Chapter 1

The Maxwell Postulates and Constitutive Relations

The action of complex arrangements of matter on electromagnetic waves lies at the heart of this book. Herein, we generally regard matter from a macroscopic perspective, which means that electromagnetic wavelengths are assumed to be large compared with interatomic distances — the atomic (and sub–atomic) nature of matter does not directly concern us. Accordingly, 'mediums' and 'materials' are referred to in this book, rather than 'assemblies of atoms and molecules'. The theoretical basis is provided by the Maxwell postulates for macroscopic fields combined with constitutive relations. The fundamental features of macroscopic electromagnetic theory, which underpin the remainder of the book, are presented in this chapter.

1.1 From microscopic to macroscopic

The electromagnetic properties of materials are conveniently characterized in macroscopic terms, wherein wavelengths are much larger than atomic length–scales (approximately 10^{-10} m). However, the macroscopic electromagnetic viewpoint is build upon a microscopic foundation [1].

Within the realm of microscopic electromagnetism, only two fields are involved: the electric field $\underline{\tilde{e}}(\underline{r}, t)$ and the magnetic field $\underline{\tilde{b}}(\underline{r}, t)$. Both of these fields vary extremely rapidly as functions of position $\underline{r}$ and time t: spatial variations occur over distances $\lesssim 10^{-10}$ m while temporal variations occur on timescales ranging from $\lesssim 10^{-13}$ s for nuclear vibrations to $\lesssim 10^{-17}$ s for electronic orbital motion [2]. The fields $\underline{\tilde{e}}(\underline{r}, t)$ and $\underline{\tilde{b}}(\underline{r}, t)$ develop due to point charges q_ℓ positioned at $\underline{r}_\ell(t)$ and moving with velocity $\underline{v}_\ell(t)$. The microscopic charge density

$$\tilde{c}(\underline{r}, t) = \sum_\ell q_\ell \, \delta \left[\underline{r} - \underline{r}_\ell(t) \right] \tag{1.1}$$

and the microscopic current density

$$\underline{\tilde{j}}(\underline{r}, t) = \sum_{\ell} q_\ell \, \underline{v}_\ell \, \delta \left[\underline{r} - \underline{r}_\ell(t) \right] \tag{1.2}$$

involve the Dirac delta function $\delta(\cdot)$.[1]

The relations between the fields $\underline{\tilde{e}}(\underline{r}, t)$ and $\underline{\tilde{b}}(\underline{r}, t)$ on the one hand and the source densities $\tilde{c}(\underline{r}, t)$ and $\underline{\tilde{j}}(\underline{r}, t)$ on the other hand are provided by the microscopic Maxwell postulates [2]

$$\left.\begin{aligned} &\nabla \times \underline{\tilde{e}}(\underline{r}, t) + \frac{\partial}{\partial t} \underline{\tilde{b}}(\underline{r}, t) = \underline{0} \\ &\nabla \times \underline{\tilde{b}}(\underline{r}, t) - \epsilon_0 \mu_0 \frac{\partial}{\partial t} \underline{\tilde{e}}(\underline{r}, t) = \mu_0 \underline{\tilde{j}}(\underline{r}, t) \\ &\nabla \bullet \underline{\tilde{e}}(\underline{r}, t) = \frac{1}{\epsilon_0} \tilde{c}(\underline{r}, t) \\ &\nabla \bullet \underline{\tilde{b}}(\underline{r}, t) = \underline{0} \end{aligned}\right\}, \tag{1.3}$$

where all quantities are in SI units, with $\epsilon_0 = 8.854 \times 10^{-12}$ F m^{-1} and $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1} being the permittivity and permeability of free space, respectively.

When the length–scales of the variation in electromagnetic fields greatly exceed atomic length–scales, the summation index ℓ in Eqs. (1.1) and (1.2) achieves enormous values. Hence, it becomes a practical necessity to consider the spatiotemporal averages of the microscopic quantities in Eqs. (1.3). In fact, spatial averaging alone suffices due to the finite universal speed $c_0 = (\epsilon_0 \mu_0)^{-1/2}$ [2]. The macroscopic counterparts of Eqs. (1.3) thus arise as

$$\left.\begin{aligned} &\nabla \times \underline{\tilde{E}}(\underline{r}, t) + \frac{\partial}{\partial t} \underline{\tilde{B}}(\underline{r}, t) = \underline{0} \\ &\nabla \times \underline{\tilde{B}}(\underline{r}, t) - \epsilon_0 \mu_0 \frac{\partial}{\partial t} \underline{\tilde{E}}(\underline{r}, t) = \mu_0 \underline{\tilde{J}}(\underline{r}, t) \\ &\nabla \bullet \underline{\tilde{E}}(\underline{r}, t) = \frac{1}{\epsilon_0} \tilde{\rho}(\underline{r}, t) \\ &\nabla \bullet \underline{\tilde{B}}(\underline{r}, t) = 0 \end{aligned}\right\}, \tag{1.4}$$

with the macroscopic fields $\underline{\tilde{E}}(\underline{r}, t)$ and $\underline{\tilde{B}}(\underline{r}, t)$ being the spatial averages of $\underline{\tilde{e}}(\underline{r}, t)$ and $\underline{\tilde{b}}(\underline{r}, t)$, respectively, and the macroscopic charge and current densities $\tilde{\rho}(\underline{r}, t)$ and $\underline{\tilde{J}}(\underline{r}, t)$ being similarly related to $\tilde{c}(\underline{r}, t)$ and $\underline{\tilde{j}}(\underline{r}, t)$.

[1] The Dirac delta function is defined in Eq. (5.4).

In matter, a distinction can be made between (i) free charges (which are externally impressed) and (ii) bound charges (which arise due to internal mechanisms). Thus, we have the externally impressed source densities

$$\left.\begin{aligned}\tilde{\rho}_{\mathrm{e}}(\underline{r},t) &= \tilde{\rho}(\underline{r},t) + \nabla \bullet \underline{\tilde{P}}(\underline{r},t)\\ \underline{\tilde{J}}_{\mathrm{e}}(\underline{r},t) &= \underline{\tilde{J}}(\underline{r},t) - \frac{\partial}{\partial t}\,\underline{\tilde{P}}(\underline{r},t) - \nabla \times \underline{\tilde{M}}(\underline{r},t)\end{aligned}\right\}, \tag{1.5}$$

where the polarization $\underline{\tilde{P}}(\underline{r},t)$ and magnetization $\underline{\tilde{M}}(\underline{r},t)$ characterize the bound source densities. Notice that $\underline{\tilde{P}}(\underline{r},t)$ and $\underline{\tilde{M}}(\underline{r},t)$ are not uniquely specified by Eqs. (1.5). That is, if $\underline{\tilde{P}}(\underline{r},t)$ were replaced by $\underline{\tilde{P}}(\underline{r},t) - \nabla \times \underline{\tilde{A}}(\underline{r},t)$ and $\underline{\tilde{M}}(\underline{r},t)$ replaced by $\underline{\tilde{M}}(\underline{r},t) + (\partial/\partial t)\,\underline{\tilde{A}}(\underline{r},t)$, then Eqs. (1.5) would still be satisfied for any differentiable vector function $\underline{\tilde{A}}(\underline{r},t)$.

The concepts of polarization and magnetization thus give rise to two further macroscopic fields defined as

$$\left.\begin{aligned}\underline{\tilde{D}}(\underline{r},t) &= \epsilon_0\,\underline{\tilde{E}}(\underline{r},t) + \underline{\tilde{P}}(\underline{r},t)\\ \underline{\tilde{H}}(\underline{r},t) &= \frac{1}{\mu_0}\,\underline{\tilde{B}}(\underline{r},t) - \underline{\tilde{M}}(\underline{r},t)\end{aligned}\right\}. \tag{1.6}$$

The basic framework for our description of electromagnetic anisotropy and bianisotropy is constructed in terms of the four macroscopic electromagnetic fields $\underline{\tilde{E}}(\underline{r},t)$, $\underline{\tilde{D}}(\underline{r},t)$, $\underline{\tilde{B}}(\underline{r},t)$ and $\underline{\tilde{H}}(\underline{r},t)$. These are piecewise differentiable vector functions of position $\underline{r}$ and time t which arise as spatial averages of microscopic fields and bound sources. The fields $\underline{\tilde{E}}(\underline{r},t)$ and $\underline{\tilde{B}}(\underline{r},t)$ are directly measurable quantities which produce the Lorentz force [2]

$$\underline{\tilde{F}}_{\mathrm{Lor}}(\underline{r},t) = q(\underline{r},t)\,\left[\underline{\tilde{E}}(\underline{r},t) + \underline{v}(\underline{r},t) \times \underline{\tilde{B}}(\underline{r},t)\right], \tag{1.7}$$

acting on a point charge $q(\underline{r},t)$ travelling at velocity $\underline{v}(\underline{r},t)$. Accordingly, $\underline{\tilde{E}}(\underline{r},t)$ and $\underline{\tilde{B}}(\underline{r},t)$ are viewed as the *primitive* fields. The fields $\underline{\tilde{D}}(\underline{r},t)$ and $\underline{\tilde{H}}(\underline{r},t)$ develop within a medium in response to the primitive fields; hence, they are considered as *induction* fields. Conventionally, $\underline{\tilde{E}}(\underline{r},t)$ and $\underline{\tilde{D}}(\underline{r},t)$ are called the electric field and the dielectric displacement, respectively. The conventional names for $\underline{\tilde{B}}(\underline{r},t)$ and $\underline{\tilde{H}}(\underline{r},t)$ — magnetic induction and magnetic field, respectively — are confusing and are avoided in this book.

The physical principles governing the behaviour of $\underline{\tilde{E}}(\underline{r},t)$, $\underline{\tilde{D}}(\underline{r},t)$, $\underline{\tilde{B}}(\underline{r},t)$ and $\underline{\tilde{H}}(\underline{r},t)$ are encapsulated by the Maxwell postulates, which — after combining Eqs. (1.4)–(1.6) — we write as two curl postulates

$$\left.\begin{aligned}\nabla \times \underline{\tilde{H}}(\underline{r},t) - \frac{\partial}{\partial t}\,\underline{\tilde{D}}(\underline{r},t) &= \underline{\tilde{J}}_{\mathrm{e}}(\underline{r},t)\\ \nabla \times \underline{\tilde{E}}(\underline{r},t) + \frac{\partial}{\partial t}\,\underline{\tilde{B}}(\underline{r},t) &= -\underline{\tilde{J}}_{\mathrm{m}}(\underline{r},t)\end{aligned}\right\} \tag{1.8}$$

and two divergence postulates

$$\left.\begin{aligned}\nabla \bullet \underline{\tilde{D}}(\underline{r},t) = \tilde{\rho}_{\rm e}(\underline{r},t)\\ \nabla \bullet \underline{\tilde{B}}(\underline{r},t) = \tilde{\rho}_{\rm m}(\underline{r},t)\end{aligned}\right\}. \tag{1.9}$$

The terms on the right sides of Eqs. (1.8) and (1.9) represent sources of fields. Whereas $\underline{\tilde{J}}_{\rm e}(\underline{r},t)$ and $\tilde{\rho}_{\rm e}(\underline{r},t)$ are the externally impressed electric current and electric charge densities, respectively, the magnetic current and magnetic charge densities — denoted by $\underline{\tilde{J}}_{\rm m}(\underline{r},t)$ and $\tilde{\rho}_{\rm m}(\underline{r},t)$ — do not represent physical quantities but are added for mathematical convenience [3]. In consonance with our macroscopic viewpoint, the source terms are also piecewise differentiable and satisfy the continuity relations

$$\left.\begin{aligned}\nabla \bullet \underline{\tilde{J}}_{\rm e}(\underline{r},t) + \frac{\partial}{\partial t}\,\tilde{\rho}_{\rm e}(\underline{r},t) = 0\\ \nabla \bullet \underline{\tilde{J}}_{\rm m}(\underline{r},t) + \frac{\partial}{\partial t}\,\tilde{\rho}_{\rm m}(\underline{r},t) = 0\end{aligned}\right\}. \tag{1.10}$$

A redundancy is implicit in Eqs. (1.8)–(1.10), from the macroscopic viewpoint. The continuity relations (1.10), when combined with the Maxwell curl postulates (1.8), yield the Maxwell divergence postulates (1.9). Therefore, under the presumption of source continuity, there is no need for us to consider explicitly the divergence postulates (1.9).

