Chapter 4

Scattering Theory

Given a hamiltonian $\hat{H}$, the goal of this chapter is to solve the Schrödinger equation for a scattering problem and derive general expressions for the S-matrix, T-matrix, and transition rate, many of whose consequences have already been examined in Vol. I.[1] We work in the abstract Hilbert space.

4.1 Interaction Picture

Assume the hamiltonian can be split into two parts $\hat{H} = \hat{H}_0 + \hat{H}_1$, the first part of which leads to an exactly solvable problem, for example, free quanta with no interactions. $\hat{H}_1$ may, or may not, have an explicit time dependence; that depends on the problem at hand.[2] We then want to solve the Schrödinger equation

$$\begin{aligned} \hat{H} &= \hat{H}_0 + \hat{H}_1 \\ i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle &= \hat{H}\,|\Psi(t)\rangle \qquad ; \text{ Schrödinger-equation} \end{aligned} \tag{4.1}$$

Define a new state vector $|\Psi_{\rm I}(t)\rangle$ by

$$\begin{aligned} |\Psi_{\rm I}(t)\rangle &\equiv e^{\frac{i}{\hbar}\hat{H}_0 t}|\Psi(t)\rangle \qquad &; \text{ interaction picture} \\ |\Psi_{\rm I}(0)\rangle &= |\Psi(0)\rangle \qquad &; \text{ coincide at } t=0 \end{aligned} \tag{4.2}$$

[1] For a comprehensive treatment of scattering theory, see [Goldberger and Watson (2004)].

[2] Scattering in a given external field, for example, may lead to an explicitly time-dependent $\hat{H}_1(t)$.

What equation of motion does this new state satisfy? Just compute

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}|\Psi_{\rm I}(t)\rangle &= -\hat{H}_0\, e^{\frac{i}{\hbar}\hat{H}_0 t}|\Psi(t)\rangle + e^{\frac{i}{\hbar}\hat{H}_0 t}\, i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle \\
&= -\hat{H}_0\,|\Psi_{\rm I}(t)\rangle + e^{\frac{i}{\hbar}\hat{H}_0 t}(\hat{H}_0+\hat{H}_1)e^{-\frac{i}{\hbar}\hat{H}_0 t}|\Psi_{\rm I}(t)\rangle \qquad (4.3)
\end{aligned}
$$

The terms in $\hat{H}_0$ cancel, and thus

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}|\Psi_{\rm I}(t)\rangle &= \hat{H}_{\rm I}(t)|\Psi_{\rm I}(t)\rangle \qquad ;\ \text{interaction picture} \\
\hat{H}_{\rm I}(t) &\equiv e^{\frac{i}{\hbar}\hat{H}_0 t}\,\hat{H}_1\, e^{-\frac{i}{\hbar}\hat{H}_0 t} \qquad (4.4)
\end{aligned}
$$

The advantage of this new formulation is that in the limit $\hat{H}_1 \to 0$, the state $|\Psi_{\rm I}(t)\rangle$ becomes time-independent; the free time variation, which can be extremely rapid, has been explicitly dealt with. Equations (4.2) and (4.4) are said to be a formulation of the problem in the *interaction picture*.

4.2 Adiabatic Approach

We will find that when we try to solve the resulting equations and generate the S-matrix, there will be infinite time integrals to carry out over oscillating integrands. In order to give the theory a well-defined mathematical meaning, we introduce an *adiabatic damping factor* $e^{-\epsilon|t|}$ with $\epsilon \geq 0$, and use the following interaction in the interaction picture

$$
\hat{H}_{\rm I}^{\epsilon}(t) \equiv e^{-\epsilon|t|}\hat{H}_{\rm I}(t) \qquad ;\ \text{adiabatic damping} \qquad (4.5)
$$

The theory is then defined to be what is obtained in the limit as $\epsilon \to 0$.[3]

This is a somewhat archaic approach, and there are more sophisticated ways of doing formal scattering theory, which, however, can easily lead to spurious results if one is not very careful and thoughtful. The great advantage of this adiabatic approach is that it *allows one to do well-defined mathematics at each step.*

One can imagine that the interaction in Eq. (4.5) is being turned on and off very slowly ("adiabatically") as the time $t \to \pm\infty$, that is, in the infinite past and infinite future.[4] This allows us to easily specify the initial and final states in any scattering process, since now as $t \to \pm\infty$,

[3]There may, or may not, be other limits — we will not go there.
[4]Explicitly dealing with the scattering of wave packets can play the same role.

the hamiltonian simply reduces to $\hat{H}_0$, and we know how to solve the non-interacting problem

$$\begin{aligned} \hat{H} &= \hat{H}_0 & ;\ t \to \pm\infty \\ i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle &= \hat{H}_0\,|\Psi(t)\rangle \\ |\Psi(t)\rangle &= e^{-\frac{i}{\hbar}E_0 t}\,|\psi\rangle \end{aligned} \tag{4.6}$$

Here $|\psi\rangle$ is simply a *solution to the free, time-independent, Schrödinger equation*

$$\hat{H}_0\,|\psi\rangle = E_0|\psi\rangle \tag{4.7}$$

The interaction-picture state vector in Eq. (4.2) is then given in this same limit by

$$\begin{aligned} |\Psi_{\rm I}(t)\rangle &= e^{\frac{i}{\hbar}\hat{H}_0 t}\,|\Psi(t)\rangle = |\psi\rangle & ;\ t \to \pm\infty \\ i\hbar\frac{\partial}{\partial t}|\Psi_{\rm I}(t)\rangle &= 0 \end{aligned} \tag{4.8}$$

Thus, in summary, with the adiabatic approach in the interaction picture, one has

$$\begin{aligned} |\Psi_{\rm I}(t)\rangle &= |\psi\rangle & ;\ t \to \pm\infty \\ \hat{H}_0\,|\psi\rangle &= E_0|\psi\rangle \end{aligned} \tag{4.9}$$

One starts with an initial state of this type, and then slowly turns on and off the interaction. The transition amplitude into a final state of this type is then calculated. The (transition probability)/(time interval the interaction is on) gives the transition rate,[5] and the path from the transition rate to a cross section was detailed in Vol. I.

It is then necessary to determine what happens when the interaction in Eq. (4.5) is turned on and off adiabatically. This is done through the construction of the *time-development operator* for the problem.

[5] We shall get more sophisticated here and actually derive a general expression for the transition rate itself.

4.3 $\hat{U}$-Operator

Let us look for an operator that develops our system in time

$$
\begin{aligned}
|\Psi_{\rm I}(t)\rangle &= \hat{U}_\epsilon(t,t_0)|\Psi_{\rm I}(t_0)\rangle \\
i\hbar\frac{\partial}{\partial t}|\Psi_{\rm I}(t)\rangle &= i\hbar\frac{\partial}{\partial t}\hat{U}_\epsilon(t,t_0)|\Psi_{\rm I}(t_0)\rangle = \hat{H}_{\rm I}^\epsilon(t)\hat{U}_\epsilon(t,t_0)|\Psi_{\rm I}(t_0)\rangle
\end{aligned} \tag{4.10}
$$

If this is to hold for all $|\Psi_{\rm I}(t_0)\rangle$, then $\hat{U}_\epsilon(t,t_0)$ must satisfy the operator relation

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\hat{U}_\epsilon(t,t_0) &= \hat{H}_{\rm I}^\epsilon(t)\hat{U}_\epsilon(t,t_0) \\
\hat{U}_\epsilon(t_0,t_0) &= 1
\end{aligned} \tag{4.11}
$$

This differential equation, with its initial condition, can be rewritten as an *integral equation*

$$
\hat{U}_\epsilon(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} e^{-\epsilon|t'|}\,\hat{H}_{\rm I}(t')\hat{U}_\epsilon(t',t_0)\,dt' \tag{4.12}
$$

It is readily verified that Eqs. (4.11) are reproduced by this expression.

We will try to find a solution to this equation as a power series in $\hat{H}_{\rm I}$.[6] Let us substitute this expression for $\hat{U}_\epsilon(t',t_0)$ in the integrand on the r.h.s.

$$
\begin{aligned}
&\hat{U}_\epsilon(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} e^{-\epsilon|t'|}\,\hat{H}_{\rm I}(t')\,dt' + \\
&\left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} e^{-\epsilon|t'|}\,\hat{H}_{\rm I}(t')\,dt' \int_{t_0}^{t'} e^{-\epsilon|t''|}\,\hat{H}_{\rm I}(t'')\hat{U}_\epsilon(t'',t_0)\,dt''
\end{aligned} \tag{4.13}
$$

This expression is still exact. Repeated application of this process leads to the following infinite series in $\hat{H}_{\rm I}$

$$
\begin{aligned}
\hat{U}_\epsilon(t,t_0) = \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n &\int_{t_0}^{t} e^{-\epsilon|t_1|}\,dt_1 \int_{t_0}^{t_1} e^{-\epsilon|t_2|}\,dt_2 \cdots \int_{t_0}^{t_{n-1}} e^{-\epsilon|t_n|}\,dt_n \times \\
&\hat{H}_{\rm I}(t_1)\hat{H}_{\rm I}(t_2)\cdots\hat{H}_{\rm I}(t_n)
\end{aligned} \tag{4.14}
$$

[6] One can only expect a power series to hold for scattering amplitudes at all energies in the absence of bound states; however, we will eventually “zip things up again” and obtain closed forms that are also valid in the presence of bound states.

