using the lifting property.

Theorem 1.2.17. *Let R be a ring. The following are equivalent:*

(1) *R is a right perfect ring.*
(2) *Every projective right R-module is a lifting module.*
(3) *Every quasi-projective right R-module is a lifting module.*
(4) *$R^{(\mathbb{N})}$ is a lifting module.*

Proof. (1) $\Leftrightarrow$ (2). This follows from Proposition 1.2.6.

(2) $\Rightarrow$ (3). Let Q_R be a quasi-projective module and let A be a submodule of Q. Consider the canonical epimorphism $f : Q \to Q/A$. Now choose

an epimorphism $g : P \to Q$, where P_R is a projective module. Then, since P is a lifting module, by Proposition 1.2.5, there is a decomposition $P = P_1 \oplus P_2$ such that $P_1 \subseteq g^{-1}(A)$ and $fg|_{P_2} : P_2 \to Q/A$ is a projective cover. Since Q is quasi-projective, the decomposition $P = P_1 \oplus P_2$ induces a direct decomposition $Q = g(P_1) \oplus g(P_2)$ by Theorem 1.1.24 (4). Thus it follows that $g(P_1) \subseteq A$ and $g(P_2) \cap A \ll g(P_2)$.

(3) $\Rightarrow$ (2) is obvious.

(1) $\Rightarrow$ (4) follows from Theorem 1.2.16.

(4) $\Rightarrow$ (1). By (4), R is a semiperfect ring and R/J is semisimple. Since $R^{(\mathbb{N})}$ is lifting, there exists a decomposition $R^{(\mathbb{N})} = X \oplus Y$ such that $X \subseteq J(R^{(\mathbb{N})})$ and $J(R^{(\mathbb{N})}) \cap Y \ll Y$. Since $J(R^{(\mathbb{N})}) = J(X) \oplus J(Y)$ and $X \subseteq J(R^{(\mathbb{N})})$, we see that $J(X) = X$, which implies $X = 0$ and $R^{(\mathbb{N})}J = J(R^{(\mathbb{N})}) \ll R^{(\mathbb{N})}$. Hence, by Theorem 1.2.15, J is right T-nilpotent. Thus R is a right perfect ring. □

Theorem 1.2.18. ([113]) *For a projective module P, the following are equivalent:*

(1) *P is a lifting module.*

(2) (*a*) $J(P) \ll P$,

(*b*) $\overline{P} = P/J(P)$ *is semisimple, and*

(*c*) *every decomposition of $\overline{P}$ lifts to a decomposition of P.*

Proof. Note that, for any indecomposable projective lifting module T, $J(T) = TJ$ is a maximal submodule.

(1) $\Rightarrow$ (2). (a) There exists a direct decomposition $P = P_1 \oplus P_2$ such that $J(P) = P_1 \oplus (J(P) \cap P_2)$ and $J(P) \cap P_2 \ll P$. Then $J(P_1) = P_1$ and hence, by Theorem 1.1.2, $P_1 = 0$. Therefore $P = P_2$ and hence $J(P) \ll P$. (b) By Theorem 1.1.22, P can be expressed as a direct sum $\oplus_{i\in I} P_i$ of cyclic indecomposable projective lifting modules $\{P_i\}_{i\in I}$. Since $J(P_i)$ is the unique maximal submodule of P_i, $\overline{P}$ is semisimple. (c) Consider a decomposition $\overline{P} = \oplus_{i\in I}\overline{A_i}$. Then $A_i = A_i^* \oplus B_i$ for some direct summand

A_i^* of P and some small submodule B_i of P. Since $J(P) \ll P$, it follows that $P = \sum_{i\in I} A_i^*$ and $\overline{P} = \oplus_{i\in I}\overline{A_i^*}$. By Theorem 1.1.23, this implies $P = \oplus_{i\in I} A_i^*$.

(2) $\Rightarrow$ (1). Let A be a submodule of P. Then $\overline{P} = \overline{A} \oplus \overline{B}$ for some submodule B by (b). From (c), this decomposition lifts to a decomposition $P = P_1 \oplus P_2$ with $\overline{P_1} = \overline{A}$ and $\overline{P_2} = \overline{B}$. Let π: $P = P_1 \oplus P_2 \to P_1$ be the projection. Then it follows from $\overline{P_1} = \overline{A}$ and (a) that $\pi(A) = P_1$. Since P is projective, there exists a direct summand A^* of P such that $A = A^* \oplus (A \cap P_2)$. Thus, by $\overline{P_1} = \overline{A}$ and (a), we see that $A \cap P_2 \ll P_2$. □

Theorem 1.2.19. ([146], cf. [70] or [188]) *Every projective right R-module over a right perfect ring is a discrete module, and hence has the exchange property.*

Proof. Let P_R be a projective module. By Theorem 1.2.17, P is lifting, and hence P is discrete from Remark 1.1.21 and Theorem 1.1.22. □

1.3 Frobenius Algebras, and Nakayama Permutations and Nakayama Automorphisms of QF-Rings

In this section, we recall the definition of Frobenius algebras and Nakayama automorphisms for Frobenius algebras, and we introduce Nakayama automorphisms for Frobenius rings for later use.

