RESENT PROGRESS ON THE QUANTITATIVE ARITHMETIC OF DEL PEZZO SURFACES

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We survey the state of affairs for the distribution of $\mathbb{Q}$-rational points on non-singular del Pezzo surfaces of low degree, highlighting the recent resolution of Manin's conjecture for a non-singular del Pezzo surface of degree 4 by la Bretèche and Browning [3].

1. Introduction

Let $V \subset \mathbb{P}^{n-1}$ be a non-singular projective variety defined over $\mathbb{Q}$. A fundamental theme in mathematics is to understand when the set of rational points $V(\mathbb{Q}) = V \cap \mathbb{P}^{n-1}(\mathbb{Q})$ is non-empty, and if it is non-empty, to provide a reasonable description of it. Our aim here is to survey recent joint work of the author with la Bretèche [3], which addresses the second question from a quantitative standpoint for a certain variety of dimension 2.

Before turning to the specifics of this work it is perhaps worth entertaining ourselves with some preliminary observations. In the classification of algebraic varieties the Fano varieties, namely those varieties V whose anticanonical divisor $-K_V$ is ample, are those on which we expect to stand the highest chance of finding rational points. The situation for curves is relatively well-understood and so we will consider here the case of Fano surfaces, beginning with some simple-minded numerics. Imagine that we are given a Fano variety V of dimension 2 and degree d, which is a non-singular complete intersection in $\mathbb{P}^{n-1}$. Quadrics are easy to analyse and so we assume that $d > 2$. Suppose that $V = W_1 \cap \cdots \cap W_t$ for hypersurfaces $W_i \subset \mathbb{P}^{n-1}$ of degree d_i and assume, furthermore, that the intersection is generically transversal. We are not interested in hyperplane sections of V, and so we will assume without loss of generality that $d_i \geq 2$. Then the following inequalities must be satisfied:

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(1) $d_1 + \cdots + d_t < n$, [Fano]
(2) $n - 1 - t = 2$, [complete intersection of dimension 2]
(3) $d = d_1 \cdots d_t \geq 3$, [Bézout]
(4) $d_t \geq \cdots \geq d_1 \geq 2$.

It follows that the only possibilities are

$$(d; d_1, \dots, d_t; n; t) \in \big\{(3; 3; 4; 1), (4; 2, 2; 5; 2)\big\}.$$

These surfaces correspond to a cubic surface in $\mathbb{P}^3$ and an intersection of 2 quadrics in $\mathbb{P}^4$, respectively. In fact these are the most famous examples of "del Pezzo surfaces". This article will focus its attention on the latter type of surfaces, which comprise the non-singular del Pezzo surfaces of degree 4.

Henceforth, we let $V \subset \mathbb{P}^4$ be such a del Pezzo surface, cut out by the intersection of two quadrics

$$\begin{cases} \Phi_1(x_0, \dots, x_4) = 0, \\ \Phi_2(x_0, \dots, x_4) = 0, \end{cases}$$

with Φ_1, Φ_2 quadratic forms defined over $\mathbb{Z}$. We will refer to such a surface as a "dP_4" for ease of notation. The geometry of dP_4s is classical and is discussed in Hartshorne [11, Section V.4], for example. The question of finding conditions under which $V(\mathbb{Q})$ is non-empty is a difficult problem in number theory that will not concern us here. We refer the interested reader to the thesis of Wittenberg [25].

Assuming that $V(\mathbb{Q}) \neq \emptyset$, it follows from work of Segre [22] that $V(\mathbb{Q})$ is dense in V under the Zariski topology. Thus it is natural to give some measure of the density of rational points on V. It is at this point that the Manin conjecture [8] enters the picture. This predicts an asymptotic formula for the growth rate of the counting function

$$N_{U,H}(B) = \#\{x \in U(\mathbb{Q}) : H(x) \leq B\},$$

as $B \to \infty$, where $U \subset V$ is the Zariski open subset formed by deleting the 16 lines from V and $H : \mathbb{P}^4(\mathbb{Q}) \to \mathbb{R}_{\geq 0}$ is defined via $H(x) = \|\mathbf{x}\|$, if $x = [\mathbf{x}]$ with $\mathbf{x} = (x_0, \dots, x_4) \in \mathbb{Z}^5$ such that $\gcd(x_0, \dots, x_4) = 1$. Here $\| \cdot \|$ is an arbitrary norm on $\mathbb{R}^5$. Let $\mathrm{Pic}_{\overline{\mathbb{Q}}}(V)$ be the geometric Picard group of V, which is just the group of Weil divisors $\mathrm{Div}(V)$ modulo linear equivalence. Suppose that K is a splitting field for the 16 lines on V. Assuming that $V(\mathbb{Q})$ is non-empty, the Manin conjecture predicts the existence of a positive constant $c_{V,H}$ such that

$$N_{U,H}(B) = c_{V,H} B (\log B)^{\mathrm{rank}\mathrm{Pic}(V) - 1}\big(1 + o(1)\big), \tag{1.1}$$

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as $B \to \infty$, where $\mathrm{Pic}(V) = \mathrm{Pic}_{\overline{\mathbb{Q}}}(V)^{\mathrm{Gal}(K/\mathbb{Q})}$. Furthermore, there is a conjecture due to Peyre [20] concerning the value of the constant $c_{V,H}$.

It should be stressed that the Manin conjecture has received a great deal of attention in the context of singular del Pezzo surfaces, an account of which can be found in the author's companion survey [4]. The situation for non-singular dP_4s is rather less satisfactory. For an arbitrary such surface the best result we have is the upper bound

$$N_{U,H}(B) = O_\varepsilon(B^{\frac{3}{2}+\varepsilon}),$$

for any $\varepsilon > 0$. This is due to Salberger, but has yet to appear in print. It is based on a far-reaching refinement of Heath-Brown's "determinant method" developed in [14]. The implied constant in Salberger's estimate is allowed to depend on ε but is uniform in the coefficients of Φ_1 and Φ_2. It seems likely that by adapting further work of Heath-Brown [13] one could hope to replace the exponent $\frac{3}{2}$ by $\frac{5}{4}$, under a certain unproven hypothesis concerning the growth rate of the rank of elliptic curves over $\mathbb{Q}$.