By convention, the first term in this series is 1. Note that it is important to keep the *ordering* of the operators $\hat{H}_{\rm I}(t)$ straight in the integrand, since they do not necessarily commute at different times. It is easy to remember the ordering since the operators are *time-ordered*, with the operator at the latest time appearing furthest to the left.

Equation (4.14) can be rewritten in the following manner

$$\hat{U}_\epsilon(t,t_0) = \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \frac{1}{n!}\int_{t_0}^{t} e^{-\epsilon|t_1|}\,dt_1 \int_{t_0}^{t} e^{-\epsilon|t_2|}\,dt_2 \cdots \int_{t_0}^{t} e^{-\epsilon|t_n|}\,dt_n \times$$
$$T\left[\hat{H}_{\rm I}(t_1)\hat{H}_{\rm I}(t_2)\cdots\hat{H}_{\rm I}(t_n)\right] \qquad ;\; t \geq t_0 \tag{4.15}$$

Here

- All the integrals are now over the full range $\int_{t_0}^{t}$;
- The "T-product" carries the instruction that the operators are to be time-ordered, with the operator at the latest time sitting to the left;
- Each term in the sum is divided by $n!$.

The proof that Eq. (4.15) reproduces Eq. (4.14) is quite simple. There are $n!$ possible orderings of the times in the multiple integral, pick one, say $t_1 > t_2 > t_3 > \cdots > t_n$. All possible time orderings of these integration variables provides a complete enumeration of the region of integration in the multiple integral. The operator in the integrand is time-ordered in each case. But now all of these contributions are *identical* by a change of dummy integration variables. Thus Eq. (4.14) is reproduced.[7]

The *scattering operator* $\hat{S}$ is now defined in the following manner

$$\hat{S} \equiv \mathrm{Lim}_{\epsilon\to 0}\ \mathrm{Lim}_{t\to+\infty}\ \mathrm{Lim}_{t_0\to-\infty}\ \hat{U}_\epsilon(t,t_0) \tag{4.16}$$

One lets the initial time $t_0 \to -\infty$, the final time $t \to +\infty$, and then, at the very end, the limit of the adiabatic damping factor $\epsilon \to 0$ is taken. Thus

$$\begin{aligned}\hat{S} &= \mathrm{Lim}_{\epsilon\to 0}\ \hat{S}_\epsilon \\ &= \mathrm{Lim}_{\epsilon\to 0}\ \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \frac{1}{n!}\int_{-\infty}^{\infty} e^{-\epsilon|t_1|}\,dt_1 \cdots \int_{-\infty}^{\infty} e^{-\epsilon|t_n|}\,dt_n \times \\ &\quad T\left[\hat{H}_{\rm I}(t_1)\hat{H}_{\rm I}(t_2)\cdots\hat{H}_{\rm I}(t_n)\right]\end{aligned} \tag{4.17}$$

Everything so far has assumed $t \geq t_0$ in Eqs. (4.11) and the subsequent development; however, one can equally well write these equations for $t \leq t_0$.

[7]The explicit demonstration of this equality for $n = 2$ is assigned as Prob. 4.1.

How is the above analysis modified? Write Eq. (4.12) in the following fashion

$$\hat{U}_\epsilon(t,t_0) = 1 + \frac{i}{\hbar}\int_t^{t_0} e^{-\epsilon|t'|}\,\hat{H}_{\rm I}(t')\hat{U}_\epsilon(t',t_0)\,dt' \tag{4.18}$$

It is readily verified that this expression reproduces Eqs. (4.11), and it is most convenient since the integral now runs in the positive direction if $t_0 \geq t$. A repetition of the above arguments in this case then leads to the following infinite series

$$\hat{U}_\epsilon(t,t_0) = \sum_{n=0}^{\infty}\left(\frac{i}{\hbar}\right)^n \frac{1}{n!}\int_t^{t_0} e^{-\epsilon|t_1|}\,dt_1 \int_t^{t_0} e^{-\epsilon|t_2|}\,dt_2 \cdots \int_t^{t_0} e^{-\epsilon|t_n|}\,dt_n \times$$
$$\overline{T}\left[\hat{H}_{\rm I}(t_1)\hat{H}_{\rm I}(t_2)\cdots\hat{H}_{\rm I}(t_n)\right] \qquad ;\ t \leq t_0 \tag{4.19}$$

The "$\overline{T}$-product" instructs the operators to be anti-time-ordered such that the operator with the *earliest* time sits to the left. A simple reversal of the limits of integration in each integral then gives the equivalent expression

$$\hat{U}_\epsilon(t,t_0) = \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \frac{1}{n!}\int_{t_0}^{t} e^{-\epsilon|t_1|}\,dt_1 \int_{t_0}^{t} e^{-\epsilon|t_2|}\,dt_2 \cdots \int_{t_0}^{t} e^{-\epsilon|t_n|}\,dt_n \times$$
$$\overline{T}\left[\hat{H}_{\rm I}(t_1)\hat{H}_{\rm I}(t_2)\cdots\hat{H}_{\rm I}(t_n)\right] \qquad ;\ t \leq t_0 \tag{4.20}$$

We are now in a position to exhibit some of the properties of $\hat{U}_\epsilon(t,t_0)$ from these series expansions:[8]

(1) Since the adjoint of a product is the product of the adjoints in the reverse order, it follows immediately from Eqs. (4.15) and (4.19) that

$$\hat{U}_\epsilon(t,t_0)^\dagger = \hat{U}_\epsilon(t_0,t) \tag{4.21}$$

which holds for both $t > t_0$ and $t < t_0$.

(2) If one ends up back at the start time, no matter whether $t > t_0$ or $t < t_0$, it must be true that

$$\hat{U}_\epsilon(t_0,t)\hat{U}_\epsilon(t,t_0) = 1 \tag{4.22}$$

This follows from the series expansions, and the explicit demonstration of this relation for $n = 2$ is left as Prob. 4.1.

(3) It follows from the results in (1) and (2) that

$$\hat{U}_\epsilon(t,t_0)^\dagger = \hat{U}_\epsilon(t,t_0)^{-1} \qquad ;\ \text{unitary} \tag{4.23}$$

[8]Note that relations (1)–(4) hold for finite ϵ.

The time-evolution operator is *unitary.* We know this must be true, since the Schrödinger equation preserves the *norm* of the states. This is now readily verified from the relation

$$
\begin{aligned}
\langle\Psi_{\mathrm{I}}(t)|\Psi_{\mathrm{I}}(t)\rangle &= \langle\Psi_{\mathrm{I}}(t_0)|\hat{U}_\epsilon(t,t_0)^\dagger\hat{U}_\epsilon(t,t_0)|\Psi_{\mathrm{I}}(t_0)\rangle \\
&= \langle\Psi_{\mathrm{I}}(t_0)|\hat{U}_\epsilon(t,t_0)^{-1}\hat{U}_\epsilon(t,t_0)|\Psi_{\mathrm{I}}(t_0)\rangle \\
&= \langle\Psi_{\mathrm{I}}(t_0)|\Psi_{\mathrm{I}}(t_0)\rangle \qquad (4.24)
\end{aligned}
$$

(4) If one propagates the system from $t_0 \to t_1$, and then from $t_1 \to t_2$, the result must be the same as propagation from $t_0 \to t_2$. Thus the time-evolution operator must obey the *group property*

$$
\hat{U}_\epsilon(t_2,t_1)\hat{U}_\epsilon(t_1,t_0) = \hat{U}_\epsilon(t_2,t_0) \qquad ; \text{ group property} \qquad (4.25)
$$

Let us demonstrate this result for $t_2 > t_1 > t_0$. The result in (2) can then be used to extend it to any relative times. For example, if $t_1 > t_2$, just write

$$
\begin{aligned}
\hat{U}_\epsilon(t_2,t_1)\hat{U}_\epsilon(t_1,t_0) &= \hat{U}_\epsilon(t_2,t_1)\hat{U}_\epsilon(t_1,t_2)\hat{U}_\epsilon(t_2,t_0) \\
&= \hat{U}_\epsilon(t_2,t_0) \qquad ; \; t_1 > t_2 \qquad (4.26)
\end{aligned}
$$

Write out the νth term in the sum on the r.h.s. of Eq. (4.25)

$$
\begin{aligned}
\hat{U}_\epsilon^{(\nu)}(t_2,t_0) = \left(-\frac{i}{\hbar}\right)^\nu \frac{1}{\nu!}\int_{t_0}^{t_2} e^{-\epsilon|t_1'|}\,dt_1'\cdots\int_{t_0}^{t_2} e^{-\epsilon|t_\nu'|}\,dt_\nu' \times \\
T\left[\hat{H}_{\mathrm{I}}(t_1')\hat{H}_{\mathrm{I}}(t_2')\cdots\hat{H}_{\mathrm{I}}(t_\nu')\right] \qquad (4.27)
\end{aligned}
$$

Now note:

- There are $\nu!/n!m!$ ways to partition the times $t_1'\cdots t_\nu'$ so that n times are greater than the intermediate time t_1, and m times are less than t_1 — pick one;
- Now integrate over all possible relative orderings of the times within this particular partition;
- Then sum over all possible choices of the times within this particular partition. This provides a complete enumeration of the regions of integration for a given (n,m);
- The contributions in the sum are identical by a change of dummy integration variables, giving $\nu!/n!m!$ equal contributions;

- Then sum over all values of (n,m) for which $m+n=\nu$. This provides a complete evaluation of the multiple integral in Eq. (4.27)

$$\hat{U}_\epsilon^{(\nu)}(t_2,t_0) = \frac{1}{\nu!}\sum_{n+m=\nu}\left(-\frac{i}{\hbar}\right)^{n+m}\frac{\nu!}{n!m!}\times \tag{4.28}$$
$$\int_{t_1}^{t_2} e^{-\epsilon|t_1'|}\,dt_1'\cdots\int_{t_1}^{t_2} e^{-\epsilon|t_n'|}\,dt_n'\;T\left[\hat{H}_\mathrm{I}(t_1')\cdots\hat{H}_\mathrm{I}(t_n')\right]\times$$
$$\int_{t_0}^{t_1} e^{-\epsilon|t_{n+1}'|}\,dt_{n+1}'\cdots\int_{t_0}^{t_1} e^{-\epsilon|t_{n+m}'|}\,dt_{n+m}'\;T\left[\hat{H}_\mathrm{I}(t_{n+1}')\cdots\hat{H}_\mathrm{I}(t_{n+m}')\right]$$

- Finally, use $\sum_\nu\sum_{n+m=\nu} = \sum_n\sum_m$. This establishes Eq. (4.25).

