

# Contents

<i>Preface</i>	vii
<i>List of Figures</i>	xix
<i>List of Tables</i>	xxi
1. Background and General Comments	1
1.1 Truly Nonlinear Functions . . . . .	1
1.2 Truly Nonlinear Oscillators . . . . .	2
1.3 General Remarks . . . . .	3
1.4 Scaling and Dimensionless Form of Differential Equations	5
1.4.1 Linear Damped Oscillator . . . . .	5
1.4.2 Nonlinear Oscillator . . . . .	6
1.4.3 $\ddot{x} + ax^p = 0$ . . . . .	7
1.4.4 $\ddot{x} + ax + bx^{1/3} = 0$ . . . . .	8
1.5 Exactly Solvable TNL Oscillators . . . . .	9
1.5.1 Antisymmetric, Constant Force Oscillator . . . . .	10
1.5.2 Particle-in-a-Box . . . . .	11
1.5.3 Restricted Duffing Equation . . . . .	12
1.5.4 Quadratic Oscillator . . . . .	14
1.6 Overview of TNL Oscillator Methods . . . . .	14
1.6.1 Harmonic Balance . . . . .	16
1.6.2 Parameter Expansion . . . . .	16
1.6.3 Averaging Methods . . . . .	17
1.6.4 Iteration Techniques . . . . .	18
1.7 Discussion . . . . .	18
Problems . . . . .	20

References . . . . .	21
2. Establishing Periodicity . . . . .	23
2.1 Phase-Space . . . . .	23
2.1.1 System Equations . . . . .	24
2.1.2 Fixed-Points . . . . .	24
2.1.3 ODE for Phase-Space Trajectories . . . . .	25
2.1.4 Null-clines . . . . .	25
2.1.5 Symmetry Transformations . . . . .	26
2.1.6 Closed Phase-Space Trajectories . . . . .	26
2.1.7 First-Integrals . . . . .	26
2.2 Application of Phase-Space Methods . . . . .	27
2.2.1 Linear Harmonic Oscillator . . . . .	27
2.2.2 Several TNL Oscillator Equations . . . . .	31
2.3 Dissipative Systems: Energy Methods . . . . .	33
2.3.1 Damped Linear Oscillator . . . . .	35
2.3.2 Damped TNL Oscillator . . . . .	35
2.3.3 Mixed-Damped TNL Oscillator . . . . .	36
2.4 Resume . . . . .	39
Problems . . . . .	39
References . . . . .	40
3. Harmonic Balance . . . . .	43
3.1 Direct Harmonic Balance: Methodology . . . . .	44
3.2 Worked Examples . . . . .	46
3.2.1 $\ddot{x} + x^3 = 0$ . . . . .	47
3.2.2 $\ddot{x} + x^{-1} = 0$ . . . . .	49
3.2.3 $\ddot{x} + x^2 \text{sgn}(x) = 0$ . . . . .	51
3.2.4 $\ddot{x} + x^{1/3} = 0$ . . . . .	54
3.2.5 $\ddot{x} + x^{-1/3} = 0$ . . . . .	57
3.3 Rational Approximations . . . . .	61
3.3.1 Fourier Expansion . . . . .	62
3.3.2 Properties of $a_k$ . . . . .	62
3.3.3 Calculation of $\ddot{x}$ . . . . .	63
3.4 Worked Examples . . . . .	63
3.4.1 $\ddot{x} + x^3 = 0$ . . . . .	63
3.4.2 $\ddot{x} + x^2 \text{sgn}(x) = 0$ . . . . .	65
3.4.3 $\ddot{x} + x^{-1} = 0$ . . . . .	66

