Chapter 1
Basic Equations

Although this book is about antiplane motions depending on two spatial variables, in this chapter we begin with the three-dimensional equations of linear piezoelectricity. This is because it is usually beneficial to look at problems from a three-dimensional point of view and that antiplane motions of ceramics are in fact exact solutions of the three-dimensional equations. We first summarize the basic theory of linear piezoelectricity based on the *IEEE Standard on Piezoelectricity* [1], the classical book on piezoelectricity by H. F. Tiersten [2] who also wrote the theoretical part of [1], and Chapter 2 of a relatively recent book by the present author [3]. Then we show that antiplane motions of ceramics are a special class of motions satisfying the three-dimensional equations and derive special equations for antiplane motions. The Cartesian tensor notation, the summation convention for repeated tensor indices, and the convention that a comma followed by an index denotes partial differentiation with respect to the coordinate associated with the index are used. A superimposed dot represents a time derivative.

1.1. Equations of Linear Piezoelectricity

The equation of linear piezoelectricity can be obtained by linearizing the nonlinear electroelastic equations [4,5] under the assumption of infinitesimal deformation and fields. The equations of motion and the charge equation of electrostatic (Gauss's law) are

$$\begin{aligned} &T_{ji,j} + f_i = \rho \ddot{u}_i, \\ &D_{i,i} = q, \end{aligned} \tag{1.1}$$

where **T** is the stress tensor, ρ is the mass density, **f** is the body force per unit volume, **u** is the displacement vector, **D** is the electric displacement vector, and q is the body free charge density which is usually zero. Constitutive relations are given by an electric enthalpy function H

$$H(\mathbf{S},\mathbf{E}) = \frac{1}{2} c_{ijkl}^{E} S_{ij} S_{kl} - e_{ijk} E_i S_{jk} - \frac{1}{2} \varepsilon_{ij}^{S} E_i E_j \tag{1.2}$$

through

$$\begin{aligned} T_{ij} &= \frac{\partial H}{\partial S_{ij}} = c^E_{ijkl} S_{kl} - e_{kij} E_k, \\ D_i &= -\frac{\partial H}{\partial E_i} = e_{ikl} S_{kl} + \varepsilon^S_{ik} E_k, \end{aligned} \tag{1.3}$$

where the strain tensor, **S**, and the electric field vector, **E**, are related to the displacement, **u**, and the electric potential, ϕ, by

$$S_{ij} = (u_{j,i} + u_{i,j})/2, \quad E_i = -\phi_{,i} \ . \tag{1.4}$$

c^E_{ijkl}, e_{ijk}, and ε^S_{ij} are the elastic, piezoelectric, and dielectric constants. The superscript, E, in c^E_{ijkl} indicates that the independent electric constitutive variable is the electric field **E**. The superscript, S, in ε^S_{ij} indicates that the mechanical constitutive variable is the strain tensor, **S**. The material constants have the following symmetries:

$$\begin{aligned} c^E_{ijkl} &= c^E_{jikl} = c^E_{klij}, \\ e_{kij} &= e_{kji}, \quad \varepsilon^S_{ij} = \varepsilon^S_{ji}. \end{aligned} \tag{1.5}$$

We also assume that the elastic and dielectric tensors are positive definite in the following sense:

$$\begin{aligned} & c^E_{ijkl} S_{ij} S_{kl} \geq 0 \quad \text{for any} \quad S_{ij} = S_{ji}, \\ & \quad \text{and} \quad c^E_{ijkl} S_{ij} S_{kl} = 0 \quad \Rightarrow \quad S_{ij} = 0, \\ & \varepsilon^S_{ij} E_i E_j \geq 0 \quad \text{for any} \quad E_i, \\ & \quad \text{and} \quad \varepsilon^S_{ij} E_i E_j = 0 \quad \Rightarrow \quad E_i = 0. \end{aligned} \tag{1.6}$$

With successive substitutions from Eqs. (1.3) and (1.4), Eq. (1.1) can be written as four equations for **u** and ϕ:

$$\begin{aligned} c_{ijkl} u_{k,lj} + e_{kij} \phi_{,kj} + f_i &= \rho \ddot{u}_i, \\ e_{ikl} u_{k,li} - \varepsilon_{ij} \phi_{,ij} &= q, \end{aligned} \tag{1.7}$$

where we have neglected the superscripts of the material constants. The linearity of Eq. (1.7) allows the superposition of solutions.

Let the region occupied by a piezoelectric body be V and its boundary surface be S, as shown in Fig. 1.1. Let the unit outward normal of S be **n**.

