Chapter 1

COMPLEX ANALYSIS

1.1. CAUCHY'S THEORM

One of the most useful applications of complex analysis is the evaluation of definite integrals. For instance, the following examples can be found in nearly all standard books on the subject:

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}, \tag{1.1.1}$$

$$\int_{-\infty}^\infty \frac{x^{\alpha-1}}{1+x} dx = \frac{\pi}{\sin \alpha\pi}, \qquad 0 < \alpha < 1, \tag{1.1.2}$$

$$\int_{-\infty}^\infty \frac{\cos x}{x^2+a^2} dx = \frac{\pi e^{-a}}{a}, \qquad a > 0, \tag{1.1.3}$$

$$\int_{-\infty}^\infty \frac{1}{1+x^4} dx = \frac{\pi}{\sqrt{2}}; \tag{1.1.4}$$

see, e.g., [5, pp. 137 & 139] and [13, pp. 231 & 235]. In this section, we wish to illustrate the method of complex integration by studying a slightly more complicated integral; namely, the integral

$$I(z) = \int_a^b \frac{\log t}{\sqrt{(t-a)(b-t)}} \frac{dt}{t-z}, \tag{1.1.5}$$

where $0 < a < b < \infty$, $z \notin (a, b)$ and $|\arg z| < \pi$, which occurred in a recent study of the asymptotic behavior of the Stieltjes-Wigert polynomials [21]. The main tools in complex integration are the following two results; see [2].

Theorem 1.1.1. (Cauchy's theorem) *Let γ be the oriented piecewise smooth boundary of a compact subset K of an open set Ω, and let $f(z)$ be an analytic function in Ω. Then*

$$\int_\gamma f(z)dz = 0. \tag{1.1.6}$$

Theorem 1.1.2. (Cauchy's integral formula) *Let γ be the positively oriented piecewise smooth boundary of a compact subset K of an open set Ω, $f(z)$ be an analytic function in Ω, and z_0 be an interior point of K. Then,*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}dz = f(z_0). \tag{1.1.7}$$

To evaluate the Cauchy-type integral $I(z)$ in (1.1.5), we first need an auxiliary result on the integral

$$I^*(z) = \int_0^\infty \frac{1}{\sqrt{(a+s)(b+s)}}\frac{ds}{s+z}. \tag{1.1.8}$$

Lemma 1.1. *For any $0 < a < b < \infty, z \notin [a,b]$ and $|\arg z| < \pi$, we have*

$$\begin{aligned} I^*(z) =& \frac{1}{\sqrt{(z-a)(z-b)}} \\ &\times \log\frac{[z+\sqrt{ab}+\sqrt{(z-a)(z-b)}]^2}{(\sqrt{a}+\sqrt{b})^2 z}, \end{aligned} \tag{1.1.9}$$

where the branches of the square root and the logarithm are taken to be positive when $z \in (b,\infty)$.

Proof. Make the change of variable

$$s = \frac{b-a}{4}\left(t+\frac{1}{t}\right) - \frac{a+b}{2}.$$

This transformation takes the s-interval $[0,\infty)$ onto the t-interval $[1/c,\infty)$, where $c = (\sqrt{b}-\sqrt{a})/(\sqrt{b}+\sqrt{a})$. Simple calculation gives

$$I^*(z) = \int_{1/c}^\infty \frac{dt}{\frac{1}{4}(b-a)(t^2+1)+(z-\frac{1}{2}(a+b))t}.$$

Let

$$t_\pm = \frac{2}{b-a}\left[-\left(z-\frac{b+a}{2}\right)\pm\sqrt{(z-a)(z-b)}\right],$$

and note that

$$t_+ - t_- = \frac{4}{b-a}\sqrt{(z-a)(z-b)}.$$

Here, we take the branch of the square roots to be positive when $z \in (b,\infty)$. By partial fractions,

$$I^*(z) = \frac{1}{\sqrt{(z-a)(z-b)}}\int_{1/c}^\infty\left(\frac{1}{t-t_+}-\frac{1}{t-t_-}\right)dt.$$

An integration then yields

$$I^*(z) = \frac{1}{\sqrt{(z-a)(z-b)}} \log \frac{z+\sqrt{ab}+\sqrt{(z-a)(z-b)}}{z+\sqrt{ab}-\sqrt{(z-a)(z-b)}}.$$

Note that $I^*(z)$ is positive when $z \in (b,\infty)$. Thus, we need to take the branch of the logarithm on the right-hand side to be also positive when $z \in (b,\infty)$. The last equation is clearly equivalent to (1.1.9). ∎

To evaluate the integral $I(z)$ in (1.1.5), we consider the contour integral

$$J(z) = \int_C \frac{\log\zeta}{\sqrt{(\zeta-a)(\zeta-b)}}\frac{d\zeta}{\zeta-z}, \qquad z \in \mathbb{C}\backslash(-\infty,0]\cup[a,b], \tag{1.1.10}$$

where C is a positively oriented contour consisting of a large circle $\Gamma_R = \{z : |z| = R\}$, two straight lines Σ_+ and Σ_-, one above and one below the cut along the negative real-axis, and a closed curve Γ embracing a cut along the interval $[a,b]$; see Figure 1.1. In (1.1.10), the square root and the logarithm take their principal values. By Cauchy's integral formula,

$$J(z) = 2\pi i \frac{\log z}{\sqrt{(z-a)(z-b)}}. \tag{1.1.11}$$

On the large circle Γ_R, it is easily seen that the integrand in (1.1.10) is dominated by $(\log R)/R^2$; thus,

$$\int_{\Gamma_R} \frac{\log\zeta}{\sqrt{(z-a)(z-b)}}\frac{d\zeta}{\zeta-z} = O\left(\frac{\log R}{R}\right) \tag{1.1.12}$$

as $R \to \infty$. We deform the curve Γ into two straight line segments joining a and b. Due to the cut along the interval $[a,b]$, we have $\sqrt{\zeta-b} = \sqrt{b-\zeta}e^{i\pi/2}$ for ζ on the upper edge of the cut and $\sqrt{\zeta-b} = \sqrt{b-\zeta}e^{-i\pi/2}$ for ζ on the lower edge of the cut. Thus,

$$\int_\Gamma \frac{\log\zeta}{\sqrt{(\zeta-a)(\zeta-b)}}\frac{d\zeta}{\zeta-z} = \frac{2}{i}\int_a^b \frac{\log t}{\sqrt{(t-a)(b-t)}}\frac{dt}{t-z}, \tag{1.1.13}$$

where the path of integration on the left-hand side is oriented in the clockwise direction. Also, since $\log\zeta = \log|\zeta| \pm i\pi$ for $\zeta \in \Sigma_\pm$, we have

$$\int_{\Sigma_+ + \Sigma_-} \frac{\log\zeta}{\sqrt{(\zeta-a)(\zeta-b)}}\frac{d\zeta}{\zeta-z} = 2\pi i \int_0^\infty \frac{1}{\sqrt{(a+s)(b+s)}}\frac{ds}{s+z} \tag{1.1.14}$$

when $R \to \infty$. A combination of (1.1.10), (1.1.11), (1.1.12) and (1.1.13) gives the following result.

Lemma 1.2. *For any* $0 < a < b < \infty, z \notin [a,b]$ *and* $|\arg z| < \pi$, *the integral* $I(z)$ *in* (1.1.5) *is given by*

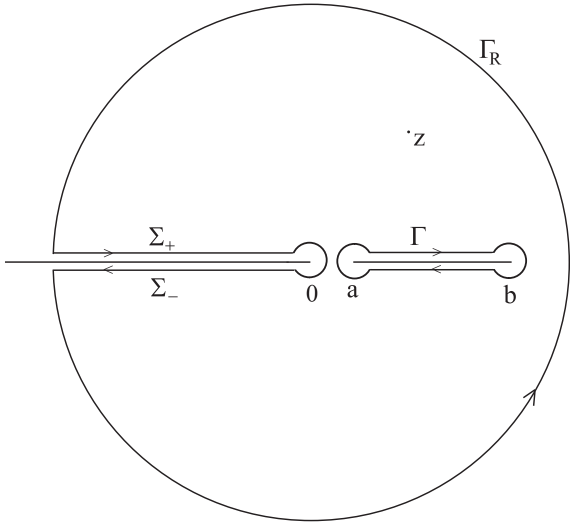

Fig. 1.1. Contour C.

$$I(z) = \frac{\pi}{\sqrt{(z-a)(z-b)}} \left\{ \log \frac{1}{z} + \log \frac{[z + \sqrt{ab} + \sqrt{(z-a)(z-b)}]^2}{(\sqrt{a}+\sqrt{b})^2 z} \right\}, \tag{1.1.15}$$

where the branches of the square root and the logarithm are taken as in Lemma 1.1.

Another important consequence of Cauchy's theorem is the residue theorem [11,13], which can quickly lead to interesting applications. The *residue* of a function $f(z)$ at an isolated singularity z_0 is the coefficient a_{-1} in its Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n. \tag{1.1.16}$$

In terms of an integral, we also have

$$\text{Res}\ (f, z_0) = \frac{1}{2\pi i} \int_\gamma f(z) dz, \tag{1.1.17}$$

where γ is a simple, closed, and positively oriented curve encircling z_0 and not any other singularity. The symbol on the left-hand side of (1.1.17) denotes the residue of f at z_0. Furthermore, from (1.1.16) we have

$$\text{Res}\ (f; z_0) = \lim_{z \to z_0} (z - z_0) f(z), \tag{1.1.18}$$

if the limit on the right-hand side exists.

Theorem 1.1.3. (Residue Theorem) *Let γ be the positively oriented piecewise smooth boundary of a compact subset K of an open set Ω, and let $z_1, \cdots, z_n$ be n distinct points in K. Let $f(z)$ be an analytic function in Ω except for isolated singularities at $z_1, \cdots, z_n$. Then,*

$$\int_\gamma f(z)dz = 2\pi i \sum_{i=1}^{n} \text{Res } (f, z_i). \tag{1.1.19}$$

As a simple example, let us evaluate the integral

$$I(\lambda) = \int_0^\infty \frac{\cos \lambda x}{1+x^2} dx, \qquad \lambda > 0, \tag{1.1.20}$$

which will be used as an illustration of "Exponential Asymptotics" in a later chapter. Since the integrand is an even function, we have

$$I(\lambda) = \frac{1}{2} \int_{-\infty}^\infty \frac{\cos \lambda x}{1+x^2} dx. \tag{1.1.21}$$

Let γ be the curve shown in Figure 1.2, where $R > 1$. By the residue theorem,

$$\int_\gamma \frac{e^{i\lambda z}}{1+z^2} dz = 2\pi i \text{ Res} \left(\frac{e^{i\lambda z}}{1+z^2}, i \right).$$

Each of the integrals on the three lines not on the real-axis can easily be shown to be $O(1/R)$. Thus, as $R \to \infty$, we have

$$\int_{-\infty}^\infty \frac{e^{i\lambda x}}{1+x^2} dx = 2\pi i \text{ Res} \left(\frac{e^{i\lambda z}}{1+z^2}, i \right).$$

Since $1 + z^2 = (z - i)(z + i)$, the residue on the right-hand side is given by

$$\lim_{z \to i} (z - i) \frac{e^{i\lambda z}}{1+z^2} = \frac{e^{-\lambda}}{2i}.$$

Coupling the two results gives

$$\int_{-\infty}^\infty \frac{e^{i\lambda z}}{1+z^2} dz = \pi e^{-\lambda}.$$

Taking just the real part, we obtain from (1.1.21)

$$I(\lambda) = \frac{\pi}{2} e^{-\lambda}, \qquad \lambda > 0. \tag{1.1.22}$$

Sometimes for purposes of computation, it is convenient to formulate the residue theorem in a different form. Suppose $f(z)$ is an analytic function

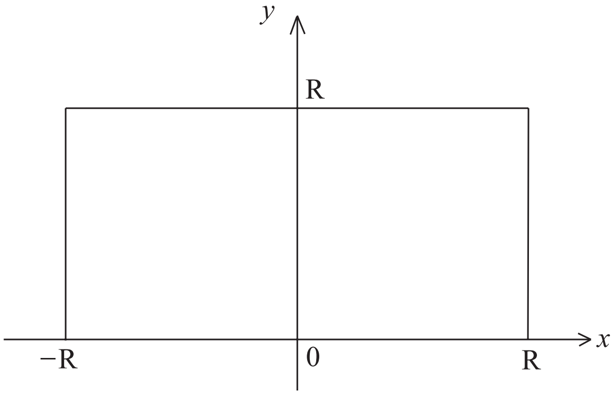

Fig. 1.2. Contour γ.

