Preface

This book explores captivating curves and related sets in the plane. Cantor sets, curves, maps, fixed points, and the mathematics of fractal geometry interact to produce remarkable examples. We present a sampling of classical results and introduce some fascinating new curves. We consider the curves as paths in the plane, examine visual images of them, and determine their topological properties. The material is presented with emphasis on the geometric intuition characteristic of the study of curves. Interest in curves has not diminished over the years. The advent of computing power to produce visual images has served to make the study of curves and their properties even more intriguing. Many examples and illustrative figures accompany our discussion of curves. We intend that the surprising features of these curves will capture your interest and motivate deeper exploration of curious curves.

Chapter 1 contains examples of curves that display intriguing properties. We visit self-similar curves, simple curves, space filling curves, self-intersecting curves, curves with zero area, simple curves with positive area, and generalized curves. Examples display that the attractor of an iterated function system may also be a curve. A generalized Cantor set C_h is constructed with a discussion of its length and the area of $C_h \times C_h$. This construction is used as a reference throughout the text where the set C_h appears in various settings.

In Chapter 2 we consider the Koch curve as a path in the plane by constructing the continuous function which defines it. We find the length of the curve and show that it has a tangent nowhere. We conclude with a discussion of the Cantor function.

In Chapter 3 we revisit some of the examples of Chapter 1 and introduce many other curves. We show that every compact set in the plane (including every curve in the plane) is a continuous image of the Cantor set. Homeomorphisms defined on Cantor sets on the unit interval are used to construct a simple curve in the unit square that has positive area at each point!

In Chapter 4 we explore two examples of generalized Koch curves, again looking at self-intersection properties. A new class of Cantor sets is used to describe the double points in one of the examples.

Chapter 5 develops enough theory of metrics to prove that the Hausdorff metric is a complete metric on the set of compact sets in the plane.

Chapter 6 introduces contraction maps and a class of contraction maps called iterated function systems (IFS). Some fractal images are shown to be the fixed points (attractors) of an IFS. We prove that a connected attractor of an IFS is a curve. We also give a necessary and sufficient condition for an attractor to be connected.

Chapter 7 contains an introduction to Hausdorff dimension. Examples of sets of various dimensions are provided by Cantor sets and curves. Tent maps are also discussed. Resident sets for tent maps are either the interval $[0, 1]$ or Cantor sets that generate simple curves with area equal to zero and specified Hausdorff dimension; tangent lines to these curves are also addressed.

Chapter 8 discusses resident sets for quadratic maps. It contains a brief look at fixed points and repellers, Julia sets, and the Mandelbrot set. A Julia set is either connected or it is a Cantor set. We conclude with a question about connected Julia sets.

The material in the appendices is a reference for mathematical properties used in this text that may be unfamiliar to you.

Appendix A contains properties of points on a line and material about sequences and convergence. A few needed details are given about representation of numbers in differing bases.

Appendix B, an intuitive discussion of length and area, provides enough information to be able to find lengths of curves and areas of some interesting and unusual sets in the plane.

In Appendix C maps and sequences of maps are defined and the properties of continuity and convergence discussed. Pertinent topological properties of sets in the plane are discussed.

Appendix D introduces infinite sets and explains why the rational numbers are countably infinite and the real numbers are uncountably infinite.

This book is suitable for a topics course, capstone course, or senior seminar; it is also intended for independent study by students and others interested in mathematics. The topics of this book have proved to be a rich source of projects for undergraduate research for students in a mathematics Research Experiences for Undergraduates (REU) program at the University of Akron. Curves can often provide a better representation of natural phenomena than do the figures of classical geometry. Thus the material is appropriate not only for people working in mathematics, but also those in other sciences. Problems play a vital role in the book. Some are routine, others are more challenging. Occasionally, easily established results used in the text have been made into problems. At other times, proofs of topics not covered in the text are sketched and you are asked to fill in details. Much of the learning of this material will be gained by working through the problems. Solutions to selected problems are provided at the end of the text.

Most notation used is either explained in the text, or else taken from calculus and set theory. A few reminders and additional explanations are collected here.

(1) Natural numbers: $\mathbb{N} = \{1, 2, 3, \cdots\}$.
(2) Integers: $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$.
(3) Real numbers: $\mathbb{R} = (-\infty, \infty)$.
(4) The plane: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\}$.
(5) Complex numbers: $\mathbb{C} = \mathbb{R}^2$ also considered as points in the plane.
(6) Intervals in $\mathbb{R}$: $(a, b) = \{x : a < x < b\}$ and $[a, b] = \{x : a \leq x \leq b\}$.
(7) Intervals in $\mathbb{R}^2$: For points $z, w \in \mathbb{R}^2$ the interval $[z, w]$ is the line segment connecting z and w with parametric representation $(1 - t)\,z + tw$, $t \in [0, 1]$.

We will use the notion of a function, the definition of a continuous function, the sum of an infinite series, the limit of a sequence, the least upper bound property for the real number system, uniformly Cauchy sequences of functions, and a bit about countable and uncountable sets. The formal prerequisite is a strong course in calculus with some experience at reading and writing proofs. Elementary analysis is extremely efficient for investigating many features of curves.

Richard B. Darst, Judith A. Palagallo, and Thomas E. Price