4.4 $\hat{U}$-Operator for Finite Times

We started from the hamiltonian

$$\hat{H}_\epsilon = \hat{H}_0 + e^{-\epsilon|t|}\,\hat{H}_1 \tag{4.29}$$

In the end, we are to take the limit $\epsilon\to 0$, which restores the proper hamiltonian. Let us assume that we have used the preceding analysis to propagate the system from its initial state at $t_0\to-\infty$ to a finite time such that

$$|t| \ll 1/\epsilon \qquad ;\ \text{finite time} \tag{4.30}$$

Now, for this time,

$$\hat{H} = \hat{H}_0 + \hat{H}_1 \qquad ;\ \text{full } \hat{H} \tag{4.31}$$

In this case, we can write a formal solution to the full Schrödinger equation as[9]

$$|\Psi_i(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\Psi_i(0)\rangle \tag{4.32}$$

Here $|\Psi_i(0)\rangle = |\Psi_\mathrm{I}^i(0)\rangle$ is the state that has propagated up to the time $t=0$ from the initial state $|\psi_i\rangle$ prepared at $t_0\to-\infty$ [see Eqs. (4.2)]. With the aid of the previous time-evolution operator, one can write this state as

$$|\Psi_i(0)\rangle = |\Psi_\mathrm{I}^i(0)\rangle = \hat{U}_\epsilon(0,-\infty)|\psi_i\rangle \equiv |\psi_i^{(+)}\rangle \tag{4.33}$$

[9]We assume here and henceforth that $\hat{H}_1$ now has no explicit time dependence.

This relation defines $|\psi_i^{(+)}\rangle$. A combination of Eqs. (4.32) and (4.33) allows the solution to the Schrödinger equation at a finite time, which satisfies Eq. (4.30), to be expressed as

$$|\Psi_i(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi_i^{(+)}\rangle \tag{4.34}$$

Some comments:

- This is the full Schrödinger state vector that develops from the state $|\psi_i\rangle$ at $t_0 \to -\infty$;
- One needs the adiabatic damping factor to bring that state vector up to finite time with $|\Psi_i(0)\rangle = \hat{U}_\epsilon(0, -\infty)|\psi_i\rangle \equiv |\psi_i^{(+)}\rangle$;
- From there, one can use the formal solution to the full Schrödinger equation in Eq. (4.34).

We note that under the conditions that one can indeed use the formal solution to the full Schrödinger equation, it follows that the interaction-picture state vector at the time t is given by

$$\begin{aligned}|\Psi_{\rm I}(t)\rangle &= e^{\frac{i}{\hbar}\hat{H}_0 t}|\Psi(t)\rangle = e^{\frac{i}{\hbar}\hat{H}_0 t}e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}|\Psi(t_0)\rangle \\ &= e^{\frac{i}{\hbar}\hat{H}_0 t}e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}e^{-\frac{i}{\hbar}\hat{H}_0 t_0}|\Psi_{\rm I}(t_0)\rangle\end{aligned} \tag{4.35}$$

Here $|\Psi_{\rm I}(t_0)\rangle$ is the interaction-picture state vector at the time t_0. But now we can immediately identify the time development operator $\hat{U}(t, t_0)$ from the first of Eqs. (4.10)!

$$\hat{U}(t, t_0) = e^{\frac{i}{\hbar}\hat{H}_0 t}e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}e^{-\frac{i}{\hbar}\hat{H}_0 t_0} \qquad ; \; |t|, |t_0| \ll 1/\epsilon \tag{4.36}$$

It is only necessary to keep careful track of the ordering of the operators, and make sure that one never interchanges factors that do not commute.

Several of our previous properties of the time-development operator follow immediately from the expression in Eq. (4.36):

$$\begin{aligned}\hat{U}(t, t_0)^\dagger &= \hat{U}(t_0, t) \\ \hat{U}(t, t_0)^\dagger &= \hat{U}(t, t_0)^{-1} \qquad ; \text{ unitary} \\ \hat{U}(t_1, t_2)\hat{U}(t_2, t_3) &= \hat{U}(t_1, t_3) \qquad ; \text{ group property}\end{aligned} \tag{4.37}$$

4.5 The *S*-Matrix

The interaction-picture state vector in the infinite future $|\Psi_{\rm I}(+\infty)\rangle$ that develops from the interaction-picture state vector in the infinite past

$|\Psi_{\rm I}(-\infty)\rangle$ is obtained with the scattering operator in Eq. (4.17)

$$|\Psi_{\rm I}(+\infty)\rangle = \hat{S}\,|\Psi_{\rm I}(-\infty)\rangle \qquad ; \text{ scattering operator} \qquad (4.38)$$

Now, with the adiabatic damping factor, the interaction state vectors in the infinite past and infinite future are simple, they are just the individual non-interacting state vectors in Eq. (4.9), or linear combinations of them. Thus, if one starts with one such prepared state $|\Psi_{\rm I}^i(-\infty)\rangle = |\psi_i\rangle$, and asks for the probability for finding a particular state $|\psi_f\rangle$ in the final state $|\Psi_{\rm I}^i(+\infty)\rangle$ that evolves, in the presence of all the interactions, from that initial prepared state, one has

$$P_{fi} = |\langle\psi_f|\Psi_{\rm I}^i(+\infty)\rangle|^2 = |\langle\psi_f|\hat{S}\,|\Psi_{\rm I}^i(-\infty)\rangle|^2 = |\langle\psi_f|\hat{S}|\psi_i\rangle|^2 \quad (4.39)$$

This is the probability of finding the initial state $|\psi_i\rangle$ in the final state $|\psi_f\rangle$ *after* the scattering has taken place. Here $|\psi_i\rangle$ and $|\psi_f\rangle$ are eigenstates of the free hamiltonian $\hat{H}_0$. The *amplitude* for this process to take place is given by the S-matrix

$$S_{fi} \equiv \langle\psi_f|\hat{S}|\psi_i\rangle \qquad ; \ S\text{-matrix} \qquad (4.40)$$

It was argued in Vol. I that the general form of the S-matrix for a scattering process is

$$S_{fi} = \delta_{fi} - 2\pi i\delta(E_f - E_i)\,\tilde{T}_{fi} \qquad (4.41)$$

where $\tilde{T}_{fi}$ is the T-matrix. There will always be an energy-conserving delta function here coming out of any calculation.[10]

The probability of making a *transition* to a state $f \neq i$ is therefore

$$P_{fi} = |2\pi i\delta(E_f - E_i)|^2|\tilde{T}_{fi}|^2 \qquad ; \text{ probability of transition} \quad (4.42)$$

It was argued in Vol. I that the square of the energy-conserving δ-function is to be interpreted as

$$\begin{aligned}|2\pi i\delta(E_f - E_i)|^2 &= 2\pi\delta(E_f - E_i)\,\frac{1}{\hbar}\int_{-\mathcal{T}/2}^{\mathcal{T}/2} dt\, e^{\frac{i}{\hbar}(E_f - E_i)t} \\ &= \frac{2\pi}{\hbar}\delta(E_f - E_i)\mathcal{T} \qquad ; \ \mathcal{T}\to\infty \qquad (4.43)\end{aligned}$$

[10]Compare Eq. (4.53) and Prob. 4.8. In Vol. I we removed some additional factors in the definition of the T-matrix element T_{fi} [see EqI. (7.36) and Eq. (7.38)].

where $\mathcal{T} \to \infty$ is the total time the interaction is turned on. The transition *rate* is then given by

$$\omega_{fi} = \frac{P_{fi}}{\mathcal{T}}$$
$$\omega_{fi} = \frac{2\pi}{\hbar}\delta(E_f - E_i)|\tilde{T}_{fi}|^2 \qquad ; \text{ transition rate } (f \neq i) \qquad (4.44)$$

This is the transition rate into *one* final state in the continuum. To get the transition rate into the *group* of states that actually get into our detectors when the states are spaced very close together, one must multiply this expression by the appropriate number of states dn_f. To get a *cross section*, one divides by the incident flux

$$d\sigma = \frac{2\pi}{\hbar}\delta(E_f - E_i)|\tilde{T}_{fi}|^2 \frac{dn_f}{I_{\text{inc}}} \qquad ; \text{ cross section} \qquad (4.45)$$

Some comments:

- All of these expressions were discussed and utilized frequently in Vol. I;
- Eq. (4.44) is the full expression for Fermi's Golden Rule, to all orders in the interaction;
- The derivation of the result for the transition rate involves some refinement when adiabatic switching is invoked, in contrast to the sudden turn-on and turn-off of the interaction in Vol. I; however, a proper derivation of the transition rate in this case, which we shall subsequently carry out, gives essentially the same result

$$S_{fi} = \delta_{fi} - 2\pi i \delta(E_f - E_i)\,\tilde{T}_{fi}$$
$$\omega_{fi} = \frac{2}{\hbar}\delta_{fi}\,\text{Im}\,\tilde{T}_{ii} + \frac{2\pi}{\hbar}\delta(E_f - E_i)|\tilde{T}_{fi}|^2 \quad ; \text{ transition rate} \quad (4.46)$$

4.6 Time-Independent Analysis

We will now perform some formal manipulations on the above results. Let us try to *explicitly carry out the time integrations* in the general term in the S-matrix in Eq. (4.17), which we rewrite in its initial time-ordered form

$$\langle\psi_f|\hat{S}_\epsilon^{(n)}|\psi_i\rangle = \left(-\frac{i}{\hbar}\right)^n \int_{-\infty}^{\infty} e^{-\epsilon|t_1|}\,dt_1 \int_{-\infty}^{t_1} e^{-\epsilon|t_2|}\,dt_2 \cdots \int_{-\infty}^{t_{n-1}} e^{-\epsilon|t_n|}\,dt_n$$
$$\times\langle\psi_f|e^{\frac{i}{\hbar}\hat{H}_0 t_1}\hat{H}_1 e^{-\frac{i}{\hbar}\hat{H}_0 t_1} e^{\frac{i}{\hbar}\hat{H}_0 t_2}\hat{H}_1 e^{-\frac{i}{\hbar}\hat{H}_0 t_2}\cdots$$
$$\cdots\hat{H}_1 e^{-\frac{i}{\hbar}\hat{H}_0 t_{n-1}} e^{\frac{i}{\hbar}\hat{H}_0 t_n}\hat{H}_1 e^{-\frac{i}{\hbar}\hat{H}_0 t_n}|\psi_i\rangle \quad (4.47)$$

Here we have simply written out $\langle\psi_f|\hat{H}_{\rm I}(t_1)\cdots\hat{H}_{\rm I}(t_n)|\psi_i\rangle$ in detail.

We will change variables in the integrals as follows

$$
\begin{aligned}
x_1 &= t_1 &&;\ t_1 = x_1 \\
x_2 &= t_2 - t_1 &&;\ t_2 = x_1 + x_2 \\
x_3 &= t_3 - t_2 &&;\ t_3 = x_1 + x_2 + x_3 \\
&\vdots && \quad\vdots \\
x_n &= t_n - t_{n-1} &&;\ t_n = x_1 + x_2 + \cdots + x_n
\end{aligned}
\tag{4.48}
$$

First, let the hamiltonians $\hat{H}_0$ on either end of the operator in Eq. (4.47) act on $|\psi_i\rangle$ and $|\psi_f\rangle$, which are eigenstates of $\hat{H}_0$ with eigenvalues E_0 and E_f respectively. Equation (4.47) then can be written as

$$
\begin{aligned}
\langle\psi_f|\hat{S}_\epsilon^{(n)}|\psi_i\rangle &= \left(-\frac{i}{\hbar}\right)^n \int_{-\infty}^{\infty} e^{-\epsilon|t_1|}\,dt_1 \int_{-\infty}^{t_1} e^{-\epsilon|t_2|}\,dt_2 \cdots \int_{-\infty}^{t_{n-1}} e^{-\epsilon|t_n|}\,dt_n \\
\times\langle\psi_f| e^{\frac{i}{\hbar}(E_f-E_0)t_1}\hat{H}_1 & e^{-\frac{i}{\hbar}\hat{H}_0(t_1-t_2)} e^{\frac{i}{\hbar}E_0(t_1-t_2)}\hat{H}_1 e^{-\frac{i}{\hbar}\hat{H}_0(t_2-t_3)} e^{\frac{i}{\hbar}E_0(t_2-t_3)}\cdots \\
&\cdots e^{-\frac{i}{\hbar}\hat{H}_0(t_{n-1}-t_n)} e^{\frac{i}{\hbar}E_0(t_{n-1}-t_n)}\hat{H}_1|\psi_i\rangle
\end{aligned}
\tag{4.49}
$$

Next, introduce the change in variables in Eqs. (4.48), starting from the right

$$
\begin{aligned}
\langle\psi_f|\hat{S}_\epsilon^{(n)}|\psi_i\rangle &= \left(-\frac{i}{\hbar}\right)^n \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}(E_f-E_0)x_1} e^{-\epsilon|x_1|}\,dx_1 \times \\
\langle\psi_f|\hat{H}_1 &\int_{-\infty}^{0} dx_2\, e^{\{-\frac{i}{\hbar}(E_0-\hat{H}_0)x_2-\epsilon|x_1+x_2|\}}\,\hat{H}_1 \times \\
&\int_{-\infty}^{0} dx_3\, e^{\{-\frac{i}{\hbar}(E_0-\hat{H}_0)x_3-\epsilon|x_1+x_2+x_3|\}}\,\hat{H}_1 \times \cdots \\
\cdots\hat{H}_1 &\int_{-\infty}^{0} dx_n\, e^{\{-\frac{i}{\hbar}(E_0-\hat{H}_0)x_n-\epsilon|x_1+\cdots+x_n|\}}\,\hat{H}_1|\psi_i\rangle
\end{aligned}
\tag{4.50}
$$

Now *do* all the integrals starting on the right, keeping all the other variables fixed while so doing.

Consider the first integral over dx_n at fixed $(x_1,\cdots,x_{n-1})$. What we really need is $\text{Lim}_{\epsilon\to 0}\,\langle\psi_f|\hat{S}_\epsilon^{(n)}|\psi_i\rangle$. Since the damping factors are just there to cut off the oscillating exponentials, we should get the same results no matter how we go to that limit, if the theory is to make sense. We claim that *in the limit*, we can replace $e^{-\epsilon|x_1+\cdots+x_n|} \doteq e^{\epsilon x_n}$ in the integral over x_n, since it is only important for very large negative x_n. Repetition of this argument, as we do the integrals from right to left, allows us to replace

$$
\begin{aligned}
&\text{Lim}_{\,\epsilon\to 0}\int\cdots\int e^{-\epsilon|x_1|}e^{-\epsilon|x_1+x_2|}\cdots e^{-\epsilon|x_1+\cdots x_n|}\cdots = \\
&\text{Lim}_{\,\epsilon\to 0}\int\cdots\int e^{-\epsilon|x_1|}e^{\epsilon x_2}\cdots e^{\epsilon x_n}\cdots
\end{aligned}
\tag{4.51}
$$

The integrals now *factor*, and they can all be immediately carried out

$$
\begin{aligned}
\langle\psi_f|\hat{S}_\epsilon^{(n)}|\psi_i\rangle &= \left(-\frac{i}{\hbar}\right)^n 2\pi\hbar\,\delta(E_f-E_0)\times \\
&\langle\psi_f|\hat{H}_1\frac{1}{-i(E_0-\hat{H}_0)/\hbar+\epsilon}\hat{H}_1\frac{1}{-i(E_0-\hat{H}_0)/\hbar+\epsilon}\hat{H}_1\cdots \\
&\qquad\cdots\frac{1}{-i(E_0-\hat{H}_0)/\hbar+\epsilon}\hat{H}_1|\psi_i\rangle
\end{aligned}
\tag{4.52}
$$

The operator $\hat{H}_1$ appears n times in this expression. This equation has meaning in terms of a complete set of eigenstates of $\hat{H}_0$ inserted between each term. With the redefinition $\epsilon\hbar \equiv \varepsilon$, one arrives at the time-independent power series expansion of the S-matrix

$$
\begin{aligned}
\text{Lim}_{\varepsilon\to 0}\,\langle\psi_f|\hat{S}_\varepsilon|\psi_i\rangle &= \langle\psi_f|\psi_i\rangle - \text{Lim}_{\varepsilon\to 0}\,2\pi i\delta(E_f-E_0)\times \\
&\qquad\langle\psi_f|\hat{H}_1\sum_{n=0}^{\infty}\left(\frac{1}{E_0-\hat{H}_0+i\varepsilon}\hat{H}_1\right)^n|\psi_i\rangle
\end{aligned}
\tag{4.53}
$$

Several comments:

- The $n=0$ term is exactly Fermi's Golden Rule (see Vol. I);
- The $+i\varepsilon$ in the denominator, with the sign coming from the correct convergence factor in the integrals, just determines the correct *boundary conditions* to put in the Green's function (see later);
- We have proceeded to take the $\varepsilon\to 0$ limit in the final factor

$$
\text{Lim}_{\,\epsilon\to 0}\int_{-\infty}^{\infty}dx_1\,e^{\{\frac{i}{\hbar}(E_f-E_0)x_1-\epsilon|x_1|\}} = 2\pi\hbar\,\delta(E_f-E_0) \tag{4.54}
$$

