Chapter 1

Background and General Comments

This chapter introduces the basic, but fundamental concepts relating to the class of oscillators we call "truly nonlinear." The two phrases "truly nonlinear oscillators" and "truly nonlinear differential equations" are used interchangeable. In Sections 1.1 and 1.2, respectively, we define truly nonlinear (TNL) functions and TNL oscillators. Section 1.3 presents general comments regarding time reversal invariant systems and odd parity oscillators. Section 1.4 discusses the important topic of the elimination of dimensional quantities in the physical nonlinear differential equations through the use of scaling parameters. The existence of and exact solutions to four TNL oscillators are given in Section 1.5; this is followed by a brief overview of four methods that can be used to construct analytic approximations to the periodic solutions for TNL oscillator differential equations. We conclude the chapter with a set of possible criteria that may be used to judge the value of a calculational method for generating approximate solutions.

1.1 Truly Nonlinear Functions

A TNL function is defined with respect to its properties in a neighborhood at a given point. For our purposes, we select $x = 0$. Thus, for a function $f(x)$, we make the following definition:

Definition 1.1. $f(x)$ is a TNL function, at $x = 0$, if $f(x)$ has no linear approximation in any neighborhood of $x = 0$.

The following are several explicit examples of TNL functions

$$f_1(x) = x^3, \quad f_2(x) = x^{1/3}, \quad f_3(x) = x + x^{1/3}. \tag{1.1.1}$$

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Note that each of these functions is defined for all real values of x, i.e., $-\infty < x < \infty$. Inspection of $f_1(x)$ and $f_2(x)$ clearly illustrate why they are TNL functions; however, the third function requires just a little more analysis to understand that it is a TNL function. We have

$$\frac{x}{x^{1/3}} = x^{2/3} \Rightarrow |x| < |x^{1/3}|, \quad 0 < |x| < 1, \tag{1.1.2}$$

and thus it follows that in a neighborhood of $x = 0$, the $x^{1/3}$ term dominates x.

Other examples of TNL functions include

$$\begin{cases} f_4(x) = \dfrac{1}{x}, \quad f_5(x) = \dfrac{1}{x^{3/5}}, \quad f_6(x) = \dfrac{x^3}{1+x^2}, \\ f_7(x) = |x|x, \quad f_8(x) + \dfrac{1}{x^{1/3}}\,. \end{cases} \tag{1.1.3}$$

In all of the above expressions, we have set possible constants, which could appear, equal to one. This does not change in any way the essential features of these functions.

1.2 Truly Nonlinear Oscillators

In this volume, we consider only one-degree-of-freedom systems that can be mathematically modeled by differential equations having (in the simplest case) the generic form

$$\ddot{x} + f(x) = 0, \tag{1.2.1}$$

where the "dots" denote time derivatives, i.e., $\dot{x} \equiv dx/dt$ and $\ddot{x} \equiv d^2x/dt^2$.

Definition 1.2. If $f(x)$ is a TNL function, then the second-order differential equation, as given in Eq. (1.2.1), is a TNL oscillator.

Using specific representations of $f(x)$, from Section 1.1, the following are particular examples of TNL oscillators:

$$\begin{cases} \ddot{x} + x^3 = 0, \\ \ddot{x} + x^{1/3} = 0, \\ \ddot{x} + x + x^{1/3} = 0, \\ \ddot{x} + \dfrac{x^3}{1+x^2} = 0, \\ \ddot{x} + |x|x = 0, \\ \ddot{x} + \dfrac{1}{x} = 0. \end{cases} \tag{1.2.2}$$

The TNL oscillator concept can also be extended to limit-cycle systems. Starting with the well-known van der Pol oscillator [1–4]

$$\ddot{x} + x = \epsilon(1 - x^2)\dot{x}, \tag{1.2.3}$$

where ϵ is a positive parameter, then the following TNL oscillator generalizations can be constructed [5, 6].

$$\begin{cases} \ddot{x} + x^3 = \epsilon(1 - x^2)\dot{x}, \\ \ddot{x} + x^{1/3} = \epsilon(1 - x^2)\dot{x}. \end{cases} \tag{1.2.4}$$

1.3 General Remarks

Let's consider in more detail the specifics of the structural properties of Eq. (1.2.1). In fact, we can also consider the more general form

$$\ddot{x} + F(x, \dot{x}) = 0, \tag{1.3.1}$$

where $F(x, \dot{x})$ depends on both x and its first derivative, $\dot{x}$.

When $F(x, \dot{x})$ depends only on x, then $F(x, \dot{x}) = f(x)$, and Eq. (1.3.1) becomes

$$\ddot{x} + f(x) = 0. \tag{1.3.2}$$

Defining $y \equiv \dot{x}$, a first-integral can be obtained [7] using

$$\frac{d^2x}{dt^2} = \frac{dy}{dt} = y\,\frac{dy}{dx}\,. \tag{1.3.3}$$

With this result, Eq. (1.3.2) can be integrated to give

$$\frac{y^2}{2} + V(x) = V(A), \tag{1.3.4}$$

where initial conditions, $x(0) = A$, $\dot{x}(0) = y(0) = 0$, were used, and $V(x)$ is the potential energy [8]. Within the domain of physics, this first integral is the total energy and the nonlinear oscillator is called a *conservative oscillator* [3, 8]. Note that this is a general result, not depending as to whether the oscillator is TNL or of the usual type. An important feature of the solutions for conservative oscillators is that they have periodic solutions that range over a continuous interval of initial values [7, 8].