except for a finite number of isolated singularities in $\mathbb{C}$, and put $z = 1/z'$. Then,

$$f(z)dz = -\frac{1}{z'^2} f\left(\frac{1}{z'}\right) dz'.$$

In view of (1.1.16), the last equation suggests that we define the residue of f at infinity to be the residue of the function

$$g(z) = -\frac{1}{z^2} f\left(\frac{1}{z}\right) \tag{1.1.23}$$

at $z = 0$. If $\sum_{n=-\infty}^{\infty} a_n z^n$ is the Laurent expansion of $f(z)$ in a neighborhood of infinity, then the residue of f at ∞ is $-a_{-1}$. In terms of an integral, one can derive from (1.1.17) and (1.1.23)

$$\text{Res}\,(f, \infty) = -\frac{1}{2\pi i} \int_\Gamma f(\zeta) d\zeta, \tag{1.1.24}$$

where Γ can be a sufficiently large positively oriented circle not containing any isolated singularities of f. It can also be shown that

$$\text{Res}\,(f, \infty) = \lim_{z \to \infty} \{-z f(z)\}, \tag{1.1.25}$$

provided the limit exists. By applying the residue theorem (Theorem 1.1.3) to the curve in Figure 1.3, we obtain the alternative formulation

$$\int_\gamma f(z)dz = -2\pi i \sum \text{Residues of } f \text{ outside } \gamma \text{ including } \infty. \tag{1.1.26}$$

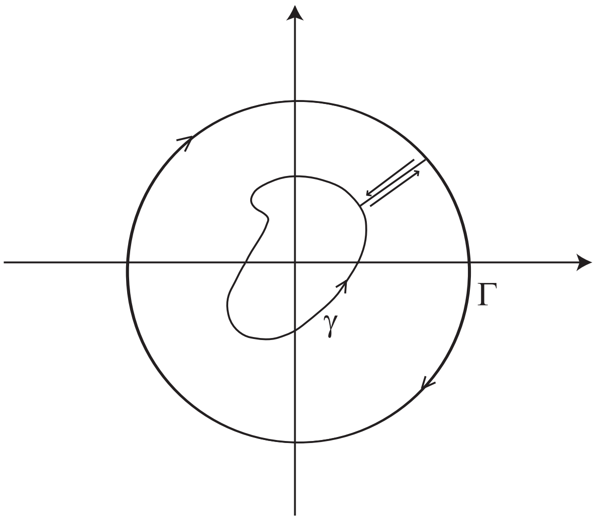

Fig. 1.3. Residues outside γ.

As an illustration, let us evaluate the integral

$$I(z) = \int_a^b \frac{\sqrt{(x-a)(b-x)}}{x} \frac{dx}{x-z}, \qquad 0 < a < b < \infty, \tag{1.1.27}$$

where z is any complex number $\neq 0$ and $\notin [a,b]$. Put

$$R(z) := \sqrt{(z-a)(z-b)} \tag{1.1.28}$$

for z in $\mathbb{C}$ cut along the line segment $[a,b]$, and define $R(x) = \sqrt{(x-a)(b-x)}e^{\pm i\pi/2}$ for x in (a,b) with $+$ and $-$ signs corresponding, respectively, to the upper and lower edges of the cut. Let $\widetilde{\gamma}$ be a clockwise oriented curve enclosing the interval $[a,b]$ but not 0 and z. Then, we can write $I(z)$ as

$$\frac{1}{\pi} I(z) = \frac{1}{2\pi i} \int_{\widetilde{\gamma}} \frac{R(\zeta)}{\zeta} \frac{d\zeta}{\zeta - z}. \tag{1.1.29}$$

It is clear that outside $\widetilde{\gamma}$, there are poles at $0, z$ and ∞. Thus, by (1.1.26), we have

$$\frac{1}{\pi} I(z) = \operatorname*{Res}_{\zeta=0}\left(\frac{R(\zeta)}{\zeta}\frac{1}{\zeta-z}\right) + \operatorname*{Res}_{\zeta=z}\left(\frac{R(\zeta)}{\zeta}\frac{1}{\zeta-z}\right) + \operatorname*{Res}_{\zeta=\infty}\left(\frac{R(\zeta)}{\zeta}\frac{1}{\zeta-z}\right); \tag{1.1.30}$$

that is,

$$I(z) = -\pi\left[1 - \frac{\sqrt{(z-a)(z-b)}}{z} - \frac{\sqrt{ab}}{z}\right]. \tag{1.1.31}$$

Since $\sqrt{(z-a)(z-b)} = -\sqrt{ab} + \frac{1}{2}\frac{a+b}{\sqrt{ab}}z + \cdots$, taking the limit as $z \to 0$ gives

$$I(0) = \pi\left[\frac{1}{2} \cdot \frac{a+b}{\sqrt{ab}} - 1\right]. \tag{1.1.32}$$

To conclude this section, we mention an expansion of the form

$$\sum_{j=0}^{\infty} f(z,j) = \frac{1}{2i}\int_{\Gamma} \cot(\pi t) f(z,t)dt, \tag{1.1.33}$$

where $f(z,t)$ depends on a real or complex parameter z, and is an analytic function of the complex variable t, and where Γ is a loop contour enclosing the points $t = 0, 1, 2, \cdots$, but not enclosing $-1, -2, -3, \cdots$ or the singularities of $f(z,t)$. This result is known as the *Watson transformation* [24, pp. 34 & 44], and can be easily verified by observing that the residue of $\cot \pi t$ at $t = j$ is $1/\pi$. In a similar manner, one can also establish

$$\sum_{j=0}^{\infty}(-1)^j f(z,j) = \frac{1}{2i}\int_{\Gamma} \csc(\pi t) f(z,t)dt. \tag{1.1.34}$$

As an illustration, we consider the sum

$$S(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{z^2 + j^2}, \tag{1.1.35}$$

where z is a complex parameter $\neq 0, \pm 1, \pm 2, \cdots$. Direct application of (1.1.34) would not lead to the result we wish to derive; instead, we make a slight modification of the method. Clearly, $S(z)$ can also be expressed as

$$S(z) = \frac{1}{2z^2} + \frac{1}{2}\sum_{j=-\infty}^{\infty} (-1)^j f(z,j), \tag{1.1.36}$$

where

$$f(z,t) = \frac{1}{z^2 + t^2}. \tag{1.1.37}$$

Let J_n denote the contour shown in Figure 1.4, where n is an arbitrary positive integer bigger than $|\mathrm{Re}(iz)|$ and c is an arbitrary positive number satisfying $c < \mathrm{Im}(iz)$.

By the residue theorem,

$$\sum_{j=-n}^{n} (-1)^j f(z,j) = \frac{1}{2i}\int_{\Gamma_n} \frac{\csc(\pi t)}{z^2 + t^2} dt.$$

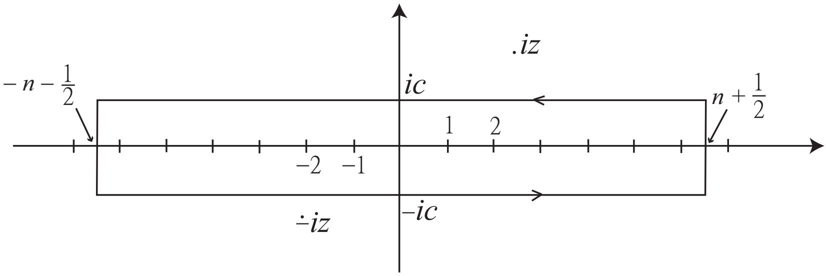

Fig. 1.4. Contour Γ_n.

It can be shown that the contribution on the two vertical parts of the contour tends to zero as $n \to \infty$. Thus,

$$S(z) = \frac{1}{2z^2} + \frac{1}{4i} \int_\Gamma \frac{\csc(\pi t)}{z^2 + t^2} dt, \tag{1.1.38}$$

where Γ consists of two infinite lines Im $t = \pm c$; cf. [14, p.303]. By moving the two lines Im $t = \pm c$ horizontally away from the real-axis (i.e., letting $c \to +\infty$), we pick up residues at the poles $t = \pm iz$, there being no other contribution from this contour deformation. Simple calculation shows that the residues at $\pm iz$ are $\csc(\pm i\pi z)/(\pm 2iz)$. Hence, from (1.1.38) it follows that

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{z^2 + j^2} = \frac{1}{2z^2} + \frac{\pi}{2z} \operatorname{csch} \pi z; \tag{1.1.39}$$

cf. [24, p.36].

1.2. ONE–TO–ONE AND ANALYTIC FUNCTIONS

In the study of conformal mappings, one is frequently required to verify whether a given function $f(z)$ is one-to-one and analytic in a region Ω; that is,

$$f(z_1) \neq f(z_2) \qquad \text{if } z_1 \neq z_2$$

for all $z_1, z_2 \in \Omega$. A synonym for such functions is *simple (schlicht)* or *univalent*.

A necessary condition for $f(z)$ to be one-to-one and analytic in a region Ω is that $f'(z) \neq 0$ for all $z \in \Omega$. To prove this, we suppose that $f'(z_0) = 0$ for some $z_0 \in \Omega$. Then, $f(z) - f(z_0)$ has a zero of order n at z_0, where

$n \geq 2$. Since $f(z)$ is not a constant, we can find a positive number ρ such that $f(z) - f(z_0)$ does not vanish on the circle $|z - z_0| = \rho$. Furthermore, $f(z) - f(z_0)$ and $f'(z)$ have no zero inside the circle other than z_0. Let m denote the minimum value of $|f(z) - f(z_0)|$ on this circle. By Rouche's theorem [11, p.218], if $0 < |a| < m$ then $f(z) - f(z_0) - a$ has the same number of zeros as $f(z) - f(z_0)$, i.e., n zeros, in $|z - z_0| \leq \rho$. Furthermore, there are no double zeros, since $f'(z) \neq 0$ in the closed disk except for $z = z_0$. But this is a contradiction, since $f(z)$ is one-to-one, i.e., it does not take any value more than once.

If $w = f(z)$ is a one-to-one mapping from Ω to Ω' and $F(w)$ is a one-to-one mapping in Ω', then $F(f(z_1)) = F(f(z_2))$ implies $f(z_1) = f(z_2)$ which, in turn, implies $z_1 = z_2$. Thus, *the composition of two one-to-one functions is again a one-to-one function.*

If $w = f(z)$ is a one-to-one mapping from Ω to Ω', then to every point $w \in \Omega'$ there is one and only one point $z \in \Omega$. This defines z as a function of w, say $z = \phi(w)$. We call ϕ the *inverse function* of $w = f(z)$. Since f is single-valued, ϕ is clearly one-to-one. To see that $\phi(w)$ is analytic, we pick a point $w_0 \in \Omega'$ and let $z_0 = \phi(w_0)$. For any given $\varepsilon > 0$ sufficiently small, we consider the disk $D = \{z : |z - z_0| < \varepsilon\}$ contained in Ω. Let γ denote the boundary of D, and Γ the image of γ under $f : \Omega \to \Omega'$; i.e., $\gamma = \partial D$ and $\Gamma = f(\gamma)$. By the argument principle [11, 13], the integral

$$I(w) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - w} dz = \frac{1}{2\pi i} \int_\Gamma \frac{1}{\zeta - w} d\zeta$$

is equal to either 0 or 1, according to whether w lies outside $f(D)$ or inside $f(D)$. Thus, for sufficiently small $\delta > 0$, each w in the disk $K = \{w : |w - w_0| < \delta\}$ is the image of a unique point z in D under $w = f(z)$; see Figure 1.5. Furthermore, we have

$$|\phi(w) - \phi(w_0)| = |z - z_0| < \varepsilon,$$

whenever $|w - w_0| < \delta$. Hence, $\phi(w)$ is continuous. (One can also reach this conclusion by using the open mapping theorem; cf. [18, p.233].) Since $f'(z_0) \neq 0$, we also have

$$\frac{z - z_0}{w - w_0} \longrightarrow \frac{1}{f'(z_0)}$$

as $w \to w_0$; that is, $\phi'(w_0)$ exits at every $w_0 \in \Omega'$. Thus, *the inverse of a one-to-one and analytic function is again a one-to-one and analytic function.*

While a necessary condition for an analytic function to be one-to-one is simple and easy to verify (as we have seen above), it is much more difficult to give a sufficient condition for such a function. The following theorem gives only a local result.