- The T-matrix can now be identified from Eqs. (4.41) and (4.53)

$$
\begin{aligned}
&\tilde{T}_{fi} \equiv \langle\psi_f|\hat{T}|\psi_i\rangle \\
\langle\psi_f|\hat{T}|\psi_i\rangle &= \langle\psi_f|\hat{H}_1\sum_{n=0}^{\infty}\left(\frac{1}{E_0-\hat{H}_0+i\varepsilon}\hat{H}_1\right)^n|\psi_i\rangle \quad ;\ \text{T-matrix}
\end{aligned}
\tag{4.55}
$$

- This last relation can be rewritten as

$$\langle\psi_f|\hat{T}|\psi_i\rangle = \langle\psi_f|\hat{H}_1|\psi_i^{(+)}\rangle$$
$$|\psi_i^{(+)}\rangle \equiv \sum_{n=0}^{\infty}\left(\frac{1}{E_0-\hat{H}_0+i\varepsilon}\hat{H}_1\right)^n|\psi_i\rangle \tag{4.56}$$

We show below that this is indeed identical to the state $|\psi_i^{(+)}\rangle$ previously introduced in Eq. (4.33). If the first term is separated out in Eq. (4.56), and the series for $|\psi_i^{(+)}\rangle$ again identified in the second, this relation can be rewritten as

$$|\psi_i^{(+)}\rangle = |\psi_i\rangle + \frac{1}{E_0-\hat{H}_0+i\varepsilon}\hat{H}_1|\psi_i^{(+)}\rangle \quad ; \text{Lippmann-Schwinger} \tag{4.57}$$

In this form, when projected into the coordinate representation, one has an *integral equation* for $|\psi_i^{(+)}\rangle$. This is the *Lippmann-Schwinger equation* [Lippmann and Schwinger (1950)], which has a meaning that extends beyond the power series expansion though which it has been derived. Note that from Eq. (4.57), one observes

$$(E_0-\hat{H}_0)|\psi_i^{(+)}\rangle = \hat{H}_1|\psi_i^{(+)}\rangle$$
$$\text{or} \qquad (E_0-\hat{H})|\psi_i^{(+)}\rangle = 0 \qquad ; \Omega\to\infty$$
$$\varepsilon\to 0 \tag{4.58}$$

Thus the state $|\psi_i^{(+)}\rangle$, in the limits as the quantization volume $\Omega\to\infty$, and as the adiabatic damping factor $\varepsilon\to 0$, is a scattering state that is an *eigenstate of the full $\hat{H}$ with eigenvalue E_0*.[11] This is the same energy we started with at $t\to-\infty$ in the interaction picture.
- One therefore does not generate all of the eigenstates of $\hat{H}$ in this manner, if there are bound states, but only the continuum scattering states.[12]
- The terms with $n\geq 1$ in Eq. (4.56) give the *higher Born approximations* for the scattering amplitude. This is just "old-fashioned" perturbation theory, except that with the $+i\varepsilon$ in them, *we now know what to do when the denominators vanish*.
- People tried to do QED with this perturbation scheme; however, by singling out the time integration, the scattering amplitude is no longer

[11] Although the dependence on Ω is not explicit, we know, for example, that with a potential $V(r)$ in a big box with rigid walls there will be a finite shift in the energy levels as the interaction is turned on; this energy shift only vanishes in the limit $\Omega\to\infty$.

[12] The completeness relation is now $\sum_i|\psi_i^{(+)}\rangle\langle\psi_i^{(+)}| + \sum_{\text{bnd states}}|\psi_b\rangle\langle\psi_b| = \hat{1}$.

explicitly covariant. Infinities arise from various sources, which are not interpretable in a non-covariant approach. We will find that by leaving the time integrations in, and starting from Eq. (4.17), we are able to maintain a covariant, gauge-invariant S-matrix, which proves essential to developing a consistent renormalization scheme.[13]

4.7 Scattering State

The *Heisenberg picture* for the state vector is defined as follows

$$|\Psi_{\rm H}\rangle \equiv e^{\frac{i}{\hbar}\hat{H}t}\,|\Psi(t)\rangle \qquad ; \text{ Heisenberg picture} \tag{4.59}$$

Correspondingly, an operator in the Heisenberg picture is defined by

$$\hat{O}_{\rm H} \equiv e^{\frac{i}{\hbar}\hat{H}t}\,\hat{O}\,e^{-\frac{i}{\hbar}\hat{H}t} \qquad ; \text{ Heisenberg picture} \tag{4.60}$$

It follows from Eq. (4.32) that the Heisenberg state vector is independent of time[14]

$$i\hbar\frac{\partial}{\partial t}|\Psi_{\rm H}\rangle = 0 \tag{4.61}$$

The interaction-picture state vector is defined in Eq. (4.2). The state vectors in all the different pictures *coincide* at $t = 0$

$$|\Psi_{\rm H}\rangle = |\Psi(0)\rangle = |\Psi_{\rm I}(0)\rangle \tag{4.62}$$

This provides further motivation for looking at the scattering state $|\psi_i^{(+)}\rangle$ defined in Eq. (4.33) by

$$|\psi_i^{(+)}\rangle \equiv \hat{U}_\epsilon(0,-\infty)|\psi_i\rangle \tag{4.63}$$

The *nth* order contribution to $\hat{U}_\varepsilon(0,-\infty)$ explicitly contains n powers of $\hat{H}_1$

$$\hat{U}_\epsilon^{(n)}(0,-\infty)|\psi_i\rangle = \left(-\frac{i}{\hbar}\right)^n \int_{-\infty}^{0} e^{\epsilon t_1}\,dt_1 \int_{-\infty}^{t_1} e^{\epsilon t_2}\,dt_2 \cdots \int_{-\infty}^{t_{n-1}} e^{\epsilon t_n}\,dt_n \times$$
$$e^{\frac{i}{\hbar}\hat{H}_0 t_1}\hat{H}_1 e^{-\frac{i}{\hbar}\hat{H}_0(t_1-t_2)}\hat{H}_1 e^{-\frac{i}{\hbar}\hat{H}_0(t_2-t_3)}\cdots e^{-\frac{i}{\hbar}\hat{H}_0(t_{n-1}-t_n)}\hat{H}_1 e^{-\frac{i}{\hbar}\hat{H}_0 t_n}|\psi_i\rangle \tag{4.64}$$

[13]See the discussion in Vol. I.

[14]We remind the reader of the assumption, at this point, that $\hat{H}$ has no explicit time dependence.

In comparing with our starting point in Eq. (4.47) from which we proceeded to explicitly carrying out the time integrations, we note two differences:

- All the times satisfy $t \leq 0$, hence the adiabatic damping factors in all cases become $e^{-\epsilon|t|} = e^{\epsilon t}$;
- There is no eigenstate $|\psi_f\rangle$ on the left, and hence the operator $\hat{H}_0$ on the left can no longer be replaced by its eigenvalue E_f.

We may proceed to change variables as in Eqs. (4.48)–(4.50). This time, instead of $e^{-\epsilon|x_1+x_2+\cdots+x_n|}$, for example, we have $e^{\epsilon(x_1+x_2+\cdots+x_n)}$ so that all the adiabatic damping factors can simply be moved to their appropriate position in the multiple integral. Thus we arrive at

$$\hat{U}_\epsilon^{(n)}(0,-\infty)|\psi_i\rangle = \left(-\frac{i}{\hbar}\right)^n \int_{-\infty}^0 dx_1\, e^{n\epsilon x_1} e^{\frac{i}{\hbar}(\hat{H}_0-E_0)x_1}\, \hat{H}_1 \times$$
$$\int_{-\infty}^0 dx_2\, e^{(n-1)\epsilon x_2} e^{\frac{i}{\hbar}(\hat{H}_0-E_0)x_2}\, \hat{H}_1 \int_{-\infty}^0 dx_3\, e^{(n-2)\epsilon x_3} e^{\frac{i}{\hbar}(\hat{H}_0-E_0)x_3}\, \hat{H}_1 \times \cdots$$
$$\cdots \hat{H}_1 \int_{-\infty}^0 dx_n\, e^{\epsilon x_n} e^{\frac{i}{\hbar}(\hat{H}_0-E_0)x_n}\, \hat{H}_1|\psi_i\rangle \tag{4.65}$$

All the integrals now *explicitly factor*, and they can immediately be done just as before with the result

$$\hat{U}_\varepsilon^{(n)}(0,-\infty)|\psi_i\rangle = \frac{1}{E_0-\hat{H}_0+in\varepsilon}\hat{H}_1\frac{1}{E_0-\hat{H}_0+i(n-1)\varepsilon}\hat{H}_1\cdots$$
$$\cdots\frac{1}{E_0-\hat{H}_0+i\varepsilon}\hat{H}_1|\psi_i\rangle \tag{4.66}$$

Again, we are interested in the limit as $\varepsilon \to 0$. Each of the $i\bar{n}\varepsilon$ in the denominators, where $\bar{n} = (1, 2, \cdots, n)$, simply serves to define how one treats the singularity in the individual Green's functions.[15] Hence, we can simply *replace them all by $i\varepsilon$ in the limit.* Thus we indeed reproduce the previously employed expression in Eq. (4.56)

$$|\psi_i^{(+)}\rangle = \hat{U}_\varepsilon(0,-\infty)|\psi_i\rangle$$
$$= \sum_{n=0}^{\infty}\left(\frac{1}{E_0-\hat{H}_0+i\varepsilon}\hat{H}_1\right)^n|\psi_i\rangle \qquad ;\ \text{scattering state} \tag{4.67}$$

Again, by separating out the first term in the second line, and then re-

[15] They serve to define a contour in the evaluation of the Green's functions (see later).

identifying the series for $|\psi_i^{(+)}\rangle$, this can be rewritten as an integral equation

$$|\psi_i^{(+)}\rangle = |\psi_i\rangle + \frac{1}{E_0 - \hat{H}_0 + i\varepsilon}\hat{H}_1|\psi_i^{(+)}\rangle \tag{4.68}$$

and the integral equation has a meaning, even when the power series solution to it does not.