We can also consider "generalized" conservative oscillators. These oscillators satisfy the condition $F(x, -\dot{x}) = F(x, \dot{x})$. A particular example is

$$\ddot{x} + (1 + \dot{x}^2)x^{1/3} = 0, \tag{1.3.5}$$

which is a TNL oscillator that can be rewritten to the form

$$y\,\frac{dy}{dx} + (1+y^2)x^{1/3} = 0. \tag{1.3.6}$$

Integrating gives

$$T(y) + V(x) = V(A),$$

where $x(0) = A$, $\dot{x}(0) = y(0) = 0$, and the generalized kinetic [8] and potential energies are

$$T(y) = \int \frac{y\,dy}{1+y^2}, \qquad V(x) = \left(\frac{3}{4}\right) x^{4/3}. \tag{1.3.7}$$

As we will show in the next chapter, all solutions to Eq. (1.3.5) are periodic.

In this volume, we will study TNL oscillators that are members of "odd-parity systems."

Definition 1.3. The differential equation

$$\ddot{x} + F(x, \dot{x}) = 0,$$

is said to be of odd-parity if this equation is invariant under $(x, \dot{x}) \to (-x, -\dot{x})$.

All of the TNL oscillators given in Eq. (1.2.2) are of odd-parity. Another example is given by Eq. (1.3.5).

The real significance of odd-parity systems is that their periodic solutions contain only odd multiples of the fundamental angular frequency in their Fourier series representations [9]. They are also important because many physically relevant systems may be modeled by nonlinear differential equations having this property [2, 7].

Except for Chapter 6, which deals with the possibility of oscillatory, but not necessarily periodic solutions, the remaining chapters will focus on constructing calculational methods for determining analytic expressions for the periodic solutions of TNL oscillators. These TNL oscillatory equations will have the properties of being invariant under time reversal, $t \to -t$, and possessing odd-parity, i.e., invariant under $x \to -x$. As an illustration as to what may occur if both conditions do not simultaneously apply, consider the three equations

$$\ddot{x} + (1 + x\dot{x})x^{1/3} = 0, \tag{1.3.8a}$$

$$\ddot{x} + (1 + \dot{x}^2)x^{1/3} = 0, \tag{1.3.8b}$$

$$\ddot{x} + (1 + \dot{x}^2)x^{1/3} = 0. \tag{1.3.8c}$$

All of these TNL equations are of odd-parity, but the first equation corresponds to a damped oscillator. This can be seen if it is rewritten to the form

$$\ddot{x} + x^{1/3} = -(xx^{1/3})\dot{x}.$$

The right-side represents a damping term and, as a consequence, the solutions are damped and oscillatory rather than periodic. (These results will be shown in Chapter 2.) Note that this equation is of odd-parity, but not invariant under $t \to -t$. The second and third equations are invariant under both $t \to -t$ and $x \to -x$, and all their solutions can be shown to be periodic.

1.4 Scaling and Dimensionless Form of Differential Equations

Differential equations modeling physical phenomena have independent and dependent variables, and parameters appearing in these equations possessing physical units such as mass, length, time, electrical charge, etc. [3, 7, 10]. Thus, the magnitude of these quantities depend on the actual physical units used, i.e., meters versus kilometers, seconds versus hours, etc. A way to eliminate this ambiguity is to reformulate the physical equations such that only dimensionless variables and parameters appear. We now demonstrate how this can be achieved by illustrating the technique on several explicit differential equations. For fuller explanations, see Mickens [3, 7] and de St. Q. Isaacson [10].

1.4.1 *Linear Damped Oscillator*

The modeling differential equation for this physical system is [7, 8]

$$m\,\frac{d^2x}{dt^2} + k_1\,\frac{dx}{dt} + kx = 0, \quad x(0) = A, \quad \frac{dx(0)}{dt} = 0, \tag{1.4.1}$$

where m is the mass, k_1 is the damping coefficient, and k is the spring constant. Each term in this equation has the physical units of force and in terms of the units mass (M), length (L) and time (T), we have

$$[x] = L, \quad [t] = T, \quad [\text{force}] = \frac{ML}{T^2}$$

$$[m] = M, \quad [k_1] = \frac{M}{T}, \quad [k] = \frac{M}{T^2}, \quad [A] = L.$$

From the parameters (m, k_1, k) two time scales may be constructed,

$$T_1 = \left(\frac{m}{k}\right)^{1/3}, \qquad T_2 = \frac{m}{k_1}. \tag{1.4.2}$$