Let R be an n-dimensional algebra over a field k; say $R = u_1k\oplus\cdots\oplus u_nk$. For any $a \in R$, there are $n \times n$ matrices $L(a)$ and $R(a) \in (k)_n$ satisfying

$$a(u_1, \cdots, u_n) = (u_1, \cdots, u_n)R(a), \quad {}^T(u_1, \cdots, u_n)a = L(a)^T(u_1, \cdots, u_n).$$

These $R(a)$ and $L(a)$ are said to be *right* and *left regular representations*, respectively. If there exists a regular $n \times n$ matrix $P \in (k)_n$ satisfying $PL(a) = R(a)P$, then $R(a)$ and $L(a)$ are said to be *equivalent*. Now R

is called a *Frobenius algebra* if, for any $a \in R$, its right and left regular representations are equivalent.

There are several characterizations of Frobenius algebras. We present three versions:

(A) Put $R^* = \mathrm{Hom}_k(R, k)$. Then R^* is an (R, R)-bimodule. Then R is a Frobenius algebra if and only if $R \cong R^*$ as right R-modules. More generally, R is called a *quasi-Frobenius algebra* if R_R and R^*_R have the same distinct representative indecomposable components, that is, for any indecomposable direct summand P_R of R_R, there exists a direct summand T_R of R^*_R such that $P \cong T$ and a similar condition holds for any indecomposable direct summand of R^*_R. In addition, if $R \cong R^*$ as (R, R)-modules, then R is called a *symmetric algebra.* If $S(eR_R) \cong T(eR_R)$ and $S({}_RRe) \cong T({}_RRe)$ for any primitive idempotent e of R, then R is called a *weakly symmetric algebra.*

(B) R is a Frobenius algebra if and only if the following hold:
For any right ideal A and any left ideal B of R,

$$rl(A) = A, \quad lr(B) = B,$$
$$\dim(A) + \dim(l(A)) = \dim(R),$$
$$\dim(B) + \dim(r(B)) = \dim(R),$$

where $l(X)$ and $r(X)$ denote the left annhilator ideal and right annihilator ideal of X, respectively, and $\dim(X)$ denotes the dimension of X over k.

(C) We arrange $Pi(R)$ as $Pi(R) = \{e_{ij}\}_{i=1,j=1}^{m \quad p(i)}$ with $e_{ij}R_R \cong e_{kl}R_R$ if $i = k$, and $e_{ij}R_R \not\cong e_{kl}R_R$ if $i \leq k$. Put $e_i = e_{i1}$. R is a Frobenius algebra if and only if the following hold:

(a) There is a pemutation of π of $\{e_1, e_2, \ldots, e_m\}$:

$$\pi = \begin{pmatrix} e_1 & e_2 & \cdots & e_m \\ e_{\pi(1)} & e_{\pi(2)} & \cdots & e_{\pi(m)} \end{pmatrix}$$

such that $S(e_iR) \cong e_{\pi(i)}R/e_{\pi(i)}J$ holds for $i = 1, \ldots, m$.

(b) $m(i) = m(\pi(i))$ holds for $i = 1, \ldots, m$.

In view of the characterization (C), the field k does not appear. From this point of view, in [133] Nakayama called R a quasi-Frobenius algebra, and introduced quasi-Frobenius rings and *Frobenius rings* as artinian rings with the condition (a) and ones with the conditions (a) and (b), respectively. It is easy to see that a QF-ring R is a Frobenius ring iff $S(R_R)_R \cong (R/J)_R$.

Remark 1.3.1.

(1) The permutation (a) above for a quasi-Frobenius algebra R or for a quasi-Frobenius ring R is called a Nakayama permutation of R. By Theorem 1.2.14, we note that a Nakayama permutation is uniquely determined up to an inner automorphism.