1.2 Boundary conditions

The set of differential Eqs. (1.8) and (1.9) apply locally, at each position $\underline{r}$ and time t. Wherever the derivatives do not exist, boundary conditions and initial conditions must be specified in order to derive unique solutions. The boundary conditions may be established by recasting the Maxwell postulates (1.8) and (1.9) in integral forms, as follows [4]: Consider a region V of finite volume, enclosed by the surface ∂V, with $\underline{\hat{s}}(\underline{r},t)$ representing the outward–pointing unit vector normal to ∂V. The application of the vector identities

$$\left.\begin{aligned}\int_V \nabla \bullet \underline{\tilde{A}}(\underline{r},t)\; d^3\underline{r} = \int_{\partial V} \underline{\hat{s}}(\underline{r},t) \bullet \underline{\tilde{A}}(\underline{r},t)\; d^2\underline{r}\\ \int_V \nabla \times \underline{\tilde{A}}(\underline{r},t)\; d^3\underline{r} = \int_{\partial V} \underline{\hat{s}}(\underline{r},t) \times \underline{\tilde{A}}(\underline{r},t)\; d^2\underline{r}\end{aligned}\right\} \tag{1.11}$$

to Eqs. (1.8) and (1.9) delivers

$$\left.\begin{aligned}\int_{\partial V} \underline{\hat{s}}(\underline{r},t) \times \underline{\tilde{H}}(\underline{r},t)\; d^2\underline{r} - \int_V \frac{\partial}{\partial t}\, \underline{\tilde{D}}(\underline{r},t)\; d^3\underline{r} = \int_V \underline{\tilde{J}}_{\mathrm{e}}(\underline{r},t)\; d^3\underline{r} \\ \int_{\partial V} \underline{\hat{s}}(\underline{r},t) \times \underline{\tilde{E}}(\underline{r},t)\; d^2\underline{r} + \int_V \frac{\partial}{\partial t}\, \underline{\tilde{B}}(\underline{r},t)\; d^3\underline{r} = -\int_V \underline{\tilde{J}}_{\mathrm{m}}(\underline{r},t)\; d^3\underline{r}\end{aligned}\right\} \tag{1.12}$$

and

$$\left.\begin{aligned}\int_{\partial V} \underline{\hat{s}}(\underline{r},t) \bullet \underline{\tilde{D}}(\underline{r},t)\; d^2\underline{r} = \int_V \tilde{\rho}_{\mathrm{e}}(\underline{r},t)\; d^3\underline{r} \\ \int_{\partial V} \underline{\hat{s}}(\underline{r},t) \bullet \underline{\tilde{B}}(\underline{r},t)\; d^2\underline{r} = \int_V \tilde{\rho}_{\mathrm{m}}(\underline{r},t)\; d^3\underline{r}\end{aligned}\right\}, \tag{1.13}$$

respectively. If the region V moves at velocity $\underline{v}(\underline{r},t)$ then, by exploiting the vector identity

$$\frac{d}{dt}\int_V \underline{\tilde{A}}(\underline{r},t)\; d^3\underline{r} = \int_V \frac{\partial}{\partial t}\underline{\tilde{A}}(\underline{r},t)\; d^3\underline{r} + \int_{\partial V} \left[\, \underline{\hat{s}}(\underline{r},t) \bullet \underline{v}(\underline{r},t)\right] \underline{\tilde{A}}(\underline{r},t)\; d^2\underline{r}, \tag{1.14}$$

Eqs. (1.12) may be expressed as

$$\left.\begin{aligned}\int_{\partial V} \left\{ \underline{\hat{s}}(\underline{r},t) \times \underline{\tilde{H}}(\underline{r},t) + \left[\, \underline{\hat{s}}(\underline{r},t) \bullet \underline{v}(\underline{r},t)\right] \underline{\tilde{D}}(\underline{r},t) \right\}\; d^2\underline{r} \\ = \frac{d}{dt}\int_V \underline{\tilde{D}}(\underline{r},t)\; d^3\underline{r} + \int_V \underline{\tilde{J}}_{\mathrm{e}}(\underline{r},t)\; d^3\underline{r} \\ \int_{\partial V} \left\{ \underline{\hat{s}}(\underline{r},t) \times \underline{\tilde{E}}(\underline{r},t) - \left[\, \underline{\hat{s}}(\underline{r},t) \bullet \underline{v}(\underline{r},t)\right] \underline{\tilde{B}}(\underline{r},t) \right\}\; d^2\underline{r} \\ = -\frac{d}{dt}\int_V \underline{\tilde{B}}(\underline{r},t)\; d^3\underline{r} - \int_V \underline{\tilde{J}}_{\mathrm{m}}(\underline{r},t)\; d^3\underline{r}\end{aligned}\right\}. \tag{1.15}$$

Now suppose that the region V has the form of a pillbox of height δ_{h}, with its lower face lying in region I and its upper face lying in region II, as illustrated in Fig. 1.1. Thus, we have the partition $V = V_{\mathrm{I}} \cup V_{\mathrm{II}}$ where V_{I} lies only in region I and V_{II} lies only in region II. Let $\underline{\hat{s}}(\underline{r},t) = \underline{\hat{n}}(\underline{r},t)$ on the upper pillbox face. In the limit $\delta_{\mathrm{h}} \to 0$, the integral forms (1.13) and (1.15) yield the boundary conditions

$$\left.\begin{aligned}\underline{\hat{n}}(\underline{r},t) \times \left[\underline{\tilde{H}}_{\mathrm{I}}(\underline{r},t) - \underline{\tilde{H}}_{\mathrm{II}}(\underline{r},t)\right] + \left[\, \underline{\hat{n}}(\underline{r},t) \bullet \underline{v}(\underline{r},t)\right] \left[\underline{\tilde{D}}_{\mathrm{I}}(\underline{r},t) - \underline{\tilde{D}}_{\mathrm{II}}(\underline{r},t)\right] \\ = \underline{\tilde{J}}^{\mathrm{S}}_{\mathrm{e}}(\underline{r},t) \\ \underline{\hat{n}}(\underline{r},t) \times \left[\underline{\tilde{E}}_{\mathrm{I}}(\underline{r},t) - \underline{\tilde{E}}_{\mathrm{II}}(\underline{r},t)\right] - \left[\, \underline{\hat{n}}(\underline{r},t) \bullet \underline{v}(\underline{r},t)\right] \left[\underline{\tilde{B}}_{\mathrm{I}}(\underline{r},t) - \underline{\tilde{B}}_{\mathrm{II}}(\underline{r},t)\right] \\ = -\underline{\tilde{J}}^{\mathrm{S}}_{\mathrm{m}}(\underline{r},t)\end{aligned}\right\} \tag{1.16}$$

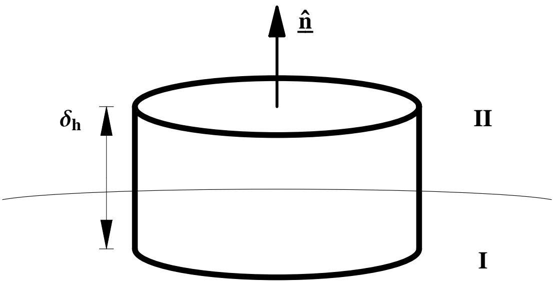

Figure 1.1 A pillbox of height $\delta_{\rm h}$, with its lower face lying in region I and its upper face lying in region II. The unit vector $\underline{\hat{n}}(\underline{r}, t)$ is normal to the upper face, pointing out of the pillbox.

and

$$\left.\begin{aligned} \underline{\hat{n}}(\underline{r}, t) \bullet \left[\underline{\tilde{D}}_{\rm I}(\underline{r}, t) - \underline{\tilde{D}}_{\rm II}(\underline{r}, t)\right] = \tilde{\rho}^{\rm S}_{\rm e}(\underline{r}, t) \\ \underline{\hat{n}}(\underline{r}, t) \bullet \left[\underline{\tilde{B}}_{\rm I}(\underline{r}, t) - \underline{\tilde{B}}_{\rm II}(\underline{r}, t)\right] = \tilde{\rho}^{\rm S}_{\rm m}(\underline{r}, t) \end{aligned}\right\}, \tag{1.17}$$

respectively. Herein,

$$\left.\begin{aligned} \underline{\tilde{A}}_{\rm I}(\underline{r}, t) = \lim_{\delta_{\rm h}\to 0} \underline{\tilde{A}}(\underline{r}, t)\Big|_{\underline{r}\in V_{\rm I}} \\ \underline{\tilde{A}}_{\rm II}(\underline{r}, t) = \lim_{\delta_{\rm h}\to 0} \underline{\tilde{A}}(\underline{r}, t)\Big|_{\underline{r}\in V_{\rm II}} \end{aligned}\right\}, \qquad (A = E, B, D, H), \tag{1.18}$$

while

$$\left.\begin{aligned} \tilde{\rho}^{\rm S}_{\ell}(\underline{r}, t) = \lim_{\delta_{\rm h}\to 0} \tilde{\rho}_{\ell}(\underline{r}, t)\,\delta_{\rm h} \\ \underline{\tilde{J}}^{\rm S}_{\ell}(\underline{r}, t) = \lim_{\delta_{\rm h}\to 0} \underline{\tilde{J}}_{\ell}(\underline{r}, t)\,\delta_{\rm h} \end{aligned}\right\}, \qquad (\ell = {\rm e},\, {\rm m})\,. \tag{1.19}$$

The surface current densities $\underline{\tilde{J}}^{\rm S}_{\rm e,m}(\underline{r}, t)$ and surface charge densities $\tilde{\rho}^{\rm S}_{\rm e,m}(\underline{r}, t)$, as defined in Eqs. (1.19), are nonzero only when the corresponding current densities and charge densities are infinite at the boundary between the regions $V_{\rm I}$ and $V_{\rm II}$. Such an eventuality may arise at the surface of a perfect conductor, for example.

Equations (1.16) and (1.17) imply that, if the boundary moves parallel to the interface, then the boundary conditions are identical to those for a stationary boundary. Furthermore, in the absence of sources, the boundary conditions for a stationary boundary, for which $\underline{\hat{n}}(\underline{r}, t) \equiv \underline{\hat{n}}(\underline{r})$, are given as

$$\left.\begin{aligned} \underline{\hat{n}}(\underline{r}) \times \underline{\tilde{H}}_{\mathrm{I}}(\underline{r}, t) = \underline{\hat{n}}(\underline{r}) \times \underline{\tilde{H}}_{\mathrm{II}}(\underline{r}, t) \\ \underline{\hat{n}}(\underline{r}) \times \underline{\tilde{E}}_{\mathrm{I}}(\underline{r}, t) = \underline{\hat{n}}(\underline{r}) \times \underline{\tilde{E}}_{\mathrm{II}}(\underline{r}, t) \end{aligned}\right\} \tag{1.20}$$

and

$$\left.\begin{aligned} \underline{\hat{n}}(\underline{r}) \bullet \underline{\tilde{D}}_{\mathrm{I}}(\underline{r}, t) = \underline{\hat{n}}(\underline{r}) \bullet \underline{\tilde{D}}_{\mathrm{II}}(\underline{r}, t) \\ \underline{\hat{n}}(\underline{r}) \bullet \underline{\tilde{B}}_{\mathrm{I}}(\underline{r}, t) = \underline{\hat{n}}(\underline{r}) \bullet \underline{\tilde{B}}_{\mathrm{II}}(\underline{r}, t) \end{aligned}\right\}. \tag{1.21}$$

Thus, the tangential components of $\underline{\tilde{E}}(\underline{r}, t)$ and $\underline{\tilde{H}}(\underline{r}, t)$ are continuous across the boundary, as are the normal components of $\underline{\tilde{D}}(\underline{r}, t)$ and $\underline{\tilde{B}}(\underline{r}, t)$.

Finally, when electromagnetic fields are considered in unbounded mediums, boundary conditions at infinity must be specified in order to derive unique solutions to the Maxwell postulates. These boundary conditions are called *radiation conditions*; they require that at infinity [4]:

- field solutions attenuate no slower than $1/|\underline{r}|$, and
- energy flow is directed outwards.

1.3 Constitutive relations

The Maxwell curl postulates (1.8) represent a system of two linear vector differential equations in terms of the two primitive vector fields $\underline{\tilde{E}}(\underline{r}, t)$ and $\underline{\tilde{B}}(\underline{r}, t)$ and the two induction vector fields $\underline{\tilde{D}}(\underline{r}, t)$ and $\underline{\tilde{H}}(\underline{r}, t)$. In order to solve these differential equations, further information — in the form of constitutive relations relating the induction fields to the primitive fields — is needed. It is these constitutive relations which characterize the electromagnetic response of a medium. The constitutive relations may be naturally expressed in the general form

$$\left.\begin{aligned} \underline{\tilde{D}}(\underline{r}, t) = \mathcal{F}\left\{\underline{\tilde{E}}(\underline{r}, t), \underline{\tilde{B}}(\underline{r}, t)\right\} \\ \underline{\tilde{H}}(\underline{r}, t) = \mathcal{G}\left\{\underline{\tilde{E}}(\underline{r}, t), \underline{\tilde{B}}(\underline{r}, t)\right\} \end{aligned}\right\}, \tag{1.22}$$

wherein $\mathcal{F}$ and $\mathcal{G}$ are linear functions of $\underline{\tilde{E}}(\underline{r}, t)$ and $\underline{\tilde{B}}(\underline{r}, t)$ for linear mediums. The case where $\mathcal{F}$ and $\mathcal{G}$ are nonlinear functions of $\underline{\tilde{E}}(\underline{r}, t)$ and $\underline{\tilde{B}}(\underline{r}, t)$ — which is relevant for nonlinear mediums — is taken up in Chap. 7.

In general, the electromagnetic response of a medium is nonlocal with respect to both space and time. Thus, the constitutive relations of a linear medium should be stated as [5]

$$\left.\begin{aligned} \underline{\tilde{D}}(\underline{r},t) &= \int_{t'}\int_{\underline{r}'}\Big[\underline{\underline{\tilde{\epsilon}}}_{\rm EB}(\underline{r}',t')\bullet\underline{\tilde{E}}(\underline{r}-\underline{r}',t-t') \\ &\qquad +\underline{\underline{\tilde{\xi}}}_{\rm EB}(\underline{r}',t')\bullet\underline{\tilde{B}}(\underline{r}-\underline{r}',t-t')\Big]\,d^3\underline{r}'\,dt' \\ \underline{\tilde{H}}(\underline{r},t) &= \int_{t'}\int_{\underline{r}'}\Big[\underline{\underline{\tilde{\zeta}}}_{\rm EB}(\underline{r}',t')\bullet\underline{\tilde{E}}(\underline{r}-\underline{r}',t-t') \\ &\qquad +\underline{\underline{\tilde{\nu}}}_{\rm EB}(\underline{r}',t')\bullet\underline{\tilde{B}}(\underline{r}-\underline{r}',t-t')\Big]\,d^3\underline{r}'\,dt' \end{aligned}\right\}, \tag{1.23}$$

where $\underline{\underline{\tilde{\epsilon}}}_{\rm EB}(\underline{r},t)$, $\underline{\underline{\tilde{\xi}}}_{\rm EB}(\underline{r},t)$, $\underline{\underline{\tilde{\zeta}}}_{\rm EB}(\underline{r},t)$ and $\underline{\underline{\tilde{\nu}}}_{\rm EB}(\underline{r},t)$ are constitutive dyadics (i.e., second–rank Cartesian tensors) that can be interpreted as 3×3 matrixes. Appendix A provides a guide to dyadic notation and algebra.