With the above indicated initial conditions, there exists only one length scale, i.e.,

$$L_1 = A. \tag{1.4.3}$$

Consequently, the following dimensionless variables can be formed

$$\bar{x} = \frac{x}{L_1} = \frac{x}{A}, \quad \bar{t} = \frac{t}{T_1}. \tag{1.4.4}$$

The particular form for $\bar{t}$ was selected because it is related to the natural frequency of the oscillators in the absence of damping, i.e., $k_1 = 0$; see Fowles [8]. Substitution of $x = A\bar{x}$ and $t = T_1\bar{t}$ into Eq. (1.4.1) gives

$$\begin{aligned}\frac{d^2x}{dt^2} + \left(\frac{k_1}{m}\right)\frac{dx}{dt} + \left(\frac{k}{m}\right)x &= \frac{d^2x}{dt^2} + \left(\frac{1}{T_2}\right)\frac{dx}{dt} + \left(\frac{1}{T_1^2}\right)x \\ &= \left(\frac{A}{T_1^2}\right)\frac{d^2\bar{x}}{d\bar{t}^2} + \left(\frac{A}{T_1T_2}\right)\frac{d\bar{x}}{d\bar{t}} + \left(\frac{A}{T_1^2}\right)\bar{x} = 0,\end{aligned}$$

which upon simplification gives

$$\frac{d^2\bar{x}}{d\bar{t}^2} + \epsilon\frac{d\bar{x}}{dt} + \bar{x} = 0, \quad \bar{x}(0) = 1, \quad \frac{d\bar{x}(0)}{d\bar{t}} = 0, \tag{1.4.5}$$

where

$$\epsilon = \frac{T_1}{T_2}. \tag{1.4.6}$$

Note that the original physical equation contains three parameters (m, k, k_1) and the initial condition parameter A, while the dimensionless equation is expressed in terms of a single parameter ϵ, which can be interpreted as the ratio of the period of the free oscillations to the damping time [7].

1.4.2 *Nonlinear Oscillator*

Consider the Duffing's equation [2, 7]

$$m\,\frac{d^2x}{dt^2} + kx + k_1x^3 = 0, \quad x(0) = A, \quad \frac{dx(0)}{dt} = 0. \tag{1.4.7}$$

The parameters (m, k, k_1, A) allow the construction of a single time scale

$$T_1 = \left(\frac{m}{k}\right)^{1/2}, \tag{1.4.8}$$

and two length scales

$$L_1 = \left(\frac{k}{k_1}\right)^{1/2}, \quad L_2 = A. \tag{1.4.9}$$

The time scale, T_1, is related to the free oscillations of the linear part of the Duffing equation. The length scale L_1 is an intrinsic, internal scale related only to the *a priori* given properties of the oscillator; it is a consequence of the oscillator being nonlinear. L_2 is the initial condition and thus is an external condition to be imposed on the nonlinear Duffing oscillator.

From the time scale and the two length scales, two dimensionless forms can be obtained; they are

$$\begin{aligned} x = L_1\bar{x} : \frac{d^2\bar{x}}{d\bar{t}^2} + \bar{x} + \bar{x}^3 = 0, \quad \bar{x}(0) = \frac{A}{L_1}, \quad \frac{d\bar{x}(0)}{d\bar{t}} = 0; \\ x = L_2\bar{x} : \frac{d^2\bar{x}}{d\bar{t}^2} + \bar{x} + \epsilon\bar{x}^3 = 0, \quad \bar{x}(0) = 1, \quad \frac{d\bar{x}(0)}{d\bar{t}} = 0, \end{aligned} \tag{1.4.10}$$

$$\epsilon = \left(\frac{L_2}{L_1}\right)^2 = \frac{k_1 A^2}{k}. \tag{1.4.11}$$

If $L_2 \ll L_1$, then Eq. (1.4.11) can provide the basis of a standard perturbation approach to solving the Duffing equation. If L_2 and L_2 are of the same order of magnitude, then nonperturbative methods must be applied [7].

1.4.3 $\ddot{x} + ax^p = 0$

Consider the following TNL oscillator

$$\frac{d^2x}{dt^2} + ax^p = 0, \quad x(0) = A, \quad \frac{dx(0)}{dt} = 0, \tag{1.4.12}$$

where

$$a > 0, \quad p = \frac{2m+1}{2n+1}, \quad (m, n) = \text{positive integers}. \tag{1.4.13}$$