(2) Let R be a basic quasi-Frobenius ring R with $Pi(R) = \{e_1, \ldots, e_m\}$. For each e_iR, we take $f_i \in Pi(R)$ such that f_iR is a projective cover of e_iR. Then the permutation

$$\begin{pmatrix} e_1 & e_2 & \cdots & e_m \\ e_{\pi(1)} & e_{\pi(2)} & \cdots & e_{\pi(m)} \end{pmatrix}$$

is a Nakayama permutation.

Next we recall Nakayama automorphisms for Frobenius algebras and introduce Nakayama automorphisms as rings for QF-rings. Let R be a Frobenius algebra and let $\varphi : R_R \to R^* = \mathrm{Hom}_k(R, k)_R$ be an isomorphism. Such φ is called a *hyper plane*. We recall the definition of the Nakayama automorphism σ of R. Firstly the map $f : R \times R \to k$ given by $(r, s) \mapsto \varphi(r)(s)$ is a nonsingular associative k-bilinear map, where "non-singular" means that $f(a, A) = 0$ implies $a = 0$, and "associative" means that $f(ab, c) = f(a, bc)$ for any a, b, $c \in R$. For each $a \in R$, there exists a unique b for which $f(a, x) = f(x, b)$ holds for any $x \in R$. Then the map $\sigma : R \to R,\ a \mapsto b$ is a k-algebra automorphism of R and is called a Nakayama automorphism of R. Though this Nakayama automorphism σ depends on the choice of the isomorphism φ, it is known that σ is uniquely determined up to an inner automorphism, that is, for another (similarly defined) Nakayama automorphism σ' with respect to other hyper plane φ', there exists a unit u in R such that $\sigma'(x) = u\sigma(x)u^{-1}$ for any x in R.

Nakayama automorphisms have a remarkable property which effects Nakayama permutations as follows:.

Theorem 1.3.2. ([133]) *Let R be a Frobenius algebra. For any its Nakayama permutation:*

$$\pi = \begin{pmatrix} e_1 & e_2 & \cdots & e_m \\ e_{\pi(1)} & e_{\pi(2)} & \cdots & e_{\pi(m)} \end{pmatrix}$$

there exists a Nakayama automorphism σ which induces this permutation, that is, $\sigma(e_i) = e_{\pi(1)}$ for $i = 1, \ldots, m$.

By adopting the property in this theorem, we introduce *Nakayama automorphisms for a Frobenius ring R* as a ring automorphism σ which induces a Nakayama permutation of R.

Remark 1.3.3.

(1) Nakayama automorphisms for Frobenius algebras are Nakayama automorphisms as QF-rings. However Nakayama automorphisms as QF-rings for Frobenius algebras are not necessarily Nakayama automorphisms as algebras (see Example 1.3.4 below). Therefore we use the term "Nakayama automorphism as a ring" in a broad sense than "Nakayama automorphism as an algebra".

(2) Frobenius algebras have always Nakayama automorphisms as algebras. But, in general QF-rings need not have Nakayama automorphisms as rings as we see in Example 5.3.2 due to Koike [98].

Example 1.3.4. Let k be a field, let p be a non-zero element in K and let $R = k\langle x, y\rangle/(xy - pyx, x^2, y^2)$. Then R is a local *Frobenius algebra.* By direct calculation, we can show that the map $\sigma : R \to R$ defined by

$$a + b\overline{x} + c\overline{y} + d\overline{xy} \ \mapsto\ a + pb\overline{y} + c\overline{x} + pd\overline{yx}$$

for any $a, b, c, d \in K$ is a Nakayama automorphism of R as an algebra with $\sigma(\overline{x}) = p\overline{y}$ and $\sigma(\overline{y}) = \overline{x}$. Since σ is neither the identity map nor an inner automorphism, the identity map is not a Nakayama automorphism of R as an algebra. Since R is a local ring, the identity map is of course a Nakayama automorphism of R as a ring.

We close this section with the following known facts:

(1) We have the following hierarchy:
symmetric algebra $\Rightarrow$ weakly symmetric algebra $\Rightarrow$ Frobenius algebra $\Rightarrow$ quasi-Frobenius algebra.

(2) Basic quasi-Frobenius algebras are Frobenius algebras.

(3) For a finite-dimensional algebra R over a field k, R is a quasi-Frobenius algebra if and only if R is a quasi-Frobenius ring.

(4) Let R be a Frobenius k-algebra and let σ be a Nakayama automorphism of R. Then, for any unit $u \in R$, the map: $R \to R$, $x \mapsto u\sigma(x)u^{-1}$ is also a Nakayama automorphism of R.