Spatial nonlocality can play a significant role when the wavelength is comparable to some characteristic length–scale in the medium [6], but it is commonly neglected and lies outside the scope of our considerations here. In contrast, temporal nonlocality is almost always a matter of central importance, because of the high speeds of electromagnetic signals. Therefore, we focus on linear, spatially local, constitutive relations of the form

$$\left.\begin{aligned} \underline{\tilde{D}}(\underline{r},t) &= \int_{t'}\Big[\underline{\underline{\tilde{\epsilon}}}_{\rm EB}(\underline{r},t')\bullet\underline{\tilde{E}}(\underline{r},t-t')+\underline{\underline{\tilde{\xi}}}_{\rm EB}(\underline{r},t')\bullet\underline{\tilde{B}}(\underline{r},t-t')\Big]\,dt' \\ \underline{\tilde{H}}(\underline{r},t) &= \int_{t'}\Big[\underline{\underline{\tilde{\zeta}}}_{\rm EB}(\underline{r},t')\bullet\underline{\tilde{E}}(\underline{r},t-t')+\underline{\underline{\tilde{\nu}}}_{\rm EB}(\underline{r},t')\bullet\underline{\tilde{B}}(\underline{r},t-t')\Big]\,dt' \end{aligned}\right\}. \tag{1.24}$$

1.4 The frequency domain

Mathematical complexities ensue from the convolution integrals appearing in the constitutive relations (1.24), when those relations are substituted into the Maxwell postulates. In order to circumvent these difficulties (often without loss of essential physics), it is common practice to introduce temporal Fourier transforms as

$$\mathcal{Z}(\underline{r},\omega) = \int_{-\infty}^{\infty}\tilde{\mathcal{Z}}(\underline{r},t)\,\exp(i\omega t)\,dt\,, \tag{1.25}$$

with $\mathcal{Z}$ standing in for $\underline{\underline{\epsilon}}_{\rm EB}$, $\underline{\underline{\xi}}_{\rm EB}$, $\underline{\underline{\zeta}}_{\rm EB}$, $\underline{\underline{\nu}}_{\rm EB}$, $\underline{E}$, $\underline{D}$, $\underline{B}$ and $\underline{H}$, while ω is called the angular frequency and $i=\sqrt{-1}$. After taking the temporal Fourier transforms of Eqs. (1.24) and implementing the convolution theorem [7], the frequency–domain constitutive relations emerge as

$$\left.\begin{aligned}\underline{D}(\underline{r},\omega) &= \underline{\underline{\epsilon}}_{\rm EB}(\underline{r},\omega) \bullet \underline{E}(\underline{r},\omega) + \underline{\underline{\xi}}_{\rm EB}(\underline{r},\omega) \bullet \underline{B}(\underline{r},\omega) \\ \underline{H}(\underline{r},\omega) &= \underline{\underline{\zeta}}_{\rm EB}(\underline{r},\omega) \bullet \underline{E}(\underline{r},\omega) + \underline{\underline{\nu}}_{\rm EB}(\underline{r},\omega) \bullet \underline{B}(\underline{r},\omega)\end{aligned}\right\}. \tag{1.26}$$

Often in electromagnetic theory, $\tilde{\underline{E}}(\underline{r},\omega)$ is partnered with $\tilde{\underline{H}}(\underline{r},\omega)$ rather than $\tilde{\underline{B}}(\underline{r},\omega)$; for example, in the formulation of boundary conditions (see Sec. 1.2) and the definition of the time–averaged Poynting vector (cf. Eq. (1.90)) [8][2]. Consequently, frequency–domain constitutive relations may be conveniently expressed as

$$\left.\begin{aligned}\underline{D}(\underline{r},\omega) &= \underline{\underline{\epsilon}}_{\rm EH}(\underline{r},\omega) \bullet \underline{E}(\underline{r},\omega) + \underline{\underline{\xi}}_{\rm EH}(\underline{r},\omega) \bullet \underline{H}(\underline{r},\omega) \\ \underline{B}(\underline{r},\omega) &= \underline{\underline{\zeta}}_{\rm EH}(\underline{r},\omega) \bullet \underline{E}(\underline{r},\omega) + \underline{\underline{\mu}}_{\rm EH}(\underline{r},\omega) \bullet \underline{H}(\underline{r},\omega)\end{aligned}\right\}. \tag{1.27}$$

Herein $\underline{\underline{\epsilon}}_{\rm EH}(\underline{r},\omega)$, $\underline{\underline{\xi}}_{\rm EH}(\underline{r},\omega)$, $\underline{\underline{\zeta}}_{\rm EH}(\underline{r},\omega)$ and $\underline{\underline{\mu}}_{\rm EH}(\underline{r},\omega)$ are temporal Fourier transforms of $\tilde{\underline{\underline{\epsilon}}}_{\rm EH}(\underline{r},t)$, $\tilde{\underline{\underline{\xi}}}_{\rm EH}(\underline{r},t)$, $\tilde{\underline{\underline{\zeta}}}_{\rm EH}(\underline{r},t)$ and $\tilde{\underline{\underline{\mu}}}_{\rm EH}(\underline{r},t)$, respectively, defined as per Eq. (1.25). The names *Boys–Post* and *Tellegen* are often associated with the constitutive relations (1.26) and (1.27), respectively [9]. A one–to–one correspondence between the Boys–Post representation and the Tellegen representation is straightforwardly established via [5]

$$\left.\begin{aligned}\underline{\underline{\epsilon}}_{\rm EB}(\underline{r},\omega) &= \underline{\underline{\epsilon}}_{\rm EH}(\underline{r},\omega) - \underline{\underline{\xi}}_{\rm EH}(\underline{r},\omega) \bullet \underline{\underline{\mu}}^{-1}_{\rm EH}(\underline{r},\omega) \bullet \underline{\underline{\zeta}}_{\rm EH}(\underline{r},\omega) \\ \underline{\underline{\xi}}_{\rm EB}(\underline{r},\omega) &= \underline{\underline{\xi}}_{\rm EH}(\underline{r},\omega) \bullet \underline{\underline{\mu}}^{-1}_{\rm EH}(\underline{r},\omega) \\ \underline{\underline{\zeta}}_{\rm EB}(\underline{r},\omega) &= -\underline{\underline{\mu}}^{-1}_{\rm EH}(\underline{r},\omega) \bullet \underline{\underline{\zeta}}_{\rm EH}(\underline{r},\omega) \\ \underline{\underline{\nu}}_{\rm EB}(\underline{r},\omega) &= \underline{\underline{\mu}}^{-1}_{\rm EH}(\underline{r},\omega)\end{aligned}\right\} \tag{1.28}$$

and

$$\left.\begin{aligned}\underline{\underline{\epsilon}}_{\rm EH}(\underline{r},\omega) &= \underline{\underline{\epsilon}}_{\rm EB}(\underline{r},\omega) - \underline{\underline{\xi}}_{\rm EB}(\underline{r},\omega) \bullet \underline{\underline{\nu}}^{-1}_{\rm EB}(\underline{r},\omega) \bullet \underline{\underline{\zeta}}_{\rm EB}(\underline{r},\omega) \\ \underline{\underline{\xi}}_{\rm EH}(\underline{r},\omega) &= \underline{\underline{\xi}}_{\rm EB}(\underline{r},\omega) \bullet \underline{\underline{\nu}}^{-1}_{\rm EB}(\underline{r},\omega) \\ \underline{\underline{\zeta}}_{\rm EH}(\underline{r},\omega) &= -\underline{\underline{\nu}}^{-1}_{\rm EB}(\underline{r},\omega) \bullet \underline{\underline{\zeta}}_{\rm EB}(\underline{r},\omega) \\ \underline{\underline{\mu}}_{\rm EH}(\underline{r},\omega) &= \underline{\underline{\nu}}^{-1}_{\rm EB}(\underline{r},\omega)\end{aligned}\right\} \tag{1.29}$$

wherein the invertibility of $\underline{\underline{\nu}}_{\rm EB}(\underline{r},\omega)$ and $\underline{\underline{\mu}}_{\rm EH}(\underline{r},\omega)$ has been assumed[3]. The Tellegen representation is largely adopted in this book, but with occasional recourse to the Boys–Post representation where appropriate.

[2] A notable exception is provided by the Lorentz transformation of fields, wherein $\tilde{\underline{E}}(\underline{r},\omega)$ is partnered with $\tilde{\underline{B}}(\underline{r},\omega)$, as described in Sec. 1.6.3.

[3] The invertibility of constitutive dyadics is an assumption rather than a fact *a priori*. An example of a bianisotropic medium characterized by singular constitutive dyadics is presented in Sec. 2.3.3.

The corresponding frequency–domain Maxwell curl postulates arise as

$$\left.\begin{aligned} \nabla\times\underline{H}(\underline{r},\omega)+i\omega\underline{D}(\underline{r},\omega)&=\underline{J}_{\mathrm{e}}(\underline{r},\omega)\\ \nabla\times\underline{E}(\underline{r},\omega)-i\omega\underline{B}(\underline{r},\omega)&=-\underline{J}_{\mathrm{m}}(\underline{r},\omega)\end{aligned}\right\}, \tag{1.30}$$

where the source terms $\underline{J}_{\mathrm{e,m}}(\underline{r},\omega)$ are the temporal Fourier transforms of $\underline{\tilde{J}}_{\mathrm{e,m}}(\underline{r},t)$, defined as in Eq. (1.25) with $\mathcal{Z}=\underline{J}_{\mathrm{e,m}}$. The constitutive relations (1.27) — or equally (1.26) — together with the Maxwell curl postulates (1.30), form a self–consistent system into which anisotropy and bianisotropy are incorporated.

The boundary conditions, derived in Sec. 1.2 from the Maxwell postulates in the time domain, carry over to the frequency domain in a straightforward manner. Thus, the frequency–domain counterparts of Eqs. (1.20) and (1.21) are provided as

$$\left.\begin{aligned} \underline{\hat{n}}(\underline{r})\times\underline{H}_{\mathrm{I}}(\underline{r},\omega)&=\underline{\hat{n}}(\underline{r})\times\underline{H}_{\mathrm{II}}(\underline{r},\omega)\\ \underline{\hat{n}}(\underline{r})\times\underline{E}_{\mathrm{I}}(\underline{r},\omega)&=\underline{\hat{n}}(\underline{r})\times\underline{E}_{\mathrm{II}}(\underline{r},\omega)\end{aligned}\right\} \tag{1.31}$$

and

$$\left.\begin{aligned} \underline{\hat{n}}(\underline{r})\bullet\underline{D}_{\mathrm{I}}(\underline{r},\omega)&=\underline{\hat{n}}(\underline{r})\bullet\underline{D}_{\mathrm{II}}(\underline{r},\omega)\\ \underline{\hat{n}}(\underline{r})\bullet\underline{B}_{\mathrm{I}}(\underline{r},\omega)&=\underline{\hat{n}}(\underline{r})\bullet\underline{B}_{\mathrm{II}}(\underline{r},\omega)\end{aligned}\right\}, \tag{1.32}$$

respectively.

The mathematical simplicity of the frequency–domain formulation in relation to the time–domain formulation is gained at a cost in terms of physical interpretation. The frequency–dependent constitutive dyadics $\underline{\underline{\epsilon}}_{\,\mathrm{EB,EH}}(\underline{r},\omega)$, $\underline{\underline{\xi}}_{\,\mathrm{EB,EH}}(\underline{r},\omega)$, $\underline{\underline{\zeta}}_{\,\mathrm{EB,EH}}(\underline{r},\omega)$, $\underline{\underline{\nu}}_{\,\mathrm{EB}}(\underline{r},\omega)$ and $\underline{\underline{\mu}}_{\,\mathrm{EH}}(\underline{r},\omega)$ are complex–valued quantities, and so also are the frequency–dependent field *phasors* $\underline{E}(\underline{r},\omega)$, $\underline{D}(\underline{r},\omega)$, $\underline{B}(\underline{r},\omega)$, $\underline{H}(\underline{r},\omega)$ and $\underline{J}_{\mathrm{e,m}}(\underline{r},\omega)$. The real–valued physical entities they represent surface only indirectly upon subjecting them to the inverse temporal Fourier transform. In this book, phasors are also called fields — in keeping with widespread usage.

Since the inverse temporal Fourier transform

$$\tilde{\mathcal{Z}}(\underline{r},t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{Z}(\underline{r},\omega)\,\exp(-i\omega t)\,d\omega \tag{1.33}$$

is necessarily real–valued for $\mathcal{Z}\in\{\underline{\underline{\epsilon}}_{\,\mathrm{EB,EH}}, \underline{\underline{\xi}}_{\,\mathrm{EB,EH}}, \underline{\underline{\zeta}}_{\,\mathrm{EB,EH}}, \underline{\underline{\nu}}_{\,\mathrm{EB}}, \underline{\underline{\mu}}_{\,\mathrm{EH}}, \underline{E}, \underline{D}, \underline{B}, \underline{H}, \underline{J}_{\mathrm{e,m}}\}$, the symmetry

$$\mathcal{Z}^*(\underline{r},\omega)=\mathcal{Z}(\underline{r},-\omega) \tag{1.34}$$

is imposed, where the superscript * indicates the complex conjugate. Therefore, the frequency–domain quantities represented by $\mathcal{Z}(\underline{r},\omega)$ are such that

$$\left.\begin{array}{l}\mathrm{Re}\,\{\mathcal{Z}(\underline{r},\omega)\} = \mathrm{Re}\,\{\mathcal{Z}(\underline{r},-\omega)\} \\ \mathrm{Im}\,\{\mathcal{Z}(\underline{r},\omega)\} = -\mathrm{Im}\,\{\mathcal{Z}(\underline{r},-\omega)\}\end{array}\right\}, \tag{1.35}$$

with the operators Re $\{\cdot\}$ and Im $\{\cdot\}$ yielding the real and imaginary parts, respectively.