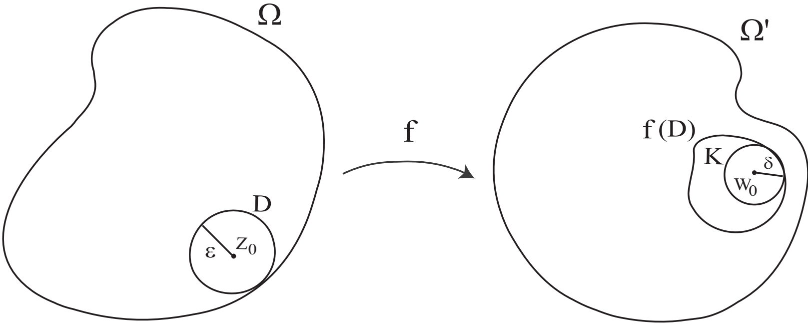

Fig. 1.5. Disks D and K.

Theorem 1.2.1. *Let $f(z)$ be analytic at $z = z_0$. If $f'(z_0) \neq 0$, then $f(z)$ is one-to-one in a neighborhood of z_0; i.e., there exists a positive number ρ such that $f(z)$ is one-to-one in the closed disk $|z - z_0| \leq \rho$.*

Proof 1. Without loss of generality, we may take $z_0 = 0$ and write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with $a_1 \neq 0$. If $f(z_1) = f(z_2)$, then

$$\sum_{n=1}^{\infty} a_n (z_1^n - z_2^n) = 0;$$

that is,

$$(z_1 - z_2) \left\{ a_1 + \sum_{n=2}^{\infty} a_n (z_1^{n-1} + z_1^{n-2} z_2 + \ldots + z_2^{n-1}) \right\} = 0.$$

If $|z_1| \leq \rho$ and $|z_2| \leq \rho$, then the absolute value of the quantity inside the curly bracket is greater than

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$$|a_1| - \sum_{n=2}^{\infty} n|a_n|\rho^{n-1},$$

which is positive if ρ is sufficiently small. Therefore, $z_1 = z_2$ and $f(z)$ is one-to-one. ∎

Proof 2. Without loss of generality, we also suppose that $f(0) = 0$. Since $f'(0) \neq 0$, 0 is a simple zero of $f(z)$. Thus, we can find a small circle C with center at $z = 0$, on which $f(z) \neq 0$, and inside which $f(z)$ has no zero except $z = 0$. Let $m = \min\{|f(z)| : z \in C\}$. Since $f(z)$ is continuous and $f(0) = 0$, we can find a closed disk $|z| \leq \rho$ in which $|f(z)| < m$. We claim that $w = f(z)$ is one-to-one in this closed disk. Let w' be any point in the disk $|w| < m$. By Rouche's theorem, the number of zeros of $f(z) - w'$ inside C is the same as the number of zeros of $f(z)$, which is one. Hence, to each w' in $|w| < m$, there is one and only one z' in C such that $w' = f(z')$; see Figure 1.6. The region that consists of these values of z' therefore corresponds to the disk $|w| < m$ in a one-to-one manner, and this region includes the disk $|z| < \rho$. ∎

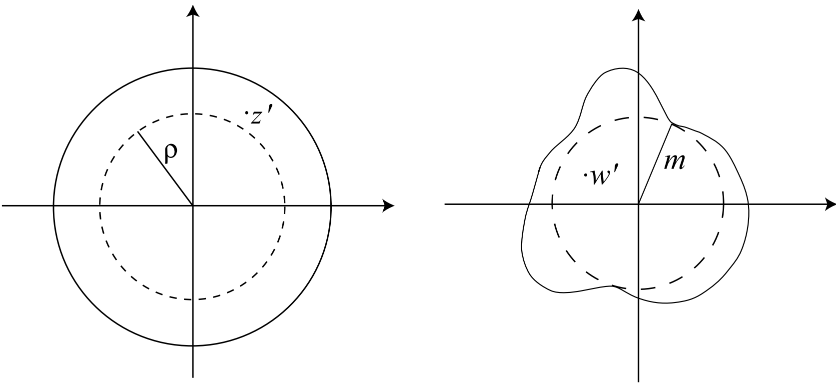

Fig. 1.6. $w' = f(z')$.

Our next result provides a sufficient condition for an analytic function to be globally one-to-one in a region Ω; cf. [12, Vol.2, Chap.4] and [20, p.201].

Theorem 1.2.2. *Let C be a simple closed piecewise smooth curve in the z-plane, and Ω denote the domain bounded by C. Let $w = f(z)$ be an analytic function in Ω and on C. If $f(z)$ is one-to-one on C, then it is also one-to-one in Ω.*

Proof. Let C' be the image of C in the w-plane. Since $f(z)$ is single-valued, C' is also a closed curve. Furthermore, since $f(z)$ is one-to-one on C, C' does not intersect itself. Let Ω' denote the domain bounded by C'.

Pick a point z_0 in Ω. By the argument principle,

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-f(z_0)}dz = m, \tag{1.2.1}$$

where m is the number of zeros of $f(z)-f(z_0)$ inside C (counting multiplicity). Since there is at least one such zero, m must be a positive integer (i.e., $m \neq 0$). At the same time, we have

$$\frac{1}{2\pi i}\int_{C'} \frac{1}{w-w_0}dw = 0,\ \pm 1, \tag{1.2.2}$$

depending on whether w_0 lies outside or inside C' and the orientation of the curve C'. Since the above two integrals are equal upon a change of variable and with $w_0 = f(z_0)$, m must be equal to 1, w_0 lies inside C', C' is positively oriented, and $f(z)$ takes the value w_0 only once in Ω. ∎

The condition in the above theorem that $f(z)$ is analytic on the curve C can be slightly weakened. For instance, it suffices to have $f(z)$ just being piecewise analytic but continuous on C. To see this, we let z_1 be a singularity of $f(z)$ on C. Form a new curve C_1 obtained from C by making a small indentation at z_1; see Figure 1.7. The number of zeros of $f(z)-f(z_0)$ in C_1 is again given by

$$\frac{1}{2\pi i}\int_{C_1} \frac{f'(z)}{f(z)-f(z_0)}\ dz.$$

As the indentation shrinks, this integral approaches the integral in (1.2.1) as long as $f(z)$ is continuous and $f'(z) = O(|z-z_1|^\alpha)$, where $\alpha > -1$. Since the argument given for Theorem 1.2.1 still applies, the result of the theorem again holds.

We may even allow $f(z)$ to have a pole on the curve C, in which case the domain Ω' will become unbounded. The result of the theorem still holds, if the order of the pole is one and the curve is sufficiently smooth. To demonstrate this, we first make a change of variable so that the pole is located at the origin and the curve is oriented in such a way that $\operatorname{Re} z \geq 0$ for all $z \in \Omega$. Let

$$w = f(z) = \frac{c}{z} + g(z),$$

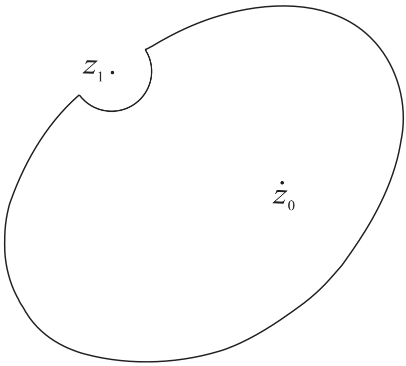

Fig. 1.7. Curve C_1.

where $c \neq 0$ is the residue of f at 0 and $g(z)$ is analytic in Ω and on C. Note that Re $(1/z) =$ Re $\{\overline{z}/|z|^2\}$, and let $a \equiv \min\{\text{Re } g(z) : z \in \Omega\}$. It is now easy to see that

$$\text{Re } w \geq \text{Re } g(z) \geq a \qquad\qquad \text{for } z \in \Omega.$$

If $b < a$, then $|w - b| \geq \text{Re}(w - b) \geq a - b > 0$ for $z \in \Omega$. Since

$$\xi = \frac{1}{w - b} = \frac{z}{1 + zg(z) - bz}$$

is analytic in Ω and on C, the theorem applies directly to ξ. Furthermore, since w is a one-to-one function of ξ, it also applies to w.

When the pole is of order 2 or above, the result of the theorem no longer holds. For an example, see [20, p.202].

As a nontrivial example of Theorem 1.2.2, we consider the mapping $z \mapsto w$ defined by

$$z - t \coth z = w - \frac{A^2(t)}{w}, \tag{1.2.3}$$

where $0 \leq t < 1$ and $A(t)$ is to be determined. This transformation is used in the derivation of an asymptotic expansion of the Laguerre polynominal $L_n^{(\alpha)}(\nu t)$, where $\nu = 4n + 2(\alpha + 1)$; see [7]. For convenience, we put

$$f(z, t) = z - t \coth z. \tag{1.2.4}$$

For the mapping $z \mapsto w$ to be analytic, we must have $dz/dw \neq 0$ or ∞. From (1.2.4), we have

$$f_z(z, t)\frac{dz}{dw} = 1 + \frac{A^2(t)}{w^2}. \tag{1.2.5}$$

Note that $f_z(z,t)$ vanishes at $z = z_+$ and z_-, where

$$z_\pm = \pm i \sin^{-1}\sqrt{t}, \qquad 0 \le t < 1. \tag{1.2.6}$$

Since the right-hand side of (1.2.5) vanishes at $w = \pm iA(t)$, we must make $z = z_+$ correspond to $w = +iA(t)$ and $z = z_-$ to $w = -iA(t)$. Substituting these into equation (1.2.3) and solving for $A(t)$, we obtain

$$A(t) = \frac{1}{2}[\sin^{-1}\sqrt{t} + \sqrt{t(1-t)}], \qquad 0 \le t < 1. \tag{1.2.7}$$

It will be shown that with this choice, the transformation $z \mapsto w(z)$ defined by (1.2.3) is one-to-one and analytic in $|\mathrm{Im}\ z| \le \pi/2$.

We first restrict ourselves to the half-strip $\{z : \mathrm{Re}\ z \le 0,\ 0 \le \mathrm{Im}\ z \le \pi/2\}$; see Figure 1.8. The point z_+ and the singularity $z = 0$ are excluded by small indentations and the strip is closed

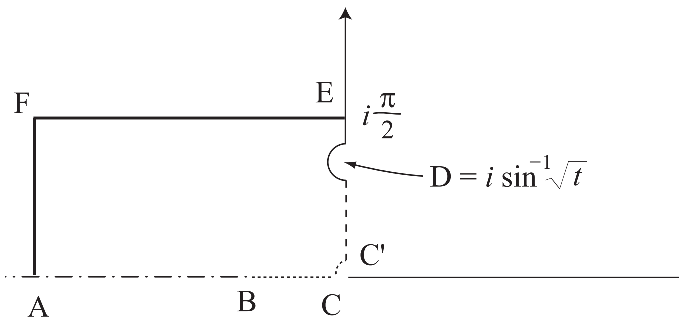

Fig. 1.8. z-plane ($t > 0$).

by a vertical line FA on the left side. The point B is a zero of $f(z,t)$. To see the properties of the mapping between z and w, it is best to introduce an intermediate variable Z defined by

$$z - t\coth z = Z = w - \frac{A^2(t)}{w}. \tag{1.2.8}$$

One can verify that as z traverses once along the indented boundary $ABCC'DEFA$ in Figure 1.8, Z also traverses exactly once along the corresponding curve in Figure 1.9. The line segments $C'D$ and DE in Figure 1.9 are considered as distinct parts of the boundary. Hence, by Theorem 1.2.2, the function $\varphi(z) = z - t\coth z$ is one-to-one and analytic in the interior of the region bounded by this curve. The image of the half-strip in the Z-plane is depicted in Figure 1.9.

