Introduction to Part 1

> One conceives the causes of all natural effects in terms of mechanical motion. This, in my opinion, we must necessarily do, or else renounce all hopes of ever comprehending anything in Physics.[5]
>
> *Christian Huygens* (1690) *Treatise on light: In which are explained the causes of that which occurs in reflection and refraction*

Our focus in this book is the description of seismic phenomena in elastic media.

The physical basis of seismic wave propagation lies in the interaction of grains within the material through which deformations propagate. It is difficult to individually describe all these interactions among the grains. However, since our experimental data are the result of a large number of such interactions, we can consider these interactions as an ensemble and describe seismic wave propagation through a granular material in terms of

[5] Readers interested in the modern view of this statement, in the context of analytical mechanics, might refer to Born, M., and Wolf, E., (1999) Principles of optics ($7th$ edition): Cambridge University Press, p. xxix. Readers interested in this statement in the context of variational mechanics, which we will discuss in Section 13.2, might refer to Poincaré, H., (1902/1968) La science et l'hypothèse: Flammarion, pp. 219 – 225.

Also, readers might refer to Einstein, A., and Infeld, L., (1938) Evolution of physics from early concepts to relativity and quanta: Simon & Schuster, p. 125:

> During the second half of the nineteenth century new and revolutionary ideas were introduced into physics; they opened the way to a new philosphical view, differing from the mechanical one.

wave propagation through a medium that is continuous. We refer to such a medium as a continuum.

Consequently, in this book, we follow the concepts of continuum mechanics where any material is described by a continuum. A continuum is formulated mathematically in terms of continuous functions representing the average properties of many microscopic objects forming the actual material. In this context, all the associated quantities become scalar, vector or tensor fields, and the formulated problems are governed by differential equations.

Using the methods of continuum mechanics, we adhere to the following statement of Kennett from his book "The seismic wavefields".

> We adopt a viewpoint in which the details of the microscopic structure of the medium through which seismic waves propagate is ignored. The material is supposed to comprise a continuum of which every subdivision possesses the macroscopic properties.

At the beginning of *Part 1*, we formulate the methods for describing deformations of continua and we introduce the concept of strain. This is followed by a description of forces acting within the continuum and the introduction of the concept of stress. We also derive the fundamental equations; namely, the equation of continuity and the equations of motion, which result from the conservation of mass and the balance of linear momentum, respectively.

To supplement these equations and, hence, to formulate a determined system that governs the behaviour of a continuum, we consider a particular class of continua that is, however, general enough to be of significance in applied seismology. Our attention focuses on elastic continua. Any continuum is characterized by its deformation in response to applied loads. In this book, we assume that this response can be adequately described by linear stress-strain equations. Also, we assume that all the energy expended on deformation is transformed into potential energy, which is stored in the deformed continuum. Consequently, upon the removal of the load, the stored energy — to which we refer to as the strain energy — allows this continuum to return to its undeformed state.

The original formulation of the theory of continuum mechanics can be dated to the second half of the eighteenth century and is associated with the work of Leonhard Euler. At the beginning of the nineteenth century, further development was achieved by Augustin-Louis Cauchy and George Green, as well as several other European scientists. The modern development of the theory of continuum mechanics is mainly associated with the work of American scientists, in particular, the work of Walter Noll, Ronald Rivlin and Clifford Truesdell, in the second half of the twentieth century.

We should also note that too literal an interpretation of the concept of continuum can lead to inaccurate conclusions. This can be illustrated by an example given by Schrödinger in his book entitled "Nature and the Greeks".

> Let a cone be cut in two by a plane parallel to its base; are the two circles, produced by the cut on the two parts equal or unequal? If unequal, then, since this would hold for any such a cut, the ascending part of the cone's surface would not be smooth but covered with indentations; if you say equal, then for the same reason, would it not mean that all these parallel sections are equal and thus the cone is a cylinder?

Also, in view of the abstract nature of continuum mechanics, we must carefully consider the definition of exactness of a solution. While exact mathematical solutions to the equations formulated in continuum mechanics exist, the equations themselves are not exact representations of nature since they rely on abstract formulations. Hardy expresses a similar thought in his book entitled "A mathematician's apology".

> It is quite common for a physicist to claim that he has found a 'mathematical proof' that the physical universe must behave in a particular way. All such claims, if interpreted literally, are strictly nonsense. It cannot be possible to prove mathematically that there will be an eclipse tomorrow, because eclipses, and other physical phenomena, do not form part of the abstract world of mathematics.

Nevertheless, the notion of continuum, as it pertains to the theory of elasticity, is particularly useful for seismological purposes because it permits convenient mathematical analysis that gives rise to scientific theory validated by experimental data.

CHAPTER 1

Deformations

> … au lieu de considérer la masse donnée comme un assemblage d'une infinité de points contigus, il faudra, suivant l'esprit du calcul infinitésimal, la considérer plutôt comme composée d'éléments infiniment petits, qui soient du même ordre de dimension que la masse entière;[1]
>
> *Joseph-Louis Lagrange (1788) Mécanique Analytique*

Preliminary Remarks

We begin our study of seismic wave propagation by considering the materials through which these waves propagate. Physical materials are composed of atoms and, hence, the fundamental treatment of this propagation would require the study of interactions among the atoms. At present, such an approach is impractical and, perhaps, impossible with the available mathematical tools. Consequently, we seek a more convenient approach. An alternative approach is offered by continuum mechanics, which allows us to obtain results consistent with observable phenomena without dealing directly with the discrete properties of the materials through which seismic waves propagate.

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[1] … instead of considering a given mass as an assembly of an infinity of neighbouring points, one shall – following the spirit of calculus – consider rather the mass as composed of infinitely small elements, which would be of the same dimension as the entire body;

As all mathematical physics, continuum mechanics utilizes abstract concepts to model physical reality.[2] In a seismological context, the Earth is regarded as a continuum that transmits mechanical disturbances. The notion of continuum allows us to describe the deformations and forces experienced by a deformable body in terms of strains and stresses within a continuum.

We begin this chapter with an explanation of the notion of continuum followed by a description of deformations within it. In particular, we derive the strain tensor, which allows us to describe both a relative change in volume and a change in shape within the continuum.

1.1. Notion of Continuum

In continuum mechanics, we choose to disregard the atomic structure of matter and the explicit interactions among particles. The notion of continuum is justified by the assumption that a material is composed of sufficiently closely spaced particles, so that its descriptive functions can be considered to be continuous. In other words, the infinitesimal elements of the material are assumed to possess the same physical properties as the properties observed in macroscopic studies. Although the microscopic structure of real materials is not consistent with the concept of continuum, this idealization provides a useful platform for mathematical analysis, which in turn permits us to model physical reality using abstract concepts.[3]

In the context of the philosophy of science, continuum mechanics is associated with the concept of emergence, which is also called methodological holism. In physics, emergence is used to describe properties, laws or phenomena that occur at macroscopic scales but not at microscopic ones, in spite of the fact that a macroscopic system can be viewed as an ensemble of microscopic ones. As an example, let us consider colour. Elementary particles, such as protons or electrons, have no colour. Colour emerges if these particles are arranged in atoms, which absorb or emit specific wave-

[2] Readers interested in the concept of models and physical understanding might refer to Weinert, F., (2005) The scientist as philosopher: Springer-Verlag, pp. 45 – 47.

[3] Readers interested in rigorous mathematical foundations of elasticity might refer to Marsden, J.E., and Hughes, T.J.R., (1983/1994) Mathematical foundations of elasticity: Dover. For general aspects of continuum-mechanics formulations, readers might refer to Malvern, L.E., (1969) Introduction to the mechanics of a continuous medium: Prentice-Hall.

lengths of light and can thus be said to have colour. Emergent concepts in continuum mechanics are elasticity, rigidity, viscosity, friction, and so on. The approach in which we invoke elementary particles to describe properties of matter belongs to the field of the condensed-matter physics, and philosophically is associated also with the concept of reductionism and so-called methodological individualism.

The concept of continuum allows us to consider materials in such a way that their descriptive functions are continuous and differentiable. In particular, we can define stress at a given point, thereby enabling us to apply calculus to the study of forces within a continuum. This definition and the subsequent application of calculus is associated with the work of Augustin-Louis Cauchy in the first half of the nineteenth century. Instead of studying atomic forces among individual particles, he introduced the notions of stress and strain in a continuum, which resulted in the equations associated with the theory of elasticity.

Using a continuum-mechanics approach to describe seismic wave propagation raises some concerns. In continuum mechanics, the behaviour of a multitude of grains in a portion of a material is discussed by studying the behaviour of the whole ensemble. Consequently, information relating to the grains themselves is lost in the averaging process. In other words, the application of continuum mechanics raises the question whether the loss of information about the granular structures of the material allows us to properly represent the macroscopic behaviour of that material. To answer this question, we state that our ability to formulate a coherent theory to accurately describe and predict observable seismic phenomena is a key criterion to justify our usage of the notion of continuum. To gain further insight into our statement, let us consider the following quote from "La science et l'hypothèse" of Poincaré.

> Dans la plupart des questions, l'analyste suppose, au debut de son calcul, soit que la matière est continue, soit, inversement, qu'elle est formée d'atomes. Il aurait fait le contraire que ses résultats n'en auraient pas été changés; il aurait eu plus de peine à les obtenir, voilà tout.[4]

[4] For most problems, the researcher assumes, at the beginning of his calculations, that either the matter is continuous or, otherwise, it is composed of atoms. If he makes the other assumption, his results do not change; they will be just more difficult to obtain.