Let

$$x = A\bar{x}, \quad t = T\bar{t},$$

and substitute these into Eq. (1.4.12) to obtain

$$\left(\frac{A}{T^2}\right)\frac{d^2\bar{x}}{d\bar{t}^2} + (aA^p)\bar{x}^p = 0,$$

or

$$\frac{d^2\bar{x}}{d\bar{t}^2} + \left[aT^2A^{(p-1)}\right]\bar{x}^p = 0.$$

Setting the coefficient of $\bar{x}^p$ equal to one gives dimensionless equation

$$\frac{d^2\bar{x}}{d\bar{t}^2} + \bar{x}^p = 0, \quad \bar{x}(0) = 1, \quad \frac{d\bar{x}(0)}{d\bar{t}} = 0, \tag{1.4.14}$$

and the time scale

$$T = \left[\frac{A^{(p-1)}}{a}\right]^{1/2}. \tag{1.4.15}$$

1.4.4 $\ddot{x} + ax + bx^{1/3} = 0$

Assume that both a and b are non-negative and consider the TNL oscillator differential equation

$$\frac{d^2x}{dt^2} + ax + bx^{1/3} = 0, \quad \bar{x}(0) = A, \quad \frac{dx(0)}{dt} = 0. \tag{1.4.16}$$

Using

$$t = T\bar{t}, \quad x = L\bar{x},$$

and substituting these expressions into the above differential equation, we find

$$\left(\frac{L}{T^2}\right)\frac{d^2\bar{x}}{d\bar{t}^2} + (aL)\bar{x} + (bL^{1/3})\bar{x}^{1/3} = 0,$$

and

$$\frac{d^2\bar{x}}{d\bar{t}^2} + (aT^2)\bar{x} + \left(\frac{bT^2}{L^{2/3}}\right)\bar{x}^{1/3} = 0.$$

The time and length scales for this particular selection may be calculated by setting to one the coefficients of the second and third terms in the last equation; doing this gives

$$T = \left(\frac{1}{a}\right)^{1/2}, \quad L = \left(\frac{b}{a}\right)^{3/2}. \tag{1.4.17}$$

Inspection of the relation for the length scale indicates that it is an intrinsic value determined by the parameters appearing in the original differential equation, while T_1 is related to the period of the free oscillations when the nonlinear term is absent. In terms of the original initial conditions, given in Eq. (1.4.16), the new initial conditions are

$$\bar{x}(0) = A\left(\frac{a}{b}\right)^{3/2}, \quad \frac{d\bar{x}(0)}{d\bar{t}} = 0. \tag{1.4.18}$$

It is of interest to investigate the case for which the length scale is taken to be $L_1 = A$. For this situation

$$t = T\bar{t}, \quad x = A\bar{x}.$$

Substituting into Eq. (1.4.16) gives

$$\left(\frac{A}{T^2}\right)\frac{d^2\bar{x}}{d\bar{t}^2} + (aA)\bar{x} + (bA^{1/3})\bar{x}^{1/3} = 0$$

and

$$\frac{d^2\bar{x}}{d\bar{t}^2} + (aT^2)\bar{x} + \frac{bT^2}{A^{1/3}}\bar{x}^{1/3} = 0.$$

Setting the coefficient of the $\bar{x}$ term equal to one gives the following expression

$$\frac{d^2\bar{x}}{d\bar{t}^2} + \bar{x} + \epsilon\bar{x}^{1/3} = 0 \tag{1.4.19}$$

where

$$\epsilon = \frac{b}{aA^{2/3}} = \left(\frac{L}{A}\right)^{2/3}. \tag{1.4.20}$$

Observe that ϵ is the ratio of the system's intrinsic length scale to the initial value $x(0) = A$, raised to the two-thirds power. The initial conditions for the dimensionless equation are

$$\bar{x}(0) = 1, \quad \frac{d\bar{x}(0)}{d\bar{t}} = 0.$$

The above example illustrates the fact that we can often eliminate all the parameters and have a nontrivial set of initial conditions or we can have one dimensionless parameter with simple initial conditions. Generally, we will opt for the first situation.

1.5 Exactly Solvable TNL Oscillators

A number of special cases of TNL oscillator differential equations exist that can be solved exactly in terms of standard known functions. These include the antisymmetric, constant force oscillator [11]; the particle-in-a-box [7]; a particular form of the Duffing equation [7]; and the quadratic oscillator [12]. In this section, we present the details of how to obtain the appropriate analytical results for each oscillator.

1.5.1 *Antisymmetric, Constant Force Oscillator* [11]

This oscillator has the following equation of motion

$$\ddot{x} + \operatorname{sgn}(x) = 0, \tag{1.5.1}$$

where the $\operatorname{sgn}(x)$ function is

$$\operatorname{sgn}(x) = \begin{cases} +1, & \text{for } x > 0, \\ 0, & \text{for } x = 0, \\ -1, & \text{for } x < 0. \end{cases} \tag{1.5.2}$$

This nonlinear equation is equivalent to the following set of linear equations

$$\ddot{x} + 1 = 0, \quad \text{for } x > 0, \tag{1.5.3a}$$

$$\ddot{x} - 1 = 0, \quad \text{for } x < 0. \tag{1.5.3b}$$

Their respective solutions are

$$x_+(t) = -\left(\frac{1}{2}\right) t^2 + A_1 t + B_1, \tag{1.5.4a}$$

$$x_-(t) = \left(\frac{1}{2}\right) t^2 + A_2 t + B_2, \tag{1.5.4b}$$

where the integration constants are denoted by (A_1, A_2, B_1, B_2). We will now obtain the required solution by using the initial conditions

$$x(0) = 0, \quad \dot{x}(0) = A > 0. \tag{1.5.5}$$

Since $A > 0$, we must use $x_+(t)$ to match these initial conditions, i.e.,

$$x_+(0) = B_1 \Rightarrow B_1 = 0,$$
$$\dot{x}_+(0) = A_1 \Rightarrow A_1 = A,$$

and

$$x_+(t) = -\left(\frac{1}{2}\right) t(t - 2A), \quad 0 \le t \le 2A. \tag{1.5.6}$$

