

Well-Posedness for a Ginzburg-Landau Model in Superfluidity

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1 Introduction

Superfluidity was discovered in 1938 by Kapitza, Allen and Misener, who observed that at a temperature close to the absolute zero, an isotope of helium, ${}^4\text{He}$ was able to flow without viscosity in narrow capillaries. More precisely, immediately below its boiling temperature $\theta = 4.21\text{K}$, ${}^4\text{He}$ behaves like an ordinary fluid with small viscosity. However it exhibits a superfluid behavior, when the temperature decreases under the critical value $\theta_\lambda = 2.17\text{K}$.

The traditional theories of superfluidity consider a superfluid as a mixture of two fluids, with the density ρ and the velocity \mathbf{v} given by

$$\rho = \rho_n + \rho_s, \quad \rho\mathbf{v} = \rho_n\mathbf{v}_n + \rho_s\mathbf{v}_s.$$

Here ρ_n , ρ_s , \mathbf{v}_n , \mathbf{v}_s are called respectively normal and superfluid density and velocity. The two fluid model was suggested by Tisza in order to explain experiments showing the twofold behavior of liquid helium: on the one hand it is able to flow frictionless through narrow channels, on the other, a body moving in ${}^4\text{He}$ shows the existence of a viscous drag.

According to Landau's point of view, the two-fluid model is just a convenient way of saying that each particle of helium is able to present two kinds of excitations. One of these motions is normal with the same properties as that of an ordinary viscous liquid, the other is superfluid ([3]).

The model that we propose is based on the interpretation of superfluidity as a second order phase transition, similar to the transition occurring in superconductivity. Indeed, as pointed out by several authors, there are many experimental phenomena (e.g., absence of dissipation, existence of a critical velocity, creation of vortices), showing the analogies between the behavior of superfluids and superconductors ([4]). In order to describe the phenomenon of superfluidity, we suggest a Ginzburg-Landau model, which initially was introduced in the framework of superconductivity, but has been recently

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adopted to study phase transitions induced by temperature variations. The resulting differential system allows to explain the main aspects typical of superfluidity. Following the idea of the Cooper pairs in the context of superconductivity, we assume that each particle behaves as a couple of particles endowed with a normal and a superfluid excitation and assume that the normal component is related to the superfluid component by means of a constitutive equation similar to London's equation in superconductivity. Accordingly the evolution equation for the superfluid component is separated from the equation of the normal component.

Thermal effects are considered by introducing the evolution equation for the temperature deduced by the first principle of thermodynamics. In particular, we assume that the heat equation involves the velocity of the superfluid component. This model agrees with the second law of thermodynamics, since the Clausius-Duhem inequality is satisfied.

We analyze the well-posedness of the differential system constituted by the Ginzburg-Landau equation, by the evolution equation of the superfluid component and by the heat equation. For this problem we have proved existence and uniqueness of the solutions.

2 A Ginzburg-Landau model for superfluidity

Let us consider a superfluid occupying a bounded domain $\Omega \subset \mathbb{R}^3$. The passage from the normal to the superfluid state induces a change in the internal structure and it is characterized as a second order phase transition, since no latent heat is released. According to Landau's theory of phase transitions, the phase is represented by the scalar parameter $f \in [-1, 1]$, related to the density n_s of the superfluid particles by the formula

$$n_s = f^2.$$

Hence the condition $f^2 = 0$ denotes the normal state, while $f^2 \in (0, 1]$ describes a superfluid state. Since superfluidity is a second order phase transition, under the transition temperature, the phase of some particles can be still equal to zero, while some particles are in the superfluid phase.

The evolution of f is ruled by the Ginzburg-Landau equation typical of second order phase transitions, interpreted as a balance law of the internal order structure (see [2]), namely

$$f_t = \frac{1}{\kappa^2} \Delta f - f(f^2 - 1 + u + \lambda p + \mathbf{v}_s^2), \quad (1)$$

where $u = \theta/\theta_c$ is the normalized temperature, p is the pressure, \mathbf{v}_s is the superfluid component of the velocity and κ, λ are positive constants.

The superfluid component \mathbf{v}_s is assumed to satisfy the equations

$$\frac{\partial \mathbf{v}_s}{\partial t} = -\frac{\mu}{\kappa^2} \nabla \phi - \mu \nabla \times \nabla \times \mathbf{v}_s - \mu f^2 \mathbf{v}_s + \nabla u + \mathbf{h} \quad (2)$$

$$\nabla \cdot (f^2 \mathbf{v}_s) = -\kappa^2 f^2 \phi, \quad (3)$$

where ϕ is a suitable scalar function and the vector \mathbf{h} is related to the body force. Equations (2) and (3) are similar to the evolution equations governing the motion of

superconducting electrons, emphasizing the evident analogies between superfluidity and superconductivity.