1.2. Rudiments of Continuum Mechanics

1.2.1. Axiomatic format

Modern continuum mechanics is a physical theory that adopts an axiomatic format rather than a historical exposition or a heuristic approach. In this format, the structure of the theory is a hypotheticodeductive system. In other words, it is a system that starts from a set of hypotheses and proceeds deductively; hence, conclusions are of no greater generality than premises, and conclusions are as certain as the premises.

To axiomatize a theory, we need to lay down a set of primitive concepts such that there is a sufficient characterization of basic ideas and a platform for the subsequent statements of the theory. To do so, we need to use the language of mathematics and the principles of logic. In other words, an axiomatic theory of physics presupposes both logic and mathematics; it requires them as a formal apparatus of description. However, this apparatus does not suffice to construct a physical theory because it is devoid of intrinsic physical meaning. Herein, we might recall that the purpose of physics is to accurately describe physical phenomena, while the purpose of logic and mathematics is to consistently define abstract concepts. To axiomatize a physical theory is to articulate its explicitness. Indeed, only explicitly articulated theories can be tested.

Among the advantages of an axiomatic format are the recognition of presupposition and assumptions contained within the theory, as well as the rigour and consistency, which provide clarity and foster coherent developments of the theory. Among the inconveniences of an axiomatic approach are the propensity for formalism at the expense of intuition, as well as the propensity for generality at the expense of concreteness. Since the purpose of this book is concrete — namely, the description of wave phenomena in elastic continua — we trust that our presentation might enjoy the advantages of the axiomatic format and avoid its inconveniences.

1.2.2. Primitive concepts of continuum mechanics

Introductory comments

The primitive basis of a physical theory is a set of formal concepts that are assigned a physical meaning. These concepts cannot be proven within a given theory but are assumed to be true.[5] The three primitive concepts upon which the continuum mechanics is formed are the material body, the manifold of physical experience, and the system of internal forces.[6] These concepts are consistent with each other since the set they form is free of contradictions. They exhibit a weak deductive completeness since all known statements within the theory can be formulated in the context of the three primitive concepts, but not every statement is entailed by these three concepts — hence, the development of the theory can continue to accommodate new ideas and observations.

Material body

The first primitive concept is the material body, $\mathscr{B}$. It is a Euclidean three-dimensional smooth manifold composed of material points $\mathbf{X}$, where we define a material point as an infinitesimal element of volume that possesses the same physical properties as the properties observed in macroscopic studies. This element of volume is sufficiently large that it contains enough discrete particles of matter to allow us to establish a concept of continuum, while it is sufficiently small to be perceived as a mathematical point.[7] The body manifold possesses the following properties. Every sufficiently smooth portion of a body is a body. Also, there is a measure called

[5] Note that, as stated on page 5, only abstract statements can be proven. Herein, the primitive concepts, although abstract, cannot be proven within the theory whose foundation they form. If they could be proven, they would be a part of the theory and not its primitive basis.

[6] Readers interested in more detail and an elegant exposition might refer to Truesdell, C., (1966) Six lectures on modern natural philosophy: Springer-Verlag, pp. 2 – 3 and pp. 96 – 97, and to Bunge, M., (1967) Foundations of physics: Springer-Verlag, pp. 143 – 157.

[7] According to Coirier, J., (1997) Mécanique des milieux continues: Concepts de base: Cours et exercices corrigés: Dunod, pp. 4 – 5, the size of such an element of volume is of the order of $10^{-15}\ m^3$ and contains of the order of 10^{10} molecules. According to Truesdell, C., and Toupin, R., in Flügge, S., (1960) Handbuch der Physik: Springer-Verlag, Vol. III/1, p. 227:

> The corpuscular theories and the field theories are mutually contradictory as direct models of nature. [...] To speak of an element of volume [...] as "a region large enough to contain many molecules but small enough to be used as an element of integration" is not only loose but also needless and bottomless.

mass, m, which is a nonnegative scalar quantity such that

$$m(\mathscr{B}_1 \cup \mathscr{B}_2) = m(\mathscr{B}_1) + m(\mathscr{B}_2),$$

where $\mathscr{B}_1$ and $\mathscr{B}_2$ are disjoint subsets of $\mathscr{B}$. In other words, the mass of a body is the sum of the masses of its parts. Furthermore, mass of a material body occupying volume V is

$$m(\mathscr{B}) = \iiint\limits_V \rho \, \mathrm{d}V,$$

where ρ is the mass density of the material composing $\mathscr{B}$.

As an abstract entity, $\mathscr{B}$ is not accessible to direct observations. We encounter its representations at particular times, t, and in particular spatial locations, $\mathbf{x}$. Points t and $\mathbf{x}$ constitute a manifold of physical experience, which is the second primitive concept of the theory of continuum mechanics.

Manifold of physical experience

The second primitive concept is the manifold of physical experience. It is a Euclidean space-time $\mathbb{E}^3 \times t$, where $\mathbb{E}^3$ is composed of points $\mathbf{x}$ and is endowed with the Euclidean metric, while t denotes time. Symbol $\times$ stands for the direct product, which is a set of all possible ordered pairs $(\mathbf{x}, t)$, which we can write explicitly as (x_1, x_2, x_3, t). We assume that a given time interval is the same for all observers — the time is absolute and universal. Also, we assume that the distance between two given locations is the same for all observers — the space is absolute. These assumptions are tantamount to limiting our study to nonrelativistic continuum mechanics. In the encounters with $\mathscr{B}$, we consider a sequence of configurations given by

$$\mathbf{x} = \chi(\mathbf{X}, t), \tag{1.2.1}$$

where χ is the motion of $\mathscr{B}$ given for a fixed t by an isomorphism: a structure-preserving continuous map between topological spaces that is injective, which means that to every element of one set there corresponds at most one element of the other set, and surjective, which means that to every element of one set there corresponds at least one element of the other set; thus the mapping between $\mathbf{X}$ and $\mathbf{x}$ is one-to-one. Hence, two distinct elements of $\mathscr{B}$ cannot occupy the same point on the manifold of physical experience, and no single element of $\mathscr{B}$ can occupy two distinct points. Since χ is an isomorphism, its inverse is continuous. Thus, we can write

$$\mathbf{X} = \chi^{-1}(\mathbf{x}, t),$$

which allows us to consider the spatial description, as opposed to the material one given by equation (1.2.1). In a material description, the observer identifies the location of points $\mathbf{X} \in \mathscr{B}$ that are immersed in $\mathbb{E}^3$; material point (X_1, X_2, X_3) and time t are the independent variables of this description. In a spatial description, the observer cannot identify particular material points but only spatial locations $\mathbf{x} \in \mathbb{E}^3$; in other words, the observer is unable to follow the displacement of a given material point. In this description, location (x_1, x_2, x_3) and time t are the independent variables. We will discuss these two descriptions in more detail in Section 1.3.

Note that the goal of theoretical sciences is to give the best possible conceptual representation of a given system. The best representation is as close as possible to an isomorphism, which is attainable only in mathematics. Isomorphism is a perfect formal analogy; herein, whatever happens in $\mathbf{X}$ has its isomorphic image in $\mathbf{x}$, and vice versa. Commonly, the same system can be represented in a variety of ways.

System of internal forces

The third primitive concept is the system of internal forces within body $\mathscr{B}$. These forces are described by establishing Cauchy's stress principle, which can be stated in the following way.

Cauchy's stress principle: The action of the material occupying a portion within the body that is exterior to the closed surface on the material within this surface is represented by vector field $\mathbf{T}$.

Vector $\mathbf{T}$ is called traction. It acts on a surface whose outward normal unit vector is $\mathbf{n}$. $\mathbf{T}$ is assumed to depend continuously on $\mathbf{n}$. Its physical dimensions are force per unit area. The concept of the stress tensor that is invoked by the system of internal forces is central to the theory of continuum mechanics. We will discuss it in detail in Section 2.3.

1.3. Material and Spatial Descriptions

1.3.1. Fundamental concepts

While using the concept of continuum, which does not involve any discrete particles, we must carefully consider methods that allow us to describe the displacement of material points within the continuum. In continuum mechanics, we can describe such a displacement in at least two ways; namely, by studying material and spatial descriptions.[8] We can observe the displacement either by following a given material point — in other words, following an infinitesimal element of the continuum, which is analogous to following a particle in particle mechanics — or by studying the flow of the continuum across a fixed position, which does not have an analogue in particle mechanics. As stated in Section 1.2.2, the first approach is called the material description of motion while the second one is called the spatial description of motion. These approaches are also known as the Lagrangian description and the Eulerian description, respectively.[9]

In global geodynamics, the fundamental laws that govern deformations of the Earth necessitate the distinction between the equations derived using the material and the spatial formulations. However, in applied seismology, we can often accurately analyze observable phenomena while ignoring the distinction between the material and the spatial descriptions. For instance, examining a recorded signal, we consider the spatial location of the seismic receiver, even though such a receiver, which is attached to the material that moves due to deformation, is an example of the material description.

To gain insight into the meaning of the material and spatial descriptions, consider a moving continuum and let the observer focus the attention on a given material point within the continuum. Suppose the position of a material point at initial time t_0 is given by vector $\mathbf{X}$. Although position vector

[8] Material and spatial descriptions correspond to the referential and spatial descriptions of Malvern, L.E., (1969) Introduction to the mechanics of a continuous medium: Prentice-Hall, p. 138, where the relative description is also discussed.

[9] Readers interested in detailed descriptions of these approaches and their consequences might refer to Malvern, L.E., (1969) Introduction to the mechanics of a continuous medium: Prentice-Hall, pp. 138 – 145. Readers interested in heuristic descriptions might refer to Snieder, R., (2004) A guided tour of mathematical methods for the physical sciences: Cambridge University Press, pp. 57 – 61.

