Chapter 2

No-Go Theorems and Graded Lie Algebras

2.1 The Theorems of Coleman-Mandula and Haag, Łopuszański, Sohnius

We now discuss the two theorems already referred to in the Introduction.[1]

2.1.1 The Theorem of Coleman–Mandula

The following theorem[2] was established by S. Coleman and J. Mandula [23]:

> Let G be a connected symmetry group of the S-matrix, *i.e.* a group whose generators commute with the S-matrix, and make the following five assumptions:
>
> (i) *Lorentz invariance:* G contains a subgroup which is locally isomorphic to the Poincaré group.
>
> (ii) *Particle finiteness:* All particle types correspond to positive-energy representations of the Poincaré group. For any finite mass M, there is only a finite number of particles with mass less than M.

[1] A.O. Barut and R. Rączka [8], p. 43, also give considerations concerning the unification of the Poincaré algebra with internal symmetry algebras. Stronger group theoretical lemmas are given on p. 629, and it is pointed out (p. 630) that the infinite-parameter Lie algebra associated with noncompact dynamical groups (which lead to infinite particle multiplets (*cf.* pp. 411, 609)) does not contradict these theorems.

[2] See also the discussion in S. Weinberg [118].

(iii) *Weak elastic analyticity:* Elastic scattering amplitudes are analytic functions of centre-of-mass energy squared s and invariant momentum transfer squared t in some neighbourhood of the physical region, except at normal thresholds.

(iv) *Occurrence of scattering:* Let $|p\rangle$ and $|p'\rangle$ be any two one-particle momentum eigenstates, and let $|p, p'\rangle$ be the two-particle state constructed from these. Then

$$T|p, p'\rangle \neq 0,$$

where T is the T-matrix defined by

$$S = \mathbb{1} - i(2\pi)^4 \delta^4(p_\mu - p'_\mu)T,$$

except, perhaps, for certain isolated values of S. In simpler terms this assumption means: Two plane waves scatter at almost any energy.

(v) *Technical assumption:* The generators of G, considered as integral operators in momentum space, have distributions for their kernels.

Then the group G is locally isomorphic to the direct product of a compact symmetry group and the Poincaré group.

We recall briefly some basic results of scattering theory. The *Hilbert space* $\mathcal{H}$ is the direct sum of an infinite number of subspaces, *i.e.*

$$\mathcal{H} = \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus \cdots .$$

Here $\mathcal{H}^{(n)}$ denotes the n-particle subspace. It is a subspace of the direct product (symmetric or antisymmetric in accordance with the generalized exclusion principle) of n Hilbert spaces, each being isomorphic to $\mathcal{H}^{(1)}$. The S-matrix is a unitary operator on $\mathcal{H}$. A unitary operator U on $\mathcal{H}$ is said to be a *symmetry transformation* of the S-matrix if

(i) U transforms one-particle states into one-particle states,

(ii) U acts on many-particle states as if they were tensor products of one-particle states,

(iii) U commutes with S.

Thus the theorem of Coleman and Mandula, stated here without proof,[3] demonstrates that the most general Lie algebra of symmetries of the S-matrix contains the energy momentum operator P_μ, the Lorentz rotation generator $M_{\mu\nu}$, and a finite number of Lorentz scalar operators B_l, *i.e.*

$$\left[P_\mu, B_l\right] = 0, \qquad \left[M_{\mu\nu}, B_l\right] = 0,$$

where the B_l constitute a *Lie algebra*,

$$\left[B_l, B_m\right] = i{c_{lm}}^k B_k,$$

and the ${c_{lm}}^k$ are the structure constants of the Lie algebra of the compact internal symmetry[4] group (*e.g.* $SU(2,\mathbb{C})$).

2.1.2 The Theorem of Haag, Łopuszański and Sohnius

Supersymmetries avoid the restrictions of the Coleman-Mandula theorem by relaxing one condition.[5] R. Haag, J.T. Łopuszański and M.F. Sohnius [54] generalize the notion of a Lie algebra to include algebraic systems whose defining relations involve in addition to the usual commutators also anti-commutators. These algebras[6] are called *superalgebras* or *graded Lie algebras*. The generalization of the Poincaré algebra to a superalgebra is obtained in its simplest version by the following procedure. One adds to the Poincaré algebra a Majorana spinor charge with components $Q_a, a = 1, \ldots, 4$. These operators are called *spinor charges*. They have the following properties:

$$\begin{aligned} \left\{Q_a, \overline{Q}_b\right\} &= 2(\gamma^\mu)_{ab} P_\mu, \\ \left[Q_a, P_\mu\right] &= 0, \\ \left[Q_a, M^{\mu\nu}\right] &= (\sigma^{4\mu\nu})_{ab} Q_b. \end{aligned} \tag{2.1}$$

We shall see later that

$$\left\{Q_a, Q_b\right\} = -2(\gamma^\mu C)_{ab} P_\mu,$$

[3] For a proof of the theorem see the book of S. Weinberg [118].