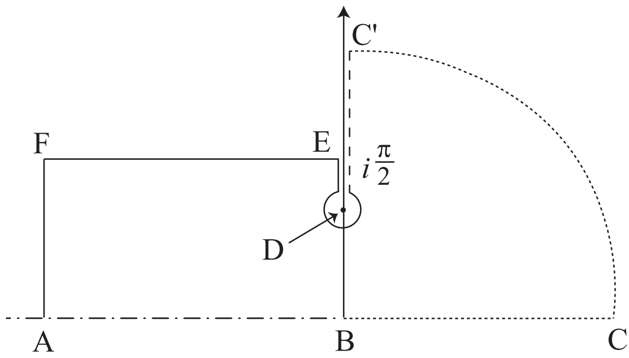

Fig. 1.9. Z-plane ($t > 0$).

Next, we consider the mapping $\psi : w \mapsto Z$ defined by $\psi(w) = w - A^2(t)/w$. By the same argument as above, when w traverses once along the boundary of the region $ABCC'DEFA$ in Figure 1.10, Z goes exactly once around the corresponding curve in the Z-plane. Hence, by

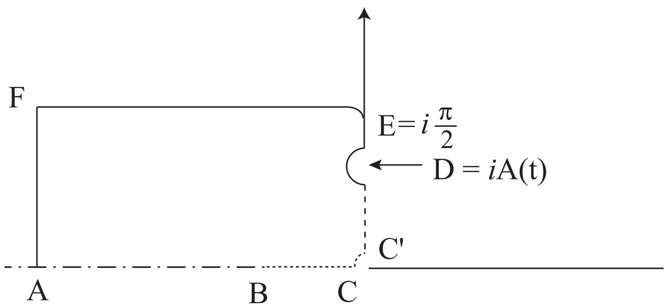

Fig. 1.10. w-plane ($t > 0$).

Theorem 1.2.2, ψ is one-to-one and analytic in the interior of the region $ABCC'DEFA$ in Figure 1.10. The equation of the boundary curve EF in Figure 1.10 is given implicitly by

$$\frac{\pi}{2} = v + \frac{A^2(t)v}{u^2 + v^2}, \qquad \text{Re } Z = u - \frac{A^2(t)u}{u^2 + v^2}, \tag{1.2.9}$$

where $w = u + iv$ and $u < 0$. Since $u \leq 0$ and Re $Z \leq 0$, we have, from the second equation above, $A^2(t) \leq u^2 + v^2$. This together with the first equation implies that $v \geq \pi/4$, i.e., the curve EF in Figure 1.10 remains in the region Im $w \geq \pi/4$. Also from (1.2.7), we have $A(0) = 0$, $A'(t) > 0$ for $0 < t < 1$ and $A(1^-) = \pi/4$. Hence, $0 < A(t) < \pi/4$ for $0 < t < 1$. Therefore, the point D in Figure 1.10 is at least a positive distance away from the curve EF.

The transformation $z \mapsto w$ is obtained by composing $\varphi : z \mapsto Z$ and $\psi^{-1} : Z \mapsto w$. Since both mappings $z \mapsto Z$ and $Z \mapsto w$ are one-to-one and analytic within the boundary $ABCC'DEFA$, so is $z \mapsto w$. Let $z = x + iy$ and $w = u + iv$. It can be shown by direct computation that the real parts of $z - t\coth z$ and $u - A^2(t)/u$ are odd in x and u, and even in y and v, respectively, and that the imaginary parts of these functions are odd in y and v, and even in x and u, respectively (Exercise). Hence, the mapping of the rest of the strip $|\text{Im } z| \leq \pi/2$ is deducible from Figure 1.8 and Figure 1.10 by reflection in the real and imaginary axes. This establishes the one-to-one and analytic nature of the function $w(z,t)$ in $|\text{Im } z| \leq \pi/2$, except possibly at $z = z_{\pm}$ and $z = 0$. From the above argument, it is also evident that neighborhoods of these points are mapped into neighborhoods of their corresponding images. Consequently, $w(z,t)$ is bounded and analytic at these points. (Note also that near $z = 0$ and $t = 0$, we have $w \sim A^2(t)z/t$ and $A^2(t) \sim t$, respectively.)

1.3. DARBOUX'S METHOD

Consider the function

$$L(z) = \frac{1}{\log(1-z)} + \frac{1}{z}. \tag{1.3.1}$$

Near the origin, it has the Maclaurin expansion

$$L(z) = \sum_{n=0}^{\infty} l_n z^n, \tag{1.3.2}$$

where $l_0 = 1/2$, $l_1 = 1/12$, $l_2 = 1/24$, $l_3 = 19/720$, $l_4 = 3/160, \cdots$. In a private correspondence (February, 2004), Donald Knuth asked Frank Olver the following question: What is the asymptotic behavior of the coefficients l_n in (1.3.2) as $n \to \infty$? The answer to the question is

$$l_n \sim \frac{1}{(n+1)\log^2(n+1)}\left[1 - \frac{2\gamma}{\log(n+1)} + \cdots\right], \tag{1.3.3}$$

where γ is Euler's constant. This result is not obvious, and has motivated us to write this section to introduce the so-called Darboux method.

Let $F(z)$ be an analytic function at $z=0$, and let its Maclaurin expansion be given by

$$F(z)=\sum_{n=0}^{\infty} f_n z^n. \tag{1.3.4}$$

We assume that $F(z)$ has a singularity at $z=1$, and is analytic within and on the contour C shown in Figure 1.11. In a neighborhood of $z=1, F(z)$ is assumed to have the form

$$F(z)=(1-z)^{\lambda-1}(\log(1-z))^{\mu}G(z), \tag{1.3.5}$$

where λ and μ are fixed complex numbers, $G(z)$ is analytic at $z=1$, and $\log(1-z)$ has its principal value which is real when z is real and <1. This version of the assumption is actually more general than the original one imposed by Darboux which only allows $F(z)$ to have a singularity of the form $(1-z)^{\lambda-1}$; see [22, p.177] or [19, p.207].

By Cauchy's theorem, we have from (1.3.4)

$$\begin{aligned} 2\pi i f_n &= \int_C z^{-n-1}F(z)dz \\ &= \int_{|z|=1+\delta} z^{-n-1}F(z)dz + \int_{|z-1|=\delta} z^{-n-1}F(z)dz. \end{aligned} \tag{1.3.6}$$

The path of integration on the large circle begins at $1+\delta$, goes around the origin in the counter-clockwise direction, and ends at $(1+\delta)e^{2\pi i}$, whereas the path of integration on the small circle begins at $(1+\delta)e^{2\pi i}$, goes around $z=1$ in the clockwise direction, and ends at $1+\delta$.

In (1.3.6), the number δ is not a fixed constant, but chosen to be

$$\delta=\delta_n=n^{-\frac{1}{2}}. \tag{1.3.7}$$

On the circle $|z|=1+\delta_n$, we assume that $F(z)$ satisfies

$$F(z)=O(n^s), \qquad \text{as } n\to\infty, \tag{1.3.8}$$

for some fixed number s. A simple estimation then gives

$$\begin{aligned} \int_{|z|=1+\delta_n} z^{-n-1}F(z)dz &= O\left(\frac{n^s}{(1+1/n^{\frac{1}{2}})^n}\right), &&\text{as } n\to\infty, \\ &= O(\exp(-\varepsilon\sqrt{n})), &&\text{as } n\to\infty, \end{aligned}$$

for some fixed $\varepsilon>0$. Since this integral is exponentially small, it is anticipated that the asymptotic behavior of f_n will be determined by the asymptotic behavior of the intgeral

$$I_n=\frac{i}{2\pi}\int_{|z-1|=\delta_n} z^{-n-1}F(z)dz, \tag{1.3.9}$$

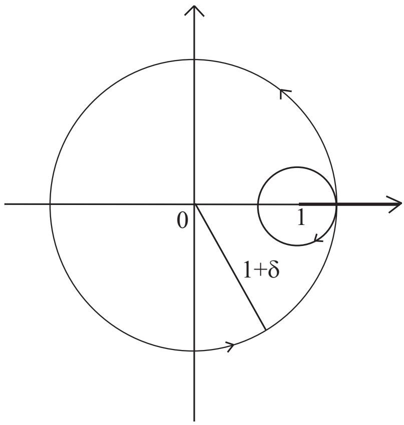

Fig. 1.11. Contour $C : \delta > 0$.

where the path of integration on $|z-1| = \delta_n$ is now oriented in the positive direction.

Before we begin the study of the behavior of the integral I_n, we shall digress briefly to discuss the function

$$M(\lambda, \mu, n) = \frac{i}{2\pi} \int_{\infty}^{(0+)} (-z)^{\lambda-1} (\log(-z))^{\mu} e^{-(n+1)z} dz, \qquad (1.3.10)$$

where the loop contour of integration and the cuts in the z-plane are illustrated in Figure 1.12 below. If $\mu = 0$, then the integral in (1.3.10) can be expressed in terms of the gamma function [5, 14]

$$\frac{1}{\Gamma(1-\lambda)} = \frac{i}{2\pi} \int_{\infty}^{(0+)} (-u)^{\lambda-1} e^{-u} du, \qquad |\arg(-u)| \le \pi. \quad (1.3.11)$$

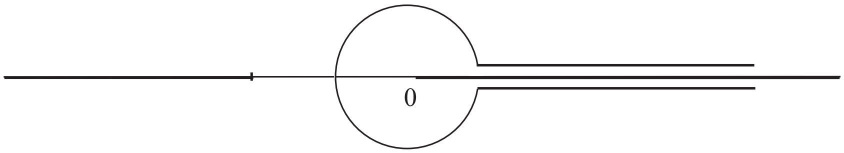

Fig. 1.12. Loop contour.

Differentiating both sides with respect to λ gives

$$D^k \left[\frac{1}{\Gamma(1-\lambda)} \right] = \frac{i}{2\pi} \int_{\infty}^{(0+)} (-u)^{\lambda-1} (\log(-u))^k e^{-u} du, \qquad (1.3.12)$$

where $D^k = d^k/d\lambda^k$.

Lemma 1.3. *For any fixed integer* $N \geq 0$,

$$M(\lambda,\mu,n) = \frac{(-\log(n+1))^\mu}{(n+1)^\lambda}\left[\sum_{k=0}^{N}\binom{\mu}{k}\frac{D^k[1/\Gamma(1-\lambda)]}{(-\log(n+1))^k} + O\left(\frac{1}{(\log(n+1))^{N+1}}\right)\right], \tag{1.3.13}$$

as $n \to \infty$.

Proof. In (1.3.10), we replace $(n+1)z$ by u to obtain

$$M(\lambda,\mu,n) = \frac{1}{(n+1)^\lambda}\frac{i}{2\pi}\int_\infty^{(0+)}(-u)^{\lambda-1}\left[\log\left(\frac{-u}{n+1}\right)\right]^\mu e^{-u}du. \tag{1.3.14}$$

Divide the loop path of integration into two parts $A+B$, where A is the portion contained within $|u| \leq (n+1)^\rho$ for some fixed ρ in $0<\rho<1$, and where B consists of two straight line portions of the loop. Since $\arg(-u) = \pm\pi$ on $B, \log(-u/n+1)$ satisfies the inequalities

$$\pi \leq \left|\log\left(\frac{-u}{n+1}\right)\right| \leq \left|\log\left|\frac{u}{n+1}\right|\right| + \pi; \tag{1.3.15}$$

see (1.3.11). Hence, $|\log(-u/(n+1))|$ is uniformly bounded away from zero. Although $|\log(-u/(n+1))|$ becomes unbounded on B, it is either bounded by $\log|u|$ or by $\log(n+1)$, depending on which is larger. An easy estimate shows that for $|u| \geq (n+1)^\rho$, there must be an $\varepsilon > 0$ such that

$$\int_B (-u)^{\lambda-1}\left[\log\left(\frac{-u}{n+1}\right)\right]^\mu e^{-u}du = O(\exp(-\varepsilon n^\rho)), \tag{1.3.16}$$

as $n \to \infty$, with λ and μ unrestricted, and the order relation holds uniformly.