$\mathbf{X}$ is not a material point, we will refer to a given material point as "material point $\mathbf{X}$", which is a concise way of referring to a material point that at time t_0 occupied position $\mathbf{X}$, as shown in Remark 1.6.1, which follows Exercise 1.1. At a later time t, the position vector of the material point $\mathbf{X}$ is given by $\mathbf{x}$. With a certain abuse of notation, we write expression (1.2.1) as $\mathbf{x}(\mathbf{X},t)$; this mapping gives position, $\mathbf{x}$, of material point $\mathbf{X}$ at time t. This is the material description, where variable $\mathbf{X}$ identifies the material point. We assumed that, for a given time t, this mapping is one-to-one and continuous, as well as possessing the continuous inverse. Also, we have to assume that this mapping and its inverse have continuous partial derivatives to whatever order is required. Since we assume that the transition of the material point from the initial position to the present one occurs in a smooth fashion, vector $\mathbf{x}$ is a continuous function of time and, by symmetry, its inverse is also continuous. Again, with a certain abuse of notation, this inverse can be written as $\mathbf{X}(\mathbf{x},t)$, which fixes our attention on a given region in space and takes position, $\mathbf{x}$, and time, t, as independent variables. To introduce the material and spatial coordinates, consider an orthonormal coordinate system, where

$$x_i = x_i(X_1, X_2, X_3, t), \qquad i \in \{1,2,3\},$$

and

$$X_i = X_i(x_1, x_2, x_3, t), \qquad i \in \{1,2,3\},$$

with the components x_i and X_i being the spatial and material coordinates, respectively.

In general, a physical quantity that characterizes a continuum can be described by a function $f(\mathbf{x},t)$, which is a spatial description of this quantity, or by a function $F(\mathbf{X},t)$, which is a material description of this quantity. The material and spatial descriptions are consistent with one another. The relation between f and F is given by $f(\mathbf{x}(\mathbf{X},t),t) = F(\mathbf{X},t)$, or by $f(\mathbf{x},t) = F(\mathbf{X}(\mathbf{x},t),t)$.

1.3.2. Material time derivative

In view of the previous section, we see that either the material or the spatial description can be used to describe the temporal variation of a given physical quantity. Let us consider time derivatives in the context of either description.

The material description consists of fixing our attention on a given material point, $\mathbf{X}$, and observing the variation of the quantity F with time. The time derivative associated with this viewpoint can be written as

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \left.\frac{\mathrm{d}F(\mathbf{X},t)}{\mathrm{d}t}\right|_{\mathbf{X}}, \tag{1.3.1}$$

where symbol $|_{\mathbf{X}}$ means that the derivative is evaluated at $\mathbf{X}$.

The spatial description consists of fixing our attention on a given spatial location, $\mathbf{x}$, and observing the variation of quantity f with time. The time derivative associated with this viewpoint can be written as

$$\frac{\partial f}{\partial t} = \left.\frac{\partial f(\mathbf{x},t)}{\partial t}\right|_{\mathbf{x}}, \tag{1.3.2}$$

where symbol $|_{\mathbf{x}}$ means that the derivative is evaluated at $\mathbf{x}$.

The material and spatial descriptions are related by the chain rule of differentiation. To see this relation, consider a three-dimensional continuum and explicitly write

$$F(X_1,X_2,X_3,t) = f(x_1(X_1,X_2,X_3,t),x_2(X_1,X_2,X_3,t),x_3(X_1,X_2,X_3,t),t). \tag{1.3.3}$$

Taking the time derivative of both sides, we get

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \left.\left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial t}\right]\right|_{\mathbf{X}}.$$

Since the derivative is evaluated for a given material point, $\mathbf{X}$, it implies that $\partial x_i/\partial t$ are the components of velocity of this point moving in space; we will write these components as v_i. Thus,

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}v_1 + \frac{\partial f}{\partial x_2}v_2 + \frac{\partial f}{\partial x_3}v_3. \tag{1.3.4}$$

As shown in expression (1.3.1), $\mathrm{d}F/\mathrm{d}t$ describes the temporal variation of a given quantity for a particular material point within the continuum, and as shown in expression (1.3.2), $\partial f/\partial t$ describes the temporal variation of this quantity at a particular point in space. To discuss the relation between $\mathrm{d}F/\mathrm{d}t$ and $\partial f/\partial t$, we write the last three terms on the right-hand side of the above equation as a scalar product to get

$$\frac{\mathrm{d}F}{\mathrm{d}t}=\frac{\partial f}{\partial t}+[v_1,v_2,v_3]\cdot\left[\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\frac{\partial f}{\partial x_3}\right].$$

Recognizing that the second term in brackets is the gradient of function f, and denoting the velocity vector by $\mathbf{v}=[v_1,v_2,v_3]$, we write

$$\frac{\mathrm{d}F}{\mathrm{d}t}=\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla f,$$

which is the relation between the time derivatives of F and f.

We can formally rewrite the right-hand side of the above equation as

$$\frac{\mathrm{d}F(\mathbf{X},t)}{\mathrm{d}t}=\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)f(\mathbf{x},t), \tag{1.3.5}$$

where the term in parentheses is an operator acting on function f, and the material and spatial coordinates are related by

$$\mathbf{x}=\mathbf{x}(\mathbf{X},t). \tag{1.3.6}$$

The term in parentheses on the right-hand side of equation (1.3.5) is called the material time-derivative operator. It can be applied to a scalar, to a vector, or to a tensor function of position and time coordinates. It is common to denote this operator by $\mathrm{D}/\mathrm{D}t$ so as to concisely write equation (1.3.5) as

$$\frac{\mathrm{d}F(\mathbf{X},t)}{\mathrm{d}t}=\frac{\mathrm{D}f(\mathbf{x},t)}{\mathrm{D}t},$$

where $\mathrm{D}/\mathrm{D}t:=\partial/\partial t+\mathbf{v}\cdot\nabla$.

Examining the left-hand side of equation (1.3.5) in view of expression (1.3.1), we conclude that the material time derivative is a rate of change associated with a particular element of the continuum. In other words, it is measured by an observer travelling with this element. Mathematically, the material time derivative is the time derivative with material coordinates held constant. To consider spatial coordinates in this context, let us examine the right-hand side of equation (1.3.5). The first term of $(\partial/\partial t+\mathbf{v}\cdot\nabla)$ describes the time rate of change at the location $\mathbf{x}$, while the second term describes the rate of change associated with the motion of material points; more explicitly, the second term describes the spatial rate of change of material point $\mathbf{X}$ moving with velocity $\mathbf{v}$. Recalling expression (1.3.3), we can write expression (1.3.4) with a certain abuse of notation as

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \left.\frac{\partial f}{\partial t}\right|_{\mathbf{x}} + \sum_{i=1}^{3} \left.\frac{\partial f}{\partial x_i}\right|_{t} v_i|_{\mathbf{X}},$$

where $\partial f/\partial t$ is evaluated at location $\mathbf{x}$, $\partial f/\partial x_i$ is evaluated at instant t for a given material point $\mathbf{X}$ moving with velocity $\mathbf{v}$.

As we see from the formulation presented above, in general, $\mathrm{d}F/\mathrm{d}t$ differs from $\partial f/\partial t$ by term $\mathbf{v}\cdot\nabla f$.[10] This term vanishes in the absence of motion, $\mathbf{v} = \mathbf{0}$, or if f does not vary spatially, $\nabla f = \mathbf{0}$. Also, if this term is negligible, we need not distinguish between the material and spatial descriptions. This is an important concept that often allows us simplify our approach; we will discuss it in the next section.

1.3.3. Conditions of linearized theory

In general, equations governing wave phenomena in elastic media are nonlinear. However, seismic experiments indicate that important aspects of wave propagation can be adequately described by linear equations, which greatly simplify mathematical formulations. The process of going from nonlinear equations to linear ones is called the linearization process and the resulting theory is the linearized theory. This linearization is achieved by the fact that, under certain assumptions that appear to be satisfied for many seismological studies, the material and spatial descriptions are equivalent to one another.

The linearization allows us to formulate mathematical statements of seismic wave phenomena in a form that is simpler than it would be otherwise possible. In this section, we briefly discuss the conditions that allow us to use linearization. A more detailed description of the linearization process is beyond the scope of this book.[11]

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[10] Readers interested in this term and its meaning as the convective derivative might refer to Sedov, L.I., (1971) A course in continuum mechanics: Wolters-Noordhoff Publishing, Vol. I, pp. 31 – 32.

[11] Readers interested in a thorough analysis of physical quantities in the material and spatial descriptions, and the subsequent linearization might refer to Achenbach, J.D., (1973) Wave propagation in elastic solids: North Holland, pp. 11 – 21 and 46 – 47, to Malvern, L.E., (1969) Introduction to the mechanics of a continuous medium: Prentice-Hall, pp. 497 – 565, and to Marsden, J.E., and Hughes, T.J.R., (1983/1994) Mathematical foundations of elasticity: Dover, pp. 9 – 10 and 226 – 246.

In applied seismology, we often assume that the displacements of material elements resulting from the propagation of seismic waves can be considered as infinitesimal. Such an assumption is used in this entire book. As a consequence of this assumption and in view of the material time derivative, discussed in Section 1.3.2, we can conclude that, while considering displacements, it is unnecessary to distinguish between the material and spatial descriptions.

To arrive at this conclusion, let us consider the notion of displacement using both the material and spatial descriptions. Displacement is the difference between the final position and the initial position. Using the material description, we can write the displacement vector as

$$\mathbf{U}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t) - \mathbf{X}, \tag{1.3.7}$$

while using the spatial description, we note that the displacement vector is

$$\mathbf{u}(\mathbf{x},t) = \mathbf{x} - \mathbf{X}(\mathbf{x},t). \tag{1.3.8}$$

Note that at the initial time, $\mathbf{x} = \mathbf{X}$.