We suppose that the normal component \mathbf{v}_n is related to the superfluid component \mathbf{v}_s through the constitutive equation

$$\mathbf{v}_n = \nabla \times \mathbf{v}_s. \quad (4)$$

Therefore the evolution equation for \mathbf{v}_n is obtained by taking the curl of (2), namely

$$\frac{\partial \mathbf{v}_n}{\partial t} = -\mu \nabla \times \nabla \times \mathbf{v}_n - \mu \nabla \times (f^2 \mathbf{v}_s) + \nabla \times \mathbf{h}.$$

Equations (1)-(4) allow to explain some aspects related to the phenomenon of superfluidity.

First, we examine the phase diagram of ^4He , represented in figure 1. We notice that

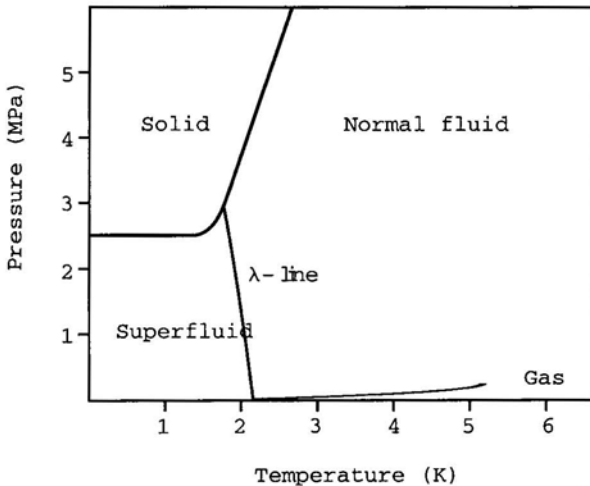


Figure 1: Phase diagram of ^4He

the curve separating the two phases, called λ -line, exhibits a high slope such that it can be approximated by a vertical line. By considering equation (1), we observe that if

$$R = u + \lambda p + \mathbf{v}_s^2 - 1 > 0,$$

then the fluid is in the normal state, since $f = 0$ is the solution of (1), with the boundary and initial conditions

$$\nabla f \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad f(x, 0) = 0.$$

Conversely, when $R < 0$ the pattern is able to describe the superfluid phase. Therefore, the phase transition occurs when

$$u + \lambda p + \mathbf{v}_s^2 = 1. \quad (5)$$

If we consider the equilibrium states, i.e. $\mathbf{v}_s = 0$, then (5) reduces to

$$u + \lambda p = 1,$$

which is a good approximation of the λ -line.

In addition, if \mathbf{v}_s^2 overcomes a threshold value, then $R > 0$, so that superfluidity is destroyed. This proves existence of a critical velocity, experimentally observed, above which superfluid properties disappear.

Henceforth, we choose $\lambda = 0$, which corresponds to the approximation of the λ -line with a vertical line. Then (1) reduces to

$$f_t = \frac{1}{\kappa^2} \Delta f - f(f^2 - 1 + u + \mathbf{v}_s^2).$$

Now we analyze the physical meaning of (2)-(4). The term ∇u accounts for the thermo-mechanical effect, according to which the particles of the superfluid flow in the same direction as the heat flux.

Since in (2) ∇u has the same sign of the acceleration of the superfluid component, the gradient of the temperature causes an increase of the superfluid component \mathbf{v}_s .

Moreover, suppose that a capillary is connected with a vessel containing the superfluid, and that owing to the narrowness of the tube the motion across the capillary satisfies $\nabla \times \mathbf{v}_s = 0$. Then, in view of (4),

$$\mathbf{v}_n = \nabla \times \mathbf{v}_s = 0,$$

so that the normal component is absent inside the tube. This means that only the superfluid component is allowed to flow across capillaries while the normal component undergoes a viscous resistance which forbids the flow to cross the tube.