On the part A of the loop, we have

$$\left[1 - \frac{\log(-u)}{\log(n+1)}\right]^\mu = \sum_{k=0}^{N}\binom{\mu}{k}\frac{(\log(-u))^k}{(-\log(n+1))^k} + O\left(\frac{(\log u)^{N+1}}{(\log(n+1))^{N+1}}\right), \tag{1.3.17}$$

as $n \to \infty$, for every fixed integer $N \geq 0$. Since $\int_A (-u)^{\lambda-1}(\log(-u))^k e^{-u}du$ exists as an absolutely convergent integral for each fixed integer $k \geq 0$, it follows that

$$\frac{i}{2\pi}\int_A (-u)^{\lambda-1}\left[\log\left(\frac{-u}{n+1}\right)\right]^\mu e^{-u}du$$

$$= (-\log(n+1))^{\mu}\Bigg[\sum_{k=0}^{N}\binom{\mu}{k}\frac{1}{(-\log(n+1))^k} \times \frac{i}{2\pi}\int_A (-u)^{\lambda-1}(\log(-u))^k e^{-u}du + O\bigg(\frac{1}{(\log(n+1))^{N+1}}\bigg)\Bigg], \qquad \text{as } n\to\infty. \tag{1.3.18}$$

By the same argument used to obtain (1.3.16), we also have

$$\frac{i}{2\pi}\int_A (-u)^{\lambda-1}(\log(-u))^k e^{-u}du = \frac{i}{2\pi}\int_{\infty}^{(0+)} (-u)^{\lambda-1}(\log(-u))^k e^{-u}du + O(\exp(-\varepsilon n^{\rho})). \tag{1.3.19}$$

Hence, on account of (1.3.12)

$$\frac{i}{2\pi}\int_A (-u)^{\lambda-1}(\log(-u))^k e^{-u}du = D^k[1/\Gamma(1-\lambda)] + O(\exp(-\varepsilon n^{\rho})), \tag{1.3.20}$$

as $n\to\infty$. A combination of the results (1.3.14), (1.3.16), (1.3.18) and (1.3.20) yields the desired expansion given in (1.3.13). ∎

The result in Lemma 1.3 will be used in a slightly different form. First, we note that

$$\exp\{-(n+1)[\log(u+1)-u]\} = \exp\bigg\{-\frac{1}{2}wu\bigg[\frac{2(\log(u+1)-u)}{u^2}\bigg]\bigg\} := E(w,u), \tag{1.3.21}$$

where

$$w = (n+1)u. \tag{1.3.22}$$

On the circle $|u| = \delta_n = n^{-\frac{1}{2}}, w$ is a bounded quantity. Now, let $P_m(w)$ be the polynomials defined by

$$G(u+1)E(w,u) = \sum_{m=0}^{\infty} P_m(w)u^m, \tag{1.3.23}$$

where $G(z)$ is the function given in (1.3.5). An explicit expression for $P_m(w)$ is given by

$$P_m(w) = \frac{1}{m!}\frac{d^m}{du^m}[G(u+1)E(w,u)]\bigg|_{u=0}. \tag{1.3.24}$$

Consider the integral

$$J_m(n) := \frac{i}{2\pi}\int_{\gamma_n}(-u)^{\lambda+m-1}(\log(-u))^{\mu}P_m((n+1)u)e^{-(n+1)u}du, \quad (1.3.25)$$

where γ_n is the contour which traverses on the circle $|u| = \delta_n$ in the positive direction, and begins and ends on the positive half of the real axis. The polynomials $P_m((n+1)u)$ may be written as

$$P_m((n+1)u) = \sum_{s=0}^{m} p_s(n+1)^s u^s, \quad (1.3.26)$$

where p_s is a fixed number. Hence,

$$J_m(n) = \sum_{s=0}^{m}(-1)^s p_s(n+1)^s \frac{i}{2\pi}\int_{\gamma_n}(-u)^{\lambda+m+s-1}(\log(-u))^{\mu}e^{-(n+1)u}du. \quad (1.3.27)$$

Since the error incurred by extending the circular paths of integration to infinite loops is exponentially small, we have

$$J_m(n) = \sum_{s=0}^{m}(-1)^s p_s(n+1)^s M(\lambda+m+s,\mu,n) + O(\exp(-\varepsilon n^{\frac{1}{2}})), \quad (1.3.28)$$

as $n \to \infty$; see (1.3.16). Hence, by Lemma 1.3,

$$J_m(n) \sim \frac{(-\log(n+1))^{\mu}}{(n+1)^{\lambda+m}}\sum_{k=0}^{\infty}\binom{\mu}{k}A_k(\lambda,m)(-\log(n+1))^{-k}, \quad (1.3.29)$$

where

$$A_k(\lambda,m) = \sum_{s=0}^{m}(-1)^s p_s D^k[1/\Gamma(1-\lambda-m-s)]. \quad (1.3.30)$$

Returning to (1.3.9), we replace $z-1$ by u and obtain

$$I_n = \frac{i}{2\pi}\int_{\gamma_n}(u+1)^{-n-1}F(u+1)du, \quad (1.3.31)$$

where γ_n is the contour desccribed in (1.3.25).

Theorem 1.3.1. *If $F(t)$ is analytic within and on the contour C shown in Figure* 1.11, *and if $F(t)$ satisfies the conditions in* (1.3.5) *and* (1.3.8), *then for any fixed integer $N \geq 0$*

$$f_n = \sum_{m=0}^{N}(-1)^m J_m(n) + O\left(\frac{(\log n)^{\mu}}{n^{\lambda+N+1}}\right) \quad (1.3.32)$$

as $n \to \infty$, where $J_m(n)$ is given in (1.3.25).

Proof. Substituting (1.3.5) into (1.3.31) gives

$$I_n = \frac{i}{2\pi}\int_{\gamma_n} (-u)^{\lambda-1}(\log(-u))^{\mu} G(u+1)(u+1)^{-n-1}du. \tag{1.3.33}$$

By (1.3.21) and (1.3.23), the factor $G(u+1)\exp\{-(n+1)[\log(u+1)-u]\}$ can be written as

$$G(u+1)\exp\{-(n+1)[\log(u+1)-u]\} = \sum_{m=0}^{N} P_m(w)u^m + R_N(n,u), \tag{1.3.34}$$

where $N \geq 0$ is any fixed integer. The error term $R_N(n,u)$ in (1.3.34) can be expressed as

$$R_N(n,u) = \left(\frac{1}{2\pi i}\int_{|\zeta|=2K/n^{\frac{1}{2}}} G(\zeta+1)E(w,\zeta)\frac{d\zeta}{\zeta^{N+1}(\zeta-u)}\right)u^{N+1}$$

on account of the Cauchy integral formula, where K is a positive constant and $E(w,\zeta)$ is given in (1.3.21). A simple estimation gives

$$R_N(n,u) = O(n^{(N+1)/2}u^{N+1}), \qquad \text{as } n \to \infty, \tag{1.3.35}$$

provided $|u| \leq K/n^{\frac{1}{2}}$. Coupling (1.3.25) and (1.3.34), we obtain

$$I_n = \sum_{m=0}^{N}(-1)^m J_m(n) + E_N(n), \tag{1.3.36}$$

where

$$E_N(n) = \frac{i}{2\pi}\int_{\gamma_n} (-u)^{\lambda-1}(\log(-u))^{\mu} R_N(n,u)e^{-(n+1)u}du. \tag{1.3.37}$$

Now choose N large enough so that Re $(\lambda+N+1) > 0$. The circular path of integration can then be replaced by two straight lines joining $u=0$ to $u=\delta_n$, one on the top side of the cut in the u-plane, and the other on the lower side of this cut. Hence,

$$E_N(n) = O\left(n^{(N+1)/2}\int_L |(-u)^{\lambda+N}(\log(-u))^{\mu}e^{-(n+1)u}du|\right), \tag{1.3.38}$$

where L is the integration path shown in Figure 1.13. By the same argument given in Lemma 1.3, it can be shown that the integral in (1.3.38) is $O((\log n)^{\mu}/n^{\lambda+N+1})$. Thus,

$$E_N(n) = O((\log n)^{\mu}/n^{\lambda+(N+1)/2}) \qquad \text{as } n \to \infty. \tag{1.3.39}$$

From (1.3.36), it follows that

$$I_n = \sum_{m=0}^{N}(-1)^m J_m(n) + O((\log n)^{\mu}/n^{\lambda+(N+1)/2}). \tag{1.3.40}$$

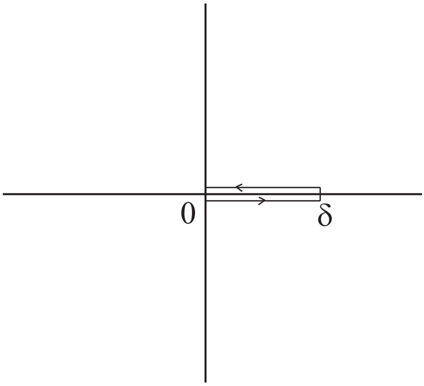

Fig. 1.13. The integration path L.

This is short of the claim in (1.3.32). However, the order of the terms $J_m(n)$, given in (1.3.29), indicates that the result in (1.3.40) can be improved to read

$$I_n = \sum_{m=0}^{N} (-1)^m J_m(n) + O((\log n)^\mu / n^{\lambda+N+1}) \tag{1.3.41}$$

as $n \to \infty$, for any fixed integer $N \geq 0$. This is essentially the statement of the theorem, on account of (1.3.6) and (1.3.9). ∎

When $\mu = 0$, the canonical form (1.3.5) reduces to the Darboux condition and our expansion (1.3.32) is equivalent to the result stated in Wong [22, p.117]; see also Szegő [19, p.207].

From (1.3.29), we have

$$J_0(n) \sim \frac{(-\log(n+1))^\mu}{(n+1)^\lambda} \sum_{k=0}^{\infty} \binom{\mu}{k} (-\log(n+1))^{-k} p_0 D^k[1/\Gamma(1-\lambda)]$$

and

$$J_1(n) \sim \frac{(-\log(n+1))^\mu}{(n+1)^{\lambda+1}} \sum_{k=0}^{\infty} \binom{\mu}{k} (-\log(n+1))^{-k} \sum_{s=0}^{1} p_s D^k[1/\Gamma(1-\lambda-s)].$$

Hence, for any integer $N \geq 0$,

$$J_0(n) - J_1(n) = G(1)\frac{(-\log(n+1))^{\mu}}{(n+1)^{\lambda}} \times \left[\sum_{k=0}^{N}\binom{\mu}{k}(-\log(n+1))^{-k}D^k[1/\Gamma(1-\lambda)] + O((\log(n+1))^{-N-1}) + O(n^{-1})\right]$$

as $n \to \infty$. Clearly, none of the terms of $J_1(n)$ can contribute to the asymptotic expansion for f_n unless the infinite asymptotic expansion for $J_0(n)$ terminates after a finite number of terms. The same will be true for $J_m(n), m \geq 1$. Hence, the general situation is

$$f_n \sim G(1)\frac{(-\log(n+1))^{\mu}}{(n+1)^{\lambda}}\sum_{k=0}^{\infty}\binom{\mu}{k}D^k[1/\Gamma(1-\lambda)](-\log(n+1))^{-k} \quad (1.3.42)$$

as $n \to \infty$.

Returning to (1.3.1), we note that

$$zL(z) - 1 = F(z),$$

where $F(z)$ is given in (1.3.5) with $\lambda = 1, \mu = -1$ and $G(z) = z$. Thus,

$$l_n = f_{n+1}, \qquad n = 0, 1, 2, \cdots.$$

Since $\Gamma(\lambda)\Gamma(1-\lambda) = \pi/\sin\pi\lambda$, a straightforward calculation gives

$$l_n \sim \frac{1}{(n+2)\log^2(n+2)}\left[1 - \frac{2\gamma}{\log(n+2)} + \cdots\right], \quad (1.3.43)$$

which is equivalent to (1.3.3). In (1.3.43), $\gamma = -\Gamma'(1)$ is Euler's constant.