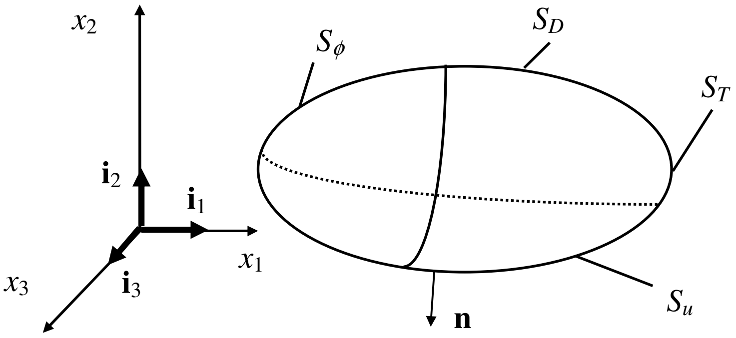

Fig. 1.1. A piezoelectric body and partitions of its boundary surface.

For boundary conditions, we consider the following partitions of S:

$$\begin{aligned} S_u \cup S_T = S_\phi \cup S_D = S, \\ S_u \cap S_T = S_\phi \cap S_D = 0, \end{aligned} \tag{1.8}$$

where S_u is the part of S on which the mechanical displacement is prescribed, and S_T is the part of S where the traction vector is prescribed. S_ϕ represents the part of S which is electroded where the electric potential is no more than a function of time, and S_D is the unelectroded part. For mechanical boundary conditions, we have prescribed displacement $\bar{u}_i$

$$u_i = \bar{u}_i \quad \text{on} \quad S_u, \tag{1.9}$$

and prescribed traction $\bar{t}_j$

$$T_{ij} n_i = \bar{t}_j \quad \text{on} \quad S_T. \tag{1.10}$$

Electrically, on the electroded portion of S,

$$\phi = \bar{\phi} \quad \text{on} \quad S_\phi, \tag{1.11}$$

where $\bar{\phi}$ does not vary spatially. On the unelectroded part of S, the charge condition can be written as

$$D_j n_j = -\bar{\sigma} \quad \text{on} \quad S_D, \tag{1.12}$$

where $\overline{\sigma}$ is free charge density per unit surface area. In the above formulation, we assume very thin electrodes. The mechanical effects like inertia and stiffness of the electrodes are neglected.

On an electrode, S_ϕ, the total free electric charge, Q_e (a scalar), can be represented by

$$Q_e = \int_{S_\phi} -n_i D_i dS \,. \tag{1.13}$$

The electric current flowing out of the electrode is given by

$$I = -\dot{Q}_e \,. \tag{1.14}$$

Sometimes there are two (or more) electrodes on a body, and the electrodes are connected to an electric circuit. In this case, circuit equation(s) will need to be considered.

Consider the following initial-boundary-value problem:

$$\begin{aligned}
&T_{ji,j} + \rho f_i = \rho \ddot{u}_i \quad \text{in} \quad V, \quad t > t_0, \\
&D_{i,i} = q \quad \text{in} \quad V, \quad t > t_0, \\
&T_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k \quad \text{in} \quad V, \quad t > t_0, \\
&D_i = e_{ijk} S_{jk} + \varepsilon_{ij} E_j \quad \text{in} \quad V, \quad t > t_0, \\
&S_{ij} = (u_{i,j} + u_{j,i})/2 \quad \text{in} \quad V, \quad t > t_0, \\
&E_i = -\phi_{,i} \quad \text{in} \quad V, \quad t > t_0,
\end{aligned} \tag{1.15}$$

and

$$\begin{aligned}
&u_i = \overline{u}_i \quad \text{on} \quad S_u, \quad t > t_0, \\
&T_{ji} n_j = \overline{t}_i \quad \text{on} \quad S_T, \quad t > t_0, \\
&\phi = \overline{\phi} \quad \text{on} \quad S_\phi, \quad t > t_0, \\
&D_i n_i = -\overline{\sigma} \quad \text{on} \quad S_D, \quad t > t_0, \\
&u_i = u_i^0 \quad \text{in} \quad V, \quad t = t_0, \\
&\dot{u}_i = v_i^0 \quad \text{in} \quad V, \quad t = t_0, \\
&\phi = \phi^0 \quad \text{in} \quad V, \quad t = t_0,
\end{aligned} \tag{1.16}$$

where u_i^0, v_i^0 and ϕ^0 are initial data. Under Eqs. (1.15) and (1.16), **S**, **E**, **T**, **D** and the velocity fields are unique. The displacement field may have an undetermined static rigid-body displacement if $S_u = 0$. The electric potential may have an undetermined constant if $S_\phi = 0$.