By virtue of the representation (1.33), the time–domain fields $\underline{\tilde{A}}(\underline{r},t)$ ($A \in \{\, E,\, B,\, D,\, H,\, J_{\mathrm{e,m}}\}$) may be regarded as continuous sums of time–harmonic components, over all angular frequencies. This leads us to a useful concept in electromagnetics — closely allied to the frequency–domain representation — namely, the representation of monochromatic fields. A monochromatic field, which oscillates at single angular frequency $\omega = \omega_{\mathrm{s}}$, may be represented by the vector

$$\underline{\tilde{A}}_{\,\mathrm{mono}}(\underline{r},t) = \mathrm{Re}\,\left\{\underline{\check{\check{A}}}\,(\underline{r},\omega_{\mathrm{s}})\,\exp\left(-i\omega_{\mathrm{s}}t\right)\right\}, \qquad (A = E, B, D, H, J_{\mathrm{e,m}}), \tag{1.36}$$

where the amplitude $\underline{\check{\check{A}}}\,(\underline{r},\omega_{\mathrm{s}}) \in \mathbb{C}^3$ in general. Taking the temporal Fourier transform of $\underline{\tilde{A}}_{\,\mathrm{mono}}(\underline{r},t)$ — which is written as $\underline{A}_{\,\mathrm{mono}}(\underline{r},\omega)$, we find that the corresponding frequency–domain representation is

$$\underline{A}_{\,\mathrm{mono}}(\underline{r},\omega) = \frac{1}{2}\,\left[\underline{\check{\check{A}}}\,(\underline{r},\omega_{\mathrm{s}})\,\delta(\omega-\omega_{\mathrm{s}}) + \underline{\check{\check{A}}}^{*}(\underline{r},\omega_{\mathrm{s}})\,\delta(\omega+\omega_{\mathrm{s}})\right]. \tag{1.37}$$

Notice that the phasor $\underline{A}_{\,\mathrm{mono}}(\underline{r},\omega)$ in Eq. (1.37) satisfies the symmetry condition (1.34). The monochromatic field amplitudes $\underline{\check{\check{A}}}\,(\underline{r},\omega_{\mathrm{s}})$ satisfy the frequency–domain constitutive relations, Maxwell postulates and boundary conditions in exactly the same way as the frequency–domain phasors $\underline{A}_{\,\mathrm{mono}}(\underline{r},\omega)$; that is, $\underline{\check{\check{A}}}\,(\underline{r},\omega_{\mathrm{s}})$ can take the place of $\underline{A}_{\,\mathrm{mono}}(\underline{r},\omega)$ in Eqs. (1.26), (1.27), (1.30), (1.31) and (1.32) for $A \in \{\, E,\, B,\, D,\, H,\, J_{\mathrm{e,m}}\}$, and with $\omega = \omega_{\mathrm{s}}$ therein.

We close this section with a note of caution. The correspondence between the time and frequency domains may not always be one–to–one: if a time–domain function is not absolutely integrable over the real axis then its Fourier transform does not exist, and therefore the transformation to the frequency domain cannot take place [10]. Further complications can arise from the non–uniqueness of the inverse Fourier transform [11].

1.5 6–vector/6×6 dyadic notation

The use of a 6–vector/6×6 dyadic notation allows the Tellegen constitutive relations (1.27) to be expressed succinctly as

$$\underline{\mathbf{C}}(\underline{r},\omega) = \underline{\underline{\mathbf{K}}}_{\,\mathrm{EH}}(\underline{r},\omega) \bullet \underline{\mathbf{F}}(\underline{r},\omega)\,, \tag{1.38}$$

with the 6–vectors

$$\underline{\mathbf{C}}(\underline{r},\omega) = \begin{bmatrix} \underline{D}(\underline{r},\omega) \\ \underline{B}(\underline{r},\omega) \end{bmatrix} \tag{1.39}$$

and

$$\underline{\mathbf{F}}(\underline{r},\omega) = \begin{bmatrix} \underline{E}(\underline{r},\omega) \\ \underline{H}(\underline{r},\omega) \end{bmatrix} \tag{1.40}$$

containing components of the electric and magnetic fields, while the 6×6 constitutive dyadic

$$\underline{\underline{\mathbf{K}}}_{\,\mathrm{EH}}(\underline{r},\omega) = \begin{bmatrix} \underline{\underline{\epsilon}}_{\,\mathrm{EH}}(\underline{r},\omega) & \underline{\underline{\xi}}_{\,\mathrm{EH}}(\underline{r},\omega) \\ \underline{\underline{\zeta}}_{\,\mathrm{EH}}(\underline{r},\omega) & \underline{\underline{\mu}}_{\,\mathrm{EH}}(\underline{r},\omega) \end{bmatrix}. \tag{1.41}$$

The result of combining the constitutive relations (1.27) with the Maxwell curl postulates (1.30) is thereby compactly expressed as

$$\left[\underline{\underline{\mathbf{L}}}(\nabla) + i\omega \underline{\underline{\mathbf{K}}}_{\,\mathrm{EH}}(\underline{r},\omega) \right] \bullet \underline{\mathbf{F}}(\underline{r},\omega) = \underline{\mathbf{Q}}(\underline{r},\omega), \tag{1.42}$$

where the source 6–vector

$$\underline{\mathbf{Q}}(\underline{r},\omega) = \begin{bmatrix} \underline{J}_{\,\mathrm{e}}(\underline{r},\omega) \\ \underline{J}_{\,\mathrm{m}}(\underline{r},\omega) \end{bmatrix} \tag{1.43}$$

and the linear differential operator

$$\underline{\underline{\mathbf{L}}}(\nabla) = \begin{bmatrix} \underline{\underline{0}} & \nabla \times \underline{\underline{I}} \\ -\nabla \times \underline{\underline{I}} & \underline{\underline{0}} \end{bmatrix}, \tag{1.44}$$

with $\underline{\underline{0}}$ and $\underline{\underline{I}}$ being the null and identity 3×3 dyadics, respectively.

In a similar fashion, the four 3×3 dyadics $\underline{\underline{\epsilon}}_{\,\mathrm{EB}}(\underline{r},\omega)$, $\underline{\underline{\xi}}_{\,\mathrm{EB}}(\underline{r},\omega)$, $\underline{\underline{\zeta}}_{\,\mathrm{EB}}(\underline{r},\omega)$ and $\underline{\underline{\nu}}_{\,\mathrm{EB}}(\underline{r},\omega)$, which specify the constitutive properties in the Boys–Post representation (1.26) may be represented by the 6×6 constitutive dyadic

$$\underline{\underline{\mathbf{K}}}_{\,\mathrm{EB}}(\underline{r},\omega) = \begin{bmatrix} \underline{\underline{\epsilon}}_{\,\mathrm{EB}}(\underline{r},\omega) & \underline{\underline{\xi}}_{\,\mathrm{EB}}(\underline{r},\omega) \\ \underline{\underline{\zeta}}_{\,\mathrm{EB}}(\underline{r},\omega) & \underline{\underline{\nu}}_{\,\mathrm{EB}}(\underline{r},\omega) \end{bmatrix}. \tag{1.45}$$

The transformations (1.29) and (1.28) may then be expressed in terms of the invertible 6×6 dyadic operator $\underline{\underline{\boldsymbol{\tau}}}$ which we define through the following relationships:[4]

$$\left.\begin{aligned} \underline{\underline{\mathbf{K}}}_{\,\mathrm{EB}}(\underline{r},\omega) &\equiv \underline{\underline{\boldsymbol{\tau}}}\left\{\underline{\underline{\mathbf{K}}}_{\,\mathrm{EH}}(\underline{r},\omega)\right\} \\ &= \begin{bmatrix} \underline{\underline{\epsilon}}_{\,\mathrm{EH}} - \underline{\underline{\xi}}_{\,\mathrm{EH}} \bullet \underline{\underline{\mu}}^{-1}_{\,\mathrm{EH}} \bullet \underline{\underline{\zeta}}_{\,\mathrm{EH}} & \underline{\underline{\xi}}_{\,\mathrm{EH}} \bullet \underline{\underline{\mu}}^{-1}_{\,\mathrm{EH}} \\ -\underline{\underline{\mu}}^{-1}_{\,\mathrm{EH}} \bullet \underline{\underline{\zeta}}_{\,\mathrm{EH}} & \underline{\underline{\mu}}^{-1}_{\,\mathrm{EH}} \end{bmatrix} \\ \underline{\underline{\mathbf{K}}}_{\,\mathrm{EH}}(\underline{r},\omega) &\equiv \underline{\underline{\boldsymbol{\tau}}}^{-1}\left\{\underline{\underline{\mathbf{K}}}_{\,\mathrm{EB}}(\underline{r},\omega)\right\} \\ &= \begin{bmatrix} \underline{\underline{\epsilon}}_{\,\mathrm{EB}} - \underline{\underline{\xi}}_{\,\mathrm{EB}} \bullet \underline{\underline{\nu}}^{-1}_{\,\mathrm{EB}}(\bullet \underline{\underline{\zeta}}_{\,\mathrm{EB}} & \underline{\underline{\xi}}_{\,\mathrm{EB}} \bullet \underline{\underline{\nu}}^{-1}_{\,\mathrm{EB}} \\ -\underline{\underline{\nu}}^{-1}_{\,\mathrm{EB}}(\bullet \underline{\underline{\zeta}}_{\,\mathrm{EB}} & \underline{\underline{\nu}}^{-1}_{\,\mathrm{EB}} \end{bmatrix} \end{aligned}\right\}. \qquad (1.46)$$

1.6 Form invariances

Under certain linear transformations of coordinates and fields, the Maxwell postulates retain their form. In this section we describe spatial and temporal invariances as well as spatiotemporal covariance. While spatiotemporal covariance is of immense theoretical importance, invariances with respect to spatial and temporal transformations are commonly applied in many practical situations. Chiral invariance, which captures the nonuniqueness of the Maxwell postulates under linear field transformations, is discussed. A recently discovered invariance of the (frequency–domain) Maxwell postulates to a certain transformation involving complex conjugates is presented. And the implications of various transformations on electromagnetic energy and momentum are also outlined.

1.6.1 *Time reversal*

Let the operation of time reversal be denoted by $\mathcal{T}$, i.e., $\mathcal{T}\{t\} = -t$. Under the presumption that electric and magnetic source densities transform as [12]

$$\left.\begin{aligned} \mathcal{T}\{\tilde{\rho}_{\mathrm{e}}(\underline{r},t)\} &= \tilde{\rho}_{\mathrm{e}}(\underline{r},-t) \\ \mathcal{T}\{\tilde{\rho}_{\mathrm{m}}(\underline{r},t)\} &= -\tilde{\rho}_{\mathrm{m}}(\underline{r},-t) \end{aligned}\right\}, \qquad (1.47)$$

[4]For compact presentation, the dependency of the 3×3 constitutive dyadics on $\underline{r}$ and ω is omitted from Eqs. (1.46).

the continuity relations (1.10) yield

$$\left.\begin{aligned}\mathcal{T}\left\{\underline{\tilde{J}}_{\mathrm{e}}(\underline{r},t)\right\} &= -\underline{\tilde{J}}_{\mathrm{e}}(\underline{r},-t)\\ \mathcal{T}\left\{\underline{\tilde{J}}_{\mathrm{m}}(\underline{r},t)\right\} &= \underline{\tilde{J}}_{\mathrm{m}}(\underline{r},-t)\end{aligned}\right\}, \tag{1.48}$$

and the electromagnetic fields are required to transform as

$$\left.\begin{aligned}\mathcal{T}\left\{\underline{\tilde{E}}(\underline{r},t)\right\} &= \underline{\tilde{E}}(\underline{r},-t), & \mathcal{T}\left\{\underline{\tilde{D}}(\underline{r},t)\right\} &= \underline{\tilde{D}}(\underline{r},-t)\\ \mathcal{T}\left\{\underline{\tilde{B}}(\underline{r},t)\right\} &= -\underline{\tilde{B}}(\underline{r},-t), & \mathcal{T}\left\{\underline{\tilde{H}}(\underline{r},t)\right\} &= -\underline{\tilde{H}}(\underline{r},-t)\end{aligned}\right\} \tag{1.49}$$

in order to preserve the form of the Maxwell postulates. From the definition of the temporal Fourier transform (1.25), we see that the frequency–domain counterparts of Eqs. (1.47), (1.48) and (1.49) are as follows:

$$\left.\begin{aligned}\mathcal{T}\left\{\rho_{\mathrm{e}}(\underline{r},\omega)\right\} &= \rho_{\mathrm{e}}^{*}(\underline{r},\omega), & \mathcal{T}\left\{\underline{J}_{\mathrm{e}}(\underline{r},\omega)\right\} &= -\underline{J}_{\mathrm{e}}^{*}(\underline{r},\omega)\\ \mathcal{T}\left\{\rho_{\mathrm{m}}(\underline{r},\omega)\right\} &= -\rho_{\mathrm{m}}^{*}(\underline{r},\omega), & \mathcal{T}\left\{\underline{J}_{\mathrm{m}}(\underline{r},\omega)\right\} &= \underline{J}_{\mathrm{m}}^{*}(\underline{r},\omega)\\ \mathcal{T}\left\{\underline{E}(\underline{r},\omega)\right\} &= \underline{E}^{*}(\underline{r},\omega), & \mathcal{T}\left\{\underline{D}(\underline{r},\omega)\right\} &= \underline{D}^{*}(\underline{r},\omega)\\ \mathcal{T}\left\{\underline{B}(\underline{r},\omega)\right\} &= -\underline{B}^{*}(\underline{r},\omega), & \mathcal{T}\left\{\underline{H}(\underline{r},\omega)\right\} &= -\underline{H}^{*}(\underline{r},\omega)\end{aligned}\right\}. \tag{1.50}$$