Observe that $x_+(t)$ has the following properties

- $x_+(0) = 0$,
- $x_+(t) > 0$, for $0 < t < 2A$,
- $x_+(2A) = 0$.

These results imply that the period, T, is

$$T = 4A. \tag{1.5.7}$$

Now, at $t = 2A$ and $t = 4A$, we require

$$x_-(2A) = 0, \quad x_-(4A) = 0.$$

From Eq. (1.5.4b) it follows that

$$A_2 = -3A, \quad B_2 = 4A^2,$$

and, as a consequence of these values $x_-(t)$ is

$$x_-(t) = \frac{t^2}{2} - (3A)t + 4A^2, \quad 2A \le t \le 4A. \tag{1.5.8}$$

Combining this information gives

$$x(t + 4A) = x(t), \tag{1.5.9}$$

$$x(t) = \begin{cases} -t(t-2A)/2, & \text{for } 0 \le t \le 2A, \\ \dfrac{t^2}{2} - (3A)t + 4A^2, & \text{for } 2A < t \le 4A. \end{cases} \tag{1.5.10}$$

The Fourier series representation for $x(t)$ can be easily calculated and is given by the following expression

$$x(t) = \left(\frac{16A^2}{\pi^3}\right) \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \sin\left[\frac{(2k+1)\pi t}{2A}\right]. \tag{1.5.11}$$

Note that only odd values of the fundamental period, $T = 4A$, appear in the expansion. Further, observe that the four coefficients have the upper bound

$$b_k \le \frac{C(A)}{k^3}, \tag{1.5.12}$$

where $C(A)$ can be determined by inspection from Eq. (1.5.11).

1.5.2 *Particle-in-a-Box*

Consider a one-dimensional box located between $x = 0$ and $x = L$, i.e., the "size" of the box is L. Let a particle be situated in the box such that at $t = 0$, it is at $x = 0$ with the velocity $v_0 > 0$, i.e., it is moving to the right. After a time T^*, where

$$T^* = \frac{L}{v_0}, \tag{1.5.13}$$

the particle hits the wall at $x = L$, reverses direction and continues to the left. Again, after a time interval of T^*, the particle collides with the wall at $x = 0$ and reverses direction. Thus, the overall motion is periodic with period

$$T = 2T^* = \frac{2L}{v_0}\,; \tag{1.5.14}$$

therefore

$$x(t + T) = x(t).$$

If we define the velocity function as $v(t) = \dot{x}(t)$, then

$$v(t + T) = v(t),$$

where

$$x(t) = \begin{cases} v_0 t, & \text{for } 0 \le t \le \dfrac{T}{2}, \\ v_0(T - t), & \text{for } \dfrac{T}{2} \le t \le T, \end{cases} \tag{1.5.15}$$

$$v(t) = \begin{cases} v_0, & \text{for } 0 < t < \dfrac{T}{2}, \\ -v_0, & \text{for } \dfrac{T}{2} < t < T. \end{cases} \tag{1.5.16}$$

If we let $L = \pi$ and $v_0 = 1$, then $T = 2\pi$ and the Fourier series for $x(t)$ and $v(t)$ are

$$x(t) = \frac{\pi}{2} - \left(\frac{4}{\pi}\right) \sum_{k=1}^{\infty} \frac{\cos(2k-1)t}{(2k-1)^2} \tag{1.5.17}$$

$$v(t) = \left(\frac{4}{\pi}\right) \sum_{k=1}^{\infty} \frac{\sin(2k-1)t}{(2k-1)}. \tag{1.5.18}$$

Again, observe that only odd multiples of the fundamental period appear in the expansion.

1.5.3 *Restricted Duffing Equation*

The full Duffing equation takes the form

$$\ddot{x} + k_1\dot{x} + kx + k_2x^3 = 0.$$

The restricted Duffing equation is (in dimensionless units)

$$\ddot{x} + x^3 = 0. \tag{1.5.19}$$

For the initial conditions

$$x(0) = A, \quad \dot{x}(0) = 0, \tag{1.5.20}$$

the exact solution is [7, 13, 14]

$$x(t) = Acn\left(At; 1/\sqrt{2}\right), \tag{1.5.21}$$

where "cn" is the Jacobi elliptic function [13, 15].