[4] In general symmetries are classified as *geometric symmetries* and *internal symmetries*. Geometric symmetry transformations operate on spacetime coordinates (such as Lorentz or Poincaré transformations), whereas internal symmetry transformations do not affect the spacetime manifold but operate on objects (*e.g.* scalar fields, spinor fields), defined on the spacetime manifold.

[5] In this sense, the Haag-Łopuszański-Sohnius theorem is a natural extension of the Coleman-Mandula theorem.

[6] In the mathematical literature, the first who considered groups with commuting and anticommuting parameters were F.A. Berezin and G.I. Kac [11].

where (*cf.* Eq. (1.207f)):

$$\sigma_4^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]$$

and

$$\overline{Q}_a = \left(Q^\dagger \gamma_0\right)_a.$$

The operators P_μ and $M_{\mu\nu}$ are the usual generators of *displacements* and *homogeneous Lorentz transformations* of spacetime. In order to incorporate an internal symmetry in a nontrivial way it is often convenient to rewrite Eqs. (2.1) in terms of two-component *Weyl spinors* Q_A and $\overline{Q}_{\dot{A}}$. Then:

$$\begin{aligned}
&\left\{Q_A, Q_B\right\} = 0, && \left\{\overline{Q}_{\dot{A}}, \overline{Q}_{\dot{B}}\right\} = 0,\\
&\left\{Q_A, \overline{Q}_{\dot{B}}\right\} = 2(\sigma^\mu)_{A\dot{B}} P_\mu, && \left[Q_A, M^{\mu\nu}\right] = i(\sigma_2^{\mu\nu})_A{}^B Q_B,\\
&\left[Q_A, P_\mu\right] = 0, && \left[\overline{Q}_{\dot{A}}, P_\mu\right] = 0,
\end{aligned} \tag{2.2}$$

the $\sigma_2^{\mu\nu}$ being defined by Eq. (1.138a). In Eq. (2.2) the dotted and undotted indices assume values $A = 1, 2$ and $\dot{A} = \dot{1}, \dot{2}$ respectively and refer to the $(0, 1/2)$ and $(1/2, 0)$ repesentations of the spinor group $SL(2, \mathbb{C})$.

We now assume that we have a set of generators, *i.e.* spinor charges[7] $Q_A^\alpha, \alpha = 1, \ldots, N$, which transform according to some representation of a compact Lie group G, such as $SU(3, \mathbb{C})$, which represents the internal symmetry group. Then the generators of G are the *Lorentz scalars* B_l. The $\overline{Q}_{\dot{A}}^\alpha$ transform according to the complex conjugate representation of this group. Then the relations (2.2) generalize as follows:

$$\left.\begin{aligned}
\left\{Q_A^\alpha, Q_B^\beta\right\} &= 0,\\
\left\{\overline{Q}_{\dot{A}}^\alpha, \overline{Q}_{\dot{B}}^\beta\right\} &= 0,\\
\left\{Q_A^\alpha, \overline{Q}_{\dot{B}}^\beta\right\} &= 2\delta^{\alpha\beta}(\sigma^\mu)_{A\dot{B}} P_\mu,\\
\left[Q_A^\alpha, P_\mu\right] &= 0,\\
\left[\overline{Q}_{\dot{A}}^\alpha, P_\mu\right] &= 0,\\
\left[Q_A^\alpha, B_l\right] &= iS_l{}^{\alpha\beta} Q_A^\beta,\\
\left[Q_A^\alpha, M^{\mu\nu}\right] &= i(\sigma_2^{\mu\nu})_A{}^B Q_B^\alpha,\\
\left[B_l, B_m\right] &= ic_{lm}{}^k B_k,
\end{aligned}\right\} \tag{2.3}$$

where the $S_l{}^{\alpha\beta}$ are the Hermitian representation matrices of the representation containing the charges Q_A^α. As mentioned before, the B_l are the generators of the internal symmetry group. The theorem of Haag, Łopuszański and

[7]Here, N denotes the dimension of the chosen representation of the internal symmetry group G.