Therefore, under time reversal, the constitutive dyadics transform as

$$\left.\begin{aligned}\mathcal{T}\left\{\underline{\underline{\epsilon}}_{\,\mathrm{EB,EH}}(\underline{r},\omega)\right\} &= \underline{\underline{\epsilon}}^{*}_{\,\mathrm{EB,EH}}(\underline{r},\omega)\\ \mathcal{T}\left\{\underline{\underline{\xi}}_{\,\mathrm{EB,EH}}(\underline{r},\omega)\right\} &= -\underline{\underline{\xi}}^{*}_{\,\mathrm{EB,EH}}(\underline{r},\omega)\\ \mathcal{T}\left\{\underline{\underline{\zeta}}_{\,\mathrm{EB,EH}}(\underline{r},\omega)\right\} &= -\underline{\underline{\zeta}}^{*}_{\,\mathrm{EB,EH}}(\underline{r},\omega)\\ \mathcal{T}\left\{\underline{\underline{\nu}}_{\,\mathrm{EB}}(\underline{r},\omega)\right\} &= \underline{\underline{\nu}}^{*}_{\,\mathrm{EB}}(\underline{r},\omega)\\ \mathcal{T}\left\{\underline{\underline{\mu}}_{\,\mathrm{EH}}(\underline{r},\omega)\right\} &= \underline{\underline{\mu}}^{*}_{\,\mathrm{EH}}(\underline{r},\omega)\end{aligned}\right\}, \tag{1.51}$$

by virtue of Eqs. (1.26) and (1.27). The time–reversal asymmetry which is exhibited by the magnetoelectric constitutive dyadics $\underline{\underline{\xi}}_{\,\mathrm{EB,EH}}(\underline{r},\omega)$ and $\underline{\underline{\zeta}}_{\,\mathrm{EB,EH}}(\underline{r},\omega)$ originates from irreversible physical processes, such as can develop through the application of quasistatic biasing fields or by means of relative motion [13]. This issue is enlarged upon in Chap. 2 in the context of Faraday chiral mediums and Lorentz–transformed constitutive dyadics.

1.6.2 *Spatial inversion*

We turn now to the inversion of space, denoted by the operator $\mathcal{P}$ as $\mathcal{P}\left\{\underline{r}\right\} = -\underline{r}$. Similarly to the time–reversal scenario presented in Sec. 1.6.1, if it is assumed that the electric and magnetic charge densities transform as [12]

$$\left.\begin{aligned} \mathcal{P}\left\{\tilde{\rho}_{\mathrm{e}}(\underline{r},t)\right\} &= \tilde{\rho}_{\mathrm{e}}(-\underline{r},t) \\ \mathcal{P}\left\{\tilde{\rho}_{\mathrm{m}}(\underline{r},t)\right\} &= -\tilde{\rho}_{\mathrm{m}}(-\underline{r},t) \end{aligned}\right\}, \tag{1.52}$$

then, by virtue of the continuity relations (1.10), we have

$$\left.\begin{aligned} \mathcal{P}\left\{\underline{\tilde{J}}_{\mathrm{e}}(\underline{r},t)\right\} &= -\underline{\tilde{J}}_{\mathrm{e}}(-\underline{r},t) \\ \mathcal{P}\left\{\underline{\tilde{J}}_{\mathrm{m}}(\underline{r},t)\right\} &= \underline{\tilde{J}}_{\mathrm{m}}(-\underline{r},t) \end{aligned}\right\}; \tag{1.53}$$

also, the form invariance of the Maxwell postulates enjoins the relationships

$$\left.\begin{aligned} \mathcal{P}\left\{\underline{\tilde{E}}(\underline{r},t)\right\} &= -\underline{\tilde{E}}(-\underline{r},t), \quad & \mathcal{P}\left\{\underline{\tilde{D}}(\underline{r},t)\right\} &= -\underline{\tilde{D}}(-\underline{r},t) \\ \mathcal{P}\left\{\underline{\tilde{B}}(\underline{r},t)\right\} &= \underline{\tilde{B}}(-\underline{r},t), \quad & \mathcal{P}\left\{\underline{\tilde{H}}(\underline{r},t)\right\} &= \underline{\tilde{H}}(-\underline{r},t) \end{aligned}\right\}. \tag{1.54}$$

Switching from the time domain to the frequency domain does not alter the action of the spatial–inversion operator $\mathcal{P}$ on the field quantities. Hence, from the constitutive relations (1.26) and (1.27) we find that

$$\left.\begin{aligned} \mathcal{P}\left\{\underline{\underline{\epsilon}}_{\mathrm{EB,EH}}(\underline{r},\omega)\right\} &= \underline{\underline{\epsilon}}_{\mathrm{EB,EH}}(-\underline{r},\omega) \\ \mathcal{P}\left\{\underline{\underline{\xi}}_{\mathrm{EB,EH}}(\underline{r},\omega)\right\} &= -\underline{\underline{\xi}}_{\mathrm{EB,EH}}(-\underline{r},\omega) \\ \mathcal{P}\left\{\underline{\underline{\zeta}}_{\mathrm{EB,EH}}(\underline{r},\omega)\right\} &= -\underline{\underline{\zeta}}_{\mathrm{EB,EH}}(-\underline{r},\omega) \\ \mathcal{P}\left\{\underline{\underline{\nu}}_{\mathrm{EB}}(\underline{r},\omega)\right\} &= \underline{\underline{\nu}}_{\mathrm{EB}}(-\underline{r},\omega) \\ \mathcal{P}\left\{\underline{\underline{\mu}}_{\mathrm{EH}}(\underline{r},\omega)\right\} &= \underline{\underline{\mu}}_{\mathrm{EH}}(-\underline{r},\omega) \end{aligned}\right\}. \tag{1.55}$$

1.6.3 *Lorentz covariance*

Suppose that an inertial reference frame Σ' moves with constant velocity $\underline{v} = v\hat{\underline{v}}$ with respect to an inertial reference frame Σ. The spacetime coordinates $(\underline{r}', t')$ in Σ' are related to the spacetime coordinates $(\underline{r}, t)$ in Σ by Lorentz transformation [8]

$$\left.\begin{aligned} \underline{r}' &= \underline{\underline{Y}} \bullet r - \gamma\, \underline{v} t \\ t' &= \gamma\left(t - \frac{\underline{r} \bullet \underline{v}}{c_0^2}\right) \end{aligned}\right\}, \tag{1.56}$$

with

$$\left.\begin{aligned} \underline{\underline{Y}} &= \underline{\underline{I}} + (\gamma - 1)\, \hat{\underline{v}}\, \hat{\underline{v}} \\ \gamma &= \left(1 - \beta^2\right)^{-\frac{1}{2}} \\ \beta &= \frac{v}{c_0} \end{aligned}\right\}, \tag{1.57}$$

and $c_0 = (\epsilon_0 \mu_0)^{-1/2}$ being the speed of light in free space (i.e., vacuum[5]).

By application of the transformations (1.56), the time–domain fields $\underline{\tilde{E}}(\underline{r}, t)$, $\underline{\tilde{B}}(\underline{r}, t)$, $\underline{\tilde{D}}(\underline{r}, t)$ and $\underline{\tilde{H}}(\underline{r}, t)$ in the inertial reference frame Σ are found to be related to their counterparts in inertial reference frame Σ', namely $\underline{\tilde{E}}'(\underline{r}', t')$, $\underline{\tilde{B}}'(\underline{r}', t')$, $\underline{\tilde{D}}'(\underline{r}', t')$ and $\underline{\tilde{H}}'(\underline{r}', t')$ as

$$\left.\begin{aligned} \underline{\tilde{E}}'(\underline{r}', t') &= \gamma \left[\underline{\underline{Y}}^{-1} \bullet \underline{\tilde{E}}(\underline{r}, t) + \underline{v} \times \underline{\tilde{B}}(\underline{r}, t) \right] \\ \underline{\tilde{B}}'(\underline{r}', t') &= \gamma \left[\underline{\underline{Y}}^{-1} \bullet \underline{\tilde{B}}(\underline{r}, t) - \frac{1}{c_0^2} \underline{v} \times \underline{\tilde{E}}(\underline{r}, t) \right] \\ \underline{\tilde{D}}'(\underline{r}', t') &= \gamma \left[\underline{\underline{Y}}^{-1} \bullet \underline{\tilde{D}}(\underline{r}, t) + \frac{1}{c_0^2} \underline{v} \times \underline{\tilde{H}}(\underline{r}, t) \right] \\ \underline{\tilde{H}}'(\underline{r}', t') &= \gamma \left[\underline{\underline{Y}}^{-1} \bullet \underline{\tilde{H}}(\underline{r}, t) - \underline{v} \times \underline{\tilde{D}}(\underline{r}, t) \right] \end{aligned}\right\} . \tag{1.58}$$

The Maxwell postulates are *Lorentz covariant*, which means that they retain their form under the spatiotemporal transformation (1.56). The Lorentz covariance of the Maxwell postulates has far–reaching implications for the constitutive relations that develop in uniformly moving reference frames, as described in Sec. 2.3.1.

1.6.4 *Chiral invariance*

In addition to being form–invariant under spatial, temporal and spatiotemporal transformations described in Secs. 1.6.1–1.6.3, the Maxwell postulates do not change their form under the following transformation of fields [3]

$$\left.\begin{aligned} \mathcal{R}_\psi \{ \underline{\tilde{E}}(\underline{r}, t) \} &= \underline{\tilde{E}}(\underline{r}, t) \cos\psi - Z \underline{\tilde{H}}(\underline{r}, t) \sin\psi \\ \mathcal{R}_\psi \{ \underline{\tilde{H}}(\underline{r}, t) \} &= Z^{-1} \underline{\tilde{E}}(\underline{r}, t) \sin\psi + \underline{\tilde{H}}(\underline{r}, t) \cos\psi \\ \mathcal{R}_\psi \{ \underline{\tilde{B}}(\underline{r}, t) \} &= \underline{\tilde{B}}(\underline{r}, t) \cos\psi + Z \underline{\tilde{D}}(\underline{r}, t) \sin\psi \\ \mathcal{R}_\psi \{ \underline{\tilde{D}}(\underline{r}, t) \} &= -Z^{-1} \underline{\tilde{B}}(\underline{r}, t) \sin\psi + \underline{\tilde{D}}(\underline{r}, t) \cos\psi \end{aligned}\right\} \tag{1.59}$$

and source densities

$$\left.\begin{aligned} \mathcal{R}_\psi \{ \tilde{\rho}_{\mathrm{e}}(\underline{r}, t) \} &= \tilde{\rho}_{\mathrm{e}}(\underline{r}, t) \cos\psi - Z^{-1} \tilde{\rho}_{\mathrm{m}}(\underline{r}, t) \sin\psi \\ \mathcal{R}_\psi \{ \tilde{\rho}_{\mathrm{m}}(\underline{r}, t) \} &= Z \tilde{\rho}_{\mathrm{e}}(\underline{r}, t) \sin\psi + \tilde{\rho}_{\mathrm{m}}(\underline{r}, t) \cos\psi \\ \mathcal{R}_\psi \{ \underline{\tilde{J}}_{\mathrm{e}}(\underline{r}, t) \} &= \underline{\tilde{J}}_{\mathrm{e}}(\underline{r}, t) \cos\psi - Z^{-1} \underline{\tilde{J}}_{\mathrm{m}}(\underline{r}, t) \sin\psi \\ \mathcal{R}_\psi \{ \underline{\tilde{J}}_{\mathrm{m}}(\underline{r}, t) \} &= Z \underline{\tilde{J}}_{\mathrm{e}}(\underline{r}, t) \sin\psi + \underline{\tilde{J}}_{\mathrm{m}}(\underline{r}, t) \cos\psi \end{aligned}\right\} . \tag{1.60}$$

[5]The classical electrodynamic approach is adopted here in which free space and vacuum are viewed as being equivalent. The nonclassical representation of vacuum is described in Sec. 7.4.

Herein the scalar Z is an impedance required to maintain dimensional integrity and ψ is a complex–valued angle. If $\psi \in \mathbb{R}$ (i.e., the set of real numbers), then the transformation operator $\mathcal{R}_\psi$ represents a rotation of the fields. For this reason, the Maxwell postulates are said to possess the property of *chiral invariance*.

The special case of $\psi = \pi/2$ is interesting: The electric and magnetic fields, and similarly the electric and magnetic charge densities, interchange under $\mathcal{R}_{\pi/2}$, which is often called the *duality transformation* [2]. By virtue of the duality of the electric charge and the magnetic charge, it is merely a matter of convention whether a particular particle is said to possess an electric charge or a magnetic charge.

Chiral invariance has an important bearing on the existence or nonexistence of magnetic monopoles. In fact, the question of the existence of magnetic monopoles is more fundamentally the question of whether all charged carriers possess the same proportion of electric charge and magnetic charge. If the answer to this question is in the affirmative, then — by applying a $\mathcal{R}_\psi$ transformation with the appropriate choice of ψ — either the magnetic monopole or the electric monopole could be said to not exist. Duality is best considered globally; i.e., for all mediums, at all times, and everywhere. Accordingly, the appropriate choice of ψ is made for physical certainty; however, that choice does not preclude the later application of duality in a local context for mathematical convenience.