Since the same quantity is given by expressions (1.3.7) and (1.3.8), we can write

$$\mathbf{U}(\mathbf{X},t) = \mathbf{u}(\mathbf{x},t), \tag{1.3.9}$$

where the material and spatial coordinates are related by equation (1.3.6).

We can develop each component of $\mathbf{U}(\mathbf{X},t)$ into Taylor's series about $\mathbf{x}$ to obtain

$$U_i(\mathbf{X},t) = U_i(\mathbf{X},t)|_{\mathbf{X}=\mathbf{x}} + \left[\left.\frac{\partial U_i(\mathbf{X},t)}{\partial X_1}\right|_{\mathbf{X}=\mathbf{x}}, \left.\frac{\partial U_i(\mathbf{X},t)}{\partial X_2}\right|_{\mathbf{X}=\mathbf{x}}, \left.\frac{\partial U_i(\mathbf{X},t)}{\partial X_3}\right|_{\mathbf{X}=\mathbf{x}} \right] \cdot (\mathbf{X} - \mathbf{x}) + \ldots,$$

where $i \in \{1,2,3\}$. Assuming that the gradient of the displacement, which is shown in brackets, is vanishingly small, we can consider only the first term of the series. Thus, we can write

$$\mathbf{U}(\mathbf{X},t) \approx \mathbf{U}(\mathbf{x},t). \tag{1.3.10}$$

Hence, expression (1.3.7) can be written as

$$\mathbf{U}(\mathbf{x},t) \approx \mathbf{x}(\mathbf{X},t) - \mathbf{X}. \tag{1.3.11}$$

Since in expression (1.3.11) $\mathbf{U}$ is a function of $\mathbf{x}$, we rewrite — using expression (1.3.8) — the displacement as a function of $\mathbf{x}$ to get

$$\mathbf{U}(\mathbf{x},t) \approx \mathbf{x} - \mathbf{X}(\mathbf{x},t). \tag{1.3.12}$$

Comparing expressions (1.3.8) and (1.3.12), we see that

$$\mathbf{U}(\mathbf{x},t) \approx \mathbf{u}(\mathbf{x},t).$$

Thus, in view of expression (1.3.10), we conclude that — for infinitesimal displacements — we can write

$$\mathbf{U}(\mathbf{X},t) \approx \mathbf{u}(\mathbf{x},t). \tag{1.3.13}$$

To gain insight into the meaning of this result, we examine equations (1.3.9) and (1.3.13). Equation (1.3.9) states that $\mathbf{U} = \mathbf{u}$, with $\mathbf{x}$ related to $\mathbf{X}$ by equation (1.3.6). Equation (1.3.13) states that $\mathbf{U} \approx \mathbf{u}$, where we can simply replace $\mathbf{x}$ by $\mathbf{X}$, without invoking equation (1.3.6). This approximation is illustrated in Exercise 1.2. Now, let us consider the velocity using both the material and spatial descriptions. To do so, let the physical quantity considered in the material time derivative be given by displacement. In such a case, expression (1.3.5) becomes

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{u}.$$

If both the gradient of the displacement $\mathbf{u}$ and the velocity $\mathbf{v}$ are infinitesimal, we can ignore the second term on the right-hand side to obtain

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} \approx \frac{\partial \mathbf{u}}{\partial t}.$$

Also, let us consider the acceleration using both the material and spatial descriptions. To do so, let the physical quantity considered in the material time derivative be given by velocity. In such a case, expression (1.3.5) becomes

$$\frac{\mathrm{d}^2\mathbf{U}}{\mathrm{d}t^2} = \frac{\partial^2 \mathbf{u}}{\partial t^2} + (\mathbf{v} \cdot \nabla)\,\frac{\partial \mathbf{u}}{\partial t}.$$

If both the gradient of $\partial \mathbf{u}/\partial t$ and the velocity $\mathbf{v}$ are infinitesimal, we can ignore the second term on the right-hand side to obtain

$$\frac{\mathrm{d}^2\mathbf{U}}{\mathrm{d}t^2} \approx \frac{\partial^2\mathbf{u}}{\partial t^2}.$$

This property of the time derivative of displacement, which results from the linearized theory, is used, for instance, in the derivation of equations of motion (2.6.3).

Thus, we can conclude that, under the assumption of infinitesimal displacements of a given element of the continuum, we do not need to distinguish between either the material and spatial coordinates or the material and spatial descriptions of displacements. In other words, $\mathbf{X} \approx \mathbf{x}$ and $\mathbf{U} \approx \mathbf{u}$. Furthermore, if we also assume that the velocities of these displacements are infinitesimal, that the gradients of these displacements are infinitesimal, and that the gradients of these velocities are also infinitesimal, there is no need to distinguish between the material and spatial descriptions while studying velocities and accelerations. In other words, $\mathrm{d}\mathbf{U}/\mathrm{d}t \approx \partial \mathbf{u}/\partial t$ and $\mathrm{d}^2\mathbf{U}/\mathrm{d}t^2 \approx \partial^2\mathbf{u}/\partial t^2$, respectively.

It is important to note that the assumptions about the properties of the displacements, gradients of displacements, velocities and gradients of velocities are independent of each other. They result from the physical context in which we consider a given mathematical formulation. For instance, in the context of applied seismology, we assume that the displacement amplitude of a material point is small compared to the wavelength. Also, we assume that the velocity of this displacement is small compared to the wave propagation velocity.[12]

In this section, we attempted to justify the linearization by *a priori* arguments. Let us also mention another approach that plays an important role in continuum mechanics and consists of an *a posteriori* justification; in other words, the results justify the assumptions. Herein, the initial approach could be to try the linear formulation by ignoring the nonlinear terms without invoking any physical reason; it is common to begin with a linear formulation and to investigate its applicability. Having developed a theory based on such a formulation, we would compare the predictions of

[12] Readers interested in details of this linearization might refer to Achenbach, J.D., (1973) Wave propagation in elastic solids: North Holland, pp. 17 – 21.

this theory with experimental results. If the agreement between the theory and experiments is satisfactory, we could accept the assumptions. In our context, many seismological measurements agree with theoretical prediction of a linearized theory, thus providing an *a posteriori* justification.

Following our decision to make no distinction between the material and spatial descriptions, we follow the customary notation to describe the coordinates as well as the displacements of a given element of the continuum using lower-case letters. Also, to avoid any confusion, we note that the velocities denoted by v and V, in Parts 2 and 3 of the book, refer to the propagation velocities; namely, phase velocity and the ray velocity, respectively. They are not directly associated with the velocities of displacements of a given element of the continuum, which we discuss herein.

1.4. Strain

1.4.1. Introductory comments[13]

Seismic waves consist of the propagation of deformations through a material. To study these waves, we wish to describe the associated deformations of the continuum in the context of infinitesimal displacements.

Deformation of a continuum is a change of positions of points within it relative to each other. If such a change occurs, a continuum is said to be strained. This strain is accompanied by stress. The produced stress resists deformation and attempts to restore the continuum to its unstrained state. The resistance of a continuum to the deformation and the continuum's tendency to restore itself to its undeformed state account for the propagation of seismic waves.

The relation between stress and strain is one of mutual dependence and is an intrinsic concept of elasticity theory. In this theory, applied forces are formulated in terms of a stress tensor, discussed in Chapter 2, while the associated deformations are formulated in terms of a strain tensor, discussed below.

[13]Readers interested in a thorough description of strain and deformation might refer to Malvern, L.E., (1969) Introduction to the mechanics of a continuous medium: Prentice-Hall, Chapter 4: Strain and Deformation.

1.4.2. Derivation of strain tensor

In this section we show that the strain tensor describes the deformation within the continuum.

The concept of deformation implies that distances among points within the continuum change. To derive the strain tensor in a three-dimensional continuum, consider two infinitesimally close points therein whose coordinates are given by $[x, y, z]$ and $[x+\mathrm{d}x, y+\mathrm{d}y, z+\mathrm{d}z]$. The square of the distance between these points is given by

$$(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 . \tag{1.4.1}$$

Let the continuum be subjected to deformation. After the deformation, which is described by displacement vector

$$\mathbf{u} = [u_x(x,y,z), u_y(x,y,z), u_z(x,y,z)],$$

the coordinates of the first point are given by

$$\left[x + u_x|_{x,y,z}, y + u_y|_{x,y,z}, z + u_z|_{x,y,z} \right], \tag{1.4.2}$$

while the coordinates of the second point are given by

$$\begin{aligned} &\Big[x + \mathrm{d}x + u_x|_{x+\mathrm{d}x,y+\mathrm{d}y,z+\mathrm{d}z}, \\ &y + \mathrm{d}y + u_y|_{x+\mathrm{d}x,y+\mathrm{d}y,z+\mathrm{d}z}, \\ &z + \mathrm{d}z + u_z|_{x+\mathrm{d}x,y+\mathrm{d}y,z+\mathrm{d}z} \Big], \end{aligned} \tag{1.4.3}$$

where the arguments in the subscripts are the values at which the components of function $\mathbf{u}$ are evaluated. Subtracting the components given in expression (1.4.2) from the corresponding components given in expression (1.4.3), we obtain the difference between the corresponding coordinates of the two points after the deformation; namely,

$$\begin{aligned} &\Big[\mathrm{d}x + u_x|_{x+\mathrm{d}x,y+\mathrm{d}y,z+\mathrm{d}z} - u_x|_{x,y,z}, \\ &\mathrm{d}y + u_y|_{x+\mathrm{d}x,y+\mathrm{d}y,z+\mathrm{d}z} - u_y|_{x,y,z}, \\ &\mathrm{d}z + u_z|_{x+\mathrm{d}x,y+\mathrm{d}y,z+\mathrm{d}z} - u_z|_{x,y,z} \Big]. \end{aligned} \tag{1.4.4}$$