One can do better when the dP_4 is assumed to have a "conic bundle structure". This simply boils down to Φ_1 taking the shape

$$\Phi_1(x_0, \dots, x_4) = x_0x_1 - x_2x_3, \tag{1.2}$$

which we now suppose to be the case. In work communicated at the conference *Higher dimensional varieties and rational points* at Budapest in 2001, Salberger has established the upper bound

$$N_{U,H}(B) = O_{\varepsilon,V}(B^{1+\varepsilon}) \tag{1.3}$$

for any $\varepsilon > 0$, where now the implied constant is allowed to depend on ε and V. Leung [17] has subsequently shown how to replace B^ε in Salberger's work by a small power of $\log B$. His work implies that

$$N_{U,H}(B) = O_{\varepsilon,V}(B(\log B)^5). \tag{1.4}$$

This is best possible whenever the 16 lines are all defined over $\mathbb{Q}$, since the geometric Picard group of a dP_4 has rank 6.

As highlighted by Swinnerton-Dyer [24, Question 15], it has become something of milestone to establish the Manin conjecture for a single example of a dP_4. The gradual improvements recorded above all point the way towards the possibility of actually establishing an asymptotic formula for the counting function associated to a well-chosen dP_4. We will see that this is indeed the case for the surface defined by the forms (1.2) and

$$\Phi_2(x_0, \dots, x_4) = x_0^2 + x_1^2 + x_2^2 - x_3^2 - 2x_4^2. \tag{1.5}$$

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Let $V_0 \subset \mathbb{P}^4$ denote the corresponding conic bundle dP_4 and let U_0 be the Zariski open subset formed by deleting the lines from V_0. It is easy to check that V_0 is non-singular and we will prove shortly that $\mathrm{Pic}(V_0) \cong \mathbb{Z}^5$, with $K = \mathbb{Q}(i)$. We are now ready to reveal the main result that is surveyed in this note, and which confirms the conjectured estimate (1.1) for the surface under consideration.

Theorem. *We have*

$$N_{U_0,H}(B) = c_{V_0,H} B(\log B)^4 + O\Big(\frac{B(\log B)^4}{\log\log B}\Big),$$

where $c_{V_0,H} > 0$ *is the constant predicted by Peyre.*

For comparison, Fok [17] also studies the surface V_0, obtaining the upper bound

$$N_{U_0,H}(B) = O\big(B(\log B)^4\big). \tag{1.6}$$

We will see in due course how our argument leads to this estimate rather easily. In the following section we will show how to compute the Picard group associated to the surface V_0. While these calculations are routine they are perhaps not entirely familiar to analytic number theorists and so we will present fuller details than appear in [3]. Section 3 comprises the bulk of this survey and contains a discussion of the proof of the theorem. A number of innocent simplifications will be made along the way and so any serious reader would do better to consult [3] directly. Finally, in Section 4 we will suggest some avenues for future exploration.

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2. Geometry of V_0

Our main task in this section is to calculate an explicit basis for the Picard group of the surface V_0 that is defined by (1.2) and (1.5). We begin by recalling some basic facts about the geometry of general dP_4s $V \subset \mathbb{P}^4$. All of the facts that we will need can be found in the book by Manin [18]. Such

V are isomorphic to the blow-up of $\mathbb{P}^2$ along a union of 5 points $p_1, \dots, p_5$, no 3 of which are collinear. Let E_i denote the exceptional divisor above p_i, for $1 \le i \le 5$, let $L_{i,j}$ denote the strict transform of the line going through p_i and p_j, for $1 \le i < j \le 5$, and let Q denote the strict transform of the unique conic going through all 5 points. These 16 divisors constitute the famous 16 lines on V.

If Λ is the strict transform of a line in $\mathbb{P}^2$ not passing through any of the 5 distinguished points, then a free basis for the geometric Picard group $\mathrm{Pic}_{\overline{\mathbb{Q}}}(V)$ is given by $\{\Lambda, E_1, \dots, E_5\}$. Here, as in all that follows, we have found it convenient to identify divisors with their classes in $\mathrm{Pic}_{\overline{\mathbb{Q}}}(V)$. In terms of this basis the anticanonical divisor class can be written $-K_V = 3\Lambda - \sum_{i=1}^5 E_i$, and furthermore, we have $L_{i,j} = \Lambda - E_i - E_j$ and $Q = 2\Lambda - \sum_{i=1}^5 E_i$. There is an intersection form $(\cdot,\cdot)$ on $\mathrm{Pic}_{\overline{\mathbb{Q}}}(V)$, which is non-degenerate, symmetric and bilinear. In terms of this intersection form the divisors $\Lambda, E_1, \dots, E_5$ satisfy the relations

$$(\Lambda,\Lambda) = 1, \quad (\Lambda, E_i) = 0, \quad (E_i, E_j) = \begin{cases} -1, & \text{if } i = j, \\ 0, & \text{if } i \ne j. \end{cases}$$

One can now check that $-K_V$ is very ample, with self-intersection number $4 = \deg V$. We are now ready to produce an explicit basis for $\mathrm{Pic}(V_0)$.

Let $\varepsilon_1, \varepsilon_2 \in \{-1, +1\}$ and let $i = \sqrt{-1}$. Then a straightforward calculation reveals that the 16 lines on V_0 are given by

$$M_1(\varepsilon_1,\varepsilon_2){:}\begin{cases} x_0 = \varepsilon_1 x_2 = \varepsilon_2 x_4, \\ x_1 = \varepsilon_1 x_3, \end{cases} \qquad M_2(\varepsilon_1,\varepsilon_2){:}\begin{cases} x_1 = \varepsilon_1 x_2 = \varepsilon_2 x_4, \\ x_0 = \varepsilon_1 x_3, \end{cases}$$
$$M_3(\varepsilon_1,\varepsilon_2){:}\begin{cases} x_3 = \varepsilon_1 i x_0 = \varepsilon_2 i x_4, \\ x_1 = \varepsilon_1 i x_2, \end{cases} \qquad M_4(\varepsilon_1,\varepsilon_2){:}\begin{cases} x_3 = \varepsilon_1 i x_1 = \varepsilon_2 i x_4, \\ x_0 = \varepsilon_1 i x_2. \end{cases}$$