1.2. Cylindrical Coordinates

Quite a few problems in this book are analyzed in cylindrical coordinates. For convenience we summarize the relevant equations below. The cylindrical coordinates (r, θ, z) are defined by

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z\,. \tag{1.17}$$

In cylindrical coordinates, we have the strain-displacement relation

$$\begin{aligned} &S_{rr} = u_{r,r}, \quad S_{\theta\theta} = \frac{1}{r}u_{\theta,\theta} + \frac{u_r}{r}, \quad S_{zz} = u_{z,z}, \\ &2S_{r\theta} = u_{\theta,r} + \frac{1}{r}u_{r,\theta} - \frac{u_\theta}{r}, \quad 2S_{\theta z} = \frac{1}{r}u_{z,\theta} + u_{\theta,z}, \\ &2S_{zr} = u_{r,z} + u_{z,r}. \end{aligned} \tag{1.18}$$

The electric field-potential relation is given by

$$E_r = -\phi_{,r}, \quad E_\theta = -\frac{1}{r}\phi_{,\theta}, \quad E_z = -\phi_{,z}\,. \tag{1.19}$$

The equations of motion are

$$\begin{aligned} &\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} + \frac{\partial T_{zr}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r} + f_r = \rho \ddot{u}_r, \\ &\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{z\theta}}{\partial z} + \frac{2}{r}T_{r\theta} + f_\theta = \rho \ddot{u}_\theta, \\ &\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{1}{r}T_{rz} + f_z = \rho \ddot{u}_z. \end{aligned} \tag{1.20}$$

The electrostatic charge equation is

$$\frac{1}{r}(rD_r)_{,r} + \frac{1}{r}D_{\theta,\theta} + D_{z,z} = q\,. \tag{1.21}$$

1.3. Matrix Notation

We now introduce a compact matrix notation [1,2]. This notation consists of replacing pairs of indices, ij or kl, by single indices, p or q, where i, j, k and l take the values of 1, 2 and 3; and p and q take the values of 1, 2, 3, 4, 5 and 6 according to

ij or kl:	11	22	33	23 or 32	31 or 13	12 or 21
p or q:	1	2	3	4	5	6

(1.22)

Thus

$$c_{ijkl} \to c_{pq}, \quad e_{ikl} \to e_{ip}, \quad T_{ij} \to T_p . \tag{1.23}$$

For the strain tensor, we introduce S_p such that

$$\begin{aligned} &S_1 = S_{11}, \quad S_2 = S_{22}, \quad S_3 = S_{33}, \\ &S_4 = 2S_{23}, \quad S_5 = 2S_{31}, \quad S_6 = 2S_{12}. \end{aligned} \tag{1.24}$$

The constitutive relations can then be written as

$$\begin{aligned} T_p &= c_{pq}^E S_q - e_{kp} E_k , \\ D_i &= e_{iq} S_q + \varepsilon_{ik}^S E_k . \end{aligned} \tag{1.25}$$

In matrix form, Eq. (1.25) becomes

$$\begin{aligned} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} &= \begin{pmatrix} c_{11}^E & c_{12}^E & c_{13}^E & c_{14}^E & c_{15}^E & c_{16}^E \\ c_{21}^E & c_{22}^E & c_{23}^E & c_{24}^E & c_{25}^E & c_{26}^E \\ c_{31}^E & c_{32}^E & c_{33}^E & c_{34}^E & c_{35}^E & c_{36}^E \\ c_{41}^E & c_{42}^E & c_{43}^E & c_{44}^E & c_{45}^E & c_{46}^E \\ c_{51}^E & c_{52}^E & c_{53}^E & c_{54}^E & c_{55}^E & c_{56}^E \\ c_{61}^E & c_{62}^E & c_{63}^E & c_{64}^E & c_{65}^E & c_{66}^E \end{pmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} \\ &\quad - \begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \\ e_{14} & e_{24} & e_{34} \\ e_{15} & e_{25} & e_{35} \\ e_{16} & e_{26} & e_{36} \end{pmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}, \\ \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} &= \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26} \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} \end{bmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} \\ &\quad + \begin{pmatrix} \varepsilon_{11}^S & \varepsilon_{12}^S & \varepsilon_{13}^S \\ \varepsilon_{21}^S & \varepsilon_{22}^S & \varepsilon_{22}^S \\ \varepsilon_{31}^S & \varepsilon_{32}^S & \varepsilon_{33}^S \end{pmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}. \end{aligned} \tag{1.26}$$