Examining expression (1.4.4), we can gain insight into the physical meaning of the displacement vector. In general, u_x, u_y and u_z are each a function of x, y and z. Thus, the length and orientation of vector $\mathbf{u} = [u_x, u_y, u_z]$ depends on position. Hence, in general, the displacement vector at $x + \mathrm{d}x$, $y + \mathrm{d}y$, $z + \mathrm{d}z$ has different length and orientation than it has at x, y, z; for instance, $u_x|_{x+\mathrm{d}x,y+\mathrm{d}y,z+\mathrm{d}z}$ differs from $u_x|_{x,y,z}$. The dependence of $\mathbf{u}$ on position results — upon application of $\mathbf{u}$ — in a relative change of positions of points within the continuum. Prior to deformation, the three coordinates of our two points were separated by $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$, respectively. After the deformation, they are separated by different amounts, as shown in expression (1.4.4). We can get further insight into $\mathbf{u}$ by considering special cases. If $\mathbf{u}$ is given by constants, then its components are the same at all positions. In such a case, expression (1.4.4) reduces to $[\mathrm{d}x, \mathrm{d}y, \mathrm{d}z]$. This means that there is no change of positions of points within the continuum relative to each other. In such a case, we can view $\mathbf{u}$ as resulting in the translation of the whole medium without deformation. Other special cases are discussed in Section 1.4.3. Let us return to our derivation. In view of infinitesimal displacements, the components of $\mathbf{u}$ that are evaluated in expression (1.4.4) at $x + \mathrm{d}x$, $y + \mathrm{d}y$, $z + \mathrm{d}z$ can be approximated by the first two terms of Taylor's series about (x, y, z); namely,

$$u_x|_{x+\mathrm{d}x,y+\mathrm{d}y,z+\mathrm{d}z} \approx u_x|_{x,y,z} + \left.\frac{\partial u_x}{\partial x}\right|_{x,y,z} \mathrm{d}x + \left.\frac{\partial u_x}{\partial y}\right|_{x,y,z} \mathrm{d}y + \left.\frac{\partial u_x}{\partial z}\right|_{x,y,z} \mathrm{d}z,$$

$$u_y|_{x+\mathrm{d}x,y+\mathrm{d}y,z+\mathrm{d}z} \approx u_y|_{x,y,z} + \left.\frac{\partial u_y}{\partial x}\right|_{x,y,z} \mathrm{d}x + \left.\frac{\partial u_y}{\partial y}\right|_{x,y,z} \mathrm{d}y + \left.\frac{\partial u_y}{\partial z}\right|_{x,y,z} \mathrm{d}z$$

and

$$u_z|_{x+\mathrm{d}x,y+\mathrm{d}y,z+\mathrm{d}z} \approx u_z|_{x,y,z} + \left.\frac{\partial u_z}{\partial x}\right|_{x,y,z} \mathrm{d}x + \left.\frac{\partial u_z}{\partial y}\right|_{x,y,z} \mathrm{d}y + \left.\frac{\partial u_z}{\partial z}\right|_{x,y,z} \mathrm{d}z.$$

Inserting these terms into expression (1.4.4) and simplifying, we obtain the approximation for the difference of the corresponding coordinates of the two points after the deformation; namely,

$$\left[\mathrm{d}x + \left.\frac{\partial u_x}{\partial x}\right|_{x,y,z}\mathrm{d}x + \left.\frac{\partial u_x}{\partial y}\right|_{x,y,z}\mathrm{d}y + \left.\frac{\partial u_x}{\partial z}\right|_{x,y,z}\mathrm{d}z,\right.$$
$$\mathrm{d}y + \left.\frac{\partial u_y}{\partial x}\right|_{x,y,z}\mathrm{d}x + \left.\frac{\partial u_y}{\partial y}\right|_{x,y,z}\mathrm{d}y + \left.\frac{\partial u_y}{\partial z}\right|_{x,y,z}\mathrm{d}z,$$
$$\left.\mathrm{d}z + \left.\frac{\partial u_z}{\partial x}\right|_{x,y,z}\mathrm{d}x + \left.\frac{\partial u_z}{\partial y}\right|_{x,y,z}\mathrm{d}y + \left.\frac{\partial u_z}{\partial z}\right|_{x,y,z}\mathrm{d}z\right].$$

Hence, the square of the distance between the two points after the deformation can be approximated by

$$(\mathrm{d}\breve{s})^2 \approx \left(\mathrm{d}x + \left.\frac{\partial u_x}{\partial x}\right|_{x,y,z}\mathrm{d}x + \left.\frac{\partial u_x}{\partial y}\right|_{x,y,z}\mathrm{d}y + \left.\frac{\partial u_x}{\partial z}\right|_{x,y,z}\mathrm{d}z\right)^2$$
$$+ \left(\mathrm{d}y + \left.\frac{\partial u_y}{\partial x}\right|_{x,y,z}\mathrm{d}x + \left.\frac{\partial u_y}{\partial y}\right|_{x,y,z}\mathrm{d}y + \left.\frac{\partial u_y}{\partial z}\right|_{x,y,z}\mathrm{d}z\right)^2$$
$$+ \left(\mathrm{d}z + \left.\frac{\partial u_z}{\partial x}\right|_{x,y,z}\mathrm{d}x + \left.\frac{\partial u_z}{\partial y}\right|_{x,y,z}\mathrm{d}y + \left.\frac{\partial u_z}{\partial z}\right|_{x,y,z}\mathrm{d}z\right)^2.$$

Squaring the parentheses on the right-hand side and — in view of infinitesimal gradients of the displacement — neglecting the terms that contain the products of two derivatives, we obtain

$$(\mathrm{d}\breve{s})^2 \approx (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 \tag{1.4.5}$$
$$+ 2\left(\left.\frac{\partial u_x}{\partial x}\right|_{x,y,z}(\mathrm{d}x)^2 + \left.\frac{\partial u_y}{\partial y}\right|_{x,y,z}(\mathrm{d}y)^2 + \left.\frac{\partial u_z}{\partial z}\right|_{x,y,z}(\mathrm{d}z)^2\right.$$
$$+ \left.\frac{\partial u_x}{\partial y}\right|_{x,y,z}\mathrm{d}x\mathrm{d}y + \left.\frac{\partial u_x}{\partial z}\right|_{x,y,z}\mathrm{d}x\mathrm{d}z + \left.\frac{\partial u_y}{\partial x}\right|_{x,y,z}\mathrm{d}x\mathrm{d}y$$
$$\left. + \left.\frac{\partial u_y}{\partial z}\right|_{x,y,z}\mathrm{d}y\mathrm{d}z + \left.\frac{\partial u_z}{\partial x}\right|_{x,y,z}\mathrm{d}x\mathrm{d}z + \left.\frac{\partial u_z}{\partial y}\right|_{x,y,z}\mathrm{d}y\mathrm{d}z\right),$$

which is the expression for the square of the distance between the two points after the deformation.

Using expressions (1.4.1) and (1.4.5), we obtain the difference in the square of the distance between the two points that results from the deformation; namely,

$$(\mathrm{d}\breve{s})^2-(\mathrm{d}s)^2\approx 2\left[\left.\frac{\partial u_x}{\partial x}\right|_{x,y,z}(\mathrm{d}x)^2+\left.\frac{\partial u_y}{\partial y}\right|_{x,y,z}(\mathrm{d}y)^2+\left.\frac{\partial u_z}{\partial z}\right|_{x,y,z}(\mathrm{d}z)^2\right.$$
$$+\left(\left.\frac{\partial u_x}{\partial y}\right|_{x,y,z}+\left.\frac{\partial u_y}{\partial x}\right|_{x,y,z}\right)\mathrm{d}x\mathrm{d}y+\left(\left.\frac{\partial u_y}{\partial z}\right|_{x,y,z}+\left.\frac{\partial u_z}{\partial y}\right|_{x,y,z}\right)\mathrm{d}y\mathrm{d}z$$
$$\left.+\left(\left.\frac{\partial u_x}{\partial z}\right|_{x,y,z}+\left.\frac{\partial u_z}{\partial x}\right|_{x,y,z}\right)\mathrm{d}x\mathrm{d}z\right].$$

Letting $x_1=x$, $x_2=y$ and $x_3=z$, we can can concisely write this expression as

$$(\mathrm{d}\breve{s})^2-(\mathrm{d}s)^2\approx\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\left.\frac{\partial u_{x_i}}{\partial x_j}\right|_{x_1,x_2,x_3}+\left.\frac{\partial u_{x_j}}{\partial x_i}\right|_{x_1,x_2,x_3}\right)\mathrm{d}x_i\mathrm{d}x_j,$$

The left-hand side is a scalar while $\mathrm{d}x_i$ and $\mathrm{d}x_j$ are components of a vector. The term in parentheses on the right-hand side is a component of a second-rank tensor[14], as shown in Exercise 1.4. In elasticity theory, the term in parentheses is the definition of the strain tensor for infinitesimal displacements; namely,

$$\varepsilon_{ij}:=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right),\qquad i,j\in\{1,2,3\},\qquad (1.4.6)$$

[15]where $u_i=u_{x_i}$, $u_j=u_{x_j}$ and the partial derivatives are evaluated at $\mathbf{x}=[x_1,x_2,x_3]$.[16] As indicated above, the strain tensor is a second-rank tensor.