In particular all of the lines split over the Gaussian field $\mathbb{Q}(i)$. Let us write $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ for the Galois group of the splitting field. Among the lines above we will need to identify 5 mutually skew lines that are globally invariant under the action of $\mathcal{G}$. Let us set

$$\begin{aligned} &E_1 = M_1(-1,1), \quad E_2 = M_1(1,-1), E_3 = M_2(-1,-1), \\ &E_4 = M_4(-1,-1), E_5 = M_4(1,1). \end{aligned}$$

These lines satisfy $(E_i, E_j) = 0$ for each $1 \le i < j \le 5$. Furthermore, E_1, E_2, E_3 are defined over $\mathbb{Q}$, and E_4, E_5 are defined over $\mathbb{Q}(i)$, but are conjugate under the action of $\mathcal{G}$. We may therefore take $\Lambda, E_1, \dots, E_5$ as a basis for $\mathrm{Pic}_{\overline{\mathbb{Q}}}(V_0)$, whence $\mathrm{Pic}(V_0) = \{\Lambda, E_1, E_2, E_3, E_4 + E_5\}$. This establishes the fact that $\mathrm{Pic}(V_0) \cong \mathbb{Z}^5$, as recorded in the introduction. Using

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the intersection form it is now routine to show that we can choose the projection of V_0 to $\mathbb{P}^2$ in such a way that we have the equalities

$$\begin{array}{lll} L_{1,2}=M_2(1,-1), & L_{1,3}=M_1(-1,-1), & L_{1,4}=M_3(-1,-1),\\ L_{1,5}=M_3(1,1), & L_{2,3}=M_1(1,1), & L_{2,4}=M_3(1,-1),\\ L_{2,5}=M_3(-1,1), & L_{3,4}=M_4(-1,1), & L_{3,5}=M_4(1,-1),\\ L_{4,5}=M_2(1,1), & Q=M_2(-1,1), & \end{array}$$

in the description of the lines.

Let us identify $\mathrm{Pic}(V_0)\otimes_{\mathbb{Z}}\mathbb{R}$ with its dual using the intersection form. Let $\Lambda_{\mathrm{eff}}(V_0)$ denote the convex cone in $\mathrm{Pic}(V_0)\otimes_{\mathbb{Z}}\mathbb{R}$ that is generated by the classes of effective divisors, and let $\Lambda_{\mathrm{eff}}^{\vee}(V_0)$ denote its dual cone, with respect to the intersection form. As is well-known, $\Lambda_{\mathrm{eff}}(V_0)$ is generated by the classes $[O]=\sum_{L\in O}L$ associated to each orbit O of the action of $\mathcal{G}$ on the 16 lines. In our setting we have 12 orbits overall, given by

$$\begin{array}{ll} O_1(\varepsilon_1,\varepsilon_2)=\{M_1(\varepsilon_1,\varepsilon_2)\}, & O_3(\varepsilon_1,\varepsilon_2)=\{M_3(\varepsilon_1,\varepsilon_2),M_3(-\varepsilon_1,-\varepsilon_2)\},\\ O_2(\varepsilon_1,\varepsilon_2)=\{M_2(\varepsilon_1,\varepsilon_2)\}, & O_4(\varepsilon_1,\varepsilon_2)=\{M_4(\varepsilon_1,\varepsilon_2),M_4(-\varepsilon_1,-\varepsilon_2)\}, \end{array}$$

for the various $\varepsilon_1,\varepsilon_2\in\{-1,+1\}$. We may therefore conclude that $\Lambda_{\mathrm{eff}}(V_0)$ is generated by the classes

$$\begin{array}{l} E_1,E_2,E_3,E_4+E_5,\\ \Lambda-E_i-E_j \text{ for } (i,j)\in\{(1,2),(1,3),(2,3),(4,5)\},\\ 2\Lambda-E_1-E_2-E_3-E_4-E_5,\\ 2\Lambda-2E_i-E_4-E_5 \text{ for } i\in\{1,2,3\}, \end{array} \tag{2.7}$$

in $\mathrm{Pic}(V_0)$. We will take advantage of this information in §3.5.

3. Overview of the proof

The proof of the theorem can be broken into a number of basic steps. The primary ingredients involved are the following:

(1) reduction to conics of low height;
(2) parametrisation of the conics;
(3) lattice point counting in the plane;
(4) divisor problem for binary forms; and
(5) comparison with Peyre's constant.

We proceed to discuss each of these steps in a little more detail.

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3.1. *Reduction to conics of low height*

Recall the definition of the quadratic forms Φ_1, Φ_2 from (1.2) and (1.5). Although this is not crucial to the success of the proof, it simplifies matters to work with the norm on $\mathbb{R}^5$ given by $\|\mathbf{x}\| = \max\{|x_0|, |x_1|, |x_2|, |x_3|, \sqrt{\frac{2}{3}}|x_4|\}$. The starting point of the investigation is the expression

$$N_{U_0,H}(B) = 8N_1(B) + O(B),$$

where

$$N_1(B) = \#\left\{\mathbf{x} \in \mathbb{N}^5 : \begin{array}{l} \gcd(x_0, \dots, x_3) = 1, \ \ \max\{x_0, \dots, x_3\} \le B, \\ \Phi_1(\mathbf{x}) = \Phi_2(\mathbf{x}) = 0, \ \ \{x_0, x_1\} \neq \{x_2, x_3\} \end{array}\right\}.$$

This amounts to showing that we can restrict the count to positive integer solutions, an argument that follows from elementary considerations. It is here that our choice of norm plays a useful role. As should be clear from Section 2, the condition $\{x_0, x_1\} \neq \{x_2, x_3\}$ ensures that we remain in U_0.