The above material is taken from Wong and Wyman [23]. It is interesting to note that a related problem can be found in Pólya's book [15]. More precisely, in Exercise (Example) 8 on page 9 of his book, Pólya gave the first few coefficients in the expansion

$$\frac{z}{\log(1+z)} = \sum_{n=0}^{\infty} A_n \frac{z^n}{n!}, \qquad |z| < 1; \quad (1.3.44)$$

namely,

$$\begin{aligned} &A_0 = 1, \quad A_1 = 1, \quad A_2 = 1, \quad A_3 = 2, \quad A_4 = 4, \\ &A_5 = 14, \quad A_6 = 38, \quad A_7 = 216, \quad A_8 = 600, \quad A_9 = 6240. \end{aligned} \quad (1.3.45)$$

Furthermore, he asked the reader to make a conjecture on A_n. Then, on page 213 in the solution section at the end of the book, he remarked that from (1.3.45) it is reasonable to conjecture that A_n is positive and increases with n. But he went on to say that this conjecture is totally false, since it can be proved that for large n, we have

$$\frac{A_n}{n!} \sim (-1)^{n-1} \frac{1}{n \log^2 n}.$$

The coefficients A_n in (1.3.44) are related to the coefficients l_n in (1.3.2). The above piece of information was provided to me by Philippe Flajolet.

1.4. RIEMANN-HILBERT PROBLEMS

Let Γ be a smooth, nonself-intersecting and oriented curve in the complex plane $\mathbb{C}$; let Γ^0 denote the interior of Γ (i.e., Γ without the endpoints). Let $F(z)$ be an analytic function in $\mathbb{C}$ except along the curve Γ. If ζ is a point on Γ^0, then we let $F_+(\zeta)$ and $F_-(\zeta)$ denote, respectively, the limiting values of $F(z)$ as $z \to \zeta$ from the left and right sides of Γ; see Figure 1.14. Let $f(\zeta)$ be an analytic function in a domain Ω containing the curve Γ. The so-called Riemann-Hilbert problem is to find a function $F(z)$ which is analytic in $\mathbb{C}$ except on Γ, where it satisfies the jump condition

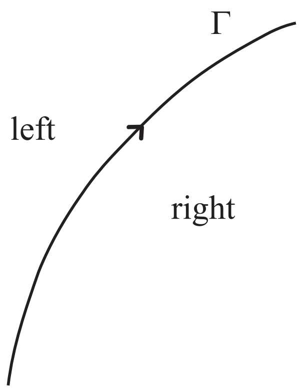

Fig. 1.14. Curve Γ.

$$F_+(\zeta) - F_-(\zeta) = f(\zeta), \qquad \zeta \in \Gamma^0. \tag{1.4.1}$$

A solution to this problem is provided by the following formulas of Plemelj.

Theorem 1.4.1. *Let* Γ *and* $f(\zeta)$ *be given as above, and let*

$$F(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta, \qquad z \notin \Gamma. \tag{1.4.2}$$

For $\zeta \in \Gamma^0$, we have

$$F_+(\zeta) - F_-(\zeta) = f(\zeta), \tag{1.4.3}$$

$$F_+(\zeta) + F_-(\zeta) = \frac{1}{\pi i} \fint_\Gamma \frac{f(t)}{t-\zeta} dt, \tag{1.4.4}$$

where the integral is understood in the sense of Cauchy principal value.

Proof. Draw a circle C_ε centered at ζ and with radius ε, and let C_ε^- denote that part of the circle which lies on the right side of Γ. Let a and b denote the endpoints of Γ, and z be a point on the left side of the indented curve shown in Figure 1.15. Then, by Cauchy's theorem

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} dt = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \left(\int_a^{\zeta-\varepsilon} + \int_{C_\varepsilon^-} + \int_{\zeta+\varepsilon}^b \right) \frac{f(t)}{t-z} dt. \tag{1.4.5}$$

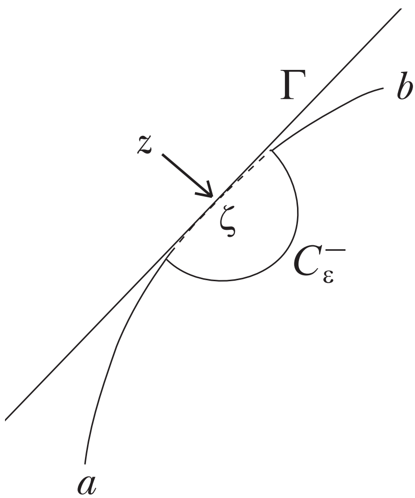

Fig. 1.15. Half-circle C_ε^-.

On the right-hand side of (1.4.5), we first let z tend to ζ. By definition,

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \left(\int_a^{\zeta-\varepsilon} + \int_{\zeta+\varepsilon}^b \right) \frac{f(t)}{t-\zeta} dt = \frac{1}{2\pi i} \fint_\Gamma \frac{f(t)}{t-\zeta} dt.$$

On the half-circle $C_\varepsilon^- = \{z : z = \zeta + \varepsilon e^{i\theta}, \theta_\varepsilon - \pi \le \theta \le -\theta_\varepsilon\}$, where $\theta_\varepsilon > 0$ is the angle between the tangent to Γ at ζ and the radial line joining ζ to the intersection point of Γ and C_ε^-, we have

$$\frac{1}{2\pi i} \int_{C_\varepsilon^-} \frac{f(t)}{t-\zeta} dt = \frac{1}{2\pi} \int_{\theta_\varepsilon - \pi}^{-\theta_\varepsilon} f(\zeta + \varepsilon e^{i\theta}) d\theta.$$

Therefore,

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{C_\varepsilon^-}\frac{f(t)}{t-\zeta}\,dt=\frac{1}{2}f(\zeta).$$

From (1.4.5), it follows that

$$F_+(\zeta)=\frac{1}{2}f(\zeta)+\frac{1}{2\pi i}\fint_\Gamma\frac{f(t)}{t-\zeta}dt. \tag{1.4.6}$$

In a similar manner, one can show that

$$F_-(\zeta)=-\frac{1}{2}f(\zeta)+\frac{1}{2\pi i}\fint_\Gamma\frac{f(t)}{t-\zeta}dt. \tag{1.4.7}$$

Substracting (1.4.7) from (1.4.6) gives (1.4.3), and adding (1.4.7) to (1.4.6) gives (1.4.4). ∎

Although the Cauchy-type integral (1.4.2) provides a solution to the Riemann-Hilbert problem (1.4.1), it is *not* unique. If Γ has no endpoint, then it is unique up to an entire function. If, in addition, the solution is required to behave like a polynomial at infinity, then it is unique up to a polynomial. To see this, we assume that $G(z)$ behaves like z^m for large z, m being a positive integer, and that $G(z)$ also satisfies the jump condition

$$G_+(\zeta)-G_-(\zeta)=f(\zeta), \qquad \zeta\in\Gamma^0.$$

By the generalized Liouville theorem [20, p.85], all we can say is that $G(z)$ differs from the Cauchy integral $F(z)$ in (1.4.2) by a polynomial $P_m(z)$ of degree m; that is,

$$G(z)=\frac{1}{2\pi i}\int_\Gamma\frac{f(\zeta)}{\zeta-z}d\zeta+P_m(z). \tag{1.4.8}$$

However, if we are looking for a solution that tends to zero as $z\to\infty$, then the solution given by (1.4.2) is unique! If Γ has an endpoint, then the solution of (1.4.1) is unique up to a function analytic in the complex plane except for a possible singularity at the endpoint of Γ.

In Theorem 1.4.1, we have assumed that $f(\zeta)$ is an analytic function in a domain containing the curve Γ. Formula (1.4.3) actually holds under much weaker conditions. For instance, it is sufficient to have $f(\zeta)$ just being piecewise analytic with isolated branch points; see [12, Vol.1, p.312]. An even weaker condition for the validity of the Plemelj formula (1.4.3) is that $f(\zeta)$ satisfies the inequality

$$|f(z)-f(\zeta)|\le C|z-\zeta|^k$$

for all $z \in \Gamma$, where C and k are positive constants.

There is a multiplicative form of the Riemann-Hilbert problem, which is to find an analytic function $H(z)$ in $\mathbb{C}\backslash\Gamma$ satisfying

$$H_+(\zeta) = H_-(\zeta)g(\zeta), \qquad \zeta \in \Gamma^0, \tag{1.4.9}$$

where $g(\zeta)$ is a given function on the curve Γ. Suppose that $g(\zeta) \neq 0$ and we can take a branch of $\log g(\zeta)$ on Γ^0. Then, taking logarithms on both sides of (1.4.9), we obtain formally

$$\log H_+(\zeta) - \log H_-(\zeta) = \log g(\zeta), \qquad \zeta \in \Gamma^0. \tag{1.4.10}$$

As long as $\log g(\zeta)$ satisfies the conditions required for the validity of formula (1.4.3), we have

$$\log H(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\log g(\zeta)}{\zeta - z}d\zeta, \tag{1.4.11}$$

from which we can obtain a solution to (1.4.9) by taking exponentiation. As in the case of the additive form of the Riemann-Hilbert problem, this solution is also *not* unique since $H(z)$ multiplied by any single-valued analytic function will again satisfy the jump condition (1.4.9). To make the solution unique, one can require $H(z) \to 1$ as $z \to \infty$ or, equivalently, $\log H(z) \to 0$ as $z \to \infty$.

A natural generalization of the multiplicative Riemann-Hilbert problem (1.4.9) is to consider a two dimensional case; that is, given a curve Γ in the complex plane $\mathbb{C}$ and a 2×2 matrix $V(z)$ of functions analytic in a domain containing Γ, we seek a 2×2 matrix $R(z)$ of functions analytic in $\mathbb{C}\backslash\Gamma$ satisfying

$$R_+(\zeta) = R_-(\zeta)V(\zeta), \qquad \zeta \in \Gamma^0. \tag{1.4.12}$$

We call $V(\zeta)$ a *jump matrix*. This problem is much harder than the one-dimensional case, and we refer the interested readers to [3]. However, if Γ is a closed curve and the jump matrix $V(\zeta)$ satisfies the condition

$$\det V(\zeta) = 1 \qquad \text{for } \zeta \in \Gamma, \tag{1.4.13}$$

then it is relatively easy to establish the uniqueness of solution $R(z)$ satisfying the *normalization condition* $R(z) \to I$ as $z \to \infty$, where I is the 2×2 identity matrix. To show this, we first consider the scalar function $\det R : \mathbb{C}\backslash\Gamma \to \mathbb{C}$. By virtue of (1.4.12) and (1.4.13),

$$(\det R)_+(\zeta) = (\det R)_-(\zeta), \qquad \zeta \in \Gamma.$$

Hence, $\det R$ is an entire function. Furthermore, by the normalization, $(\det R)(z) \to 1$ as $z \to \infty$. The Liouville theorem then infers that

$(\det R)(z) = 1$ for all $z \in \mathbb{C}$. Now, suppose that we have another solution $\widetilde{R}(z)$ satisfying (1.4.12), (1.4.13) and the normalization condition. Since the determinant of $R(z)$ is equal to 1, $R(z)$ has an inverse. Put $X(z) := \widetilde{R}(z)R^{-1}(z)$. Clearly, $X(z)$ is analytic in $\mathbb{C}\backslash\Gamma$ and

$$\begin{aligned} X_+(\zeta) &= \widetilde{R}_+(\zeta)R_+^{-1}(\zeta) = \widetilde{R}_-(\zeta)V(\zeta)[R_-(\zeta)V(\zeta)]^{-1} \\ &= \widetilde{R}_-(\zeta)R_-^{-1}(\zeta) = X_-(\zeta) \end{aligned}$$

for $\zeta \in \Gamma$. Therefore, $X(z)$ is an entire function. Since $X(z) \to I$ as $z \to \infty$, by Liouville's theorem $X(z)$ is the identity matrix for $z \in \mathbb{C}$. This, of course, implies that $\widetilde{R}(z) = R(z)$ for $z \in \mathbb{C}\backslash\Gamma$.