2.1 Thermodynamical consistence of the model

The evolution equation for the temperature is deduced from energy balance law

$$E_t = \mathcal{P}^f + \mathcal{P}^{v_s} + h, \quad (6)$$

where E is the total energy, h is the rate at which heat is absorbed and $\mathcal{P}^f, \mathcal{P}^{v_s}$, are the internal powers related to the variables f, \mathbf{v}_s , defined as

$$\begin{aligned} \mathcal{P}^f &= f_t^2 + \frac{1}{2\kappa^2} [(\nabla f)^2]_t + \frac{1}{4} [(1 - f^2)^2]_t + f f_t (u + \mathbf{v}_s^2) \\ \mathcal{P}^{v_s} &= \frac{1}{2} (\mathbf{v}_n^2)_t + \mu^{-1} \left(\frac{\partial \mathbf{v}_s}{\partial t} + \frac{\mu}{\kappa^2} \nabla \phi - \nabla u \right)^2 + f^2 \mathbf{v}_s \cdot \frac{\partial \mathbf{v}_s}{\partial t} + u \nabla \cdot (f^2 \mathbf{v}_s) \\ &\quad + \mu f^2 \phi^2 \end{aligned}$$

We define the internal energy as

$$e = C(u) + \frac{1}{4}(1 - f^2)^2 + \frac{1}{2\kappa^2}(\nabla f)^2,$$

where C is a function of the temperature such that $c(u) = C'(u)$ is the specific heat.

Moreover, we consider each particle of the superfluid as a pair endowed with two excitements represented by the components \mathbf{v}_n and $|f|\mathbf{v}_s$. Accordingly, the kinetic energy of the particle is defined as

$$T = \frac{1}{2}(\mathbf{v}_n^2 + f^2\mathbf{v}_s^2).$$

The previous representation of the kinetic energy is in accordance with the representation of the velocity

$$\mathbf{v} = \mathbf{v}_n + |f|\mathbf{v}_s,$$

when the components \mathbf{v}_n and \mathbf{v}_s are orthogonal.

Hence, the total energy is given by

$$E = T + e = \frac{1}{2}\mathbf{v}_n^2 + \frac{1}{2}f^2\mathbf{v}_s^2 + C(u) + \frac{1}{4}(1 - f^2)^2 + \frac{1}{2\kappa^2}(\nabla f)^2,$$

Substitution into (6) leads to

$$h = c(u)u_t - f_t^2 - \mu^{-1} \left(\frac{\partial \mathbf{v}_s}{\partial t} + \frac{\mu}{\kappa^2} \nabla \phi - \nabla u \right)^2 - u f f_t - u \nabla \cdot (f^2 \mathbf{v}_s) - \mu f^2 \phi^2. \quad (7)$$

The heat balance is expressed by the equation

$$h = -\nabla \cdot \mathbf{q} + r,$$

from which we deduce the entropy equation

$$\frac{h}{u} - \frac{1}{u^2} \mathbf{q} \cdot \nabla u = -\nabla \cdot \left(\frac{\mathbf{q}}{u} \right) + \frac{r}{u}. \quad (8)$$

Here the heat flux \mathbf{q} is supposed to obey the constitutive Fourier law

$$\mathbf{q} = -k(u)\nabla u, \quad (9)$$

where the conductivity k depends on the absolute temperature.

By substituting (7), (9) into (8) and neglecting the quadratic terms we obtain

$$\frac{c(u)}{u} u_t - f f_t = \nabla \cdot \left(\frac{k(u)}{u} \nabla u \right) + \nabla \cdot (f^2 \mathbf{v}_s) + \frac{r}{u}. \quad (10)$$

Since the thermal conductivity and the specific heat are very small when the absolute temperature is close to zero, in this framework we will consider the linear approximations

$$k(u) = k_0 u, \quad c(u) = c_0 u, \quad k_0, c_0 > 0.$$

Moreover we assume that $\hat{r} = r/u$ is a known function of x, t . Accordingly (10) reduces to

$$c_0 u_t - f f_t = k_0 \Delta u + \nabla \cdot (f^2 \mathbf{v}_s) + \hat{r}. \quad (11)$$

We show that equations (2), (3) and (11) allow to prove the existence of second sound effect, revealing that superfluids are able to propagate thermal waves. For this problem, let us suppose that f is constant and that \mathbf{h}, \hat{r} vanish. Therefore the heat equation (11) reads

$$c_0 u_t = k_0 \Delta u + f^2 \nabla \cdot \mathbf{v}_s. \quad (12)$$

Besides the heat flux due to the Fourier conduction law, we observe that (12) contains a flux proportional to the superfluid component \mathbf{v}_s . In addition, the velocity \mathbf{v}_s satisfies the equation

$$\frac{\partial \mathbf{v}_s}{\partial t} = -\frac{\mu}{\kappa^2} \nabla \phi - \mu \nabla \times \nabla \times \mathbf{v}_s - \mu f^2 \mathbf{v}_s + \nabla u, \quad (13)$$

which has a similar structure of the heat propagation law in the Cattaneo-Maxwell regime. Since the heat propagation in a superfluid is not primarily due to the thermal conductivity of the material, we suppose $k_0 \ll 1$. Then, the effect of the superfluid component is prevailing, so that the second sound is expected. Indeed, (3) and (13) lead to

$$(\nabla \cdot \mathbf{v}_s)_t = -\frac{\mu}{\kappa^2} \Delta \phi + \mu \kappa^2 f^2 \phi + \Delta u$$

Therefore, by neglecting thermal conductivity, from (12) we deduce

$$c_0 u_{tt} - f^2 \Delta u = f^2 (\kappa^2 \mu f^2 \phi - \frac{\mu}{\kappa^2} \Delta \phi),$$

which means that the temperature satisfies an equation able to provide a new kind of wave.