The next stage of the argument involves parametrising the solutions to the equation $\Phi_1(\mathbf{x}) = 0$ and then substituting this parametrisation into the second equation $\Phi_2(\mathbf{x}) = 0$. Without labouring the details, the outcome is that $N_1(B)$ is the number of $(a, b, x, y, z) \in \mathbb{N}^5$ such that

$$C_{a,b}: \quad (a^2 - b^2)x^2 + (a^2 + b^2)y^2 = 2z^2, \tag{3.8}$$

with

$$\gcd(a, b) = \gcd(x, y) = 1, \quad ab, xy \neq 1, \quad \max\{a, b\} \max\{x, y\} \le B.$$

Fundamental to our argument is the observation

$$(x, y, z) \in C_{a,b} \cap \mathbb{N}^3 \Leftrightarrow (b, a, z) \in C_{y,x} \cap \mathbb{N}^3. \tag{3.9}$$

For fixed coprime a, b such that $ab \neq 1$ it is clear that $C_{a,b} \subset \mathbb{P}^2$ defines a non-singular plane conic with discriminant equal to $-2(a^4 - b^4)$. In Figure 1 we have sketched the affine part of the system of fibres $\{C_{x,1}\}_{x \in \mathbb{A}^1}$. We let $M_{a,b}(B)$ denote the number of $(x, y, z) \in C_{a,b} \cap \mathbb{N}^3$ for which $\gcd(x, y) = 1$, $\max\{a, b\} < \max\{x, y\}$ and $\max\{a, b\} \max\{x, y\} \le B$. Note that we must automatically have $xy \neq 1$ in this definition. Using (3.9) we find that

$$N_1(B) = \sum_{\substack{a,b \le B \\ \gcd(a,b)=1, ab \neq 1}} M_{a,b}(B) = 2 \sum_{\substack{a,b < \sqrt{B} \\ \gcd(a,b)=1, ab \neq 1}} M_{a,b}(B) + E(B),$$

where $E(B)$ denotes the contribution from a, b, x, y, z for which $\max\{a, b\}$ is equal to $\max\{x, y\}$.

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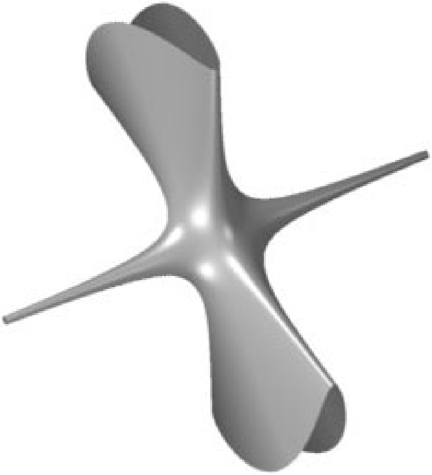

Fig. 1. The fibres $\{C_{x,1}\}_{x\in\mathbb{A}^1}$

With a little work it is possible to show that $E(B) = O(B(\log B)^3)$, which is satisfactory. Thus we have succeeded in reducing the problem to one that involves a family of conics of relatively low height. In fact there are merely $O(B)$ conics to consider.

Were we charged instead with establishing an upper bound like (1.3) or (1.4) instead of our theorem, our analysis would now be relatively straightforward, thanks to the control over the growth rate of rational points on conics found in the author's joint work with Heath-Brown [5, Corollary 2]. This implies that

$$M_{a,b}(B) \ll \tau(|a^4-b^4|)\Big(1+\frac{B}{\max\{a,b\}^{5/3}|a-b|^{1/3}}\Big),$$

where τ is the divisor function, since the corresponding conic has underlying matrix with determinant $-2(a^4-b^4)$ and 2×2 minors with greatest common divisor $O(1)$. Here the fact that $\gcd(a,b)=1$ and $ab\neq 1$ is crucial. Furthermore, we have employed the lower bound $|a^4-b^4| \gg \max\{a,b\}^3|a-b|$ to simplify the second term in this upper bound.

The upper bound (1.6) is now an easy consequence of taking $x=\sqrt{B}$ in the pair of estimates

$$\sum_{a,b\leq x}\tau(|a^4-b^4|) \ll x^2(\log x)^3, \qquad \sum_{a<b\leq x}\frac{\tau(|a^4-b^4|)}{b^{5/3}|a-b|^{1/3}} \ll (\log x)^4.$$

Although we will not prove them here, both of these bounds make essential use of joint work of the author with la Bretèche [1] concerning uniform upper bounds for general sums of the shape

$$\sum_{n_1 \leq N_1} \sum_{n_2 \leq N_2} \varphi(|F(n_1, n_2)|),$$

with $\varphi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ a suitable multiplicative arithmetic function and F a suitable binary form. This investigation extends work by Nair [19] on the corresponding situation for polynomials in only one variable.

3.2. *Parametrisation of the conics*

Turning the upper bound (1.6) into an asymptotic formula requires substantial further work. Recall the definition (3.8) of $C_{a,b}$. Fundamental to the proof is the observation that for each $a, b \in \mathbb{N}$ the conic $C_{a,b}$ always contains the rational point

$$\xi = [1, -1, a].$$

In the classical manner we can use this point to parametrise $C_{a,b}(\mathbb{Q})$, by considering the residual intersection with $C_{a,b}$ of an arbitrary line through ξ. Once carried out this eventually leads to the conclusion that

$$M_{a,b}(B) = \#\left\{(s,t) \in \mathbb{Z}^2 : \begin{array}{l} \gcd(s,t) = 1,\ |s|, t > 0,\ 0 < Q_3(s,t), \\ 0 < Q_1(s,t), Q_2(s,t) \leq \frac{\lambda B}{\max\{a,b\}} \end{array}\right\} + O(1),$$

where

$$\begin{aligned} Q_1(s,t) &= -2s^2 - (a^2 - b^2)t^2 + 4ast, \\ Q_2(s,t) &= 2s^2 - (a^2 - b^2)t^2, \\ Q_3(s,t) &= 2as^2 - 2(a^2 - b^2)st + a(a^2 - b^2)t^2, \end{aligned}$$

and

$$\lambda = \gcd(Q_1, Q_2) = \gcd(Q_1 - Q_2, Q_2) = \gcd(4s(s - at), 2s^2 - (a^2 - b^2)t^2).$$

We have almost arrived at a lattice point counting problem, but there are still some annoying coprimality conditions to eliminate from the expression.