[14]Both terms "rank" and "order" are commonly used to describe the number of indices of a tensor. In this book, we use the former term since it does not appear in any other context, while the latter term is used in the context of differential equations. Note that although the term "rank" also has a specific meaning in matrix algebra, we do not use it in such a context in this book.

[15]In this book, symbol := refers to a definition. In particular, ":=" is read as "is defined by" and "=:" as "defines".

[16]Readers interested in formulation of the strain tensor leading to its form that is valid for curvilinear coordinates might refer to Synge, J.L., and Schild, A., (1949/1978) Tensor calculus: Dover, pp. 202 – 205. Readers interested in a concise formulation of the strain

Thus, if we suppose that a continuum is deformed in such a way that points are displaced by vector $\mathbf{u}(\mathbf{x})$, then, the strain tensor is defined by expression (1.4.6). Considering infinitesimal displacements, the components of this tensor allow us to describe the deformation associated with any such a displacement. Examining expression (1.4.6), we see that in the particular case discussed on page 24, where $\mathbf{u}$ is given by constants, $\varepsilon_{ij} = 0$ for all i and j. In other words, according to the strain tensor, the continuum is not deformed, as expected in view of our discussion on page 24.

In view of its definition, the strain tensor is symmetric; namely, $\varepsilon_{ij} = \varepsilon_{ji}$. Consequently, in a three-dimensional continuum, there are only six independent components. Also, in view of its definition, the strain tensor is dimensionless.

Note the following analogy between vector calculus and tensor calculus. The gradient operator applied to the scalar field $f(x_1, x_2, x_3)$ results in a vector field described by three components; namely,

$$\nabla f(x_1, x_2, x_3) := \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right].$$

As shown in the derivation of expression (1.4.6), the gradient operator applied to the vector field $\mathbf{u} = [u_1, u_2, u_3]$ results in a second-rank tensor field described by nine components of the form $\partial u_i / \partial x_j$, where $i, j \in \{1, 2, 3\}$.

1.4.3. Physical meaning of strain tensor

Introductory comments

The strain tensor describes two types of deformation. Firstly, the sides of a volume element within a continuum can change in length. This can result in a change of volume without, necessarily, a change in shape. Components of the strain tensor, which we use to describe such deformations, are dimensionless quantities given by a change in length per unit length. Secondly, the sides of an element within a continuum can change orientation with respect to each other. This results in a change of shape without, necessarily, a change in volume. Components of the corresponding strain tensor are measured in radians and describe the change in angles before

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tensor that invokes tensorial properties might refer to Semay, C., and Silvestre-Brac, B., (2007) Introduction au calcul tensoriel: Applications à la physique: Dunod, pp. 203 – 204.

and after the deformation. Thus, the strain tensor describes relative linear displacement and relative angular displacement.

Relative change in length

To illustrate a length change expressed by a strain tensor, we revisit the derivation shown in Section 1.4.2 and consider the one-dimensional case.

Let $\mathbf{x} = [x_1, 0, 0]$ and $\mathbf{x} + \mathrm{d}\mathbf{x} = [x_1 + \mathrm{d}x_1, 0, 0]$ be two close points on the x_1-axis prior to deformation. During deformation, these points may be removed from the x_1-axis, however, their coordinates along this axis after the deformation are

$$\check{x}_1 = x_1 + u_1|_{x_1,0,0}, \qquad (1.4.7)$$

and

$$\check{x}_1 + \mathrm{d}\check{x}_1 = x_1 + \mathrm{d}x_1 + u_1|_{x_1+\mathrm{d}x_1,0,0}. \qquad (1.4.8)$$

The distance between their components along the x_1-axis after the deformation is given by the difference between expressions (1.4.7) and (1.4.8); namely,

$$\mathrm{d}\check{x}_1 = \mathrm{d}x_1 + u_1|_{x_1+\mathrm{d}x_1,0,0} - u_1|_{x_1,0,0}. \qquad (1.4.9)$$

Taylor's series of the middle term on the right-hand side can be written as

$$u_1|_{x_1+\mathrm{d}x_1,0,0} = u_1|_{x_1,0,0} + \left.\frac{\partial u_1}{\partial x_1}\right|_{x_1,0,0} \mathrm{d}x_1 + \frac{1}{2}\left.\frac{\partial^2 u_1}{\partial x_1^2}\right|_{x_1,0,0} (\mathrm{d}x_1)^2 + \ldots.$$

Using the approximation consisting of the first two terms, we can write expression (1.4.9) as

$$\mathrm{d}\check{x}_1 \approx \mathrm{d}x_1 + \left.\frac{\partial u_1}{\partial x_1}\right|_{x_1,0,0} \mathrm{d}x_1,$$

which can be restated as

$$\mathrm{d}\check{x}_1 \approx \left(1 + \frac{\partial u_1}{\partial x_1}\right) \mathrm{d}x_1.$$

Hence, in view of definition (1.4.6), we can write the distance between the two points after deformation as

$$\mathrm{d}\check{x}_1 \approx (1 + \varepsilon_{11})\, \mathrm{d}x_1, \qquad (1.4.10)$$

where $\mathrm{d}x_1$ is the distance between these two points prior to deformation.

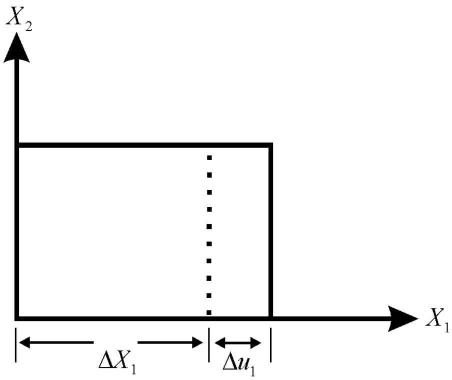

Fig. 1.4.1 Deformation: Uniaxial extension in the X_1-axis direction.

Thus, ε_{11} is a relative elongation or contraction along the x_1-axis. Similarly, $\partial u_2/\partial x_2 = \varepsilon_{22}$ and $\partial u_3/\partial x_3 = \varepsilon_{33}$ correspond to relative elongations or contractions along the x_2-axis and the x_3-axis, respectively.

To pictorially see the meaning of ε_{ii}, where $i \in \{1,2,3\}$, consider Figure 1.4.1 with axes defined in terms of the material coordinates that correspond to the configuration of the element of the continuum before deformation. The relative elongation along the X_1-axis can be written as

$$\frac{\Delta X_1 + \Delta u_1}{\Delta X_1} = 1 + \frac{\Delta u_1}{\Delta X_1}. \tag{1.4.11}$$

Considering infinitesimal gradients of the displacement, discussed in Section 1.3.3, and in view of Exercise 1.5, we can restate expression (1.4.11) as

$$1 + \frac{\partial u_1}{\partial x_1}. \tag{1.4.12}$$

Expression (1.4.12) is a relative change in length due to deformation. Now, recall equation (1.4.10), which we can restate as

$$\frac{\mathrm{d}\check{x}_1}{\mathrm{d}x_1} \approx 1 + \varepsilon_{11}, \tag{1.4.13}$$

to describe a relative change in length due to deformation. Hence, examining expressions (1.4.12) and (1.4.13), we conclude that $\varepsilon_{11} \equiv \partial u_1/\partial x_1$, as expected.

Relative change in volume

Having formulated the relative change in length, we can express a relative change in volume.

Consider a rectangular box with edge lengths Δx_1, Δx_2, and Δx_3, along the x_1-axis, the x_2-axis and the x_3-axis, respectively. Its volume is

$$V = \Delta x_1 \Delta x_2 \Delta x_3. \tag{1.4.14}$$

After the deformation, following expression (1.4.10), the edge lengths become $(1+\varepsilon_{11})\Delta x_1$, $(1+\varepsilon_{22})\Delta x_2$ and $(1+\varepsilon_{33})\Delta x_3$, respectively, and, the volume becomes

$$\breve{V} = (1+\varepsilon_{11})(1+\varepsilon_{22})(1+\varepsilon_{33})V. \tag{1.4.15}$$

Note that to state $\breve{V}$ as written in expression (1.4.15), we require that after the deformation the original rectangular box remains rectangular. In the context of our study, where the deformations are assumed to be small, the errors resulting from departing from this requirement are considered to be negligible. In other words, we use expression (1.4.15) even if Δx_1, Δx_2 and Δx_3 are no longer parallel to the corresponding axes.

Assuming small deformations and, consequently, retaining only first-order strain-component terms resulting from the triple product, the volume of the deformed rectangular box can be written as

$$\breve{V} \approx (1+\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})V. \tag{1.4.16}$$

Thus, using expressions (1.4.14) and (1.4.16), we can state the relative change in volume as

$$\frac{\breve{V}-V}{V} \approx \varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33} =: \varphi. \tag{1.4.17}$$

We refer to φ as dilatation.

Using vector calculus, we can conveniently state the relative change in volume in terms of the displacement vector, $\mathbf{u}$. In view of definition (1.4.6), expression (1.4.17) can be stated as divergence, since we can write

$$\varphi = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \nabla \cdot \mathbf{u}. \tag{1.4.18}$$

To gain insight into the physical meaning of φ, we can revisit a special case discussed on page 24. If $\mathbf{u}$ is given by constants, there is no deformation and, hence, no change in volume, as expected. Also, if $\mathbf{u} = [u_1(x_2,x_3), u_2(x_1,x_3), u_3(x_1,x_2)]$, then $\varphi = 0$. Such a displacement vector causes an infinitesimal deformation that results in change of shape, as discussed in the next section, but not in change of volume.

The dilatation will appear in stress-strain equations (5.12.4). It will also appear in the wave equation for P waves given in expression (6.1.12). Since dilatation is associated with a change in volume, P waves can be viewed as the propagation of compression within the continuum.