We conclude this section, showing that the model proposed is consistent with the second law of thermodynamics. To this purpose, let us write the Clausius-Duhem inequality in the non-local formulation

$$\frac{\partial \eta}{\partial t} \geq \frac{h}{u} - \frac{1}{u^2} \mathbf{q} \cdot \nabla u + \nabla \cdot \mathbf{N}. \quad (14)$$

where η is the entropy and \mathbf{N} is a suitable flux vector such that $\mathbf{N} \cdot \mathbf{n}|_{\partial\Omega} = 0$. In view of the definitions (7), (9) of h, \mathbf{q} and with the choices

$$\eta(u, f) = c_0 u - \frac{1}{2} f^2, \quad \mathbf{N} = f^2 \mathbf{v}_s,$$

inequality (14) implies

$$0 \geq -f_t^2 - \mu^{-1} (\mathbf{v}_{st} + \nabla \phi - \nabla u)^2 - \mu f^2 \phi^2,$$

which proves the thermodynamical consistence of the model.

2.2 Rotation of a superfluid

Liquid helium has a peculiar behavior when it is set into rotation. Let us consider a cylindrical bucket of radius R , containing Helium II rotating around its axis with angular velocity ω . When ω exceeds a critical value ω_{c_1} , parallel straight vortex lines appear and their number grows with the increasing of the angular velocity. However, if ω overcomes a second threshold value ω_{c_2} , the cores of the vortices overlaps and superfluidity is destroyed.

The phenomenon is identical to the creation of vortices observed by Abrikosov in a type II superconductor: if the intensity of an impressed magnetic field is between two critical values H_{c_1} and H_{c_2} , vortex lines occur. By letting $\mathbf{h} = 0$, the stationary problem in isothermal conditions is ruled by the system

$$\frac{1}{\kappa^2} \Delta f - f(f^2 - 1 + u + \mathbf{v}_s^2) = 0 \quad (15)$$

$$-\frac{1}{\kappa^2} \nabla \phi - \nabla \times \nabla \times \mathbf{v}_s - f^2 \mathbf{v}_s = 0 \quad (16)$$

$$\nabla \cdot (f^2 \mathbf{v}_s) + \kappa^2 f^2 \phi = 0 \quad (17)$$

We associate the boundary conditions

$$\nabla f \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad (\nabla \times \mathbf{v}_s) \times \mathbf{n}|_{\partial\Omega} = \omega, \quad \mathbf{v}_s \cdot \mathbf{n}|_{\partial\Omega} = 0. \quad (18)$$

Since (15)-(18) coincide with the Ginzburg-Landau equations studied by Abrikosov for superconductors of type II, with the same analysis, we may prove the existence of vortex lines in a superfluid.

We restrict our attention to the case $f^2 = \lambda^2 = \text{const}$. Due to the symmetry of the domain, we introduce a system of cylindrical coordinates (r, θ, z) with versors of the axes $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ respectively and look for solutions of (16)-(17) with

$$\mathbf{v}_s = v_s(r) \mathbf{e}_z.$$

Then v_s satisfies the differential equation

$$v_s'' + \frac{1}{r} v_s' - \lambda^2 v_s = 0, \quad (19)$$

with boundary condition

$$v_s'(R) = \omega.$$

Equation (19) is a Bessel equation, whose bounded solution is

$$v_s(r) = \frac{\omega}{\lambda I_1(\lambda R)} I_0(\lambda r),$$

where I_0, I_1 are the modified Bessel functions of the first kind.