The constitutive relations (1.24) retain their form under the transformation of fields (1.59) provided that the constitutive dyadics transform as

$$\left.\begin{aligned}
\mathcal{R}_\psi\left\{\underline{\underline{\tilde{\epsilon}}}_{\mathrm{EB}}(\underline{r},t)\right\} &= \cos^2\psi\,\underline{\underline{\tilde{\epsilon}}}_{\mathrm{EB}}(\underline{r},t) + Z^{-2}\sin^2\psi\,\underline{\underline{\tilde{\mu}}}_{\mathrm{EB}}(\underline{r},t)\\
&\quad -Z^{-1}\sin\psi\cos\psi\left(\underline{\underline{\tilde{\xi}}}_{\mathrm{EB}}(\underline{r},t) + \underline{\underline{\tilde{\zeta}}}_{\mathrm{EB}}(\underline{r},t)\right)\\
\mathcal{R}_\psi\left\{\underline{\underline{\tilde{\xi}}}_{\mathrm{EB}}(\underline{r},t)\right\} &= \sin\psi\cos\psi\left(Z\,\underline{\underline{\tilde{\epsilon}}}_{\mathrm{EB}}(\underline{r},t) - Z^{-1}\,\underline{\underline{\tilde{\mu}}}_{\mathrm{EB}}(\underline{r},t)\right)\\
&\quad +\cos^2\psi\,\underline{\underline{\tilde{\xi}}}_{\mathrm{EB}}(\underline{r},t) - \sin^2\psi\,\underline{\underline{\tilde{\zeta}}}_{\mathrm{EB}}(\underline{r},t)\\
\mathcal{R}_\psi\left\{\underline{\underline{\tilde{\zeta}}}_{\mathrm{EB}}(\underline{r},t)\right\} &= \sin\psi\cos\psi\left(Z\,\underline{\underline{\tilde{\epsilon}}}_{\mathrm{EB}}(\underline{r},t) - Z^{-1}\,\underline{\underline{\tilde{\mu}}}_{\mathrm{EB}}(\underline{r},t)\right)\\
&\quad +\cos^2\psi\,\underline{\underline{\tilde{\zeta}}}_{\mathrm{EB}}(\underline{r},t) - \sin^2\psi\,\underline{\underline{\tilde{\xi}}}_{\mathrm{EB}}(\underline{r},t)\\
\mathcal{R}_\psi\left\{\underline{\underline{\tilde{\mu}}}_{\mathrm{EB}}(\underline{r},t)\right\} &= Z^2\sin^2\psi\,\underline{\underline{\tilde{\epsilon}}}_{\mathrm{EB}}(\underline{r},t) + \cos^2\psi\,\underline{\underline{\tilde{\mu}}}_{\mathrm{EB}}(\underline{r},t)\\
&\quad +Z\sin\psi\cos\psi\left(\underline{\underline{\tilde{\xi}}}_{\mathrm{EB}}(\underline{r},t) + \underline{\underline{\tilde{\zeta}}}_{\mathrm{EB}}(\underline{r},t)\right)
\end{aligned}\right\}. \tag{1.61}$$

1.6.5 *Conjugate invariance*

The frequency–domain Maxwell curl postulates (1.30) are invariant under a further transformation, namely the conjugate transformation which is effected by the operator $\mathcal{C}$. The conjugate–transformed fields are specified as [14]

$$\left.\begin{aligned} \mathcal{C}\left\{\underline{E}(\underline{r},\omega)\right\} &= \underline{E}^*(\underline{r},\omega) \\ \mathcal{C}\left\{\underline{H}(\underline{r},\omega)\right\} &= \underline{H}^*(\underline{r},\omega) \\ \mathcal{C}\left\{\underline{D}(\underline{r},\omega)\right\} &= -\underline{D}^*(\underline{r},\omega) \\ \mathcal{C}\left\{\underline{B}(\underline{r},\omega)\right\} &= -\underline{B}^*(\underline{r},\omega) \end{aligned}\right\}, \tag{1.62}$$

while the source densities transform as

$$\left.\begin{aligned} \mathcal{C}\left\{\rho_{\mathrm{e}}(\underline{r},\omega)\right\} &= -\rho_{\mathrm{e}}^*(\underline{r},\omega) \\ \mathcal{C}\left\{\rho_{\mathrm{m}}(\underline{r},\omega)\right\} &= -\rho_{\mathrm{m}}^*(\underline{r},\omega) \\ \mathcal{C}\left\{\underline{J}_{\,\mathrm{e}}(\underline{r},\omega)\right\} &= \underline{J}_{\,\mathrm{e}}^*(\underline{r},\omega) \\ \mathcal{C}\left\{\underline{J}_{\,\mathrm{m}}(\underline{r},\omega)\right\} &= \underline{J}_{\,\mathrm{m}}^*(\underline{r},\omega) \end{aligned}\right\}. \tag{1.63}$$

When applied to linear materials, the Maxwell postulates remain invariant, provided that the 3×3 constitutive dyadics undergo the following transformations:

$$\left.\begin{aligned} \mathcal{C}\left\{\underline{\underline{\epsilon}}_{\,\mathrm{EB}}(\underline{r},\omega)\right\} &= -\underline{\underline{\epsilon}}_{\,\mathrm{EB}}^*(\underline{r},\omega) \\ \mathcal{C}\left\{\underline{\underline{\xi}}_{\,\mathrm{EB}}(\underline{r},\omega)\right\} &= \underline{\underline{\xi}}_{\,\mathrm{EB}}^*(\underline{r},\omega) \\ \mathcal{C}\left\{\underline{\underline{\zeta}}_{\,\mathrm{EB}}(\underline{r},\omega)\right\} &= \underline{\underline{\zeta}}_{\,\mathrm{EB}}^*(\underline{r},\omega) \\ \mathcal{C}\left\{\underline{\underline{\nu}}_{\,\mathrm{EB}}(\underline{r},\omega)\right\} &= -\underline{\underline{\nu}}_{\,\mathrm{EB}}^*(\underline{r},\omega) \end{aligned}\right\} \tag{1.64}$$

and

$$\left.\begin{aligned} \mathcal{C}\left\{\underline{\underline{\epsilon}}_{\,\mathrm{EH}}(\underline{r},\omega)\right\} &= -\underline{\underline{\epsilon}}_{\,\mathrm{EH}}^*(\underline{r},\omega) \\ \mathcal{C}\left\{\underline{\underline{\xi}}_{\,\mathrm{EH}}(\underline{r},\omega)\right\} &= -\underline{\underline{\xi}}_{\,\mathrm{EH}}^*(\underline{r},\omega) \\ \mathcal{C}\left\{\underline{\underline{\zeta}}_{\,\mathrm{EH}}(\underline{r},\omega)\right\} &= -\underline{\underline{\zeta}}_{\,\mathrm{EH}}^*(\underline{r},\omega) \\ \mathcal{C}\left\{\underline{\underline{\mu}}_{\,\mathrm{EH}}(\underline{r},\omega)\right\} &= -\underline{\underline{\mu}}_{\,\mathrm{EH}}^*(\underline{r},\omega) \end{aligned}\right\}. \tag{1.65}$$

The conjugation symmetry represented in Eqs. (1.62)–(1.65) arises as a generalization of the transformation which reverses the sign of the real–valued permittivity and permeability scalars for isotropic dielectric–magnetic mediums [14]. The effect of the conjugate transformation (1.64) would be observable in, for example, planewave propagation through a material slab, and may be fruitfully applied in determining the reflection and transmission characteristics of isotropic dielectric–magnetic materials whose permittivity and permeability scalars have negative real parts. Such materials are considered to be of technological promise because they may support negative refraction, as discussed in Secs. 2.1.2 and 4.8.1 [14].

1.6.6 *Energy and momentum*

We now turn to more practical matters by considering how spatial, temporal and field transformations influence measurable quantities. Let us introduce the energy flow density, as given by the instantaneous Poynting vector

$$\underline{\tilde{S}}(\underline{r},t) = \underline{\tilde{E}}(\underline{r},t) \times \underline{\tilde{H}}(\underline{r},t); \tag{1.66}$$

the total energy density

$$\tilde{W}(\underline{r},t) = \frac{1}{2}\left[\underline{\tilde{D}}(\underline{r},t) \bullet \underline{\tilde{E}}(\underline{r},t) + \underline{\tilde{B}}(\underline{r},t) \bullet \underline{\tilde{H}}(\underline{r},t)\right]; \tag{1.67}$$

and the Maxwell stress tensor

$$\begin{aligned}\underline{\underline{\tilde{T}}}(\underline{r},t) = &-\frac{1}{2}\left[\underline{\tilde{D}}(\underline{r},t) \bullet \underline{\tilde{E}}(\underline{r},t) + \underline{\tilde{B}}(\underline{r},t) \bullet \underline{\tilde{H}}(\underline{r},t)\right] \underline{\underline{I}} \\ &+\underline{\tilde{D}}(\underline{r},t)\underline{\tilde{E}}(\underline{r},t) + \underline{\tilde{B}}(\underline{r},t)\underline{\tilde{H}}(\underline{r},t).\end{aligned} \tag{1.68}$$

A straightforward application of the field transformations (1.49) and (1.54), respectively, reveals that

$$\left.\begin{aligned}\mathcal{T}\left\{\underline{\tilde{S}}(\underline{r},t)\right\} &= -\underline{\tilde{S}}(\underline{r},-t) \\ \mathcal{T}\left\{\tilde{W}(\underline{r},t)\right\} &= \tilde{W}(\underline{r},-t) \\ \mathcal{T}\left\{\underline{\underline{\tilde{T}}}(\underline{r},t)\right\} &= \underline{\underline{\tilde{T}}}(\underline{r},-t)\end{aligned}\right\} \tag{1.69}$$

and

$$\left.\begin{aligned}\mathcal{P}\left\{\underline{\tilde{S}}(\underline{r},t)\right\} &= -\underline{\tilde{S}}(-\underline{r},t) \\ \mathcal{P}\left\{\tilde{W}(\underline{r},t)\right\} &= \tilde{W}(-\underline{r},t) \\ \mathcal{P}\left\{\underline{\underline{\tilde{T}}}(\underline{r},t)\right\} &= \underline{\underline{\tilde{T}}}(-\underline{r},t)\end{aligned}\right\}. \tag{1.70}$$

The chiral invariance of the Maxwell postulates carries over to measurable quantities. It follows immediately from Eqs. (1.59) that

$$\left.\begin{aligned}\mathcal{R}_\psi\left\{\underline{\tilde{S}}(\underline{r},t)\right\} &= \underline{\tilde{S}}(\underline{r},t) \\ \mathcal{R}_\psi\left\{\tilde{W}(\underline{r},t)\right\} &= \tilde{W}(\underline{r},t) \\ \mathcal{R}_\psi\left\{\underline{\underline{\tilde{T}}}(\underline{r},t)\right\} &= \underline{\underline{\tilde{T}}}(\underline{r},t)\end{aligned}\right\}. \tag{1.71}$$

A notable consequence is that electromagnetic fields cannot be thus uniquely determined from measurements of electromagnetic energy and/or momentum.

1.7 Constitutive dyadics

Let us now examine more closely the constitutive dyadics which characterize the electromagnetic response of a medium. In the most general linear scenario, the 6×6 constitutive dyadic $\underline{\underline{\mathbf{K}}}_{\,\mathrm{EH}}(\underline{r},\omega)$ is assembled from 36 complex–valued scalar parameters. This vast parameter space may be reduced through the imposition of physical constraints which require the constitutive dyadics to satisfy certain symmetries. Also, our attention is often restricted to special cases and idealizations which manifest as symmetries of the constitutive dyadics.

Note that the constitutive dyadics of *homogeneous* mediums are not functions of $\underline{r}$.

1.7.1 *Constraints*

1.7.1.1 *Causality and Kramers–Kronig relations*

The formulations of constitutive relations for any realistic material must conform to the principle of causality; i.e., 'effect' must appear *after* the 'cause'. Hence, neither can a cause and its effect be simultaneous nor can an effect precede its cause. The principle of causality is most transparently implemented in the time domain for constitutive relations of the form given in Eqs. (1.22).

The induced fields $\underline{\tilde{D}}(\underline{r},t)$ and $\underline{\tilde{H}}(\underline{r},t)$ develop in response to the primitive fields $\underline{\tilde{E}}(\underline{r},t)$ and $\underline{\tilde{B}}(\underline{r},t)$, such that

$$\left.\begin{aligned} \underline{\tilde{D}}(\underline{r},t) &= \epsilon_0\underline{\tilde{E}}(\underline{r},t) + \underline{\tilde{P}}(\underline{r},t) \\ \underline{\tilde{H}}(\underline{r},t) &= \frac{1}{\mu_0}\underline{\tilde{B}}(\underline{r},t) - \underline{\tilde{M}}(\underline{r},t) \end{aligned}\right\}. \tag{1.72}$$

The polarization $\underline{\tilde{P}}(\underline{r},t)$ and the magnetization $\underline{\tilde{M}}(\underline{r},t)$ indicate the electromagnetic response of a medium, and must therefore be causally connected to the primitive fields.[6]

[6] For classical vacuum — which is not a material medium — the polarization and magnetization vectors are null–valued. In this case there is no distinction between primitive fields and induction fields. For a description of the nonclassical vacuum, see Sec. 7.4.

With regard to the time–domain linear constitutive relations (1.24), causality dictates that

$$\left.\begin{aligned}&\underline{\underline{\tilde{\epsilon}}}_{\mathrm{EB}}(\underline{r},t)-\epsilon_0\delta(\underline{r})\underline{\underline{I}}\equiv\underline{\underline{0}}\\&\underline{\underline{\tilde{\xi}}}_{\mathrm{EB}}(\underline{r},t)\equiv\underline{\underline{0}}\\&\underline{\underline{\tilde{\zeta}}}_{\mathrm{EB}}(\underline{r},t)\equiv\underline{\underline{0}}\\&\mu_0^{-1}\delta(\underline{r})\underline{\underline{I}}-\underline{\underline{\tilde{\nu}}}_{\mathrm{EB}}(\underline{r},t)\equiv\underline{\underline{0}}\end{aligned}\right\}\qquad\text{for}\quad t\leq 0. \tag{1.73}$$

When translated into the frequency domain, the causality requirement (1.73) gives rise to integral relations between the real and imaginary parts of the frequency–dependent constitutive parameters, as we now outline.