Sohnius now states that the maximal symmetry of the S-matrix is the direct product of an internal symmetry with the superalgebra given by relations (2.3). The only allowed extension is the possible appearance of so-called *central charges* in the anticommutator of two undotted spinors. Instead of the first relation of Eq. (2.3) one would then have

$$\left\{Q_A^\alpha, Q_B^\beta\right\} = \epsilon_{AB} Z^{\alpha\beta}, \quad Z^{\alpha\beta} = -Z^{\beta\alpha},$$

where ϵ_{AB} is given by Eq. (1.66). Furthermore, the Z's commute with the B's, *i.e.*

$$\left[Z^{\alpha\beta}, B_l\right] = 0,$$

which is the reason why the quantities $Z^{\alpha\beta}$ are called *central charges*.

2.2 Graded Lie Algebras

2.2.1 Lie Algebras

Before we consider graded Lie algebras it is worthwhile to recapitulate the definition of a Lie algebra (see *e.g.* [19, Chap. 1], [115, Chap. 2], or [72, Chap. 5]).

Definition (Lie algebra)

A Lie algebra consists of a vector space L over a field (here $\mathbb{R}$ or $\mathbb{C}$) with a composition rule called product, denoted by $\circ$, defined as follows:

$$\circ \; : \; L \times L \;\longrightarrow\; L.$$

If $v_1, v_2, v_3 \in L$, then the following properties define the Lie algebra:

(i) $v_1 \circ v_2 \in L$ (closure of L under $\circ$),

(ii) $v_1 \circ (v_2 + v_3) = v_1 \circ v_2 + v_1 \circ v_3$ (linearity),

(iii) $v_1 \circ v_2 = -v_2 \circ v_1$ (antisymmetry),

(iv) $v_1 \circ (v_2 \circ v_3) + v_3 \circ (v_1 \circ v_2) + v_2 \circ (v_3 \circ v_1) = 0$ (Jacobi identity).

Example: The matrix space of complex 2×2-matrices which are traceless and anti-Hermitian forms a Lie algebra, $su(2,\mathbb{C})$, the Lie algebra of the Lie group $SU(2,\mathbb{C})$, provided we define the product as

$$a \circ b := \left[a, b\right] = ab - ba, \quad \forall\; a, b \in su(2,\mathbb{C}).$$

A basis of the vector space L is given by the three matrices

$$\tau_i := \frac{i}{2}\sigma_i, \quad \tau_i^\dagger = -\tau_i, \quad i = 1, 2, 3,$$

where the σ_i are the three *Pauli matrices* (*cf.* Eq. (1.103b)).

We verify that the algebra $su(2, \mathbb{C})$ satisfies items (i) to (iv) of the definition of a Lie algebra.

(i) *Closure.* The product is defined as (see Chapter 1, Eq. (1.104b))

$$\tau_i \circ \tau_i := \big[\tau_i, \tau_k\big] = -\epsilon_{ijk}\tau_k,$$

where ϵ_{ijk} is totally antisymmetric in its indices i, j, k and $\epsilon_{123} = 1$. Hence, the vector space is closed under the product $\circ = [\ ,\ ]$.

(ii) *Linearity* is demonstrated as follows:

$$\begin{aligned}\tau_i \circ \big(\tau_j + \tau_k\big) &= \big[\tau_i, \tau_j + \tau_k\big] \\ &= \tau_i(\tau_j + \tau_k) - (\tau_j + \tau_k)\tau_i \\ &= \tau_i\tau_j + \tau_i\tau_k - \tau_j\tau_i - \tau_k\tau_i \\ &= \tau_i\tau_j - \tau_j\tau_i + \tau_i\tau_k - \tau_k\tau_i \\ &= \big[\tau_i, \tau_j\big] + \big[\tau_i, \tau_k\big] \\ &= \tau_i \circ \tau_j + \tau_j \circ \tau_k.\end{aligned}$$

This verifies the linearity of the product $\circ = [\ ,\ ]$.

(iii) The *antisymmetry* is shown as follows:

$$\begin{aligned}\tau_i \circ \tau_j = \big[\tau_i, \tau_j\big] &= \tau_i\tau_j - \tau_j\tau_i \\ &= -(\tau_j\tau_i - \tau_i\tau_j) \\ &= -\big[\tau_j, \tau_i\big] \\ &= -\tau_j \circ \tau_i.\end{aligned}$$

(iv) It is cumbersome to verify the *Jacobi identity*; however, it is well known that the following relation holds which expresses the same property:

$$\big[\tau_1, [\tau_2, \tau_3]\big] + \big[\tau_3, [\tau_1, \tau_2]\big] + \big[\tau_2, [\tau_3, \tau_1]\big] = 0.$$