Let k and k' satisfy the relation

$$(k')^2 + k^2 = 1.$$

Define the complete elliptical integral of the first kind to be [13, 15]

$$F(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}.$$

Define $q(k)$ as

$$q(k) \equiv \exp\left[-\frac{\pi F(k')}{F(k)}\right],$$

and take $v(k, u)$ to be

$$v(k, u) = \left[\frac{\pi}{2F(k)}\right] u.$$

Based on the above quantities, the Jacobi cosine elliptic function is given by the formula

$$cn(u, k) = \left[\frac{2\pi}{kF(k)}\right] \sum_{m=0}^{\infty} \left(\frac{q^{m+\frac{1}{2}}}{1 + q^{2m+1}}\right) \cos(2m+1)v. \tag{1.5.22}$$

For our case, i.e., the restricted Duffing equation, we have

$$k = \frac{1}{\sqrt{2}}, \quad F\left(\frac{1}{\sqrt{2}}\right) = 1.854\,074\ldots$$

$$q\left(\frac{1}{\sqrt{2}}\right) = 0.043\,213\ldots.$$

If we write $q\left(1/\sqrt{2}\right)$ as

$$q\left(1/\sqrt{2}\right) = e^{-a}$$

then

$$a = 3.141592\ldots$$

and it is easy to show that the Fourier coefficients are bounded by an exponential function of m, i.e.,

$$\begin{aligned} a_m &\equiv \left[\frac{2\sqrt{2}\pi}{F\left(\frac{1}{\sqrt{2}}\right)}\right]\left[\frac{e^{-(m+\frac{1}{2})a}}{1+e^{-(2m+1)a}}\right], \\ &< Ce^{-am}, \end{aligned} \tag{1.5.23}$$

where C can be easily found from the above expression. This result implies that the Fourier coefficients decrease rapidly and, consequently, the use of just a few terms in the expansion of $cn(u,k)$ may provide an accurate analytical representation of the periodic solution [7, 16, 17].

1.5.4 *Quadratic Oscillator*

The quadratic oscillator differential equation is

$$\ddot{x} + |x|x = 0 \quad \text{or} \quad \ddot{x} + x^2\text{sgn}(x) = 0. \tag{1.5.24}$$

In Section 2.2.2, we show that all solutions for this TNL oscillator are periodic with period given by the expression [12]

$$T(A) = \frac{2^{1/6}\left[\Gamma\left(\frac{1}{3}\right)\right]^3}{\pi} \cdot \frac{1}{A^{1/2}}, \tag{1.5.25}$$

where $x(0) = A$ and $\dot{x}(0) = 0$ are the initial conditions. Further, we find that the solution is

$$x(t) = A\left[\frac{\left(\sqrt{3}+1\right)cn(t,k) - \left(\sqrt{3}-1\right)}{1+cn(t,k)}\right], \tag{1.5.26}$$

where

$$cn(t,k) \equiv cn\left[\left(\frac{4A^2}{3}\right)^{1/4} t, k\right], \quad k^2 = \frac{2+\sqrt{3}}{4}. \tag{1.5.27}$$

Consequently, the periodic solution is expressed as a rational function of the Jacobi cosine function.

1.6 Overview of TNL Oscillator Methods

Nonlinear oscillations occurring in one-degree-of-freedom systems have been studied intensely for almost two centuries [2–4, 18–25]. The general form that those equations take is

$$\ddot{x} + x = \epsilon f(x, \dot{x}), \quad 0 < \epsilon \ll 1, \tag{1.6.1}$$

where ϵ is a small parameter. These classical methods are based on expansions in terms of ϵ which are taken to be asymptotic series. Each particular perturbation method is distinguished by how this feature is accomplished. If from *a priori* considerations it can be determined that periodic solutions exist, then a major task, for each method, is to eliminate the so-called secular terms. Secular terms are expressions in the solutions that are oscillatory, with increasing, time dependent amplitudes [3, 4, 7], i.e., for an odd-parity system

$$\text{secular term} : t^n \cos[(2k+1)\Omega t], \tag{1.6.2}$$

where (n, k) are integers, with $n \geq 1$ and $k \geq 0$. For all of the standard methods, procedures have evolved to resolve this issue.

Inspection of Eq. (1.6.1) shows that each of the classical methods has at its foundation the explicit assumption that when $\epsilon = 0$ the resulting "core" equation is the linear harmonic oscillator differential equation, namely,

$$\ddot{x}_0 + x_0 = 0, \tag{1.6.3}$$

where the zero indicates $\epsilon = 0$. This fact presents an immediate difficulty for TNL oscillators, where Eq. (1.6.1) is replaced by, for example,

$$\ddot{x} + x^p = \epsilon f(x, \dot{x}), \quad p \neq 1. \tag{1.6.4}$$

We observe that when $\epsilon = 0$, this equation reduces to the nonlinear equation

$$\ddot{x}_0 + x_0^p = 0, \tag{1.6.5}$$

and this type of equation would, at the very least, greatly complicate any solution construction based on expansions in the parameter ϵ. The general conclusion is that the standard classical perturbation procedures cannot be applied to TNL equations.