Moreover the normal component is

$$\mathbf{v}_n = \nabla \times \mathbf{v}_s = -v_s'(r) \mathbf{e}_\theta = v_n(r) \mathbf{e}_\theta,$$

with

$$v_n(r) = -\frac{\omega}{I_1(\lambda R)} I_1(\lambda r) \rightarrow 0, \quad \text{as } r \rightarrow 0.$$

3 Well-posedness of the problem

The model proposed to describe superfluidity considers as state variables the order parameter f , the superfluid component \mathbf{v}_s and the absolute temperature θ , satisfying the differential system

$$\begin{aligned} f_t &= \frac{1}{\kappa^2} \Delta f - f(f^2 - 1 + u + \mathbf{v}_s^2) \\ (\mathbf{v}_s)_t &= -\nabla \phi - \mu \nabla \times \nabla \times \mathbf{v}_s - f^2 \mathbf{v}_s + \nabla u + \mathbf{h} \\ c_0 u_t - f f_t &= \nabla \cdot (f^2 \mathbf{v}_s + k_0 \nabla u) + r \\ \nabla \cdot (f^2 \mathbf{v}_s) &= -\kappa^2 f^2 \phi \end{aligned}$$

We associate the boundary and initial conditions

$$\begin{aligned} \nabla f \cdot \mathbf{n}|_{\partial\Omega} &= 0, \quad (\nabla \times \mathbf{v}_s) \times \mathbf{n}|_{\partial\Omega} = \boldsymbol{\omega}, \quad \mathbf{v}_s \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad u|_{\partial\Omega} = u_b \\ f(x, 0) &= f_0(x), \quad \mathbf{v}_s(x, 0) = \mathbf{v}_{s0}(x), \quad u(x, 0) = u_0(x). \end{aligned}$$

Existence and uniqueness of solutions are proved by means of a suitable decomposition of the unknown \mathbf{v}_s, ϕ , i.e.

$$\mathbf{v}_s = -\mathbf{A} + \frac{1}{\kappa} \nabla \varphi, \quad \phi_s = \nabla \cdot \mathbf{A} - \frac{1}{\kappa} \varphi_t, \quad \mathbf{A} \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

By letting $\psi = f e^{i\varphi}$, the problem can be written by means of the variables ψ, \mathbf{A}, u in the form

$$\begin{aligned} \psi_t &= \frac{1}{\kappa^2} \Delta \psi - \frac{2i}{\kappa} \mathbf{A} \cdot \nabla \psi - \psi |\mathbf{A}|^2 + i(\kappa - \frac{1}{\kappa})(\nabla \cdot \mathbf{A}) \psi \\ &\quad - \psi (|\psi|^2 - 1 + u) \end{aligned} \quad (20)$$

$$\begin{aligned} \mathbf{A}_t &= \nabla(\nabla \cdot \mathbf{A}) - \mu \nabla \times \nabla \times \mathbf{A} - |\psi|^2 \mathbf{A} + \frac{i}{2\kappa} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) \\ &\quad - \nabla u - \mathbf{h} \end{aligned} \quad (21)$$

$$\begin{aligned} c_0 u_t &= \frac{1}{2} (\psi_t \bar{\psi} + \psi \bar{\psi}_t) + k_0 \Delta u \\ &\quad + \nabla \cdot \left[-|\psi|^2 \mathbf{A} + \frac{i}{2\kappa} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) \right] + \hat{r} \end{aligned} \quad (22)$$

with boundary and initial conditions

$$\nabla \psi \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad (\nabla \times \mathbf{A}) \times \mathbf{n}|_{\partial\Omega} = -\boldsymbol{\omega}, \quad \mathbf{A} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad u|_{\partial\Omega} = u_b \quad (23)$$

$$\psi(x, 0) = \psi_0(x) \quad \mathbf{A}(x, 0) = \mathbf{A}_0(x) \quad u(x, 0) = u_0(x). \quad (24)$$

For sake of simplicity, $\mathbf{h}, \hat{r}, \boldsymbol{\omega}$ and u_b are assumed to be time independent.

Theorem 1. *Let $\psi_0, \mathbf{A}_0 \in H^1(\Omega), u_0 \in L^2(\Omega), \mathbf{h}, \hat{r} \in L^2(\Omega)$. Then for every $T > 0$, there exists a unique solution (ψ, \mathbf{A}, u) to (20)-(24) such that*

$$\begin{aligned} \psi, \mathbf{A} &\in L^2(0, T, H^2(\Omega)) \cap H^1(0, T, L^2(\Omega)) \\ u &\in L^2(0, T, H^1(\Omega)) \cap H^1(0, T, H^1(\Omega)'. \end{aligned}$$

Moreover $\psi, \mathbf{A} \in C(0, T, H^1(\Omega)), u \in C(0, T, L^2(\Omega))$.

Existence of a solution can be established, first by proving that system (20)-(24) admits at least a solution defined in a small time interval $[0; \tau)$. Next, by means of uniform estimates, such a solution can be extended in the whole time interval $[0; T)$ for every $T > 0$. Uniqueness is proved with standard techniques. The details of the proof of Theorem 1 can be found in [1].

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