It is quite straightforward to discover that $\lambda = 2^\nu \lambda_1 \lambda_2$, with

$$\begin{aligned} \lambda_1 &= \gcd(s, 2s^2 - (a^2 - b^2)t^2)_\flat = \gcd(s, a^2 - b^2)_\flat, \\ \lambda_2 &= \gcd(s - at, 2s^2 - (a^2 - b^2)t^2)_\flat = \gcd(s - at, a^2 + b^2)_\flat, \end{aligned}$$

and a certain explicit expression for ν. Here the symbol $\gcd(\cdot,\cdot)_\flat$ is used to denote the greatest odd common divisor of two integers. In order to simplify the exposition we will not discuss the value of ν here. This depends

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intimately on a, b, s, t and is calculated in [3, Lemma 13]. To rid ourselves of the coprimality relations we employ Möbius inversion. This gives

$$M_{a,b}(B) = \sum_{\substack{k_i\lambda_i \mid a^2+(-1)^i b^2 \\ 2\nmid k_1k_2\lambda_1\lambda_2}} \mu(k_1)\mu(k_2) \sum_{\ell=1}^{\infty} \mu(\ell) \#(\mathsf{\Lambda} \cap \mathcal{R}) + O(1) \tag{3.10}$$

where $\mathcal{R} \subset \mathbb{R}^2$ is the set of $(s,t) \in \mathbb{R}^2$ such that

$$|s|, t > 0, \quad 0 < Q_3(s,t), \quad 0 < Q_1(s,t), Q_2(s,t) \le \frac{2^\nu \lambda_1\lambda_2 B}{\max\{a,b\}}$$

and

$$\mathsf{\Lambda} = \left\{(s,t) \in \mathbb{Z}^2 : \, k_1\lambda_1 \mid s, \; k_2\lambda_2 \mid s - at, \; \ell \mid (s,t)\right\}$$

Clearly $\mathsf{\Lambda}$ is a sublattice of $\mathbb{Z}^2$ and $\mathcal{R} \subset \mathbb{R}^2$ is a region defined by a piecewise continuous boundary. Thus we have succeeded in translating the problem into a lattice point counting problem in which special attention needs to be paid to the question of uniformity in all of the parameters a, b, λ_i, k_i and ℓ.

3.3. *Lattice point counting in the plane*

In an ideal world we would like to apply the classic estimate

$$\#(\mathsf{\Lambda} \cap \mathcal{R}) = \frac{\mathrm{vol}(\mathcal{R})}{\det \mathsf{\Lambda}} + O(\partial\mathcal{R} + 1), \tag{3.11}$$

where $\partial\mathcal{R}$ denotes the perimeter of $\mathcal{R}$, in order to handle the summand in (3.10). In our setting

$$\mathrm{vol}(\mathcal{R}) = \frac{v(a,b)}{\sqrt{|a^2-b^2|}} \cdot \frac{2^\nu \lambda_1\lambda_2 B}{\max\{a,b\}}, \quad \partial\mathcal{R} \asymp \sqrt{\frac{2^\nu \lambda_1\lambda_2 B}{\max\{a,b\}}},$$

with $v(a,b)$ a real-valued continuous function that is bounded above and below by absolute constants that do not depend on any of the parameters.

The overall contribution from the error term above is clearly

$$\begin{aligned} &\gg \sum_{a,b} \sum_{\substack{k_i\lambda_i \mid a^2+(-1)^i b^2 \\ 2\nmid k_1k_2\lambda_1\lambda_2}} |\mu(k_1)\mu(k_2)| \sum_{\ell=1}^{\infty} |\mu(\ell)| \left(1 + \sqrt{\frac{\lambda_1\lambda_2 B}{\max\{a,b\}}}\right) \\ &\ge \sum_{a,b} \left(1 + \sqrt{\frac{B}{\max\{a,b\}}}\right) \\ &\gg B^{\frac{5}{4}}, \end{aligned}$$

which is disastrously big! Thus the classic lemma is too crude and we must work rather harder to make things succeed. Our approach involves using exponential sums to rewrite the divisibility information in the definition of Λ. In this way we can make the error terms more explicit and obtain cancellation in the summation over a, b and λ_1, λ_2. This part of the argument is long and technical and will be suppressed from the present exposition in order to maintain morale.

Ignoring the contribution from the error term, let us proceed to analyse the contribution from the obvious main term in the classic estimate above. Now an easy calculation reveals that $\det \Lambda = \ell^2 k_1 k_2 \lambda_1 \lambda_2 / \gcd(\ell, k_1 k_2 \lambda_1 \lambda_2)$. Hence our expected main term takes the shape

$$cB \sum_{\substack{a,b<\sqrt{B} \\ \gcd(a,b)=1, ab\neq 1}} \frac{v(a,b)}{\sqrt{|a^2-b^2|}\max\{a,b\}} g(|a^4-b^4|), \tag{3.12}$$

for a certain constant $c > 0$ and a certain arithmetic function $g : \mathbb{N} \to \mathbb{Z}_{\geq 0}$, obtained by carrying out the summation over the k_i, λ_i and ℓ. In fact one finds that g is given multiplicatively by

$$g(p^\nu) = \begin{cases} \max\{1, \nu-1\}, & \text{if } p = 2, \\ 1 + \nu\left(\frac{1-\frac{1}{p}}{1+\frac{1}{p}}\right), & \text{if } p > 2. \end{cases} \tag{3.13}$$

Recalling that $\tau(p^\nu) = 1 + \nu$ we see that g can be written as the Dirichlet convolution $g = h * \tau$ for a “small” arithmetic function h. For our purposes an arithmetic function h is said to be small if the infinite sum $\sum_{d=1}^{\infty} d^{-\frac{1}{4}} |h(d)|$ is convergent.

3.4. *Divisor problem for binary forms*

The next step is to analyse the main term (3.12). The factor $v(a,b)/(\sqrt{|a^2-b^2|}\max\{a,b\})$ is basically harmless and can be reintroduced later via an application of partial summation. We therefore need to investigate sums of the shape

$$S_*(X, \varphi) = \sum_{\substack{a,b\leq X \\ \gcd(a,b)=1}} \varphi(F(a,b)),$$

for suitable arithmetic functions $\varphi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ and suitable integral binary forms F. Let $S(X, \varphi)$ denote the corresponding sum without the condition that $\gcd(a,b) = 1$ in the summation.