Let us also consider the fully interacting state $|\psi_f^{(-)}\rangle \equiv |\Psi_{\rm I}^f(0)\rangle$ that as $t \to +\infty$ reduces to the state $|\psi_f\rangle$, so that $|\psi_f^{(-)}\rangle = \hat{U}_\epsilon(0,+\infty)|\psi_f\rangle$. If we go back to Eq. (4.19), and go through the arguments leading from Eq. (4.64) to (4.67), we see that the only changes are the replacements $E_0 \to E_f$ and $\varepsilon \to -\varepsilon$ (see Prob. 4.3). Thus

$$\begin{aligned}|\psi_f^{(-)}\rangle &\equiv \hat{U}_\varepsilon(0,+\infty)|\psi_f\rangle \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{E_f - \hat{H}_0 - i\varepsilon}\hat{H}_1\right)^n|\psi_f\rangle \qquad ;\ \text{scattering state}\end{aligned} \tag{4.69}$$

This can again be written as an integral equation, which has meaning even when the power series solution for it does not

$$|\psi_f^{(-)}\rangle = |\psi_f\rangle + \frac{1}{E_f - \hat{H}_0 - i\varepsilon}\hat{H}_1|\psi_f^{(-)}\rangle \tag{4.70}$$

The state $|\psi^{(+)}\rangle$ is known as the *outgoing* scattering state, and $|\psi^{(-)}\rangle$ as the *incoming* scattering state.[16]

There are some important properties of these scattering states that follow immediately:

(1) The unitarity of the $\hat{U}_\varepsilon$ operator implies that

$$\langle\psi_{i'}^{(+)}|\psi_i^{(+)}\rangle = \langle\psi_{i'}|\hat{U}_\varepsilon(0,-\infty)^\dagger\hat{U}_\varepsilon(0,-\infty)|\psi_i\rangle = \langle\psi_{i'}|\psi_i\rangle = \delta_{i'i} \tag{4.71}$$

Similarly[17]

$$\langle\psi_{f'}^{(-)}|\psi_f^{(-)}\rangle = \delta_{f'f} \tag{4.72}$$

[16]The Green's function in the former case has outgoing scattered waves, while in the latter case they are incoming [compare Eq. (4.107) and Prob. 4.9].

[17]The completeness relation can also be written $\sum_f |\psi_f^{(-)}\rangle\langle\psi_f^{(-)}| + \sum_{\rm bnd\ states} |\psi_b\rangle\langle\psi_b| = \hat{1}$.

(2) Furthermore, from Eq. (4.21) and the group property of $\hat{U}_\varepsilon$, it follows that

$$\begin{aligned}\langle\psi_f^{(-)}|\psi_i^{(+)}\rangle &= \langle\psi_f|\hat{U}_\varepsilon(0,+\infty)^\dagger\hat{U}_\varepsilon(0,-\infty)|\psi_i\rangle \\ &= \langle\psi_f|\hat{U}_\varepsilon(+\infty,0)\hat{U}_\varepsilon(0,-\infty)|\psi_i\rangle \\ &= \langle\psi_f|\hat{U}_\varepsilon(+\infty,-\infty)|\psi_i\rangle \qquad (4.73)\end{aligned}$$

Thus the inner product of $|\psi_f^{(-)}\rangle$ and $|\psi_i^{(+)}\rangle$ is just the S-matrix!

$$\langle\psi_f^{(-)}|\psi_i^{(+)}\rangle = \langle\psi_f|\hat{S}|\psi_i\rangle \qquad ;\ S\text{-matrix} \qquad (4.74)$$

(3) Since taking the adjoint merely reverses the order of the operators and changes the sign of the $i\varepsilon$, the T-matrix in Eq. (4.55) can also be written in the case $E_f = E_0$ as

$$\begin{aligned}\langle\psi_f|\hat{T}|\psi_i\rangle &= \langle\psi_f|\hat{H}_1\sum_{n=0}^{\infty}\left(\frac{1}{E_0-\hat{H}_0+i\varepsilon}\hat{H}_1\right)^n|\psi_i\rangle \\ &= \langle\psi_f|\left[\sum_{n=0}^{\infty}\left(\frac{1}{E_0-\hat{H}_0-i\varepsilon}\hat{H}_1\right)^n\right]^\dagger\hat{H}_1|\psi_i\rangle \\ &= \langle\psi_f^{(-)}|\hat{H}_1|\psi_i\rangle \qquad ;\ E_f = E_0 \qquad (4.75)\end{aligned}$$

(4) Thus, in *summary*, in addition to the explicit power-series expansions in Eqs. (4.53) and (4.55), we have expressions for the S-matrix and T-matrix in terms of the incoming and outgoing scattering states that are more general than the power-series solutions through which they were derived

$$\begin{aligned}\langle\psi_f|\hat{S}|\psi_i\rangle &= \langle\psi_f^{(-)}|\psi_i^{(+)}\rangle \qquad ;\ S\text{-matrix} \\ &= \langle\psi_f|\psi_i\rangle - 2\pi i\delta(E_f-E_0)\langle\psi_f|\hat{T}|\psi_i\rangle \\ \langle\psi_f|\hat{T}|\psi_i\rangle &= \langle\psi_f|\hat{H}_1|\psi_i^{(+)}\rangle = \langle\psi_f^{(-)}|\hat{H}_1|\psi_i\rangle \qquad ;\ T\text{-matrix} \qquad (4.76)\end{aligned}$$

4.8 Transition Rate

We now calculate the transition rate directly, in the presence of the adiabatic switching. The derivation is from [Gell-Mann and Goldberger (1953)], in their classic paper on scattering theory. The only subtlety in the calculation is identifying those expressions that are well-defined in the limit $\varepsilon \to 0$, and knowing when to take that limit. This takes a little experience.

The Schrödinger state vector at the finite time t for a system that started as $|\psi_i\rangle$ at $t \to -\infty$ is

$$|\Psi_i(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\Psi_i(0)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}\hat{U}_\varepsilon(0,-\infty)|\psi_i\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi_i^{(+)}\rangle \quad (4.77)$$

The states that one *observes experimentally* in scattering, decays, *etc.* are the free-particle states

$$|\Phi_f(t)\rangle = e^{-\frac{i}{\hbar}E_f t}|\psi_f\rangle \quad (4.78)$$

From the general principles of quantum mechanics, the probability of finding the system in the state $|\Phi_f(t)\rangle$ at the time t, if it started in $|\psi_i\rangle$ at $t \to -\infty$, is then

$$P_{fi}(t) = |\langle\Phi_f(t)|\Psi_i(t)\rangle|^2 \equiv |M_{fi}(t)|^2 \quad (4.79)$$

This is the probability of having made a transition to the state $|\Phi_f(t)\rangle$ at the time t. The transition *rate* is the time derivative of this quantity

$$\omega_{fi} = \frac{d}{dt}P_{fi}(t) = M_{fi}^\star(t)\frac{d}{dt}M_{fi}(t) + \text{c.c.} \quad (4.80)$$

We will show that this transition rate is independent of time for times such that $|t| \ll 1/\varepsilon$. In the end, we will again let $\Omega \to \infty$, and $\varepsilon \to 0$, where Ω is the quantization volume. Let us proceed to calculate the transition rate.

From Eqs. (4.77)–(4.79) one has

$$\begin{aligned} M_{fi}(t) &= \langle\Phi_f(t)|\Psi_i(t)\rangle \\ &= \langle\psi_f|e^{\frac{i}{\hbar}E_f t}e^{-\frac{i}{\hbar}\hat{H}t}|\psi_i^{(+)}\rangle \end{aligned} \quad (4.81)$$

This relation may be differentiated with respect to time to give

$$\frac{d}{dt}M_{fi}(t) = -\frac{i}{\hbar}\langle\psi_f|(\hat{H}-E_f)e^{\frac{i}{\hbar}E_f t}e^{-\frac{i}{\hbar}\hat{H}t}|\psi_i^{(+)}\rangle \quad (4.82)$$

The observation that $(\hat{H}-E_f)|\psi_f\rangle = (\hat{H}_0+\hat{H}_1-E_f)|\psi_f\rangle = \hat{H}_1|\psi_f\rangle$ gives

$$\frac{d}{dt}M_{fi}(t) = -\frac{i}{\hbar}e^{\frac{i}{\hbar}E_f t}\langle\psi_f|\hat{H}_1\,e^{-\frac{i}{\hbar}\hat{H}t}|\psi_i^{(+)}\rangle \quad (4.83)$$

Now Eq. (4.58) states that in the above limit

$$(E_0-\hat{H})|\psi_i^{(+)}\rangle = 0 \qquad ;\ \Omega \to \infty,\ \ \varepsilon \to 0 \quad (4.84)$$

Use of this relation in Eqs. (4.83) and (4.81) then gives

$$\frac{d}{dt}M_{fi}(t) = -\frac{i}{\hbar}e^{\frac{i}{\hbar}(E_f - E_0)t}\langle\psi_f|\hat{H}_1|\psi_i^{(+)}\rangle$$
$$M_{fi}(t) = e^{\frac{i}{\hbar}(E_f - E_0)t}\langle\psi_f|\psi_i^{(+)}\rangle \tag{4.85}$$