(5) A Nakayama automorphism of a symmetric algebra is the identity map of R.

(6) For a finite group G and a field k, the group algebra kG is a symmetric algebra. Therefore its Nakayama automorphisms as an algebra or a ring are the identity map.

For more detailed information on these algebras, the reader is referred to Lam [107], Nagao-Tushima [130] and Yamagata [189].

1.4 Notation in Matrix Representations of Rings

Let R be a ring and $\{e_i\}_{i=1}^n$ a complete set of orthogonal idempotents of R. We may represent R as

$$R = \begin{pmatrix} e_1Re_1 & \cdots & e_1Re_n \\ & \cdots & \\ e_nRe_1 & \cdots & e_nRe_n \end{pmatrix}.$$

In this representation, for each $x \in e_iRe_j$, we let $\langle\, x\,\rangle_{ij}$ denote the matrix whose (i,j) entry is x and all other entries are 0. Moreover, for any $X \subseteq e_iRe_j$, we let $\langle\, X\,\rangle_{ij} = \{\,\langle\, x\,\rangle_{ij} \mid x \in X\,\}$. We also use these notations when we consider other generalized matrix rings.

Let R and T be rings with matrix representations:

$$R = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ & \cdots & \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}, \quad T = \begin{pmatrix} B_{11} & \cdots & B_{1n} \\ & \cdots & \\ B_{n1} & \cdots & B_{nn} \end{pmatrix}.$$

A ring homomorphism

$$\tau = \begin{pmatrix} \tau_{11} & \cdots & \tau_{1n} \\ & \cdots & \\ \tau_{n1} & \cdots & \tau_{nn} \end{pmatrix} : \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ & \cdots & \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \to \begin{pmatrix} B_{11} & \cdots & B_{1n} \\ & \cdots & \\ B_{n1} & \cdots & B_{nn} \end{pmatrix}$$

is said to be a *matrix ring homomorphism* when, for each i and j, τ_{ij} is a map of A_{ij} to B_{ij} and $\tau(\langle\, x\,\rangle_{ij}) = \langle\, \tau_{ij}(x)\,\rangle_{ij}$ for any $x \in A_{ij}$.

Let R be a ring with a matrix representation:

$$R = \begin{pmatrix} Q_1 & A_{12} & \cdots & A_{1s} & \cdots & A_{1n} \\ A_{21} & Q_2 & \ddots & & & A_{2n} \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ A_{s1} & & \ddots & Q_s & \ddots & A_{sn} \\ \vdots & & & \ddots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{ns} & \cdots & Q_n \end{pmatrix}.$$

Let T be any ring and let $Q_s \in \{Q_1, \ldots, Q_n\}$. If there exists a ring isomorphism $\rho : T \to Q_s$, then, by replacing Q_s with T in the matrix representation, we can make a new ring R' which is canonically isomorphic to R. We represent R' by

$$R' = \begin{pmatrix} Q_1 & A_{12} & \cdots & \cdots & A_{1s} & \cdots & \cdots & A_{1n} \\ A_{21} & Q_2 & \ddots & & & & & A_{2n} \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ \vdots & & \ddots & Q_{s-1} & \ddots & & & \vdots \\ A_{s1} & & & \ddots & T^{\rho} & \ddots & & A_{sn} \\ \vdots & & & & \ddots & Q_{s+1} & \ddots & \vdots \\ \vdots & & & & & \ddots & \ddots & A_{n-1,n} \\ A_{n1} & A_{n2} & \cdots & \cdots & A_{ns} & \cdots & A_{n,n-1} & Q_n \end{pmatrix}$$

and identify R with R'.

COMMENTS

Extending modules and lifting modules are important as generalizations of injective modules and supplemented projective modules, respectively, and these modules have been extensively studied as mentioned in the preface. (A module M is called a supplemented module if, for any submodule X, there exists a submodule Y such that $X \cap Y \ll M$. Right projective modules over right perfect rings are supplemented modules.) Relative generalized injectivity and relative generalized projectivity are introduced in the study of the following open problems: When is the direct sum of extending modules extending? And dually, when is the direct sum of lifting modules lifting? Theorems 1.1.18 and 1.1.20 give partial answers for these problems. More work on semiperfect rings and perfect rings by using lifting modules in Section 1.2 is presented in Oshiro [146]. For Morita duality, the reader is referred to Anderson-Fuller [5], Faith [47], Lam [107], and Xue [184], and, in addition, to Nicholson-Yousif [142], and Tachikawa [172] for QF-rings.