The 2-dimensional version of the Riemann-Hilbert problem that we intend to introduce in this section concerns orthogonal polynomials. Let Γ be a simple smooth curve in the complex plane *without endpoints*, and let $w(z)$ be a weight function defined on Γ. If Γ is an infinite curve, then we assume that $w(z)$ decays sufficiently fast along Γ so that all moments

$$\int_\Gamma \zeta^k w(\zeta)d\zeta, \qquad\qquad k = 0, 1, 2, \cdots,$$

exist. Let $\pi_0(z), \pi_1(z), \pi_2(z), \cdots$ denote the monic polynomials orthogonal with respect to $w(z)$ on Γ; that is, $\pi_n(z) = z^n + \cdots$ and

$$\int_\Gamma \pi_n(\zeta)\pi_m(\zeta)w(\zeta)d\zeta = 0 \qquad\qquad \text{if } n \neq m. \tag{1.4.14}$$

The following Riemann-Hilbert problem was formulated by Fokas, Its and Kitaev [6]: Find a 2×2 matrix-valued function $Y : \mathbb{C}\backslash\Gamma \to \mathbb{C}^{2\times 2}$ satisfying

(Y_a) $Y(z)$ is analytic in $\mathbb{C}\backslash\Gamma$;

(Y_b) for $\zeta \in \Gamma$,

$$Y_+(\zeta) = Y_-(\zeta)\begin{pmatrix} 1 & w(\zeta) \\ 0 & 1 \end{pmatrix}; \tag{1.4.15}$$

(Y_c) as $z \to \infty$,

$$Y(z) = \left[I + O\left(\frac{1}{z}\right)\right]\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}. \tag{1.4.16}$$

Note that $Y(z) \not\to I$ as $z \to \infty$, so this problem does not satisfy the normalization condition mentioned previously.

Theorem 1.4.2. (Fokas, Its and Kitaev) *The Riemann-Hilbert problem* $(Y_a) - (Y_c)$ *for* Y *has the unique solution given by*

$$Y(z) = \begin{pmatrix} \pi_n(z) & C(\pi_n w)(z) \\ c_n \pi_{n-1}(z) & c_n C(\pi_{n-1} w)(z) \end{pmatrix}, \tag{1.4.17}$$

where $\pi_n(z)$ *and* $\pi_{n-1}(z)$ *are the monic orthogonal polynomials of degree* n *and* $n-1$*, respectively,* $C(\pi_j w)(z)$ *denotes the Cauchy transform of* $\pi_j w$*,*

$$C(\pi_j w)(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\pi_j(\zeta) w(\zeta)}{\zeta - z} d\zeta. \tag{1.4.18}$$

and c_n *is an explicitly given constant.*

Proof. (cf. [8]) Let $Y_{ij}(z), i, j = 1, 2$, denote the entries in the matrix $Y(z)$. Condition (Y_b) gives

$$(Y_{11})_+(\zeta) = (Y_{11})_-(\zeta), \qquad \zeta \in \Gamma.$$

Since Γ has no endpoints, $Y_{11}(z)$ is an entire function. From condition (Y_c), we also have

$$Y_{11}(z) = z^n + O(z^{n-1}) \qquad \text{as } z \to \infty.$$

Thus, by the generalized Liouville theorem, $Y_{11}(z)$ must be a monic polynomial of degree n. Let us denote it by $P_n(z)$.

Next, we examine the entry $Y_{12}(z)$. The jump condition (Y_b) gives

$$(Y_{12})_+(\zeta) = (Y_{12})_-(\zeta) + (Y_{11})_-(\zeta) w(\zeta), \qquad \zeta \in \Gamma. \tag{1.4.19}$$

Since $Y_{11}(z) = P_n(z)$, we may write (1.4.19) as

$$(Y_{12})_+(\zeta) = (Y_{12})_-(\zeta) + P_n(\zeta) w(\zeta), \qquad \zeta \in \Gamma.$$

Condition (Y_c) implies

$$Y_{12}(z) = O(z^{-n-1}) \qquad \text{as } z \to \infty. \tag{1.4.20}$$

By the Plemelj formula (Theorem 1.4.1),

$$Y_{12}(z) = C(P_n w)(z) = \frac{1}{2\pi i} \int_\Gamma \frac{P_n(\zeta) w(\zeta)}{\zeta - z} d\zeta. \tag{1.4.21}$$

In general, Cauchy transforms decay like $1/z$ as $z \to \infty$. To satisfy (1.4.20), we must impose additional conditions on the polynomial $P_n(z)$ in (1.4.21). To this end, we write

$$\frac{1}{\zeta - z} = -\sum_{k=0}^{n-1} \frac{\zeta^k}{z^{k+1}} + \frac{\zeta^n}{z^n(\zeta - z)}. \tag{1.4.22}$$

Inserting this in (1.4.21), we obtain

$$Y_{12}(z) = -\frac{1}{2\pi i}\sum_{k=0}^{n-1}\frac{1}{z^{k+1}}\int_\Gamma P_n(\zeta)\zeta^k w(\zeta)d\zeta + O(z^{-n-1}),$$

which reduces to (1.4.20) if we require

$$\int_\Gamma P_n(\zeta)\zeta^k w(\zeta)d\zeta = 0 \qquad \text{for } k = 0, 1, \cdots, n-1;$$

that is, $\{P_n(\zeta)\}$ is an orthogonal sequence with respect to $w(\zeta)$ on Γ. Since $P_n(z)$ is monic, we conclude that $P_n(z) = \pi_n(z)$. The entries $Y_{21}(z)$ and $Y_{22}(z)$ can be obtained in a similar manner. ∎

As an example of Theorem 1.4.2, we consider the Laguerre polynomials

$$L_n^{(\alpha)}(z) = \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-z)^k}{k!}. \tag{1.4.23}$$

By using Leibniz's rule, one can show that

$$L_n^{(\alpha)}(z) = \frac{1}{n!}z^{-\alpha}e^z\left(\frac{d}{dz}\right)^n(z^{n+\alpha}e^{-z}), \tag{1.4.24}$$

which is known as the Rodrigues formula. When $\alpha > -1$, these polynomials are orthogonal with respect to the weight function $x^\alpha e^{-x}$ on the interval $(0,\infty)$; that is,

$$\int_0^\infty L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)x^\alpha e^{-x}dx = 0 \qquad \text{if } n \neq m. \tag{1.4.25}$$

The Laguerre polynomial is one of the so-called classical orthogonal polynomials, the others being Hermite, Legendre, ultraspherical, and Jacobi. A definitive piece of work on this topic is the book "Orthogonal Polynomials" by G. Szegő [19]. But, one can find the Rodrigues formula (1.4.24) and the orthogonality relation (1.4.25) in any book on Special Functions (e.g., [10] and [17]) or even elementary books on Ordinary Differential Equations (e.g., [4] and [16]).

If $\alpha < -1$, then formula (1.4.25) no longer holds, i.e., the Laguerre polynomials $L_n^{(\alpha)}(z)$ are no longer orthogonal on the positive real-axis. However, as we shall see, they satisfy a kind of non-hermitian orthogonality in the complex plane. Let $\mathfrak{F}$ denote the set of all simple smooth curves Σ in $\mathbb{C}\backslash[0,\infty)$, which cross the real axis once on the negative real-axis and are symmetric with respect to the real axis. We express an arbitrary point on Σ in the upper half-plane as $z = x + iy(x)$ with $y(x) > 0$, and require

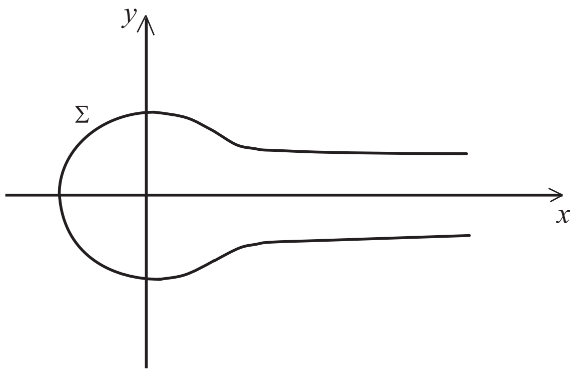

Fig. 1.16. Contour Σ.

$\lim_{x\to\infty} y(x) = M$, where M is a positive real number and may be 0; see Figure 1.16. The following result is given in Kuijlaars and McLaughlin [9].

Lemma 1.4. *Let* $\Sigma \in \mathfrak{F}, n \in \mathbb{N}$, *and* $\alpha \in \mathbb{R}$. *Then,*

$$\int_\Sigma L_n^{(\alpha)}(z) z^k z^\alpha e^{-z} dz = 0 \qquad \text{for } k = 0, 1, \cdots, n-1. \tag{1.4.26}$$

If, in addition, $\alpha + n + 1 \notin \mathbb{N}$, *then*

$$\int_\Sigma L_n^{(\alpha)}(z) z^k z^\alpha e^{-z} dz \neq 0 \qquad \text{for } k = n. \tag{1.4.27}$$

Proof. The orthogonal relation (1.4.26) is proved by using the Rodrigues formula (1.4.24) and repeated integration by parts. In the same manner, we have

$$\begin{aligned}\int_\Sigma L_n^{(\alpha)}(z) z^n z^\alpha e^{-z} dz =& \frac{1}{n!}\int_\Sigma \left(\frac{d}{dz}\right)^n (z^{\alpha+n} e^{-z}) z^n dz \\ =& (-1)^n \int_\Sigma z^{\alpha+n} e^{-z} dz.\end{aligned} \tag{1.4.28}$$

If $\alpha + n > -1$, then we can deform the contour Σ into the two edges of the cut along the positive real axis and obtain

$$\begin{aligned}\int_\Sigma L_n^{(\alpha)}(z) z^n z^\alpha e^{-z} dz =& (-1)^n (1 - e^{2\pi i\alpha}) \int_0^\infty x^{\alpha+n} e^{-x} dx \\ =& (-1)^{n+1} 2ie^{\pi i\alpha} \sin(\pi\alpha)\Gamma(n + \alpha + 1),\end{aligned} \tag{1.4.29}$$

where Γ is the gamma function. The restriction $\alpha + n > -1$ can be removed by analytic continuation, and this proves the lemma. ∎

Now, we consider the monic polynomials

$$\pi_n(z) = \frac{(-1)^n n!}{n^n} L_n^{(\alpha)}(nz), \qquad n = 0, 1, 2, \cdots. \tag{1.4.30}$$

From (1.4.26) and (1.4.27), we have the orthogonality relation

$$\int_\Sigma \pi_n(z) z^k z^\alpha e^{-nz} dz \begin{cases} = 0 & \text{for } k = 0, 1, \cdots, n-1, \\ \neq 0 & \text{for } k = n, \end{cases} \tag{1.4.31}$$

for every contour $\Sigma \in \mathfrak{F}$, provided that $\alpha + n + 1 \notin \mathbb{N}$. The following result is a corollary to Theorem 1.4.2, and is given in [9].

Theorem 1.4.3. *In the Riemann-Hilbert problem (RHP) for $Y(z)$ stated in Theorem 1.4.2, we take Γ to be the contour $\Sigma \in \mathfrak{F}$ (see Figure 1.16) and specify the weight function*

$$w(z) = z^\alpha e^{-nz}, \qquad z \in \Sigma. \tag{1.4.32}$$

If $\alpha \in \mathbb{R}$ and $\alpha + n \notin \mathbb{N}$, then the unique solution to this problem is given by

$$Y(z) = \begin{pmatrix} \pi_n(z) & C(\pi_n w)(z) \\ Q_{n-1}(z) & C(Q_{n-1} w)(z) \end{pmatrix}, \tag{1.4.33}$$

where

$$Q_{n-1}(z) = \frac{(-1)^{n+1} n^{n+\alpha} \pi e^{-\pi i \alpha}}{\sin(\pi\alpha)\Gamma(n+\alpha)} L_{n-1}^{(\alpha)}(nz) \tag{1.4.34}$$

and

$$C(f)(z) = \frac{1}{2\pi i} \int_\Sigma \frac{f(\zeta)}{\zeta - z} d\zeta, \qquad z \in \mathbb{C} \backslash \Sigma, \tag{1.4.35}$$

is the Cauchy transform of f.