When $\varphi = \tau$ is the ordinary divisor function the problem of estimating $T(X) = S(X, \tau)$ has enjoyed considerable attention in the literature.

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Dealing with forms of degree at most 2 is straightforward. Greaves [10] pioneered the study of higher degree forms by showing the existence of constants c_F, c'_F, with $c_F > 0$, such that

$$T(X) = c_F X^2 \log X + c'_F X^2 + O_{\varepsilon,F}(X^{\frac{27}{14}+\varepsilon}), \tag{3.14}$$

for any $\varepsilon > 0$, when F is an irreducible cubic form with non-zero discriminant. Greaves' proof uses exponential sums. It remained a significant open problem to deal with forms of degree 4 until Daniel [6] was able to show that

$$T(X) = c_F X^2 \log X + O_F(X^2 \log\log X),$$

for a constant $c_F > 0$, when F is an irreducible quartic form with non-zero discriminant. Daniel also sharpens Greaves' asymptotic formula, showing that the error term in (3.14) may be replaced by $O_{\varepsilon,F}(X^{\frac{15}{8}+\varepsilon})$ for cubics. Daniel's argument avoids using exponential sums and is based on some ideas from the geometry of numbers instead.

In our work we need to extend the work of Daniel to an asymptotic formula for $T(X)$, but with

- $F = L_1L_2Q$ reducible, where L_1, L_2 are non-proportional integral binary linear forms and Q is an integral binary quadratic form which is irreducible over $\mathbb{Q}$;
- τ replaced by $\tau * h$ for small h; and
- a summation over coprime a, b.

In other words we need to investigate $S_*(X; \tau * h)$ for small arithmetic functions h. All of the necessary estimates are corralled into a separate investigation [2]. As a special case we are able to show in [2, Théorème 1] that

$$T(X) = c_{L_1,L_2,Q} X^2 (\log X)^3 + O_{\varepsilon,L_1,L_2,Q}(X^2 (\log X)^{2+\varepsilon}),$$

for any $\varepsilon > 0$, and furthermore, in [2, Corollaire 2] an asymptotic formula of similar strength for $S_*(X, \tau * h)$. We also investigate the asymptotic behaviour of the more general sum

$$S_*(X, h; Y, V) = \sum_{\substack{\mathbf{x}=(x_1,x_2)\in\mathbb{Z}^2\cap X\mathcal{B} \\ \gcd(x_1,x_2)=1}} h\big(L_1, L_2, Q; Y, V\big),$$

for $Y \geq 2$ and $\mathcal{B} \subseteq [-1,1]^2$ a convex region whose boundary is defined by a piecewise continuously differentiable function, with $L_i(\mathbf{x}) > 0$ and $Q(\mathbf{x}) > 0$

for every $\mathbf{x} \in \mathcal{B}$. Furthermore, $V \subseteq [0,1]^4$ is a region cut out by a finite number of hyperplanes with absolutely bounded coefficients, and the summand is defined to be

$$h(L_1, L_2, Q; Y; V) = \sum_{\substack{d \mid L_1(\mathbf{x})L_2(\mathbf{x})Q(\mathbf{x}) \\ d_i = \gcd(d, L_i(\mathbf{x})),\ d_3 = \gcd(d, Q(\mathbf{x})) \\ (\frac{\log d_1}{\log Y}, \frac{\log d_2}{\log Y}, \frac{\log d_3}{2\log Y}, \frac{\log \max\{|x_1|,|x_2|\}}{\log Y}) \in V}} (1 * h)(d). \quad (3.15)$$

Suppose for simplicity that $L_i(\mathbf{x}) > 0$ and $Q(\mathbf{x}) > 0$ for every $\mathbf{x} \in [0,1]^2$. If one takes $V = [0,1]^4$ and $\mathcal{B} = [0,1]^2$, together with

$$h(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

and Y sufficiently large in terms of X, then $S_*(X, h; Y, V)$ simplifies to become the sum $S_*(X, \tau)$ that was introduced above. It might seem strange to study such a general function, but it turns out that this is what is required for the theorem, as we will discuss below.

The proofs in [2] are based on the approach using the geometry of numbers that was developed by Daniel in his study of $T(X)$ for irreducible forms of degree at most 4. The need to maintain complete uniformity in all of the relevant parameters causes considerable extra labour.

3.5. *Comparison with Peyre's constant*

According to the conjecture of Peyre [20], the constant $c_{V_0,H}$ in the theorem should be a product of two constants $\alpha(V_0)$ and $\tau_H(V_0)$.

Let $L(s, V_0) = \zeta(s)^5 L(s, \chi)$, where χ is the real non-principal character modulo 4. It turns out that this is the Hasse–Weil L-function associated to our non-singular dP_4. The constant $\tau_H(V_0)$ is basically a product of local densities and convergence factors. Specifically, if $\omega_{H,v}$ denotes the usual v-adic density of points on $V_0(\mathbb{Q}_v)$ for any place v of $\mathbb{Q}$, then

$$\begin{aligned} \tau_H(V_0) &= \lim_{s \to 1} \big((s-1)^5 L(s, V_0)\big) \omega_{H,\infty} \prod_p \frac{\omega_{H,p}}{L_p(1, V_0)} \\ &= \omega_{H,\infty} \prod_p \Big(1 - \frac{1}{p}\Big)^5 \omega_{H,p}. \end{aligned}$$

Checking that the predicted local factors $\omega_{H,v}$ really do match up with the factors that emerge in the theorem is quite involved.

The constant $\alpha(V_0)$ is a rational number defined as the volume of a certain polytope obtained by intersecting the cone of effective divisors of

V_0 with a hyperplane. Determining its value for non-singular del Pezzo surfaces of low degree can be a very challenging problem in its own right. For example, for the non-singular del Pezzo surface of degree 1 in which all of the 240 exceptional divisors are defined over $\mathbb{Q}$ one would need to calculate the volume of a polytope with 19440 generators! An alternative approach has been developed by Derenthal, Joyce and Teitler [7]. For V_0 we have

$$\alpha(V_0) = \frac{1}{36}, \tag{3.16}$$

as calculated by Derenthal in [3, Appendix].