3.5	Third-Order Equations . . . . .	67
3.5.1	Castor Model . . . . .	68
3.5.2	TNL Castor Models . . . . .	69
3.6	Resume . . . . .	70
3.6.1	Advantages . . . . .	70
3.6.2	Disadvantages . . . . .	70
	Problems . . . . .	71
	References . . . . .	72
4.	Parameter Expansions . . . . .	75
4.1	Introduction . . . . .	75
4.2	Worked Examples . . . . .	76
4.2.1	$\ddot{x} + x^3 = 0$ . . . . .	76
4.2.2	$\ddot{x} + x^{-1} = 0$ . . . . .	78
4.2.3	$\ddot{x} + x^3/(1 + x^2) = 0$ . . . . .	80
4.2.4	$\ddot{x} + x^{1/3} = 0$ . . . . .	81
4.2.5	$\ddot{x} + x^3 = \epsilon(1 - x^2)\dot{x}$ . . . . .	84
4.2.6	$\ddot{x} + \text{sgn}(x) = 0$ . . . . .	85
4.3	Discussion . . . . .	86
4.3.1	Advantages . . . . .	87
4.3.2	Difficulties . . . . .	87
	Problems . . . . .	87
	References . . . . .	88
5.	Iteration Methods . . . . .	89
5.1	General Methodology . . . . .	89
5.1.1	Direct Iteration . . . . .	89
5.1.2	Extended Iteration . . . . .	91
5.2	Worked Examples: Direct Iteration . . . . .	92
5.2.1	$\dot{x} + x^3 = 0$ . . . . .	92
5.2.2	$\ddot{x} + x^3/(1 + x^2) = 0$ . . . . .	97
5.2.3	$\ddot{x} + x^{-1} = 0$ . . . . .	100
5.2.4	$\ddot{x} + \text{sgn}(x) = 0$ . . . . .	103
5.2.5	$\ddot{x} + x^{1/3} = 0$ . . . . .	105
5.2.6	$\ddot{x} + x^{-1/3} = 0$ . . . . .	108
5.2.7	$\ddot{x} + x + x^{1/3} = 0$ . . . . .	110
5.3	Worked Examples: Extended Iteration . . . . .	112
5.3.1	$\ddot{x} + x^3 = 0$ . . . . .	113

5.3.2	$\ddot{x} + x^{-1} = 0$ . . . . .	115
5.4	Discussion . . . . .	117
5.4.1	Advantages of Iteration Methods . . . . .	118
5.4.2	Disadvantages of Iteration Methods . . . . .	119
	Problems . . . . .	120
	References . . . . .	121
6.	Averaging Methods . . . . .	123
6.1	Elementary TNL Averaging Methods . . . . .	124
6.1.1	Mickens-Oyedeki Procedure . . . . .	124
6.1.2	Combined Linearization and Averaging Method . . . . .	126
6.2	Worked Examples . . . . .	129
6.2.1	$\ddot{x} + x^3 = -2\epsilon\dot{x}$ . . . . .	129
6.2.2	$\ddot{x} + x^3 = -\epsilon\dot{x}^3$ . . . . .	131
6.2.3	$\ddot{x} + x^3 = \epsilon(1 - x^2)\dot{x}$ . . . . .	132
6.2.4	$\ddot{x} + x^{1/3} = -2\epsilon\dot{x}$ . . . . .	133
6.2.5	$\ddot{x} + x^{1/3} = \epsilon(1 - x^2)\dot{x}$ . . . . .	134
6.2.6	$\ddot{x} + x = -2\epsilon(\dot{x})^{1/3}$ . . . . .	135
6.2.7	General Comments . . . . .	137
6.3	Cveticanin's Averaging Method . . . . .	138
6.3.1	Exact Period . . . . .	139
6.3.2	Averaging Method . . . . .	140
6.3.3	Summary . . . . .	142
6.4	Worked Examples . . . . .	142
6.4.1	$\ddot{x} + x x ^{\alpha-1} = -2\epsilon\dot{x}$ . . . . .	142
6.4.2	$\ddot{x} + x x ^{\alpha-1} = -2\epsilon(\dot{x})^3$ . . . . .	144
6.4.3	$\ddot{x} + x x ^{\alpha-1} = \epsilon(1 - x^2)\dot{x}$ . . . . .	145
6.5	Chronology of Averaging Methods . . . . .	147
6.6	Comments . . . . .	149
	Problems . . . . .	151
	References . . . . .	152
7.	Comparative Analysis . . . . .	155
7.1	Purpose . . . . .	155
7.2	$\ddot{x} + x^3 = 0$ . . . . .	156
7.2.1	Harmonic Balance . . . . .	156
7.2.2	Parameter Expansion . . . . .	158
7.2.3	Iteration . . . . .	158