1.4. Constitutive Relations of Polarized Ceramics

In this section we obtain the constitutive relations of polarized ceramics using tensor invariants in the manner of [6]. Polarized ceramics are transversely isotropic. Let **a**, a constant unit vector, represent the direction of the axis of rotational symmetry or the poling direction of the ceramics. For linear constitutive relations we need a quadratic electric enthalpy function H. For transversely isotropic materials, a quadratic H is a function of the following invariants of degrees one and two (higher degree invariants are not included):

$$\begin{aligned} &I_1 = \mathbf{a}\cdot\mathbf{S}\cdot\mathbf{a}, \quad I_2 = \mathrm{tr}\mathbf{S}, \quad I_3 = \mathbf{a}\cdot\mathbf{E}, \\ &II_1 = \mathbf{a}\cdot\mathbf{S}^2\cdot\mathbf{a}, \quad II_2 = \mathrm{tr}\mathbf{S}^2, \\ &II_3 = \mathbf{E}\cdot\mathbf{E}, \quad II_4 = \mathbf{a}\cdot\mathbf{S}\cdot\mathbf{E} + \mathbf{E}\cdot\mathbf{S}\cdot\mathbf{a}. \end{aligned} \tag{1.27}$$

A complete quadratic function of the above seven invariants can be written as

$$\begin{aligned} H = {} & c_1 I_1^2 + c_2 I_2^2 + c_3 I_1 I_2 + c_4 II_1 + c_5 II_2 \\ & + \varepsilon_1 I_3^2 + \varepsilon_2 II_3 \\ & + e_1 I_1 I_3 + e_2 I_2 I_3 + e_3 II_4, \end{aligned} \tag{1.28}$$

where c_1, c_2, c_3, c_4 and c_5 are elastic constants, ε_1 and ε_2 are dielectric constants, and e_1, e_2 and c_3 are piezoelectric constants. Differentiation of Eq. (1.28) yields

$$\begin{aligned} \mathbf{T} = \frac{\partial H}{\partial \mathbf{S}} = {} & \frac{\partial H}{\partial I_1}\mathbf{a}\otimes\mathbf{a} + \frac{\partial H}{\partial I_2}\mathbf{1} + \frac{\partial H}{\partial II_1}(\mathbf{a}\otimes\mathbf{S}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{S}\otimes\mathbf{a}) \\ & + 2\frac{\partial H}{\partial II_2}\mathbf{S} + \frac{\partial H}{\partial II_4}(\mathbf{a}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{a}) \\ = {} & (2c_1 I_1 + c_3 I_2 + e_1 I_3)\mathbf{a}\otimes\mathbf{a} + (2c_2 I_2 + c_3 I_1 + e_2 I_3)\mathbf{1} \\ & + c_4(\mathbf{a}\otimes\mathbf{S}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{S}\otimes\mathbf{a}) + 2c_5\mathbf{S} + e_3(\mathbf{a}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{a}), \end{aligned} \tag{1.29}$$

and

$$\begin{aligned} \mathbf{D} = -\frac{\partial H}{\partial \mathbf{E}} = {} & -\frac{\partial H}{\partial I_3}\mathbf{a} - 2\frac{\partial H}{\partial II_3}\mathbf{E} - 2\frac{\partial H}{\partial II_4}\mathbf{S}\cdot\mathbf{a} \\ = {} & -(2\varepsilon_1 I_3 + e_1 I_1 + e_2 I_2)\mathbf{a} - 2\varepsilon_2\mathbf{E} - 2e_3\mathbf{S}\cdot\mathbf{a}. \end{aligned} \tag{1.30}$$

Let **a** = **i**$_3$, and rearrange Eqs. (1.29) and (1.30) in the form of Eq. (1.25). The following matrices will result:

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{31} & c_{31} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}, \tag{1.31}$$

where $c_{66} = (c_{11} - c_{12})/2$. The matrices in Eq. (1.31) have the same structures as those of crystals class C_{6v} (or 6mm) [7]. The elements of the matrices in Eq. (1.31) are related to the material constants in Eq. (1.28) by

$$\begin{aligned} & c_1 = c_{11} - 2c_{13} + c_{33} - 4c_{44}, \quad c_2 = c_{12}/2, \\ & c_3 = c_{13} - c_{12}, \quad c_4 = -c_{11} + c_{12} + 2c_{44}, \\ & c_5 = (c_{11} - c_{12})/2, \\ & \varepsilon_1 = (\varepsilon_{11} - \varepsilon_{22})/2, \quad \varepsilon_2 = -\varepsilon_{11}/2, \\ & e_1 = e_{31} + 2e_{15} - e_{33}, \quad e_2 = -e_{31}, \quad e_3 = -e_{15}. \end{aligned} \tag{1.32}$$