Suppose that the scalar function $\tilde{f}(\underline{r},t)$ represents an arbitrary component of a Boys–Post constitutive dyadic; i.e., $\tilde{f}(\underline{r},t)$ is a component of $\underline{\underline{\tilde{\epsilon}}}_{\mathrm{EB}}(\underline{r},t)-\epsilon_0\delta(\underline{r})\underline{\underline{I}}$, $\underline{\underline{\tilde{\xi}}}_{\mathrm{EB}}(\underline{r},t)$, $\underline{\underline{\tilde{\zeta}}}_{\mathrm{EB}}(\underline{r},t)$ or $\mu_0^{-1}\delta(\underline{r})\underline{\underline{I}}-\underline{\underline{\tilde{\nu}}}_{\mathrm{EB}}(\underline{r},t)$. The temporal Fourier transform of $\tilde{f}(\underline{r},t)$ may be expressed as

$$f(\underline{r},\omega)=\int_0^\infty \tilde{f}(\underline{r},t)\,\exp(i\omega t)\,dt\,, \tag{1.74}$$

wherein the causality constraint (1.73) has been applied to set the lower limit of integration equal to zero. The analytic continuation of $f(\underline{r},\omega)$ in the upper complex–ω plane is provided by the Cauchy integral formula

$$f(\underline{r},\omega)=\frac{1}{2\pi i}\oint\frac{f(\underline{r},s)}{s-\omega}\,ds\,, \tag{1.75}$$

where the integration contour extends around the upper half plane. The integrand in Eq. (1.75) vanishes as $|\,s\,|\to\infty$ for $\mathrm{Im}\,\{s\}>0$ due to the $\exp(i\omega t)$ factor occurring in the integral representation (1.74). Hence, the contour integral specified in Eq. (1.75) reduces to an integral along the real axis. Counting the single pole on the real axis at $\omega=s$ as a half residue, we have

$$f(\underline{r},\omega)=\frac{1}{\pi i}\mathsf{P}\int_{-\infty}^{\infty}\frac{f(\underline{r},s)}{s-\omega}\,ds\,, \tag{1.76}$$

where P indicates the Cauchy principal value. Hence, we have the Hilbert transforms

$$\left.\begin{aligned}\mathrm{Re}\,\{f(\underline{r},\omega)\}&=\frac{1}{\pi}\mathsf{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}\,\{f(\underline{r},s)\}}{s-\omega}\,ds\\\mathrm{Im}\,\{f(\underline{r},\omega)\}&=-\frac{1}{\pi}\mathsf{P}\int_{-\infty}^{\infty}\frac{\mathrm{Re}\,\{f(\underline{r},s)\}}{s-\omega}\,ds\end{aligned}\right\}. \tag{1.77}$$

In addition, since $\tilde{f}(\underline{r}, t)$ is real–valued, the symmetry condition (cf. Eq. (1.34))

$$f(\underline{r}, -\omega) = f^*(\underline{r}, \omega) \tag{1.78}$$

relates f to its complex conjugate f^*. Thus, Eqs. (1.77) yield the *Kramers–Kronig* relations [13]

$$\left.\begin{aligned} \mathrm{Re}\,\{f(\underline{r}, \omega)\} &= \frac{2}{\pi}\mathsf{P}\int_0^\infty \frac{s\,\mathrm{Im}\,\{f(\underline{r}, s)\}}{s^2-\omega^2}\,ds \\ \mathrm{Im}\,\{f(\underline{r}, \omega)\} &= -\frac{2}{\pi}\mathsf{P}\int_0^\infty \frac{\omega\,\mathrm{Re}\,\{f(\underline{r}, s)\}}{s^2-\omega^2}\,ds \end{aligned}\right\}. \tag{1.79}$$

Although the relations (1.79) are presented here for components of the Boys–Post constitutive dyadics, analogous relations hold for components of the Tellegen constitutive dyadics by virtue of Eqs. (1.29).

An alternative approach to the derivation of the Kramers–Kronig relations, exploiting the properties of Herglotz functions, has recently been reported [15].

The Kramers–Kronig relations represent a particular example of *dispersion relations*[7] that apply generally to frequency–dependent, causal, linear systems [16]. Often these are usefully employed in experimental determinations of constitutive parameters [17].

1.7.1.2 *Post constraint*

A structural constraint — called the Post constraint — is available for those linear mediums that exhibit magnetoelectric coupling [13]. This constraint may be expressed in terms of Boys–Post constitutive dyadics as

$$\mathrm{tr}\left[\underline{\underline{\zeta}}_{\mathrm{EB}}(\underline{r}, \omega) - \underline{\underline{\xi}}_{\mathrm{EB}}(\underline{r}, \omega)\right] = 0, \tag{1.80}$$

or, equivalently, in terms of Tellegen constitutive dyadics as

$$\mathrm{tr}\left\{\underline{\underline{\mu}}_{\mathrm{EH}}^{-1}(\underline{r}, \omega) \bullet \left[\underline{\underline{\zeta}}_{\mathrm{EH}}(\underline{r}, \omega) + \underline{\underline{\xi}}_{\mathrm{EH}}(\underline{r}, \omega)\right]\right\} = 0. \tag{1.81}$$

Hence, under the Post constraint, only 35 independent complex–valued parameters are needed to characterize the most general linear medium.

The origins of the Post constraint lie in the microscopic nature of the primitive electromagnetic fields and the Lorentz covariance of the Maxwell equations [18]. While Post established his eponymous constraint more than

[7] These causal dispersion relations should be distinguished from the planewave dispersion relations described in Sec. 4.2.

40 years ago [13], two more recent, independent, proofs — one based on a uniqueness requirement [19, 20] and another based on multipole considerations [21] — further secured the standing of the Post constraint. On the other hand, recent experimental evidence that the Post constraint is violated at low frequencies has been reported [22], but there is no microscopic understanding as yet of this evidence [23]. The incorporation of the hitherto–undiscovered axion will lead to a re–evaluation of the Post constraint even for free space [24, 25].

1.7.1.3 *Onsager relations*

The Onsager relations are a set of reciprocity relations that are applicable generally to coupled linear phenomenons at macroscopic length–scales [26–28]. While the Onsager relations were originally established for instantaneous phenomenons, their scope may be extended by means of the fluctuation–dissipation theorem [29] to include time–harmonic phenomenons too [30].

The assumption of microscopic reversibility is central to the Onsager relations. As a consequence, in order to apply the Onsager relations to electromagnetic constitutive relations, the contribution of free space must be excluded because microscopic processes cannot occur in free space. The frequency–dependent vector quantities $\underline{P}(\underline{r},\omega)$ and $\underline{M}(\underline{r},\omega)$, which are the temporal Fourier transforms of the polarization $\underline{\tilde{P}}(\underline{r},t)$ and the magnetization $\underline{\tilde{M}}(\underline{r},t)$, represent the electromagnetic response of a medium relative to the electromagnetic response of free space. For linear homogeneous mediums, the bianisotropic constitutive relations (1.26) reduce to

$$\left.\begin{aligned} \underline{P}(\underline{r},\omega) &= \left[\underline{\underline{\epsilon}}_{\,\mathrm{EB}}(\omega) - \epsilon_0 \underline{\underline{I}}\right] \bullet \underline{E}(\underline{r},\omega) + \underline{\underline{\xi}}_{\,\mathrm{EB}}(\omega) \bullet \underline{B}(\underline{r},\omega) \\ \underline{M}(\underline{r},\omega) &= -\underline{\underline{\zeta}}_{\,\mathrm{EB}}(\omega) \bullet \underline{E}(\underline{r},\omega) + \left[\frac{1}{\mu_0}\underline{\underline{I}} - \underline{\underline{\nu}}_{\,\mathrm{EB}}(\omega)\right] \bullet \underline{B}(\underline{r},\omega) \end{aligned}\right\}. \tag{1.82}$$

When a linear homogeneous medium is subjected to an external, spatially uniform, magnetostatic field $\underline{B}_{\,\mathrm{dc}}$, application of the Onsager relations to the Boys–Post constitutive relations (1.82) delivers the constraints [31]

$$\left.\begin{aligned} \underline{\underline{\epsilon}}_{\,\mathrm{EB}}(\omega)\Big|_{\underline{B}_{\,\mathrm{dc}}} &= \underline{\underline{\epsilon}}^{\mathrm{T}}_{\,\mathrm{EB}}(\omega)\Big|_{-\underline{B}_{\,\mathrm{dc}}} \\ \underline{\underline{\xi}}_{\,\mathrm{EB}}(\omega)\Big|_{\underline{B}_{\,\mathrm{dc}}} &= \underline{\underline{\zeta}}^{\mathrm{T}}_{\,\mathrm{EB}}(\omega)\Big|_{-\underline{B}_{\,\mathrm{dc}}} \\ \underline{\underline{\nu}}_{\,\mathrm{EB}}(\omega)\Big|_{\underline{B}_{\,\mathrm{dc}}} &= \underline{\underline{\nu}}^{\mathrm{T}}_{\,\mathrm{EB}}(\omega)\Big|_{-\underline{B}_{\,\mathrm{dc}}} \end{aligned}\right\}, \tag{1.83}$$

where the superscript 'T' denotes the transpose operation. The equivalent contraints for the Tellegen constitutive dyadics follow straight from Eqs. (1.28) as

$$\left.\begin{aligned} \underline{\underline{\epsilon}}_{\rm EH}(\omega)\Big|_{\underline{B}_{\rm dc}} &= \underline{\underline{\epsilon}}^{\rm T}_{\rm EH}(\omega)\Big|_{-\underline{B}_{\rm dc}} \\ \underline{\underline{\xi}}_{\rm EH}(\omega)\Big|_{\underline{B}_{\rm dc}} &= -\underline{\underline{\zeta}}^{\rm T}_{\rm EH}(\omega)\Big|_{-\underline{B}_{\rm dc}} \\ \underline{\underline{\mu}}_{\rm EH}(\omega)\Big|_{\underline{B}_{\rm dc}} &= \underline{\underline{\mu}}^{\rm T}_{\rm EH}(\omega)\Big|_{-\underline{B}_{\rm dc}} \end{aligned}\right\}. \tag{1.84}$$

1.7.2 *Specializations*

1.7.2.1 *Lorentz reciprocity*

Lorentz reciprocity – which is closely related to the topics of time reversal and the Onsager relations — is a topic that frequently crops up in theoretical analyses involving complex mediums [32]. It is often presented in terms of the interchangeability of transmitters and receivers [4, 33].

Let us consider two frequency–domain electric source current densities, namely $\underline{J}^{\rm p}_{\rm e}(\underline{r},\omega)$ and $\underline{J}^{\rm q}_{\rm e}(\underline{r},\omega)$, and two frequency–domain magnetic source current densities, namely $\underline{J}^{\rm p}_{\rm m}(\underline{r},\omega)$ and $\underline{J}^{\rm q}_{\rm m}(\underline{r},\omega)$. The sources labelled 'p' generate fields denoted as $\underline{E}^{\rm p}(\underline{r},\omega)$ and $\underline{H}^{\rm p}(\underline{r},\omega)$, whereas the sources labelled 'q' generate fields denoted as $\underline{E}^{\rm q}(\underline{r},\omega)$ and $\underline{H}^{\rm q}(\underline{r},\omega)$. The interaction of the 'p' sources with the fields generated by the 'q' sources is gauged by the *reaction* [4, 33]

$$\langle\langle {\rm p},\, {\rm q}\rangle\rangle = \int_{V_{\rm p}} \left[\, \underline{J}^{\rm p}_{\rm e}(\underline{r},\omega) \bullet \underline{E}^{\rm q}(\underline{r},\omega) - \underline{J}^{\rm p}_{\rm m}(\underline{r},\omega) \bullet \underline{H}^{\rm q}(\underline{r},\omega)\,\right]\, d^3\underline{r}\,, \tag{1.85}$$

where the integration region $V_{\rm p}$ contains the 'p' sources. Similarly, the interaction of the 'q' sources with field generated by the 'p' sources is represented by the reaction $\langle\langle {\rm q},\, {\rm p}\rangle\rangle$. If the medium which supports $\underline{J}^{\rm p,q}_{\rm e,m}(\underline{r},\omega)$, $\underline{E}^{\rm p,q}(\underline{r},\omega)$ and $\underline{H}^{\rm p,q}(\underline{r},\omega)$ is such that

$$\langle\langle {\rm p},\, {\rm q}\rangle\rangle = \langle\langle {\rm q},\, {\rm p}\rangle\rangle, \tag{1.86}$$

then it is called Lorentz–reciprocal.

