Preface

In the early 1960s, using techniques from the model theory of first order logic, Robinson gave a rigorous formulation and extension of Leibniz infinitesimal calculus. Since then the methodology has found application in a wide spectrum of areas in mathematics, with particular success in probability theory and functional analysis. In the latter, fruitful results were produced with Luxemburg's invention of the nonstandard hull construction.

There is so far no publication of a coherent and self-contained treatment of functional analysis using methods from nonstandard analysis. Therefore, this publication seeks to fill such a gap.

In a way, writing a book like this is tantamount to writing a fantasy novel on a plausible alternative evolution of mathematics: *What if rigorous nonstandard analysis were invented and become popularized before the development of Banach space theory?*

However, by adhering to such a theme too dogmatically, one misses out lots of excitement—as it is unwise to prescribe a methodology before investigating the problems. For that reason, the purpose of this book is simply to demonstrate how intuition and methods from both the classical camp and the nonstandard camp are brought together to create the fundamental concepts and results in functional analysis.

Readership

This book aims at both senior/graduate level students and researchers in functional analysis. For the former, it can be used as a self-study aid or a textbook for a course in functional analysis. For the latter, the book can be used as a reference for techniques from nonstandard analysis applicable to functional analysis, as well as for directions to further research.

PREREQUISITE

Undergraduate level courses covering naive set theory, real analysis, complex analysis and preferably some basic measure theory.

SYNOPSIS

Chapter 1 (p.1–75): A brief introduction to the logical and set-theoretical framework needed for nonstandard analysis is first presented in §1.1. Then two constructions of the nonstandard universe are given in §1.2 and §1.3. In §1.4, as a warm-up exercise, elementary calculus is used as a testing ground for convincing the reader that nonstandard analysis is indeed a collection of simple, effective and intuitive mathematical tools. §1.5 and §1.6 continue to serve the same purpose and present in a nonstandard manner all measure-theoretical and topological background required in later chapters.

§1.1–1.4 can be skipped by those with requisite skills in nonstandard analysis, namely being fluent with the transfer principle, the saturation principle, internal sets and internal extensions. §1.5 and §1.6 can be skipped by those who already had a thorough introduction to measure theory and topology, although they may still enjoy browsing through the nonstandard treatment of such topics.

Chapter 2 (p.77–180): In §2.1 and §2.2 basic results concerning normed linear spaces, Banach spaces, linear operators and the nonstandard hull construction are given. In §2.3, Helly's Theorem is placed in the nonstandard context and regarded as the most fundamental result in Banach space theory. The nonstandard hull construction is no doubt the most central notion in this book. A version of this construction is applied in §2.4. to represent the bidual of a Banach space. Reflexive spaces are perhaps the most studied class of Banach spaces. They are dealt with in §2.5. In §2.6 Hilbert spaces are given just sufficient coverage for the operator theory in the next chapter. §2.7 consists of a selection of topics, including the invariant subspace problem which witnessed the success of nonstandard analysis in the early days of its development.

Chapter 3 (p.181–275): In §3.1 Banach algebras and spectra are introduced. The nonstandard hull construction is extended in this context. C^*-algebras, Gelfand transform and the GNS construction are handled in §3.2. The norm-nonstandard hull of a C^*-algebra is the topic of §3.3. As a prominent type of C^*-algebras, von Neumann algebras are featured in §3.4 with a study of the effect of various kinds of nonstandard hull constructions on them. §3.5 is about some aspects and usage of projections.

Chapter 4 (p.277–301): This chapter is made up of some new results in Banach space and Banach algebras, such as isometric identities for Hilbert space-valued integrals, fixed point theorems, representation of Arens product on a bidual and a noncommutative version of Loeb measures. Some tangible open problems and questions are listed with the intention of inviting the serious reader to make contribution to the advancement of nonstandard methods.

Course topics

One-semester senior level course: Chapter 1 (§1.3 may substitute part of §1.2.2), §2.1–2.6.

Two-semester senior level course or *one-semester graduate level course*: The above, together with §2.7, §3.1–3.4 and any part of §3.5 and Chapter 4. Possibly with more detail on logic and ultraproduct supplemented by Chang and Keisler (1990).

Whenever possible, exercise problems should be done by using methods from nonstandard analysis.

Attributions

Other than the author's negligence, the absence of ascribing credit for a result means that it is easy or folklore or due to the author.

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