With Eq. (1.31), the constitutive relations of ceramics poled in the x_3 direction take the following form:

$$\begin{aligned} T_{11} &= c_{11}u_{1,1} + c_{12}u_{2,2} + c_{13}u_{3,3} + e_{31}\phi_{,3}, \\ T_{22} &= c_{12}u_{1,1} + c_{22}u_{2,2} + c_{13}u_{3,3} + e_{31}\phi_{,3}, \\ T_{33} &= c_{13}u_{1,1} + c_{13}u_{2,2} + c_{33}u_{3,3} + e_{33}\phi_{,3}, \\ T_{23} &= c_{44}(u_{2,3} + u_{3,2}) + e_{15}\phi_{,2}, \\ T_{31} &= c_{44}(u_{3,1} + u_{1,3}) + e_{15}\phi_{,1}, \\ T_{12} &= c_{66}(u_{1,2} + u_{2,1}), \end{aligned} \tag{1.33}$$

and

$$\begin{aligned} D_1 &= e_{15}(u_{3,1} + u_{1,3}) - \varepsilon_{11}\phi_{,1}, \\ D_2 &= e_{15}(u_{2,3} + u_{3,2}) - \varepsilon_{11}\phi_{,2}, \\ D_3 &= e_{31}(u_{1,1} + u_{2,2}) + e_{33}u_{3,3} - \varepsilon_{33}\phi_{,3}. \end{aligned} \tag{1.34}$$

The equations of motion and charge are

$$\begin{aligned}
&c_{11}u_{1,11}+(c_{12}+c_{66})u_{2,12}+(c_{13}+c_{44})u_{3,13}+c_{66}u_{1,22}\\
&\quad+c_{44}u_{1,33}+(e_{31}+e_{15})\phi_{,13}=\rho\ddot{u}_1,\\
&c_{66}u_{2,11}+(c_{12}+c_{66})u_{1,12}+c_{11}u_{2,22}+(c_{13}+c_{44})u_{3,23}\\
&\quad+c_{44}u_{2,33}+(e_{31}+e_{15})\phi_{,23}=\rho\ddot{u}_2,\\
&c_{44}u_{3,11}+(c_{44}+c_{13})u_{1,31}+c_{44}u_{3,22}+(c_{13}+c_{44})u_{2,23}\\
&\quad+c_{33}u_{3,33}+e_{15}(\phi_{,11}+\phi_{,22})+e_{33}\phi_{,33}=\rho\ddot{u}_3,\\
&e_{15}u_{3,11}+(e_{15}+e_{31})u_{1,13}+e_{15}u_{3,22}+(e_{15}+e_{31})u_{2,32}\\
&\quad+e_{31}u_{3,33}-\varepsilon_{11}(\phi_{,11}+\phi_{,22})-\varepsilon_{33}\phi_{,33}=0.
\end{aligned} \tag{1.35}$$

Sometimes a piezoelectric device is heterogeneous, with ceramics poled in different directions in different parts. In this case it is not always possible to orient the x_3 axis along the poling directions unless a few local coordinate systems are introduced. Therefore, material matrices of ceramics poled along other axes are useful. They can be obtained by tensor transformations or effectively from the matrices in Eq. (1.31) by rotating rows and columns properly. For ceramics poled in the x_1 direction, we have

$$\begin{gathered}
\begin{pmatrix}
c_{33} & c_{13} & c_{13} & 0 & 0 & 0\\
c_{13} & c_{11} & c_{12} & 0 & 0 & 0\\
c_{13} & c_{12} & c_{11} & 0 & 0 & 0\\
0 & 0 & 0 & c_{66} & 0 & 0\\
0 & 0 & 0 & 0 & c_{44} & 0\\
0 & 0 & 0 & 0 & 0 & c_{44}
\end{pmatrix},\\
\begin{pmatrix}
e_{33} & e_{31} & e_{31} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & e_{15}\\
0 & 0 & 0 & 0 & e_{15} & 0
\end{pmatrix},
\begin{pmatrix}
\varepsilon_{33} & 0 & 0\\
0 & \varepsilon_{11} & 0\\
0 & 0 & \varepsilon_{11}
\end{pmatrix}.
\end{gathered} \tag{1.36}$$

For ceramics poled in the x_2 direction, we obtain

$$\begin{pmatrix} c_{11} & c_{13} & c_{12} & 0 & 0 & 0 \\ c_{13} & c_{33} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{13} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{pmatrix}, \tag{1.37}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & e_{15} \\ e_{31} & e_{33} & e_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{33} & 0 \\ 0 & 0 & \varepsilon_{11} \end{pmatrix}.$$