Substitution of these relations into Eq. (4.80) then expresses the transition rate as

$$\omega_{fi} = \frac{2}{\hbar}\mathrm{Im}\,\langle\psi_f|\hat{H}_1|\psi_i^{(+)}\rangle\langle\psi_f|\psi_i^{(+)}\rangle^\star \tag{4.86}$$

This expression now has the following properties:

- It is independent of time;
- It is *well-defined in the limit* $\Omega \to \infty$, $\varepsilon \to 0$.[18]

From our previous analysis in Eqs. (4.76) and (4.68), we have

$$\langle\psi_f|\hat{H}_1|\psi_i^{(+)}\rangle = \langle\psi_f|\hat{T}|\psi_i\rangle = \tilde{T}_{fi}$$
$$|\psi_i^{(+)}\rangle = |\psi_i\rangle + \frac{1}{E_0 - \hat{H}_0 + i\varepsilon}\hat{H}_1|\psi_i^{(+)}\rangle \tag{4.87}$$

The inner product of the second relation with $|\psi_f\rangle$ gives

$$\langle\psi_f|\psi_i^{(+)}\rangle = \langle\psi_f|\psi_i\rangle + \frac{1}{E_0 - E_f + i\varepsilon}\tilde{T}_{fi} \tag{4.88}$$

Substitution of this relation and the first of Eqs. (4.87) into Eq. (4.86) then gives

$$\omega_{fi} = \frac{2}{\hbar}\delta_{fi}\,\mathrm{Im}\,\tilde{T}_{ii} + \frac{2}{\hbar}\mathrm{Im}\,\frac{1}{E_0 - E_f - i\varepsilon}|\tilde{T}_{fi}|^2 \tag{4.89}$$

Finally, we make use of the relation

$$\frac{1}{E_0 - E_f - i\varepsilon} = \mathcal{P}\frac{1}{E_0 - E_f} + i\pi\delta(E_0 - E_f) \tag{4.90}$$

Here $\mathcal{P}$ denotes the Cauchy principal value, defined by deleting an infinitesimal symmetric region of integration through the singularity, and then letting the size of that region go to zero. Equation (4.90) is a statement on

[18] Here we will simply justify this observation *a posteriori*, through the many applications of the final expression. Note that by taking this limit too early in the derivation, one can arrive at spurious results [for example, try substituting the second of Eqs. (4.85) into Eq. (4.79)].

contour integration; it is derived in Prob. B.4. With the use of this relation, Eqs. (4.89) and (4.76) become

$$\begin{aligned}\omega_{fi} &= \frac{2}{\hbar}\delta_{fi}\,\mathrm{Im}\,\tilde{T}_{ii} + \frac{2\pi}{\hbar}\delta(E_0 - E_f)|\tilde{T}_{fi}|^2 && ;\ \text{transition rate} \\ S_{fi} &= \delta_{fi} - 2\pi i\delta(E_0 - E_f)\tilde{T}_{fi} && ;\ S\text{-matrix}\end{aligned} \tag{4.91}$$

These expressions are exact. They are the results quoted in Eqs. (4.46) and used extensively in Vol. I.

4.9 Unitarity

The first term on the r.h.s. of ω_{fi} in Eq. (4.91) only contributes if $f = i$; it is there to take into account the depletion of the initial state. Return to Eq. (4.79). With the completeness of the states $|\Phi_f(t)\rangle$, and the normalization of the state $|\Psi_i(t)\rangle$, a sum over all final states gives[19]

$$\begin{aligned}\sum_f P_{fi}(t) &= \sum_f \langle\Psi_i(t)|\Phi_f(t)\rangle\langle\Phi_f(t)|\Psi_i(t)\rangle \\ &= \langle\Psi_i(t)|\Psi_i(t)\rangle = 1\end{aligned} \tag{4.92}$$

This is the statement of conservation of probability—the initial state must end up *somewhere*. The time derivative of this sum then vanishes

$$\frac{d}{dt}\sum_f P_{fi}(t) = \sum_f \frac{d}{dt}P_{fi}(t) = \sum_f \omega_{fi} = 0 \tag{4.93}$$

Here the transition rate has been identified from Eq. (4.80). A substitution of the expression for the transition rate in Eq. (4.91) into this relation then gives

$$-\frac{2}{\hbar}\mathrm{Im}\,\tilde{T}_{ii} = \sum_f \frac{2\pi}{\hbar}\delta(E_f - E_0)|\tilde{T}_{fi}|^2 \qquad ;\ \text{unitarity} \tag{4.94}$$

This relation for the imaginary part of the elastic T-matrix, reflecting conservation of probability and depletion of the initial state, is known as *unitarity*.

[19]This sum now *includes* the state $f = i$; the reader should note that there is no sum over the repeated index i implied in Eqs. (4.91) and (4.94).

4.10 Example: Potential Scattering

To see one practical application of the preceding scattering theory, consider the elastic scattering of a non-relativistic particle of mass m from a spherically symmetric potential $\hat{H}_1 = V(|\hat{\mathbf{x}}|)$ in three dimensions. First we calculate the Green's function, or propagator.

4.10.1 *Green's Function (Propagator)*

The Green's function in this case is defined by the following matrix element taken between eigenstates of position

$$G_0(\mathbf{x}-\mathbf{y}) = \langle\mathbf{x}|\frac{1}{\hat{H}_0 - E_0 - i\varepsilon}|\mathbf{y}\rangle \qquad ; \text{ Green's function} \tag{4.95}$$

Here

$$\hat{H}_0 = \frac{\hat{\mathbf{p}}^2}{2m} \qquad ; \ E_0 \equiv \frac{\hbar^2\mathbf{k}^2}{2m} \tag{4.96}$$

As usual, we start in a big cubical box of volume Ω where the eigenstates of momentum are plane waves satisfying periodic boundary conditions

$$\begin{aligned}
\hat{\mathbf{p}}\,|\mathbf{t}\rangle &= \hbar\mathbf{t}\,|\mathbf{t}\rangle \\
\langle\mathbf{x}|\mathbf{t}\rangle &= \phi_{\mathbf{t}}(\mathbf{x}) = \frac{1}{\sqrt{\Omega}}e^{i\mathbf{t}\cdot\mathbf{x}} \qquad ; \text{ p.b.c.}
\end{aligned} \tag{4.97}$$

The eigenstates of momentum satisfy the completeness relation

$$\sum_{\mathbf{t}} |\mathbf{t}\rangle\langle\mathbf{t}| = \hat{1} \tag{4.98}$$

Insert this expression in Eq. (4.95), and use Eqs. (4.96) and (4.97)

$$\begin{aligned}
G_0(\mathbf{x}-\mathbf{y}) &= \frac{2m}{\hbar^2}\sum_{\mathbf{t}}\langle\mathbf{x}|\mathbf{t}\rangle\frac{1}{t^2-k^2-i\varepsilon}\langle\mathbf{t}|\mathbf{y}\rangle \\
&= \frac{2m}{\hbar^2}\frac{1}{\Omega}\sum_{\mathbf{t}} e^{i\mathbf{t}\cdot(\mathbf{x}-\mathbf{y})}\frac{1}{t^2-k^2-i\varepsilon}
\end{aligned} \tag{4.99}$$

We have redefined $(2m/\hbar^2)\varepsilon \to \varepsilon$ in this expression.

Now take the limit as the volume $\Omega \to \infty$, in which case the sum over states becomes an integral, in the familiar fashion, $\sum_{\mathbf{t}} \to \Omega(2\pi)^{-3}\int d^3t$. In this limit

$$G_0(\mathbf{x}-\mathbf{y}) = \frac{2m}{\hbar^2}\frac{1}{(2\pi)^3}\int d^3t\; e^{i\mathbf{t}\cdot(\mathbf{x}-\mathbf{y})}\frac{1}{t^2-k^2-i\varepsilon} \tag{4.100}$$

It remains to do this integral. Take $\mathbf{r} \equiv \mathbf{x} - \mathbf{y}$ to define the z-axis. Then $\mathbf{t} \cdot (\mathbf{x} - \mathbf{y}) = tr \cos\theta$ and $d^3t = t^2 dt\, d\phi \sin\theta\, d\theta$. The angular integrations are then immediately performed

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, e^{itr\cos\theta} = 2\pi \int_{-1}^{1} dx\, e^{itrx} = 4\pi \frac{\sin tr}{tr} \tag{4.101}$$

We are left with

$$G_0(\mathbf{x} - \mathbf{y}) = \frac{2m}{\hbar^2} \frac{4\pi}{(2\pi)^3} \frac{1}{r} \int_0^{\infty} tdt\, \sin tr \frac{1}{t^2 - k^2 - i\varepsilon} \tag{4.102}$$

Now write the integral as

$$\begin{aligned} \int_0^{\infty} tdt\, \sin tr \cdots &= \int_0^{\infty} tdt\, \frac{1}{2i}(e^{itr} - e^{-itr}) \cdots \\ &= \frac{1}{2i} \int_{-\infty}^{\infty} tdt\, e^{itr} \cdots \end{aligned} \tag{4.103}$$

Here we have simple changed variables $t \to -t$ in the second term, and combined it with the first (the rest of the integrand is a function of t^2). The required integral is then reduced to

$$G_0(\mathbf{x} - \mathbf{y}) = \frac{2m}{\hbar^2} \frac{4\pi}{(2\pi)^3} \frac{1}{2ir} \int_{-\infty}^{\infty} tdt\, e^{itr} \frac{1}{t^2 - k^2 - i\varepsilon} \quad ; \mathbf{r} \equiv \mathbf{x} - \mathbf{y} \tag{4.104}$$

where the integral now runs along the entire real t-axis. There is sufficient convergence in the integrand that closing the contour with a semi-circle in the upper-1/2 t-plane makes a vanishing contribution to the integral in the limit as the radius R of that semi-circle becomes infinite.[20] Thus the free Green's function has been reduced to a *contour integral* where the contour C is that illustrated in Fig. 4.1.