Proof. The entries in the first row of the matrix $Y(z)$ have already been established in Theorem 1.4.2. What needs to be proved is that the entry $Y_{21}(z)$ is the polynomial $Q_{n-1}(z)$ given in (1.4.34). Note that

$$Y_{22}(z) = \frac{1}{2\pi i} \int_\Sigma \frac{Y_{21}(\zeta) \zeta^\alpha e^{-n\zeta}}{\zeta - z} d\zeta$$

by (1.4.17), and that

$$Y_{22}(z) = z^{-n} + O(z^{-n-1}), \qquad \text{as } z \to \infty,$$

by (1.4.16). Hence, using (1.4.22), we have

$$\frac{1}{2\pi i} \int_\Sigma Y_{21}(\zeta) \zeta^j \zeta^\alpha e^{-n\zeta} d\zeta = 0, \qquad 0 \leq j \leq n-2, \tag{1.4.36}$$

and

$$\frac{1}{2\pi i}\int_{\Sigma} Y_{21}(\zeta)\zeta^{n-1}\zeta^{\alpha}e^{-n\zeta}d\zeta = -1. \tag{1.4.37}$$

A comparison of (1.4.26) and (1.4.36) suggests that

$$Y_{21}(z) = c_n L_{n-1}^{(\alpha)}(nz),$$

where c_n is a constant. Substituting this into (1.4.37) gives

$$\frac{1}{2\pi i}\int_{\Sigma} c_n L_{n-1}^{(\alpha)}(n\zeta)\zeta^{n-1}\zeta^{\alpha}e^{-n\zeta}d\zeta = -1,$$

or, equivalently,

$$\frac{1}{2\pi i}\int_{\Sigma} c_n L_{n-1}^{(\alpha)}(\zeta)\zeta^{n-1}\zeta^{\alpha}e^{-\zeta}d\zeta = -n^{n+\alpha}.$$

The integral on the left-hand side can be evaluated explicitly as in (1.4.29), and we have

$$c_n = \frac{(-1)^{n+1}n^{n+\alpha}\pi e^{-\pi i\alpha}}{\sin(\alpha\pi)\Gamma(n+\alpha)},$$

thus completing the proof. ∎

1.5. EXERCISES

1. Show that

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

2. Show that for a real,

$$\int_{-\infty}^\infty \frac{\cos x - \cos a}{x^2 - a^2}dx = -\frac{\pi \sin a}{a}.$$

3. Evaluate the integrals

(i) $\displaystyle\int_0^\infty \log(1+x^2)\frac{dx}{x^{1+\alpha}}\ (0<\alpha<2)$, (ii) $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2}$,

(iii) $\displaystyle\int_{-\infty}^\infty \frac{\cos x - 1}{x^2(x^2+a^2)}dx\ (a>0)$, (iv) $\displaystyle\int_0^{2\pi} \frac{\cos 3t}{5-4\cos t}dt$.

4. Show that

$$\int_{-1}^1 \frac{\log(x+1)}{\sqrt{1-x^2}}\frac{dx}{z-x} = \frac{\pi}{\sqrt{z^2-1}}\log\left(\frac{z+1}{z+\sqrt{z^2-1}}\right), \quad z\in\mathbb{C}\setminus[-1,1].$$

5. For any integer $m \geq 1$ and $z \in \mathbb{C} \setminus [a, b]$ with $b > a$, show that

$$\int_a^b \frac{s^m}{\sqrt{(b-s)(s-a)}} \frac{1}{s-z}\, ds$$
$$= \pi \Bigg[\sum_{j=0}^{m-1} z^{m-1-j} \sum_{k=0}^{[j/2]} \binom{j}{2k} \left(\frac{b-a}{2}\right)^{2k} \left(\frac{b+a}{2}\right)^{j-2k} A_k$$
$$- \frac{z^m}{\sqrt{(z-a)(z-b)}} \Bigg],$$

where $A_0 = 1$,

$$A_k = \prod_{j=1}^{k} \frac{2j-1}{2j}$$

for $k \geq 1$, and $[x]$ stands for the largest integer $\leq x$. The function $\sqrt{(z-a)(z-b)}$ is analytic in $\mathbb{C} \setminus [a, b]$ and behaves like z as $z \to \infty$.

6. A function $f(z)$ is called *elliptic* if it is meromorphic in the complex plane and doubly periodic, i.e.,

$$f(z) = f(z + \omega_1) = f(z + \omega_2),$$

where ω_1 and ω_2 are two periods and ω_2/ω_1 is not real. Such a function is determined by its value in any period parallelogram

$$\Lambda_a = \{z \mid z = a + s\omega_1 + t\omega_2, \qquad 0 \leq s, t < 1\}.$$

Show that if an elliptic function is entire, then it must be a constant; otherwise it has at least two poles, counting multiplicity.

7. Show that the function $g(z) = \dfrac{i-z}{i+z}$ maps the upper-half plane $\mathbb{C}_+ = \{z : \Im(z) > 0\}$ onto the unit disc in a one-to-one manner.

8. Construct a conformal mapping from the region $\Omega = \{z : |z| < 1, |z + 1| > \sqrt{2}\}$ onto the unit disk.

9. Construct a conformal mapping from the interior of the unit disk onto the infinite strip $-\pi < \operatorname{Im} z < \pi$.

10. Consider the mapping $Z = z - t \cosh z$, t real, and write $Z = X + iY$ and $z = x + iy$. Show that X is odd in x and even in y, and that Y is odd in y and even in x. Similarly, if $Z = w - A^2/w$, A^2 real, and $w = u + iv$, show that X is odd in u and even in v, whereas Y is odd in v and even in u.

11. Let $\alpha > 0$ and $b = \left(\frac{3}{2}(\alpha\cosh\alpha - \sinh\alpha)\right)^{1/3} > 0$. Consider the sequence of mappings $z \leftrightarrow Z$, $Z \leftrightarrow \varphi$, and $\varphi \leftrightarrow u$ defined by

$$Z = z\cosh\alpha - \sinh z, \qquad \frac{2}{3}b^3 \sin\varphi = Z, \qquad u = 2b\sin\frac{1}{3}\varphi.$$

Find the images of the half-strip $\{z : \text{Re } z > 0,\ 0 \le \text{Im } z \le \pi\}$ in the Z-, φ-, and u-plane. Show that the mapping $z \leftrightarrow u$ is analytic and one-to-one from the region $|\text{Im } z| \le \pi$ to its image in the u-plane; see [5].

12. The Legendre polynomials are generated by

$$\frac{1}{(1 - 2z\cos\theta + z^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(\cos\theta) z^n, \qquad |z| < 1.$$

Use Darboux's method to show that for $0 < \theta < \pi$,

$$P_n(\cos\theta) \sim \left(\frac{2}{\sin\theta}\right)^{1/2} \sum_{\nu=0}^{\infty} \binom{-\frac{1}{2}}{\nu} \binom{\nu - \frac{1}{2}}{n} \frac{\cos\theta_{n,\nu}}{(2\sin\theta)^{\nu}}$$

as $n \to \infty$, where

$$\theta_{n,\nu} = \left(n - \nu + \frac{1}{2}\right)\theta + \left(n - \frac{1}{2}\nu - \frac{1}{4}\right)\pi.$$

13. Let $\{b_n\}$ be defined by $b_0 = b_1 = 1$ and

$$b_n = b_{n-1} + \frac{1}{2n} b_{n-2}, \qquad n \ge 2.$$

Prove that

(i) the generating function $B(z) = \sum_{n=0}^{\infty} b_n z^n$ satisfies

$$2B'(z) = 2zB'(z) + (2 + z)B(z).$$

(ii) Use Darboux's method to show that

$$b_n = (\pi e n)^{-1/2}\left[2n + \frac{5}{4} + O(n^{-1})\right].$$

14. Consider the Maclaurin expansion

$$\phi(t) = \sum_{n=0}^{\infty} a_n t^{2n+1}$$

of the analytic function $\phi(t)$ defined implicitly by the equation

$$\phi(t) - \sin\phi(t) = \frac{t^3}{6}$$

with the condition $\phi(t) \sim t$ as $t \to 0$. Use Darboux's method to show that

$$a_n = \frac{1}{(18)^{1/3}\Gamma(2/3) n^{4/3} (12\pi)^{2n/3}} \left[1 - \frac{1}{3n} + O\left(\frac{1}{n^{4/3}}\right)\right];$$

see [14].

15. Let

$$\phi(z) = z + \sqrt{z^2 - 1}, \qquad z \in \mathbb{C} \setminus [-1, 1],$$

where the branch cut of $\sqrt{z^2-1}$ is chosen along $[-1, 1]$ and positive for $z > 1$. Show that

   (i) $\phi(z)$ is a one-to-one map from $\mathbb{C} \setminus [-1, 1]$ onto the exterior of the unit disk.
   (ii) $\phi(z) = 2z + O(1/z)$ as $z \to \infty$.
   (iii) $\phi_+(x)\phi_-(x) = 1$ for $x \in (-1, 1)$.

16. Find all solutions $F(z)$ of the following Riemann-Hilbert problem:

   (i) $F(z)$ is analytic in $\mathbb{C} \setminus [-1, 1]$ (with orientation from -1 to 1);
   (ii) $F_+(x) + F_-(x) = 0$ for $x \in (-1, 1)$;
   (iii) $F(z) = O(1/\sqrt{z-1})$ as $z \to 1$ and $F(z) = O(1/\sqrt{z+1})$ as $z \to -1$;
   (iv) $F(z) = O(1/z)$ as $z \to \infty$.

17. Let $0 < c < 1$, $a = \dfrac{1-\sqrt{c}}{1+\sqrt{c}}$, and $b = \dfrac{1+\sqrt{c}}{1-\sqrt{c}}$. Consider the following scalar Riemann-Hilbert problem:

   (i) $F(z)$ is analytic for $z \in \mathbb{C} \setminus [0, b]$ (with orientation from 0 to b);
   (ii) $F(z)$ satisfies the following jump conditions:

$$F_+(x) - F_-(x) = -2\pi i, \qquad 0 < x < a,$$
$$F_+(x) + F_-(z) = -\log c, \qquad a < x < b;$$

   (iii) $F(z) = O(\log z)$ near $z = 0$ and $F(z) = O(1)$ near $z = b$;
   (iv) $F(z) = \dfrac{1}{z} + O(z^{-2})$ as $z \to \infty$.

   Show that its unique solution is given by

$$F(z) = -\log \frac{z(b+a) - 2 + 2\sqrt{(z-a)(z-b)}}{z(b-a)} - \frac{\log c}{2}.$$

18. Let $Y(z)$ be given as in Theorem 1.4.2. It is easily seen that as $z \to \infty$, there are 2×2 matrices Y_1, Y_2 such that

$$Y(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I + \frac{Y_1}{z} + \frac{Y_2}{z^2} + O\Big(\frac{1}{z^3}\Big).$$

Denote by $p_n(z) := \gamma_n \pi_n(z)$, $\gamma_n > 0$, the polynomials orthonormal with respect to the weight function $w(z)$ on Γ; that is,

$$\int_I p_m(x) p_n(x) w(x) dx = \delta_{m,n},$$

where $\delta_{m,n}$ is Kronecker's delta symbol defined by

$$\delta_{m,n} = 0\,(m \neq n), \qquad \delta_{m,m} = 1.$$

Show that

$$\gamma_n = (-2\pi i (Y_1)_{12})^{-1/2}.$$

19. Consider the 2×2 matrix Riemann-Hilbert problem defined on the rays $\frac{j\pi}{3}$, $1 \leq j \leq 6$ (see Figure 1.17.):

$$\Phi_{j+1} = \Phi_j\, J_j, \qquad 1 \leq j \leq 5; \qquad \Phi_1 = \Phi_6\, J_6,$$
$$\Phi_j \to 0 \qquad \text{as } z \to \infty, \qquad 1 \leq j \leq 6.$$

We assume that $J_j \in L_2 \cap L_\infty$, and that

(i) $J_2(z) = J_4^*(\bar{z}), \qquad J_1(z) = J_5^*(\bar{z})$;
(ii) $\dfrac{J_3 + J_3^*}{2}$ and $\dfrac{J_6 + J_6^*}{2}$ are both positive definite;
(iii) $\prod_{j=1}^{6} J_j = I$.

Show that the only solution of this Riemann-Hilbert problem is zero; see [1].

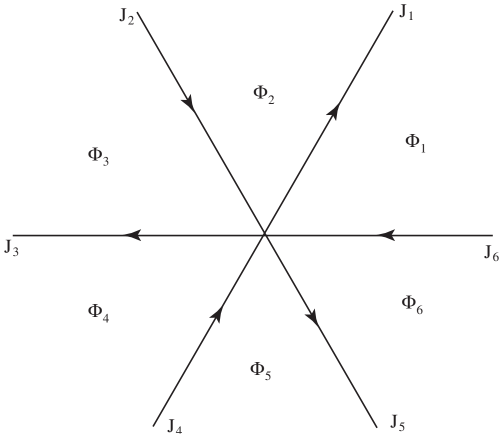

Fig. 1.17. The contours in Problem 19.

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