1.5. Antiplane Problems

For motions independent of x_3, Eq. (1.35) reduces to:

$$\begin{aligned} &c_{11}u_{1,11} + (c_{12} + c_{66})u_{2,12} + c_{66}u_{1,22} = \rho\ddot{u}_1, \\ &c_{66}u_{2,11} + (c_{12} + c_{66})u_{1,12} + c_{11}u_{2,22} = \rho\ddot{u}_2, \\ &c_{44}(u_{3,11} + u_{3,22}) + e_{15}(\phi_{,11} + \phi_{,22}) = \rho\ddot{u}_3, \\ &e_{15}(u_{3,11} + u_{3,22}) - \varepsilon_{11}(\phi_{,11} + \phi_{,22}) = 0. \end{aligned} \tag{1.38}$$

Equation (1.38) shows that u_1 and u_2 are coupled but they do not interact with the electric field. They form the usual plane-strain problem of linear elasticity. In this book we are interested in the so-called antiplane or shear-horizontal (SH) motions described by u_3 which is coupled to ϕ. In the rest of the book we limit ourselves to

$$u_1 = u_2 = 0, \quad u_3 = u(x_1, x_2, t), \quad \phi = \phi(x_1, x_2, t). \tag{1.39}$$

Corresponding to Eq. (1.39), the nonzero components of the strain tensor S_{ij} and electric field E_i are

$$\begin{Bmatrix} S_5 \\ S_4 \end{Bmatrix} = \begin{Bmatrix} 2S_{31} \\ 2S_{23} \end{Bmatrix} = \nabla u, \quad \begin{Bmatrix} E_1 \\ E_2 \end{Bmatrix} = -\nabla\phi, \tag{1.40}$$

where $\nabla = \mathbf{i}_1\partial_1 + \mathbf{i}_2\partial_2$ is the two-dimensional gradient operator. The nontrivial components of the stress tensor T_{ij} and the electric displacement vector D_i are

$$\begin{Bmatrix} T_5 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} T_{31} \\ T_{23} \end{Bmatrix} = c\nabla u + e\nabla\phi, \quad \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = e\nabla u - \varepsilon\nabla\phi, \tag{1.41}$$

where we have denoted the relevant elastic, piezoelectric, and dielectric constants by

$$c = c_{44}, \quad e = e_{15}, \quad \varepsilon = \varepsilon_{11}. \tag{1.42}$$

We will consider source-free problems with $q = 0$ and $f_3 = 0$. The equation of motion and the charge equation can be written as

$$\begin{aligned} &c\nabla^2 u + e\nabla^2 \phi = \rho \ddot{u}, \\ &e\nabla^2 u - \varepsilon\nabla^2 \phi = 0, \end{aligned} \tag{1.43}$$

where $\nabla^2 = \partial_1^2 + \partial_2^2$ is the two-dimensional Laplacian.

1.6. Bleustein's Formulation

Equation (1.43) can be decoupled [8]. We introduce

$$\psi = \phi - \frac{e}{\varepsilon} u. \tag{1.44}$$

Then, in terms of u and ψ,

$$\begin{aligned} T_{31} &= \bar{c} u_{,1} + e\psi_{,1}, \\ T_{23} &= \bar{c} u_{,2} + e\psi_{,2}, \end{aligned} \tag{1.45}$$

$$\begin{aligned} D_1 &= -\varepsilon\psi_{,1}, \\ D_2 &= -\varepsilon\psi_{,2}, \end{aligned} \tag{1.46}$$

and

$$\begin{aligned} &v_T^2 \nabla^2 u = \ddot{u}, \\ &\nabla^2 \psi = 0, \end{aligned} \tag{1.47}$$

where

$$v_T^2 = \frac{\bar{c}}{\rho}, \quad \bar{c} = c + \frac{e^2}{\varepsilon} = c(1 + k^2), \quad k^2 = \frac{e^2}{\varepsilon c}. \tag{1.48}$$

Equation (1.47) is defined over a two-dimensional domain (see Fig. 1.2). Let **n** and **s** be the unit normal and tangent of the boundary. Typical boundary conditions are the specifications of

$$\begin{aligned} &u \quad \text{or} \quad T_{n3}, \\ &\psi + \frac{e}{\varepsilon} u \quad \text{or} \quad D_n. \end{aligned} \tag{1.49}$$

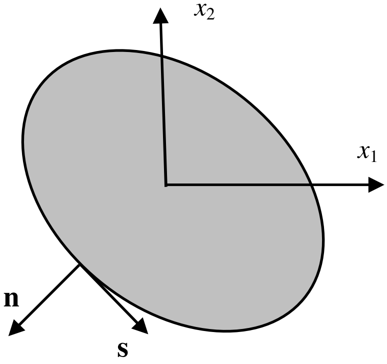

Fig. 1.2. Cross section of a cylindrical body of polarized ceramics.