Generally, four techniques can be used to determine the approximations to the periodic solutions of nonlinear oscillator differential equations. In general, they may be applied to both standard and TNL equations, i.e., those that can be expressed as in Eqs. (1.6.1) and (1.6.4). All of these procedures, except for one, set up a methodology that converts the problem of solving a single, second-order, nonlinear differential equation to one of solving, in sequence, an infinite set of linear, inhomogeneous equations. A brief discussion of each procedure will now be given. The relevant details will appear in the chapter devoted to each particular method.

1.6.1 *Harmonic Balance*

The harmonic balance method is based on the use of an assumed truncated trigonometric expansion for the periodic solution. The n-th order approximation takes a form such as

$$x_n(t) = a_1 \cos\theta + a_2 \cos(3\theta) + \cdots + a_n \cos(2n-1)\theta, \tag{1.6.6}$$

where $\theta = \Omega_n t$, and the n-coefficients and Ω_n are to be determined. For a conservative system with initial conditions, $x(0) = A$ and $\dot{x}(0) = 0$, the basic strategy is to substitute Eq. (1.6.6) into the differential equation and expand the resulting expression into a trigonometric series, but only including terms from $\cos\theta$ to $\cos(2n-1)\theta$; doing this gives the following type of relation

$$\begin{aligned} &H_1(a_1, a_2, \ldots, a_n, \Omega_n)\cos\theta + H_2(a_1, a_2, \ldots, a_n, \Omega_n)\cos(3\theta) \\ &\quad + \cdots + H_n(a_1, a_2, \ldots, \Omega_n)\cos(2n-1)\theta + \text{HOH} \simeq 0, \end{aligned} \tag{1.6.7}$$

where HOH stands for higher-order-harmonics, and for a given differential equation the $H_i(a_1, a_2, \ldots, a_n, \Omega_n)$ are completely specified. The harmonic balancing procedure consists in setting the coefficients of the cosine terms to zero, i.e.,

$$H_i(a_1, a_2, \ldots, a_n, \Omega_n) = 0; \quad i = 1, 2, 3, \ldots, n. \tag{1.6.8}$$

These n-relations, along with the initial conditions, can be solved to give all the coefficients and Ω_n as functions of A, i.e.,

$$\begin{cases} a_i = a_i(A); & i = 1, 2, \ldots, n; \\ \Omega_n = \Omega_n(A). \end{cases} \tag{1.6.9}$$

1.6.2 *Parameter Expansion*

The parameter expansion method is an extension and generalization of standard perturbation methods. The basic idea is to take a parameter occurring in the differential equation and represent it as an expansion in terms of a fictitious or artificial parameter. Such an expansion may provide a valid (asymptotic) solution when the parameter has small values. If this is correct, the resulting expression is then evaluated for some large value of the parameter (usually selected to be one) for the actual problem of interest.

To illustrate the method, consider the TNL equation

$$\ddot{x} + x^3 = 0, \tag{1.6.10}$$

it can be rewritten as

$$\ddot{x} + \Omega^2 x = \Omega^2 x - x^3, \tag{1.6.11}$$

where Ω^2 is, for the present, unknown. Now a parameter p is introduced in the following way

$$\ddot{x} + \Omega^2 x = p(\Omega^2 x - x^3). \tag{1.6.12}$$

The basic idea is now to assume p to be small and treat the latter equation using standard perturbation methods. After this is done, one sets $p = 1$ in the resulting expressions for the approximations to the solutions. In general, the angular frequency, $\Omega = 2\pi/T$, is determined by the requirement that no secular terms be present in the solutions.

1.6.3 *Averaging Methods*

Averaging methods generally start with the following representation for the periodic solutions

$$x(t) = a(t)\cos\psi(t), \tag{1.6.13}$$

and, through a series of assumptions and mathematical manipulations, derive first-order differential equations for unknown functions $a(t)$ and $\psi(t)$, i.e.,

$$\frac{da}{dt} = \epsilon F_1(a), \quad \frac{d\psi}{dt} = \epsilon F_2(a), \tag{1.6.14}$$

where ϵ is, in general, some small parameter appearing in the original differential equation. These equations are solved, in the order presented, to obtain $a(t,\epsilon)$ and $\psi(t,\epsilon)$.

In somewhat more detail, the "a" and "ψ" in Eq. (1.6.14) are not, strictly speaking the same as the "a" and "ψ" in Eq. (1.6.13). The latter equations are averaged equations, with the averaging done on an interval of 2π in the variable ψ. The details as to what exactly is needed to obtain the expressions in Eq. (1.6.14) will be given later in the chapter devoted to this topic.

This method has the distinct advantage, as compared with all the other procedures, of allowing oscillatory, but not necessarily periodic solutions, to be calculated.