For the sake of general edification, we present here an alternative proof of (3.16) using the method developed by Peyre and Tschinkel [21]. The constant $\alpha(V_0)$ is defined by Peyre [20] as the volume of the polytope

$$\{\mathbf{t} = (t_0, \dots, t_4) \in \Lambda_{\mathrm{eff}}^{\vee}(V_0) : (\mathbf{t}, -K_{V_0}) = 1\}.$$

Here $\mathbf{t} \in \mathrm{Pic}(V_0) \otimes_{\mathbb{Z}} \mathbb{R}$ is understood to mean $t_0\Lambda + \cdots + t_3E_3 + t_4(E_4+E_5)$, with $t_0, \dots, t_4 \in \mathbb{R}$. Recall that the effective cone $\Lambda_{\mathrm{eff}}(V_0)$ is generated by the classes (2.7) in $\mathrm{Pic}(V_0)$. Then it follows that $\alpha(V_0)$ is the volume of the polytope

$$P = \left\{ \mathbf{t} \in \mathbb{R}^5 : \begin{array}{l} 3t_0 - t_1 - t_2 - t_3 - t_4 = 1, \\ t_0 - t_4, t_1, t_2, t_3, t_4 > 0, \\ t_0 - t_i - t_j > 0 \text{ for } (i,j) \in \{(1,2),(1,3),(2,3)\}, \\ 2t_0 - t_1 - t_2 - t_3 - t_4 > 0, \\ 2t_0 - 2t_i - t_4 > 0 \text{ for } i \in \{1,2,3\} \end{array} \right\}.$$

Eliminating t_4, we are left with the the non-simplicial polytope

$$P = \left\{ \mathbf{t} \in \mathbb{R}^4 : \begin{array}{l} -1 + 3t_0 - t_1 - t_2 - t_3 > 0, \\ 1 - t_0, t_1, t_2, t_3 > 0, \\ t_0 - t_i - t_j > 0 \text{ for } (i,j) \in \{(1,2),(1,3),(2,3)\}, \\ 1 - 2t_0 + t_1 + t_2 + t_3 > 0, \\ 1 - t_0 - t_i + t_j + t_k > 0 \text{ for } \{i,j,k\} = \{1,2,3\} \end{array} \right\}. \tag{3.17}$$

We can calculate the volume of this using the software package `polymake` [9], for example, the outcome being that (3.16) holds.

We now arrive at one of the most subtle aspects of the proof of the theorem. In Section 3.3 we considered the effect of working with the main term in the classic estimate (3.11), which in turn led us to estimate (3.12), as discussed in Section 3.4. However, if one were to follow this course of action exactly, one would ultimately be led to a final main term of the shape

$$\frac{1}{32}\cdot\tau_H(V_0)\cdot B(\log B)^4,$$

where $\tau_H(V_0)$ is the product of local densities introduced above. One notes immediately that $\frac{1}{32}>\frac{1}{36}=\alpha(V_0)$ and so somewhere along the way we have over-counted things!

What is at fault here is our initial idea of approximating $\#(\Lambda\cap\mathcal{R})$ in (3.10) by the volume of $\mathcal{R}$ divided by the determinant of Λ. We have failed to observe that for some ranges of the parameters λ_1,λ_2 that appear in (3.10) we will have $\#(\Lambda\cap\mathcal{R})=0$ even if the approximation $\mathrm{vol}(\mathcal{R})/\det\Lambda$ is non-zero. Thus a further reduction on the set of allowable parameters λ_1,λ_2 is necessary. Once taken into account this will eventually lead to the expected main term that we see in the statement of the theorem.

For any $R>0$ and coprime $a,b<\sqrt{B}$ with $ab\neq 1$, define the region

$$V_{a,b}(R)=\left\{(t_1,t_2)\in\mathbb{R}^2_{\geq 1}:\begin{array}{l}\max\{a,b\}t_1\leq Rt_2,\\ \max\{a,b\}t_2\leq Rt_1,\\ \max\{a,b\}^3/R\leq t_1t_2,\\ t_1t_2\leq\max\{a,b\}R\end{array}\right\}. \tag{3.18}$$

It can then be shown that the summation over λ_1,λ_2 is necessarily restricted to $(\lambda_1,\lambda_2)\in V_{a,b}(R)$ for a suitable choice of R depending on B. Doing so requires one to take into account all of the available symmetry in our counting problem. Since we have $\lambda_i\mid a^2+(-1)^ib^2$, so it follows that there exist $\mu_1,\mu_2\in\mathbb{Z}$ such that $\lambda_i\mu_i=a^2+(-1)^ib^2$. Switching between the λ_i and the μ_i is a crucial ingredient in deriving some of the inequalities present in the definition of $V_{a,b}(R)$. To illustrate a simple case let us indicate how the first inequality $\max\{a,b\}\lambda_1\leq R\lambda_2$ arises in (3.18). We have already recorded an expression for the volume of $\mathcal{R}$ in Section 3.3. With a little extra work one is able to show that

$$\mathcal{R}\subset[-c\sqrt{X},c\sqrt{X}]\times(0,c\sqrt{X/|a^2-b^2|}],$$

where $c>0$ is an absolute constant and $X=2^\nu\lambda_1\lambda_2B/\max\{a,b\}$. Given that $s\neq 0$ and $\lambda_1\mid s$ for any $(s,t)\in\Lambda\cap\mathcal{R}$, it therefore follows that $\lambda_1\ll\sqrt{X}$, whence the first inequality in (3.18) is satisfied by (λ_1,λ_2) for any $R\gg 2^\nu B$.