Note that, in terms of tensor algebra, expression (1.4.17) is the trace of the strain tensor, $tr(\varepsilon_{ij})$; namely, the sum of the diagonal terms. The trace of a second-rank tensor is a scalar; hence, it is invariant under the coordinate transformations, as proved in Exercise 1.6. Thus, as expected, the description of the change in volume is independent of the choice of the coordinate system. Relative change in volume in the context of material properties is shown in Exercise 5.10.

Change in shape

The strain tensor also describes deformations leading to a change in shape. To gain geometrical insight, consider Figure 1.4.2 with axes defined in terms of the material coordinates that correspond to the configuration of the element of the continuum before deformation. A rectangular element of the continuum is deformed into a parallelogram. In other words, the original right angle is reduced to angle α. We can write this reduction as

$$\frac{\pi}{2} - \alpha = \beta_1 + \beta_2,$$

where β_1 and β_2 are the angles measured with respect to the X_1-axis and the X_2-axis, respectively. Assuming that angles β_1 and β_2 are small and measured in radians, we can approximate them by the corresponding tangents. Hence, examining Figure 1.4.2, we can write

$$\beta_1 + \beta_2 \approx \frac{\Delta u_2}{\Delta X_1} + \frac{\Delta u_1}{\Delta X_2}. \tag{1.4.19}$$

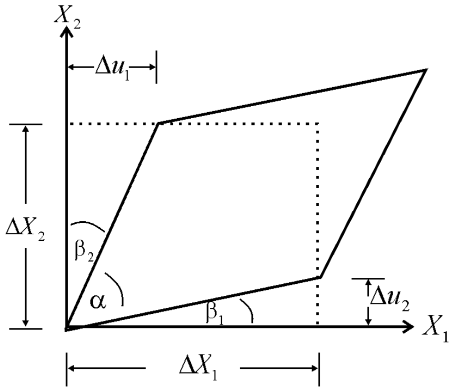

Fig. 1.4.2 Deformation: Relative change in angles.

Considering infinitesimal displacements, discussed in Section 1.3.3, and in view of Exercise 1.5, we can write equation (1.4.19) as

$$\beta_1 + \beta_2 \approx \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} = 2\varepsilon_{21} = 2\varepsilon_{12}, \qquad (1.4.20)$$

where we assume the equivalence between X_i and x_i. In other words, a function of coordinates that is evaluated at a point corresponding to the original configuration is approximately equal to this function evaluated at a point corresponding to the final position.[17]

Examining Figure 1.4.2, we see that equation (1.4.20) implies that the original segments are deviated by small angles β_1 and β_2 that can be stated as $\partial u_2/\partial x_1$ and $\partial u_1/\partial x_2$, respectively. Consequently, the initial right angle between segments, coinciding with the two axes, is changed by the sum of these two angles.[18]

[17] Readers interested in more details associated with the strain tensor in the context of the material and the spatial coordinates might refer to Malvern, L.E., (1969) Introduction to the mechanics of a continuous medium: Prentice-Hall, pp. 120 – 135.

[18] Readers interested in a geometrical interpretation of the strain-tensor components might refer to Fung, Y.C., (1977) A first course in continuum mechanics: Prentice-Hall, Inc., pp. 129 – 130.

1.5. Rotation Tensor and Rotation Vector

In Section 1.4.3, we defined dilatation, φ, which allows us to describe a relative change in volume using the divergence operator and the displacement vector, as shown in expression (1.4.18). In this section, we will associate a change in shape with the displacement vector by using the curl operator.

Let us define a tensor given by

$$\xi_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right), \qquad i, j \in \{1,2,3\}. \tag{1.5.1}$$

In view of definition (1.5.1), $\xi_{11} = \xi_{22} = \xi_{33} = 0$, and tensor ξ_{ij} has only three independent components; namely, $\xi_{23} = -\xi_{32}$, $\xi_{13} = -\xi_{31}$ and $\xi_{12} = -\xi_{21}$. Thus, ξ_{ij} is an antisymmetric tensor. We refer to ξ_{ij} as the rotation tensor. As discussed in Section 1.4.3 and illustrated in Figure 1.4.2, the quantities $\partial u_i/\partial x_j$, where $i \neq j$, are tantamount to the small deviation angles. Following the properties of the curl operator, we can associate tensor (1.5.1) with a vector given by

$$\Psi = \nabla \times \mathbf{u}, \tag{1.5.2}$$

as shown in Exercise 1.7. We refer to Ψ as the rotation vector.[19]

Rotation vector (1.5.2) will be used in formulating the wave equation involving S waves, as shown in expression (6.1.16). In other words, S waves can be viewed as the propagation of rotation within the continuum.

Note that we can use tensor calculus to relate the components of the strain tensor, the components of the rotation tensor and the components of the gradient of the displacement vector. Using expressions (1.4.6) and (1.5.1), we can write the partial derivative of a component of displacement as

$$\frac{\partial u_i}{\partial x_j} = \varepsilon_{ij} + \xi_{ij}, \qquad i, j \in \{1,2,3\}. \tag{1.5.3}$$

Equation (1.5.3) corresponds to the fact that any second-rank tensor can be written as a sum of symmetric and antisymmetric tensors.

[19] Readers interested in a relation between the rotation tensor and rotation vector might also refer to Fung, Y.C., (1977) A first course in continuum mechanics: Prentice-Hall, Inc., pp. 130 – 132.

Closing Remarks

Formulations of continuum mechanics allow us to describe deformation in a three-dimensional continuum. In subsequent chapters, these formulations will allow us to study and describe phenomena associated with wave propagation. In this study, we will use the linearized theory of elasticity. Although linearization results in a loss of subtle details, the agreement between the theory and experiments is satisfactory for our purposes.

1.6. Exercises

EXERCISE 1.1. [20]Given a material description of motion,

$$\mathbf{x}(\mathbf{X},t)=\begin{cases} x_1 = X_1 e^t + X_3\left(e^t - 1\right) \\ x_2 = X_2 + X_3\left(e^t - e^{-t}\right) \\ x_3 = X_3 \end{cases}, \tag{1.6.1}$$

verify that the transformation between the material, $\mathbf{X}$, and spatial, $\mathbf{x}$, coordinates exists, and find the spatial description of this motion.[21]

SOLUTION 1.1. The transformation between the material and spatial coordinates exists if and only if the Jacobian, which is given by

$$J := \det\begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix}, \tag{1.6.2}$$

does not vanish. Using equations (1.6.1), we obtain

[20] See also Section 1.3.1.

[21] In this book, $e^{(\cdot)}$ and $\exp(\cdot)$ are used as synonymous notations.

$$J = \det \begin{bmatrix} e^t & 0 & e^t - 1 \\ 0 & 1 & e^t - e^{-t} \\ 0 & 0 & 1 \end{bmatrix} = e^t \neq 0.$$

Thus, the transformation exists. Since, in this exercise, mapping $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ is linear, we can write it using matrix notation $\mathbf{x} = \mathbf{AX}$. We can explicitly write,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} e^t & 0 & e^t - 1 \\ 0 & 1 & e^t - e^{-t} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix},$$

where $\mathbf{A}$ is the transformation matrix. Since $\det \mathbf{A} = J \neq 0$, transformation matrix $\mathbf{A}$ has an inverse. Thus, the spatial description of motion, namely, $\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$, is $\mathbf{X} = \mathbf{A}^{-1}\mathbf{x}$. In other words,

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 & e^{-t}(1 - e^t) \\ 0 & 1 & e^{-t}\left(1 - e^{2t}\right) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

REMARK 1.6.1. Note that at $t = 0$, $\mathbf{A} = \mathbf{A}^{-1} = \mathbf{I}$; hence, $\mathbf{X}(0) = \mathbf{x}(0)$. In other words, at the initial time, both material and spatial descriptions of motion coincide. At a later time, the material point that occupied position $\mathbf{X}$ at time $t = 0$, occupies position $\mathbf{x}$.

EXERCISE 1.2. [22]Consider

$$F(X) = a \sin \frac{X}{b}, \tag{1.6.3}$$

where a and b are constants. Let the change of variables be given by $X = x - u(x)$. Show that if both a and $u(x)$ are infinitesimal while b is finite, we obtain

$$F(X) = F(x).$$

[22] See also Section 1.3.3.

SOLUTION 1.2. Considering the given change of variables, we can write expression (1.6.3) as

$$F\left(X\left(x\right)\right)=a\sin\frac{x-u\left(x\right)}{b}$$
$$=a\left(\sin\frac{x}{b}\cos\frac{u\left(x\right)}{b}-\sin\frac{u\left(x\right)}{b}\cos\frac{x}{b}\right).$$

Since $u\left(x\right)$ is an infinitesimal quantity and b is finite

$$\lim_{(u/b)\to 0}\cos\frac{u\left(x\right)}{b}=1$$

and

$$\lim_{(u/b)\to 0}\sin\frac{u\left(x\right)}{b}=0.$$

Thus, we can write

$$F\left(X\left(x\right)\right)\approx a\sin\frac{x}{b}=F\left(x\right),$$

as required.

REMARK 1.6.2. The result of Exercise 1.2, as well as the equivalence of the material and spatial coordinates for the infinitesimal displacements, is quite intuitive. In other words, considering the change of variables given by $X=x-u\left(x\right)$, we get $X\approx x$, for infinitesimal values of $u\left(x\right)$.

EXERCISE 1.3. A bar of length l would have an elongation u_1 due to strain ε_1, that is, $u_1=\varepsilon_1 l$. The same bar would have another elongation u_2 due to strain ε_2, that is, $u_2=\varepsilon_2 l$. Show that considering only linear terms, under the assumption of small strains, the total elongation due to both strains is equal to the sum of both elongations.