Although Eq. (1.47) is decoupled, electromechanical coupling still exists in the constitutive relations in Eq. (1.45) and may appear in boundary conditions.

1.7. A Static General Solution in Polar Coordinates

For static problems Eq. (1.43) reduces to

$$\begin{aligned} c\nabla^2 u + e\nabla^2\phi &= 0, \\ e\nabla^2 u - \varepsilon\nabla^2\phi &= 0. \end{aligned} \tag{1.50}$$

When $c\varepsilon + e^2 \neq 0$ which we always assume, Eq. (1.50) is equivalent to

$$\begin{aligned} \nabla^2 u &= 0, \\ \nabla^2\phi &= 0. \end{aligned} \tag{1.51}$$

In polar coordinates, we have

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= 0, \\ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} &= 0. \end{aligned} \tag{1.52}$$

For problems periodic in θ, by separation of variables, the general solution for u and ϕ is

$$u = A_0(P_0 + Q_0 \ln r) + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)(P_n r^n + Q_n r^{-n}), \tag{1.53}$$

$$\phi = C_0(R_0 + S_0 \ln r) + \sum_{n=1}^{\infty} (C_n \cos n\theta + D_n \sin n\theta)(R_n r^n + S_n r^{-n}), \tag{1.54}$$

where A_n, B_n, C_n, D_n, P_n, Q_n, R_n and S_n are undetermined constants. The corresponding stress and electric displacement components are determined from

$$T_{rz} = c\frac{\partial u}{\partial r} + e\frac{\partial \phi}{\partial r}, \quad T_{\theta z} = c\frac{1}{r}\frac{\partial u}{\partial \theta} + e\frac{1}{r}\frac{\partial \phi}{\partial \theta}, \tag{1.55}$$

$$D_r = e\frac{\partial u}{\partial r} - \varepsilon\frac{\partial \phi}{\partial r}, \quad D_\theta = e\frac{1}{r}\frac{\partial u}{\partial \theta} - \varepsilon\frac{1}{r}\frac{\partial \phi}{\partial \theta}. \tag{1.56}$$

1.8. A Time-harmonic General Solution in Polar Coordinates

For time-harmonic problems with an $\exp(i\omega t)$ factor where “i” is the imaginary unit, we will drop the factor very often for simplicity. In polar coordinates, from Eq. (1.47) we have

$$v_T^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = -\omega^2 u, \quad \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0. \tag{1.57}$$

Consider the possibility of the following fields:

$$u = u(r) \times \begin{Bmatrix} \cos \nu\theta \\ \sin \nu\theta \end{Bmatrix}, \quad \psi = \psi(r) \times \begin{Bmatrix} \cos \nu\theta \\ \sin \nu\theta \end{Bmatrix}, \tag{1.58}$$

which are consequences of the method of separation of variables. Substitution of Eq. (1.58) into Eq. (1.57) results in

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + (\xi^2 - \frac{\nu^2}{r^2})u = 0, \tag{1.59}$$

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{\nu^2}{r^2}\psi = 0, \tag{1.60}$$

where we have denoted

$$\xi = \frac{\omega}{v_T}. \tag{1.61}$$

Equation (1.59) can be written as Bessel's equation of order ν. Equation (1.60) allows a simpler power function solution. The general solution can be written as:

$$\begin{aligned} u &= C_0 J_0(\xi r) + D_0 Y_0(\xi r) \\ &\quad + \sum_{m=1}^{\infty} [C_m J_{\nu_m}(\xi r) + D_m Y_{\nu_m}(\xi r)]\cos\nu_m\theta, \end{aligned} \tag{1.62}$$

$$\begin{aligned} \psi &= F_0 \ln r + G_0 \\ &\quad + \sum_{m=1}^{\infty} [F_m r^{\nu_m} + G_m r^{-\nu_m}]\cos\nu_m\theta, \end{aligned} \tag{1.63}$$

where C_m, D_m, F_m and G_m are undetermined constants. The electric potential, the stress and the electric displacement components can be obtained as

$$\begin{aligned} \phi &= \psi + \frac{e}{\varepsilon}u \\ &= \frac{e}{\varepsilon}C_0 J_0(\xi r) + \frac{e}{\varepsilon}D_0 Y_0(\xi r) + F_0 \ln r + G_0 \\ &\quad + \sum_{m=1}^{\infty} [\frac{e}{\varepsilon}C_m J_{\nu_m}(\xi r) + \frac{e}{\varepsilon}D_m Y_{\nu_m}(\xi r) \\ &\qquad + F_m r^{\nu_m} + G_m r^{-\nu_m}]\cos\nu_m\theta, \end{aligned} \tag{1.64}$$