The outcome of this analysis is that the function g in (3.12) should be replaced by

$$\tilde{g}(a,b;R) = \sum_{\substack{d \mid a^4-b^4 \\ d_i=\gcd(d,a+(-1)^i b) \\ d_3=\gcd(d,a^2+b^2)}} (1*h)(d)\chi_{d_1d_2,d_3}(R),$$

for a suitable value of $R \geq 1$, where $\chi_{t_1,t_2}(R)$ is the characteristic function of the set (3.18) and h is such that $g = \tau * h$ in (3.13). Define the region

$$W = \left\{ \mathbf{w} \in \mathbb{R}^4 : \begin{array}{l} w_i \geq 0 \text{ for } i \in \{1,2,3,4\}, \\ w_1 + w_2 + w_4 \leq 1 + 2w_3, \\ 2w_3 + w_4 \leq 1 + w_1 + w_2, \\ 3w_4 \leq 1 + w_1 + w_2 + 2w_3, \\ w_1 + w_2 + 2w_3 \leq 1 + w_4 \end{array} \right\} \subset [0,1]^4.$$

One easily checks that $\tilde{g}(a,b;R) = h(L_1, L_2, Q; R; W)$, in the notation of (3.15), with the appropriate choices of L_i, Q.

Once these changes are carried through one is ultimately led to a main term of the shape $\mathrm{vol}(W_0) \cdot \tau_H(V_0) \cdot B(\log B)^4$, where

$$W_0 = \left\{ (w_1, w_2, \frac{w_3}{2}, w_4) \in W : \begin{array}{l} \max\{w_1, w_2, \frac{1}{2}w_3\} \leq w_4, \\ 2w_4 \leq 1 \end{array} \right\}.$$

We can calculate the volume of W_0 using `polymake` [9], the outcome being that $\mathrm{vol}(W_0) = \frac{1}{36} = \alpha(W_0)$. The reader is invited to compare W_0 with P in (3.17). Both are polytopes in $\mathbb{R}^4$ defined by a minimal set of 12 inequalities.

4. Further exploration

It remains to discuss a few of the ways in which the ideas in [3] and [2] could be developed further. While it still seems hopeless to deal with the divisor problem for binary forms of degree 5 or more, one could try to handle the remaining binary quartic forms. In view of [2] and [6] it remains to handle reducible binary quartic forms which factor as a linear form and a cubic form, or as the product of two quadratic forms, or split completely as the product of four linear forms. All of this seems doable and would no doubt prove useful in related counting problems of the nature considered in our theorem.

Problem 4.1. *Establish analogues of [2] and [6] for other reducible binary quartic forms.*

Returning to the Manin conjecture, it is very natural to wonder how far the ideas behind the proof of the theorem can be extended to other surfaces.

Problem 4.2. *Establish the Manin conjecture for other conic bundle del Pezzo surfaces of degree* 4.

A good starting point would be to consider the family of surfaces $V \subset \mathbb{P}^4$ defined by (1.2) and

$$\Phi_2(x_0, \dots, x_4) = c_0 x_0^2 + c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2 + c_4 x_4^2.$$

It is easy to check that V will be non-singular if and only if $c_0 \cdots c_4 \neq 0$ and $c_0 c_1 \neq c_2 c_3$. In attacking Problem 4.2 one might also begin by considering only those V for which the corresponding conic bundle morphisms have a section such as for the surface considered in the theorem. This is equivalent to the map $V(\mathbb{Q}) \to \mathbb{P}^1(\mathbb{Q})$ being surjective. Assuming this to be the case there will still be considerable work to be done, since the corresponding fibres take the shape

$$f_1(a,b)x^2 + f_2(a,b)y^2 + c_4 z^2 = 0,$$

for binary forms $f_1(a,b) = c_0 a^2 + c_3 b^2$ and $f_2(a,b) = c_2 a^2 + c_1 b^2$. Thus one is led to Problem 4.1 for the case in which $f_1 f_2$ factors over $\mathbb{Q}$ as the product of two quadratic forms or splits completely.

It would be even more interesting to handle a non-singular del Pezzo surface of degree 4 with a conic bundle structure, for which the morphism $V \to \mathbb{P}^1$ does not possess a section.

Problem 4.3. *Establish the Manin conjecture for a conic bundle del Pezzo surface of degree* 4 *whose morphisms to* $\mathbb{P}^1$ *do not have a section.*

Thus far we have dealt only with non-singular del Pezzo surfaces of degree 4. One might ask how far things can be pushed for the corresponding surfaces of degree 3: ie. non-singular cubic surfaces $V \subset \mathbb{P}^3$ defined over $\mathbb{Q}$. In this setting one should take $U = V \setminus \{27 \text{ lines}\}$. The following table summarises some of our progress to date:

who?	$N_{U,H}(B)$	conditions?
Slater & Swinnerton-Dyer [23]	$\gg B(\log B)^{\mathrm{rankPic}(V)-1}$	2 skew lines $/\mathbb{Q}$
Heath-Brown [12]	$\ll B^{\frac{4}{3}+\varepsilon}$	3 coplanar lines $/\mathbb{Q}$
Hooley [16]	$\ll B^{\frac{5}{3}+\varepsilon}$	V diagonal & 1 line $/\mathbb{Q}$
Salberger (in preparation)	$\ll B^{\frac{12}{7}+\varepsilon}$	—

It should be remarked that Heath-Brown's result covers the Fermat cubic surface $x_0^3+x_1^3+x_2^3+x_3^3=0$. This particular surface has already been the object of much study. Previously, Hooley [15] had used exponential sums to show that $N_{U,H}(B)=O_\varepsilon(B^{\frac{5}{3}+\varepsilon})$ for any $\varepsilon>0$. Later, a much shorter proof of this estimate was provided by Wooley [26], and it is the ideas in this argument that provided the inspiration behind Heath-Brown's sharper bound in [12]. With the latter result in mind we end with the following challenging problem.

Problem 4.4. *Prove an upper bound of the shape* $N_{U,H}(B)=O_V(B^\theta)$*, with* $\theta<4/3$*, for a single non-singular cubic surface* $V\subset\mathbb{P}^3$*.*

In attacking Problem 4.4 it makes sense to start by analysing a cubic surface with a conic bundle structure. The example

$$x_0x_1(x_0+x_1)=x_2x_3(x_2+x_3)$$

seems worthy of attention.

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