The integral is then evaluated using the complex-variable techniques summarized in appendix B. The integrand is an analytic function of t except at the poles where the denominator vanishes. That denominator can be rewritten as

$$\frac{1}{t^2 - k^2 - i\varepsilon} = \frac{1}{(t - k - i\varepsilon)(t + k + i\varepsilon)} \tag{4.105}$$

where we have again redefined $\varepsilon \to 2k\varepsilon$ (here $k > 0$), and neglected $O(\varepsilon^2)$. The integrand thus has simple poles at $t = k + i\varepsilon$ and $t = -k - i\varepsilon$, only the first of which lies inside C.

[20] See Prob. 4.4.

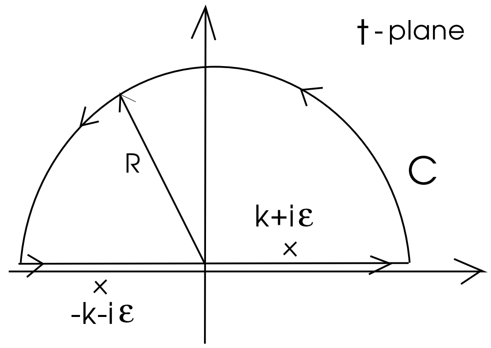

Fig. 4.1 Contour for the evaluation of the Green's function $G_0(\mathbf{x}-\mathbf{y})$ in the complex t-plane, together with the singularity structure arrived at with adiabatic damping. Here $R \to \infty$.

The integral is then given by $2\pi i \times$ (residue at k). Thus

$$G_0(\mathbf{x}-\mathbf{y}) = \frac{2m}{\hbar^2}\frac{4\pi}{(2\pi)^3}\frac{1}{2ir}2\pi i\left(\frac{e^{ikr}}{2}\right) \tag{4.106}$$

Hence we arrive at our final result for the free Green's function in potential scattering

$$G_0(\mathbf{x}-\mathbf{y}) = \frac{2m}{\hbar^2}\frac{e^{ikr}}{4\pi r} \qquad ; \ \mathbf{r} \equiv \mathbf{x}-\mathbf{y} \tag{4.107}$$

This is recognized as the familiar Green's function for the scalar Helmholtz equation (see [Fetter and Walecka (2003)]).

4.10.2 *Scattering Wave Function*

The scattering state $|\psi_i^{(+)}\rangle$ can be similarly projected onto eigenstates of position. With the use of the completeness relation for these eigenstates, and the definition of the Green's function in Eq. (4.95), one has[21]

$$\langle\mathbf{x}|\psi_i^{(+)}\rangle = \langle\mathbf{x}|\psi_i\rangle - \int d^3y\,\langle\mathbf{x}|\frac{1}{\hat{H}_0 - E_0 - i\varepsilon}|\mathbf{y}\rangle\,V(y)\,\langle\mathbf{y}|\psi_i^{(+)}\rangle \tag{4.108}$$

With the definition $\langle\mathbf{x}|\psi_i^{(+)}\rangle \equiv \psi_i^{(+)}(\mathbf{x})/\sqrt{\Omega}$, this becomes an integral equation for the scattering wave function

$$\langle\mathbf{x}|\psi_i^{(+)}\rangle \equiv \frac{1}{\sqrt{\Omega}}\psi_i^{(+)}(\mathbf{x})$$

$$\psi_i^{(+)}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} - \int d^3y\,G_0(\mathbf{x}-\mathbf{y})\,V(y)\,\psi_i^{(+)}(\mathbf{y}) \tag{4.109}$$

[21] We have used $V(|\hat{\mathbf{x}}|)\,|\mathbf{y}\rangle = V(y)\,|\mathbf{y}\rangle$ where $y \equiv |\mathbf{y}|$; note the sign of the second term.

4.10.3 *T-matrix*

The T-matrix can also be expressed in the coordinate representation as

$$\tilde{T}_{fi} = \int d^3y \, \langle \mathbf{k}_f | \mathbf{y} \rangle \, V(y) \, \langle \mathbf{y} | \psi_i^{(+)} \rangle \tag{4.110}$$

where the final state is now written explicitly as an eigenstate of momentum $|\psi_f\rangle \equiv |\mathbf{k}_f\rangle$. With the introduction of the corresponding wave functions, one has

$$\tilde{T}_{fi} = \frac{1}{\Omega} \int d^3y \, e^{-i\mathbf{k}_f \cdot \mathbf{y}} \, V(y) \, \psi_i^{(+)}(\mathbf{y}) \tag{4.111}$$

4.10.4 *Cross Section*

The differential cross section follows from the transition rate according to Eq. (4.45)

$$d\sigma = \frac{2\pi}{\hbar} \delta(E_f - E_0) |\tilde{T}_{fi}|^2 \frac{dn_f}{I_{\text{inc}}} \tag{4.112}$$

In this expression:

(1) The incident wave function is $\psi_i(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}/\sqrt{\Omega}$. This yields an incident probability flux of

$$I_{\text{inc}} = \frac{1}{\Omega} \frac{\hbar k}{m} \tag{4.113}$$

(2) The number of final states in a big box with periodic boundary conditions is

$$dn_f = \frac{\Omega}{(2\pi)^3} d^3k_f = \frac{\Omega}{(2\pi)^3} k_f^2 dk_f \, d\Omega_f \tag{4.114}$$

(3) The integral over the energy-conserving delta function gives

$$\int \delta(E_f - E_0) k_f^2 dk_f = \frac{2m}{\hbar^2} \frac{k_f}{2} \qquad ; \ |\mathbf{k}_f| = |\mathbf{k}| \tag{4.115}$$

(4) A combination of the results in Eqs. (4.112)–(4.115) gives

$$\frac{d\sigma}{d\Omega_f} = \frac{2\pi}{\hbar} \left[\frac{\Omega}{(2\pi)^3} \frac{mk}{\hbar^2} \right] \left[\frac{\Omega m}{\hbar k} \right] |\tilde{T}_{fi}|^2 \tag{4.116}$$

The factors of Ω cancel, as they must, and the final result for the differential cross section for elastic scattering of a particle of energy

$E_0 = \hbar^2k^2/2m$ from the potential $V(|\mathbf{x}|)$ takes the form

$$\frac{d\sigma}{d\Omega_f} = |f(k,\theta)|^2$$
$$f(k,\theta) \equiv -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3y\, e^{-i\mathbf{k}_f\cdot\mathbf{y}}\, V(y)\, \psi_i^{(+)}(\mathbf{y}) \qquad (4.117)$$

The minus sign is conventional.

(5) The scattering wave function $\psi_i^{(+)}(\mathbf{x})$ in this expression is the solution to the integral equation

$$\psi_i^{(+)}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{2m}{\hbar^2}\int d^3y\, \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}\, V(y)\, \psi_i^{(+)}(\mathbf{y}) \qquad (4.118)$$

4.10.5 *Unitarity*

The scattering amplitude $f(k,\theta)$ and the T-matrix are related through Eqs. (4.117) and (4.111), and thus

$$-\frac{2}{\hbar}\text{Im}\,\tilde{T}_{ii} = \frac{4\pi}{\Omega}\frac{\hbar}{m}\text{Im}\,f(k,0) \qquad (4.119)$$

The unitarity relation in Eq. (4.94) states that

$$-\frac{2}{\hbar}\text{Im}\,\tilde{T}_{ii} = \sum_f \frac{2\pi}{\hbar}\delta(E_f - E_0)|\tilde{T}_{fi}|^2 \qquad (4.120)$$

Within a factor of the incident flux, the r.h.s. of this relation is just the total cross section σ_{tot}. Thus Eq. (4.120) can be rewritten as

$$-\frac{2}{\hbar}\text{Im}\,\tilde{T}_{ii} = I_{\text{inc}}\,\sigma_{\text{tot}} = \frac{1}{\Omega}\frac{\hbar k}{m}\sigma_{\text{tot}} \qquad (4.121)$$

A comparison of Eqs. (4.119) and (4.121) then leads to the *optical theorem* relating the imaginary part of the forward elastic scattering amplitude and the total cross section[22]

$$\text{Im}\,f(k,0) = \frac{k}{4\pi}\sigma_{\text{tot}} \qquad ;\ \text{optical theorem} \qquad (4.122)$$

The analysis of potential scattering in this section provides the underlying basis for the study of scattering in quantum mechanics, as presented, for example, in [Schiff (1968)].[23]

[22]So far, there is only elastic scattering in this potential model, but the optical theorem is more general and holds in the presence of additional inelastic processes.

[23]Problems 1.1–1.5 in [Walecka (2004)] take the reader through the essentials of the partial-wave analysis of the scattering problem.