1.6.4 *Iteration Techniques*

Iteration techniques start with a given nonlinear (regular or TNL) oscillator differential equation and, through a series of manipulations, transform it into a set of linear, second-order, inhomogeneous equations that must be solved sequentially. For example, consider

$$\ddot{x} + x^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0,$$

and add $\Omega^2 x$ to both sides to obtain

$$\ddot{x} + \Omega^2 x = \Omega^2 x - x^3. \tag{1.6.15}$$

Define $x_0(t)$ to be

$$x_0(t) = A\cos(\Omega_0 t),$$

where Ω_0 is currently known. Now define the sequence of functions

$$x_0(t), x_1(t), \ldots, x_n(t), \ldots,$$

which are solutions to

$$\begin{cases} \ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = \Omega_k^2 x_k - x_k^3, \\ x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \end{cases} \tag{1.6.16}$$

The Ω_k are determined by the requirement that x_{k+1} does not contain secular terms. Note that under these conditions, $x_1(t)$ can be calculated from a knowledge of $x_0(t)$; $x_2(t)$, likewise, can be determined from $x_1(t)$; etc. In practice, the hope is that $x_k(t)$, for small values of k, will provide an accurate representation to the actual periodic solutions.

1.7 Discussion

We end this chapter with a brief overview of some of the criteria or qualities expected of a method that can be used to calculate analytical approximations to the periodic solutions to a TNL oscillator. However, in the final analysis, the validity and value of a particular method and the solutions that it produces depend heavily on what we intend to do with the results obtained from the calculations. However, the following four items/issues are of prime importance:

1) The method of calculation should be rather direct to understand and easy to implement.

In particular, this means that higher-order approximations should be capable of being straightforwardly obtained, although in practice it may be algebraically intensive to carry out this process.

2) The method should allow accurate estimates to be made for the period of the oscillations.

Again, what is to be considered "accurate enough" will clearly be a function of what we intend to do with this result.

3) The calculational procedure should produce trigonometric approximations to the actual Fourier expansions such that *a priori* known restrictions on the expansion coefficients (bounds, rates of decrease, etc.) should be generally satisfied by those of the approximate solutions.

A wide range of theorems exist on the general properties of Fourier coefficients [16, 17]. To the degree that this is possible, the coefficients appearing in the approximate expressions should also satisfy these restrictions. In fact, these limitations on the coefficients may be used as a measure of the "quality" of the solutions produced by a given calculational scheme.

4) The approximate solutions obtained from a particular calculational scheme should have the appropriate mathematical forms, structures, and properties known to be possessed by the exact solutions.

Thus, for example, the TNL equation is of odd-parity, then the approximate solutions should only contain odd multiples of the fundamental period. The occurrence of terms having even periods would indicate that the scheme is incorrect. In a similar fashion, if the TNL oscillator is conservative and if the initial conditions, $x(0) = A$ and $\dot{x}(0) = 0$, are selected, then only cosine terms should appear in its trigonometric approximation expansion.

While we realize that the criteria presented above are vague and heuristic in nature, it is clear that this situation is as it is because approximation procedures in practice, and in the results they produce, are by their essence never fully based on rigorous mathematics. In almost all cases, some mathematical requirement that was used to justify the procedure is violated. Therefore, the real value of a (practical) approximation method is whether it provides a suitable resolution of some set of issues related to a problem that is formulated in the language of mathematics. This means that the user of such schemes must have deep fundamental insight into the original (in most cases) physical problem, while also understanding and acknowledging the limitations of a purely mathematical approach. This realization has provided much of the existing tension between "pure" and "applied" mathematics.

Problems

1.1 Which of the following functions are TNL (at $x = 0$)?

(i) x^2
(ii) $x^{1/2} + x$
(iii) $x + \frac{\text{sgn}(x)}{|x|^{1/2}}$
(iv) $x + x^{5/3}$
(v) $x + |x|$.

1.2 Are any of the following differential equations TNL equations? Why?

(i) $\ddot{x} + \frac{1}{x} = 0$
(ii) $\ddot{x} - x + x^3 = 0$
(iii) $\ddot{x} + x + \frac{1}{x^{3/5}} = 0$
(iv) $\ddot{x} + \frac{x^{5/3}}{1+x^2} = 0$.

1.3 Prove that

$$\frac{d}{dx}|x| = \text{sgn}(x).$$

1.4 Is x^3 the same as $|x|^2 x$. Explain your answer. What about x and $|x|\text{sgn}(x)$?

1.5 Transform the following differential equations to dimensionless forms.

(i) $m\ddot{x} + ax^2 + bx^3 = 0$
(ii) $\ddot{x} + \omega^2 x + kx^{1/3} = (a - bx^2)\dot{x}$
(iii) $\ddot{x} + \frac{\lambda x^3}{1+fx^2} = 0$
(iv) $\ddot{x} + g|x|^2\text{sgn}(x) + hx^3 = 0$.

Assume the initial conditions are $x(0) = A$ and $\dot{x}(0) = 0$.

- For a given equation, are the scales unique?
- If not, discuss the differences between the scales.
- What physical interpretation can be associated with each set of scales when a particular differential equation has more than one set of scales.

1.6 Are any of the differential equations listed in Problems 1.2 and 1.5 of odd-parity? Which are invariant under $t \to -t$?

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