SOLUTION 1.3. Assume that ε_1 is applied first. This results in the elongation,

$$u_1=\varepsilon_1 l.$$

Hence, the new length of the bar is

$$l+u_1=l+\varepsilon_1 l=l\left(1+\varepsilon_1\right).$$

Subsequently, applying strain, ε_2, we obtain the final elongation,

$$u_f = u_1 + \varepsilon_2 l (1 + \varepsilon_1) = u_1 + \varepsilon_2 l + \varepsilon_1 \varepsilon_2 l = u_1 + u_2 + \varepsilon_1 \varepsilon_2 l.$$

Assuming that the value of the product, $\varepsilon_1 \varepsilon_2 l$, is small compared with the values of both $\varepsilon_1 l$ and $\varepsilon_2 l$ — in other words, both ε_1 and ε_2 are much smaller than unity — we obtain

$$u_f \approx u_1 + u_2.$$

REMARK 1.6.3. The same result is obtained if the order is reversed, or if ε_1 and ε_2 are applied simultaneously. This is the illustration of the fact that the principle of superposition is applicable to linear systems — a commonly used property in mathematical physics.[23]

EXERCISE 1.4. [24]Using definition (1.4.6) and considering orthonormal coordinate systems, show that strain, ε_{ij}, which is given in terms of first partial derivatives of a vector, is a second-rank tensor.

NOTATION 1.6.4. The repeated-index summation notation is used in this solution. Any term in which an index appears twice stands for the sum of all such terms as the index assumes values 1, 2 and 3.

SOLUTION 1.4. Following definition (1.4.6), consider $\partial \hat{u}_i / \partial \hat{x}_j$,where $\hat{u}_i$ are the components of the displacement vector, $\mathbf{u}$, in the transformed coordinates $\hat{x}_j$. The transformation rule of the coordinate points is given by

$$\hat{x}_j = a_{jl} x_l, \qquad j \in \{1,2,3\}, \tag{1.6.4}$$

where the entries of matrix $\mathbf{a}$ are the projections between the transformed and original axes. Matrix $\mathbf{a}$ is an orthogonal matrix, which means that its inverse is equal to its transpose. Hence,

$$x_j = a_{lj} \hat{x}_l, \qquad j \in \{1,2,3\}.$$

[23] Readers interested in the epistemological justification of linearity and the principle of superposition might refer to Steiner, M., (1998) The applicability of mathematics as a philosophical problem: Harvard University Press , pp. 30 – 32.
[24] See also Sections 1.4.2, 5.1.1 and 5.2.3.

Consequently, we obtain

$$\frac{\partial x_j}{\partial \hat{x}_l} = a_{lj}. \tag{1.6.5}$$

Since $\mathbf{u}$ is a vector, its components transform according to the rule

$$\hat{u}_i = a_{ik} u_k, \qquad i \in \{1,2,3\}.$$

Thus, we can write

$$\frac{\partial \hat{u}_i}{\partial \hat{x}_l} = a_{ik} \frac{\partial u_k}{\partial \hat{x}_l}, \qquad i,l \in \{1,2,3\},$$

which can be restated as

$$\frac{\partial \hat{u}_i}{\partial \hat{x}_l} = a_{ik} \frac{\partial u_k}{\partial x_j} \frac{\partial x_j}{\partial \hat{x}_l}, \qquad i,l \in \{1,2,3\}.$$

Hence, in view of equation (1.6.5), we can write

$$\frac{\partial \hat{u}_i}{\partial \hat{x}_l} = a_{ik} a_{lj} \frac{\partial u_k}{\partial x_j}, \qquad i,l \in \{1,2,3\}, \tag{1.6.6}$$

which is a transformation rule for the second-rank tensors. Consequently, since the sum of second-rank tensors is a second-rank tensor, an entity given by $\varepsilon_{ij} := (\partial u_i / \partial x_j + \partial u_j / \partial x_i)/2$ is a second-rank tensor.

EXERCISE 1.5. [25]Considering the one-dimensional case and assuming infinitesimal displacement gradients, in view of expressions (1.3.7) and (1.3.8), show that

$$\frac{\partial u}{\partial x} \approx \frac{\partial U}{\partial X}. \tag{1.6.7}$$

SOLUTION 1.5. Consider the one-dimensional case of expressions (1.3.7) and (1.3.8), namely

$$\begin{cases} U(X,t) = x(X,t) - X \\ \\ u(x,t) = x - X(x,t) \end{cases}.$$

[25] See also Section 1.4.3.

Taking partial derivatives with respect to the first arguments, we obtain

$$\begin{cases} \frac{\partial U}{\partial X} = \frac{\partial x}{\partial X} - 1 \\ \\ \frac{\partial u}{\partial x} = 1 - \frac{\partial X}{\partial x} \end{cases} . \tag{1.6.8}$$

Since $x(X,t)$ and $X(x,t)$ are inverses of one another, we use the properties of the derivative of an inverse to obtain

$$\frac{\partial x}{\partial X} = \frac{1}{\frac{\partial X}{\partial x}}.$$

Hence, we can write expression (1.6.8) as

$$\begin{cases} \frac{\partial U}{\partial X} = \frac{1}{\frac{\partial X}{\partial x}} - 1 \\ \\ \frac{\partial u}{\partial x} = 1 - \frac{\partial X}{\partial x} \end{cases} .$$

Solving both equations for $\partial X/\partial x$, we obtain

$$\begin{cases} \frac{\partial X}{\partial x} = \frac{1}{\frac{\partial U}{\partial X} + 1} \\ \\ \frac{\partial X}{\partial x} = 1 - \frac{\partial u}{\partial x} \end{cases} .$$

Equating the right-hand sides and solving for $\partial u/\partial x$, we get

$$\frac{\partial u}{\partial x} = \frac{\frac{\partial U}{\partial X}}{\frac{\partial U}{\partial X} + 1}. \tag{1.6.9}$$

Examining equation (1.6.9), we notice that for the infinitesimal displacement gradients, namely, $\partial U/\partial X << 1$, we can write $\partial u/\partial x \approx \partial U/\partial X$, which is expression (1.6.7), as required.

EXERCISE 1.6. [26]Prove the following theorem.

THEOREM 1.6.5. *The sum of diagonal elements of a second-rank tensor is invariant under orthogonal transformations of the coordinate system.*

[26] See also Section 1.4.3.

SOLUTION 1.6. PROOF. By definition, the components of the second-rank tensor ε_{lm} transform to the components $\hat{\varepsilon}_{ik}$, which are expressed in another coordinate system, according to the rule given by

$$\hat{\varepsilon}_{ik} = \sum_{l=1}^{3} \sum_{m=1}^{3} b_{il} b_{km} \varepsilon_{lm}, \qquad i,k \in \{1,2,3\},$$

where $\mathbf{b}$ is an orthogonal transformation matrix. Setting $k = i$, we obtain the sum of the components along the main diagonal; namely,

$$\sum_{i=1}^{3} \hat{\varepsilon}_{ii} = \sum_{i=1}^{3} \sum_{l=1}^{3} \sum_{m=1}^{3} b_{il} b_{im} \varepsilon_{lm}.$$

Hence, by orthogonality of $\mathbf{b}$, we have

$$\sum_{i=1}^{3} b_{il} b_{im} = \delta_{lm}, \qquad l,m \in \{1,2,3\}.$$

Thus, we can write

$$\sum_{i=1}^{3} \hat{\varepsilon}_{ii} = \sum_{l=1}^{3} \sum_{m=1}^{3} \delta_{lm} \varepsilon_{lm} = \sum_{m=1}^{3} \varepsilon_{mm}.$$

Since both i and m are the summation indices, we are allowed to write

$$\sum_{j=1}^{3} \hat{\varepsilon}_{jj} = \sum_{j=1}^{3} \varepsilon_{jj}.$$

This implies that the value of the sum of the diagonal elements is invariant under orthogonal transformations of the coordinate system. The sum of the diagonal elements of a second-rank tensor is a scalar. □

REMARK 1.6.6. Following Exercise 1.4, we can see that dilatation, φ, defined by expression (1.4.18) is the sum of diagonal elements of the second-rank tensor, namely, the trace of the strain tensor,

$$\varepsilon_{ij} := \left(\partial u_i / \partial x_j + \partial u_j / \partial x_i \right) / 2.$$

Consequently, as shown in Exercise 1.6, we can prove that dilatation is a scalar quantity. This is expected because of the physical meaning of dilatation. In other words, the change of volume must be independent of the coordinate system.

EXERCISE 1.7. [27]In view of the properties of vector operators, show that the components of the second-rank tensor, given by expression (1.5.1), namely,

$$\xi_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right), \qquad i,j \in \{1,2,3\},$$

are associated with rotation vector (1.5.2).

SOLUTION 1.7. Consider the displacement vector $\mathbf{u} = [u_1, u_2, u_3]$. We can write its curl as

$$\nabla \times \mathbf{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)\mathbf{e}_2 + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\mathbf{e}_3,$$

where $\mathbf{e}_i$ denotes a unit vector along the x_i-axis. Following expression (1.5.1), we can rewrite the curl as

$$\nabla \times \mathbf{u} = [2\xi_{32}, 2\xi_{13}, 2\xi_{21}].$$

Thus, ξ_{ij} can be viewed as the components of the vector that results from the rotation of $\mathbf{u}/2$. Denoting $\Psi = [2\xi_{32}, 2\xi_{13}, 2\xi_{21}]$, we obtain definition (1.5.2).

REMARK 1.6.7. The association between the components of the second-rank tensor ξ_{ij} and the components of vector Ψ is due to the antisymmetry of this tensor that results in only three independent components.

[27] See also Section 1.5.