7.2.4	Comments . . . . .	159
7.3	$\ddot{x} + x^{1/3} = 0$ . . . . .	160
7.3.1	Harmonic Balance . . . . .	160
7.3.2	Parameter Expansion . . . . .	161
7.3.3	Iteration . . . . .	162
7.3.4	Comment . . . . .	162
7.4	$\ddot{x} + x^3 = -2\epsilon\dot{x}$ . . . . .	163
7.4.1	Mickens-Oyedèjì . . . . .	163
7.4.2	Combined-Linearization-Averaging . . . . .	165
7.4.3	Cveticanin's Method . . . . .	166
7.4.4	Discussion . . . . .	167
7.5	$\ddot{x} + x^{1/3} = -2\epsilon\dot{x}$ . . . . .	167
7.5.1	Combined-Linearization-Averaging . . . . .	167
7.5.2	Cveticanin's Method . . . . .	168
7.5.3	Discussion . . . . .	170
7.6	$\ddot{x} + x^3 = \epsilon(1 - x^2)\dot{x}$ . . . . .	170
7.6.1	Mickens-Oyedèjì . . . . .	170
7.6.2	Cveticanin's Method . . . . .	171
7.6.3	Discussion . . . . .	172
7.7	$\ddot{x} + x^{1/3} = \epsilon(1 - x^2)\dot{x}$ . . . . .	175
7.8	General Comments and Calculation Strategies . . . . .	175
7.8.1	General Comments . . . . .	176
7.8.2	Calculation Strategies . . . . .	177
7.9	Research Problems . . . . .	179
	References . . . . .	181
Appendix A Mathematical Relations . . . . .		183
A.1	Trigonometric Relations . . . . .	183
A.1.1	Exponential Definitions of Trigonometric Functions . . . . .	183
A.1.2	Functions of Sums of Angles . . . . .	183
A.1.3	Powers of Trigonometric Functions . . . . .	183
A.1.4	Other Trigonometric Relations . . . . .	184
A.1.5	Derivatives and Integrals of Trigonometric Functions . . . . .	185
A.2	Factors and Expansions . . . . .	186
A.3	Quadratic Equations . . . . .	187
A.4	Cubic Equations . . . . .	187
A.5	Differentiation of a Definite Integral with Respect to a Pa- rameter . . . . .	188

A.6	Eigenvalues of a $2 \times 2$ Matrix . . . . .	188
	References . . . . .	189
Appendix B Gamma and Beta Functions . . . . .		191
B.1	Gamma Function . . . . .	191
B.2	The Beta Function . . . . .	191
B.3	Two Useful Integrals . . . . .	192
Appendix C Fourier Series . . . . .		193
C.1	Definition of Fourier Series . . . . .	193
C.2	Convergence of Fourier Series . . . . .	194
	C.2.1 Examples . . . . .	194
	C.2.2 Convergence Theorem . . . . .	194
C.3	Bounds on Fourier Coefficients . . . . .	195
C.4	Expansion of $F(a \cos x, -a \sin x)$ in a Fourier Series . . . . .	195
C.5	Fourier Series for $(\cos \theta)^\alpha$ and $(\sin \theta)^\alpha$ . . . . .	196
	References . . . . .	198
Appendix D Basic Theorems of the Theory of Second-Order Differential Equations . . . . .		199
D.1	Introduction . . . . .	199
D.2	Existence and Uniqueness of the Solution . . . . .	200
D.3	Dependence of the Solution on Initial Conditions . . . . .	200
D.4	Dependence of the Solution on a Parameter . . . . .	201
	References . . . . .	202
Appendix E Linear Second-Order Differential Equations . . . . .		203
E.1	Basic Existence Theorem . . . . .	203
E.2	Homogeneous Linear Differential Equations . . . . .	203
	E.2.1 Linear Combination . . . . .	204
	E.2.2 Linear Dependent and Linear Independent Functions . . . . .	204
	E.2.3 Theorems on Linear Second-Order Homogeneous Differential Equations . . . . .	204
E.3	Inhomogeneous Linear Differential Equations . . . . .	205
	E.3.1 Principle of Superposition . . . . .	206
	E.3.2 Solutions of Linear Inhomogeneous Differential Equations . . . . .	207

E.4	Linear Second-Order Homogeneous Differential Equations with Constant Coefficients . . . . .	207
E.5	Linear Second-Order Inhomogeneous Differential Equations with Constant Coefficients . . . . .	208
E.6	Secular Terms . . . . .	210
	References . . . . .	211
Appendix F	Lindstedt-Poincaré Perturbation Method	213
	References . . . . .	216
Appendix G	A Standard Averaging Method	217
	References . . . . .	220
Appendix H	Discrete Models of Two TNL Oscillators	221
H.1	NSFD Rules . . . . .	221
H.2	Discrete Energy Function . . . . .	222
H.3	Cube-Root Equation . . . . .	223
H.4	Cube-Root/van der Pol Equation . . . . .	225
	References . . . . .	226
	<i>Bibliography</i>	227
	<i>Index</i>	237