$$\begin{aligned} T_{rz} &= \bar{c}\frac{\partial u}{\partial r} + e\frac{\partial \psi}{\partial r} \\ &= -\bar{c}\xi C_0 J_1(\xi r) - \bar{c}\xi D_0 Y_1(\xi r) + eF_0\frac{1}{r} \\ &\quad + \sum_{m=1}^{\infty} [\bar{c}\xi C_m J'_{\nu_m}(\xi r) + \bar{c}\xi D_m Y'_{\nu_m}(\xi r) \\ &\quad + e\nu_m F_m r^{\nu_m - 1} - e\nu_m G_m r^{-\nu_m - 1}]\cos\nu_m\theta, \end{aligned} \tag{1.65}$$

$$\begin{aligned} T_{\theta z} &= \bar{c}\frac{1}{r}\frac{\partial u}{\partial \theta} + e\frac{1}{r}\frac{\partial \psi}{\partial \theta} \\ &= \frac{1}{r}\sum_{m=1}^{\infty} [\bar{c}C_m J_{\nu_m}(\xi r) + \bar{c}D_m Y_{\nu_m}(\xi r) \\ &\quad + eF_m r^{\nu_m} + eG_m r^{-\nu_m}](-\nu_m)\sin\nu_m\theta, \end{aligned} \tag{1.66}$$

$$\begin{aligned} D_r &= -\varepsilon\frac{\partial \psi}{\partial r} \\ &= -\varepsilon F_0\frac{1}{r} + \sum_{m=1}^{\infty} [-\varepsilon\nu_m F_m r^{\nu_m - 1} + \varepsilon\nu_m G_m r^{-\nu_m - 1}]\cos\nu_m\theta, \end{aligned} \tag{1.67}$$

$$\begin{aligned} D_\theta &= -\varepsilon\frac{1}{r}\frac{\partial \psi}{\partial \theta} \\ &= -\varepsilon\frac{1}{r}\sum_{m=1}^{\infty} [eF_m r^{\nu_m} + eG_m r^{-\nu_m}](-\nu_m)\sin\nu_m\theta. \end{aligned} \tag{1.68}$$

For solutions periodic in θ, we must have

$$\nu_m = m, \quad m = 1, 2, 3, \cdots . \tag{1.69}$$

1.9. Boundary Integral Equation Formulation

For time-harmonic motions, with Eq. (1.61), we can write Eq. (1.57) as

$$\begin{aligned} &\nabla^2 u + \xi^2 u = 0, \\ &\nabla^2 \psi = 0. \end{aligned} \tag{1.70}$$

Let u^* and ψ^* be the fundamental solutions of the following differential operators:

$$\begin{aligned} &-\nabla^2 u^* - \xi^2 u^* = \delta, \\ &-\nabla^2 \psi^* = \delta, \end{aligned} \tag{1.71}$$

where δ is the Dirac delta function. For any two functions over a two-dimensional domain A with a boundary curve C, there exist the following Green's identities:

$$\begin{aligned} &\int_A [u(\nabla^2 u^* + \xi^2 u^*) - u^*(\nabla^2 u + \xi^2 u)]dA \\ &= \int_C \left[u\frac{\partial u^*}{\partial n} - u^*\frac{\partial u}{\partial n}\right]dL, \end{aligned} \tag{1.72}$$

$$\int_A [\psi\nabla^2\psi^* - \psi^*\nabla^2\psi]dA = \int_C \left[\psi\frac{\partial\psi^*}{\partial n} - \psi^*\frac{\partial\psi}{\partial n}\right]dL, \tag{1.73}$$

where *dA* and *dL* are differential area and line elements. From Eqs. (1.70) through (1.73), we obtain the following boundary integral equations for u and ψ:

$$\begin{aligned}\frac{1}{2}u(P) &= -\int_C \frac{\partial u^*(P,Q)}{\partial n(Q)}u(Q)dL(Q)\\ &\quad + \int_C u^*(P,Q)\frac{\partial u(Q)}{\partial n(Q)}dL(Q),\end{aligned} \tag{1.74}$$

$$\begin{aligned}\frac{1}{2}\psi(P) &= -\int_C \frac{\partial\psi^*(P,Q)}{\partial n(Q)}\psi(Q)dL(Q)\\ &\quad + \int_C \psi^*(P,Q)\frac{\partial\psi(P,Q)}{\partial n(Q)}dL(Q).\end{aligned} \tag{1.75}$$

The fundamental solutions can be found in [9].