Combining the Tellegen constitutive relations (1.27) with the Maxwell curl postulates (1.30) and integrating thereafter, we obtain the reaction difference

$$\begin{aligned}\langle\langle \mathrm{p},\, \mathrm{q}\rangle\rangle - \langle\langle \mathrm{q},\, \mathrm{p}\rangle\rangle = &\\ -i\omega \int_{V_\mathrm{p}\cup V_\mathrm{q}} \Big\{ &\underline{E}^{\mathrm{q}}(\underline{r},\omega) \bullet \left[\underline{\underline{\epsilon}}_{\,\mathrm{EH}}(\underline{r},\omega) - \underline{\underline{\epsilon}}^{\mathrm{T}}_{\,\mathrm{EH}}(\underline{r},\omega)\right] \bullet \underline{E}^{\mathrm{p}}(\underline{r},\omega)\\ &+\underline{H}^{\mathrm{p}}(\underline{r},\omega) \bullet \left[\underline{\underline{\mu}}_{\,\mathrm{EH}}(\underline{r},\omega) - \underline{\underline{\mu}}^{\mathrm{T}}_{\,\mathrm{EH}}(\underline{r},\omega)\right] \bullet \underline{H}^{\mathrm{q}}(\underline{r},\omega)\\ &+\underline{E}^{\mathrm{q}}(\underline{r},\omega) \bullet \left[\underline{\underline{\xi}}_{\,\mathrm{EH}}(\underline{r},\omega) + \underline{\underline{\zeta}}^{\mathrm{T}}_{\,\mathrm{EH}}(\underline{r},\omega)\right] \bullet \underline{H}^{\mathrm{p}}(\underline{r},\omega)\\ &+\underline{H}^{\mathrm{p}}(\underline{r},\omega) \bullet \left[\underline{\underline{\zeta}}_{\,\mathrm{EH}}(\underline{r},\omega) + \underline{\underline{\xi}}^{\mathrm{T}}_{\,\mathrm{EH}}(\underline{r},\omega)\right] \bullet \underline{E}^{\mathrm{q}}(\underline{r},\omega)\Big\}\, d^3\underline{r},\end{aligned} \tag{1.87}$$

where the integration region $V_\mathrm{p} \cup V_\mathrm{q}$ contains both the sources ‘p’ and sources ‘q’. Thus, Lorentz reciprocity is signalled by [34]

$$\left.\begin{aligned}\underline{\underline{\epsilon}}_{\,\mathrm{EH}}(\underline{r},\omega) &= \underline{\underline{\epsilon}}^{\mathrm{T}}_{\,\mathrm{EH}}(\underline{r},\omega)\\ \underline{\underline{\xi}}_{\,\mathrm{EH}}(\underline{r},\omega) &= -\underline{\underline{\zeta}}^{\mathrm{T}}_{\,\mathrm{EH}}(\underline{r},\omega)\\ \underline{\underline{\mu}}_{\,\mathrm{EH}}(\underline{r},\omega) &= \underline{\underline{\mu}}^{\mathrm{T}}_{\,\mathrm{EH}}(\underline{r},\omega)\end{aligned}\right\}. \tag{1.88}$$

The corresponding symmetries for the Boys–Post representation follow immediately from Eqs. (1.29) as

$$\left.\begin{aligned}\underline{\underline{\epsilon}}_{\,\mathrm{EB}}(\underline{r},\omega) &= \underline{\underline{\epsilon}}^{\mathrm{T}}_{\,\mathrm{EB}}(\underline{r},\omega)\\ \underline{\underline{\xi}}_{\,\mathrm{EB}}(\underline{r},\omega) &= \underline{\underline{\zeta}}^{\mathrm{T}}_{\,\mathrm{EB}}(\underline{r},\omega)\\ \underline{\underline{\nu}}_{\,\mathrm{EB}}(\underline{r},\omega) &= \underline{\underline{\nu}}^{\mathrm{T}}_{\,\mathrm{EB}}(\underline{r},\omega)\end{aligned}\right\}. \tag{1.89}$$

Notice that the Lorentz–reciprocity conditions (1.88) and (1.89) coincide with the Onsager relations (1.84) and (1.83), respectively, in the absence of a magnetostatic field.

Many frequently encountered anisotropic and bianisotropic mediums satisfy the Lorentz reciprocity conditions. Lorentz–reciprocal mediums arise commonly as dielectric and magnetic crystals, whereas plasmas and mediums moving at uniform velocity are Lorentz–nonreciprocal mediums. These topics are discussed further in Chap. 2 in the context of specific types of anisotropic and bianisotropic mediums.

1.7.2.2 *Dissipative, nondissipative and active mediums*

No passive medium — with the unique exception of free space (which is not a material) — responds instantaneously to an applied electromagnetic field, which characteristic is enshrined as the principle of causality [8, 35]. Dissipation is therefore exhibited by all passive material mediums. However, occasionally it can be expedient to neglect dissipation, especially if attention is confined to a narrow range of angular frequencies wherein dissipation is very small over the length–scales of interest.

In order to concentrate on dissipation, we introduce the time–averaged Poynting vector for monochromatic fields

$$\langle \check{\underline{S}}(\underline{r},\omega)\rangle_{\mathrm{t}} = \frac{1}{2}\mathrm{Re}\left\{ \check{\underline{E}}(\underline{r},\omega) \times \check{\underline{H}}^*(\underline{r},\omega) \right\}, \tag{1.90}$$

which may be interpreted as the time–averaged power per unit area. The complex–valued fields amplitudes $\check{\underline{E}}(\underline{r},\omega)$ and $\check{\underline{H}}(\underline{r},\omega)$ were introduced in Eq. (1.36). Of particular relevance to us here is the divergence of Eq. (1.90), representing the time–averaged power density. Using the Maxwell curl postulates (1.30) in the absence of sources (i.e., $\underline{J}_{\mathrm{e,m}}(\underline{r},\omega) \equiv \underline{0}$), together with the Tellegen constitutive relations for a bianisotropic medium (1.27), we find that the divergence of Eq. (1.90) yields [4]

$$\langle \nabla \bullet \check{\underline{S}}(\underline{r},\omega)\rangle_{\mathrm{t}} = \frac{\omega}{4}\left[\check{\underline{E}}(\underline{r},\omega)\ \check{\underline{H}}(\underline{r},\omega) \right]^* \bullet \underline{\underline{\mathbf{m}}}(\underline{r},\omega) \bullet \begin{bmatrix} \check{\underline{E}}(\underline{r},\omega) \\ \check{\underline{H}}(\underline{r},\omega) \end{bmatrix}, \tag{1.91}$$

wherein the 6×6 Hermitian dyadic

$$\underline{\underline{\mathbf{m}}}(\underline{r},\omega) = i \begin{bmatrix} \underline{\underline{\epsilon}}_{\mathrm{EH}}(\underline{r},\omega) - \underline{\underline{\epsilon}}^{\dagger}_{\mathrm{EH}}(\underline{r},\omega) & \underline{\underline{\xi}}_{\mathrm{EH}}(\underline{r},\omega) - \underline{\underline{\zeta}}^{\dagger}_{\mathrm{EH}}(\underline{r},\omega) \\ \underline{\underline{\zeta}}_{\mathrm{EH}}(\underline{r},\omega) - \underline{\underline{\xi}}^{\dagger}_{\mathrm{EH}}(\underline{r},\omega) & \underline{\underline{\mu}}_{\mathrm{EH}}(\underline{r},\omega) - \underline{\underline{\mu}}^{\dagger}_{\mathrm{EH}}(\underline{r},\omega) \end{bmatrix}, \tag{1.92}$$

with the superscript † indicating the conjugate transpose.

A medium is nondissipative provided that $\langle \nabla \bullet \check{\underline{S}}(\underline{r},\omega)\rangle_{\mathrm{t}} = 0$. Thus, dissipation is neglected by enforcing the equalities [8]

$$\left.\begin{array}{l} \underline{\underline{\epsilon}}_{\mathrm{EH}}(\underline{r},\omega) = \underline{\underline{\epsilon}}^{\dagger}_{\mathrm{EH}}(\underline{r},\omega) \\ \underline{\underline{\xi}}_{\mathrm{EH}}(\underline{r},\omega) = \underline{\underline{\zeta}}^{\dagger}_{\mathrm{EH}}(\underline{r},\omega) \\ \underline{\underline{\mu}}_{\mathrm{EH}}(\underline{r},\omega) = \underline{\underline{\mu}}^{\dagger}_{\mathrm{EH}}(\underline{r},\omega) \end{array}\right\}, \tag{1.93}$$

or, equivalently,

$$\left.\begin{array}{l} \underline{\underline{\epsilon}}_{\mathrm{EB}}(\underline{r},\omega) = \underline{\underline{\epsilon}}^{\dagger}_{\mathrm{EB}}(\underline{r},\omega) \\ \underline{\underline{\xi}}_{\mathrm{EB}}(\underline{r},\omega) = -\underline{\underline{\zeta}}^{\dagger}_{\mathrm{EB}}(\underline{r},\omega) \\ \underline{\underline{\nu}}_{\mathrm{EB}}(\underline{r},\omega) = \underline{\underline{\nu}}^{\dagger}_{\mathrm{EB}}(\underline{r},\omega) \end{array}\right\}. \tag{1.94}$$

The distinction between the conditions for the neglect of dissipation and for Lorentz reciprocity should be noted. These conditions are summarized

Table 1.1 Conditions imposed by Lorentz reciprocity and neglect of dissipation.

$\underline{\underline{\mathbf{K}}}_{\text{EH}}$	Lorentz reciprocity	Neglect of dissipation
$\begin{bmatrix} \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix} & \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi \end{pmatrix} \\ \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta \end{pmatrix} & \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \end{bmatrix}$	$\xi = -\zeta$	$\epsilon = \epsilon^*$ $\xi = \zeta^*$ $\mu = \mu^*$
$\begin{bmatrix} \begin{pmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{pmatrix} & \begin{pmatrix} \xi_{11} & 0 & 0 \\ 0 & \xi_{22} & 0 \\ 0 & 0 & \xi_{33} \end{pmatrix} \\ \begin{pmatrix} \zeta_{11} & 0 & 0 \\ 0 & \zeta_{22} & 0 \\ 0 & 0 & \zeta_{33} \end{pmatrix} & \begin{pmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{pmatrix} \end{bmatrix}$	$\xi_{\ell\ell} = -\zeta_{\ell\ell}$	$\epsilon_{\ell\ell} = \epsilon^*_{\ell\ell}$ $\xi_{\ell\ell} = \zeta^*_{\ell\ell}$ $\mu_{\ell\ell} = \mu^*_{\ell\ell}$
$\begin{bmatrix} \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix} & \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{pmatrix} \\ \begin{pmatrix} \zeta_{11} & \zeta_{12} & \zeta_{13} \\ \zeta_{21} & \zeta_{22} & \zeta_{23} \\ \zeta_{31} & \zeta_{32} & \zeta_{33} \end{pmatrix} & \begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix} \end{bmatrix}$	$\epsilon_{\ell m} = \epsilon_{m\ell}$ $\xi_{\ell m} = -\zeta_{m\ell}$ $\mu_{\ell m} = \mu_{m\ell}$	$\epsilon_{\ell m} = \epsilon^*_{m\ell}$ $\xi_{\ell m} = \zeta^*_{m\ell}$ $\mu_{\ell m} = \mu^*_{m\ell}$

Three different forms of the 6×6 Tellegen constitutive dyadic $\underline{\underline{\mathbf{K}}}_{\text{EH}}$ for a passive medium are represented. Notice that for the medium represented in the first example, the Lorentz–reciprocity condition $\xi = -\zeta$ must be satisfied in order to comply with the Post constraint.

in Table 1.1 for three often–encountered forms of the constitutive dyadic $\underline{\underline{\mathbf{K}}}_{\text{EH}}(\underline{r}, \omega)$.

A medium is dissipative provided that $\langle \nabla \bullet \check{\underline{\check{S}}}(\underline{r}, \omega) \rangle_{\text{t}} < 0$. By inspection of Eq. (1.91), the dyadic $\underline{\underline{\mathbf{m}}}(\underline{r}, \omega)$ must be negative definite in order to satisfy this dissipative condition [36]. Accordingly, for a medium to be dissipative, it is necessary and sufficient that the two 3×3 dyadics [37]

$$\left.\begin{aligned}
\underline{\underline{m}}_1(\underline{r},\omega) &= i\left[\underline{\underline{\epsilon}}_{\rm EH}(\underline{r},\omega) - \underline{\underline{\epsilon}}^\dagger_{\rm EH}(\underline{r},\omega)\right] \\
\underline{\underline{m}}_2(\underline{r},\omega) &= i\Big\{\underline{\underline{\mu}}_{\rm EH}(\underline{r},\omega) - \underline{\underline{\mu}}^\dagger_{\rm EH}(\underline{r},\omega) \\
&\quad - \left[\underline{\underline{\zeta}}_{\rm EH}(\underline{r},\omega) - \underline{\underline{\xi}}^\dagger_{\rm EH}(\underline{r},\omega)\right] \bullet \underline{\underline{m}}_1^{-1}(\underline{r},\omega) \bullet \left[\underline{\underline{\xi}}_{\rm EH}(\underline{r},\omega) - \underline{\underline{\zeta}}^\dagger_{\rm EH}(\underline{r},\omega)\right]\Big\}
\end{aligned}\right\} \tag{1.95}$$

are negative definite. Equivalently, if the two 3×3 dyadics [37]

$$\left.\begin{aligned}
\underline{\underline{m}}_3(\underline{r},\omega) &= i\left[\underline{\underline{\mu}}_{\rm EH}(\underline{r},\omega) - \underline{\underline{\mu}}^\dagger_{\rm EH}(\underline{r},\omega)\right] \\
\underline{\underline{m}}_4(\underline{r},\omega) &= i\Big\{\underline{\underline{\epsilon}}_{\rm EH}(\underline{r},\omega) - \underline{\underline{\epsilon}}^\dagger_{\rm EH}(\underline{r},\omega) \\
&\quad - \left[\underline{\underline{\xi}}_{\rm EH}(\underline{r},\omega) - \underline{\underline{\zeta}}^\dagger_{\rm EH}(\underline{r},\omega)\right] \bullet \underline{\underline{m}}_3^{-1}(\underline{r},\omega) \bullet \left[\underline{\underline{\zeta}}_{\rm EH}(\underline{r},\omega) - \underline{\underline{\xi}}^\dagger_{\rm EH}(\underline{r},\omega)\right]\Big\}
\end{aligned}\right\} \tag{1.96}$$

are negative definite, then the necessary and sufficient conditions for dissipation are also satisfied.

A medium which is characterized by $\langle \nabla \bullet \underline{S}(\underline{r},\omega)\rangle_{\rm t} > 0$ is an active medium.

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