2.2.2 Graded Algebras

We now define graded algebras. In the simplest case a graded algebra consists of a vector space L which is the direct sum of two subspaces L_0 and L_1; *i.e.*

$$L = L_0 \oplus L_1,$$

and a product $\circ$,

$$\circ \; : \; L \times L \; \longrightarrow \; L$$

with the following properties:

(i) $u_1 \circ u_2 \in L_0, \quad \forall\, u_1, u_2 \in L_0,$

(ii) $u \circ v \in L_1, \quad \forall\, u \in L_0, v \in L_1,$

(iii) $v_1 \circ v_2 \in L_0, \quad \forall\, v_1, v_2 \in L_1.$

This algebra is called a $\mathbb{Z}_2$-*graded algebra.*

More generally, L is the direct sum of $N+1, N \geq 1$, subspaces L_k, *i.e.*

$$L = \bigoplus_{k=0}^{N} L_k,$$

with a product $\circ$,

$$\circ \; : \; L \times L \; \longrightarrow \; L,$$

such that if $u_k \in L_k$, then

$$u_j \circ u_k \in L_{j+k \bmod (N+1)}.$$

A product $\circ$ with such a property is called a *grading.*

2.2.3 Graded Lie Algebras

A graded algebra becomes a *graded Lie algebra* if one modifies the product denoted by $\circ$ in the following way. For simplicity we consider a $\mathbb{Z}_2$-graded algebra.[8] Let L_0 and L_1 be vector spaces and

$$L := L_0 \oplus L_1.$$

Thus, the vector space L can be written as the direct sum of L_0 and L_1. We define the product $\circ$

$$\circ \; : \; L \times L \; \longrightarrow \; L$$

with the following properties:

[8] An elementary presentation of graded Lie algebras, their classification and some properties of their representations is given in [96]. A mathematical treatment of these topics can be found in the monograph by B. DeWitt [26], Chap. 3. A general classification of super Lie groups can be found in [26], Chap.4. See also the dictionary [43].

(i) *Grading:* For all $x_i \in L_i,\ i = 0, 1,$

$$x_i \circ x_j \in L_{i+j \bmod 2}. \tag{2.4}$$

Then L becomes a graded algebra according to Sec. 2.2.2.

(ii) *Supersymmetrization:* For all $x_i \in L_i,\ x_j \in L_j,\ i, j = 0, 1,$

$$x_i \circ x_j = -(-1)^{ij} x_j \circ x_i. \tag{2.5}$$

(iii) *Generalized Jacobi identities:* For all $x_k \in L_k,\ x_l \in L_l,\ x_m \in L_m,\ k, l,$ $m \in \mathbb{Z}_2,$

$$x_k \circ (x_l \circ x_m)(-1)^{km} + x_l \circ (x_m \circ x_k)(-1)^{lk} + x_m \circ (x_k \circ x_l)(-1)^{ml} = 0. \tag{2.6}$$

With this definition of the product, L as a vector space becomes a *graded Lie algebra*. It is important to note that L is not a Lie algebra, since, as defined in Eq. (2.5), the product is in general not antisymmetric. To see this, it is advantageous to write out Eq. (2.5) explicitly:

(a) $i = j = 0$, *i.e.* $x_0, y_0 \in L_0$, then:

$$x_0 \circ y_0 = -(-1)^0 y_0 \circ x_0 = -y_0 \circ x_0.$$

Hence in the subspace L_0 the product $\circ$ is antisymmetric.

(b) $i = 0, j = 1$, *i.e.* $x_0 \in L_0, y_1 \in L_1$, then:

$$x_0 \circ y_1 = -(-1)^0 y_1 \circ x_0 = -y_1 \circ x_0.$$

(c) $i = j = 1$, *i.e.* $x_1, y_1 \in L_1$, then:

$$x_1 \circ y_1 = -(-1)^1 y_1 \circ x_1 = y_1 \circ x_1.$$

Hence in the subspace L_1 the product $\circ$ is symmetric.

With the above definition of a graded Lie algebra the subspace L_0 spans an ordinary Lie algebra, because the pair $(L_0, \circ)$ satisfies the definition of an ordinary Lie algebra as given in Sec. 2.2.1. The subspace L_1 is not even an algebra, since according to Eq. (2.4) L_1 is not closed under the product $\circ$, *i.e.* if $x_1, y_1 \in L_1$, then

$$x_1 \circ y_1 \in L_{1+1 \bmod 2} = L_0.$$

2.3 The Graded Lie Algebra of $SU(2,\mathbb{C})$

As an example we discuss the $\mathbb{Z}_2$ grading of $su(2,\mathbb{C})$, the Lie algebra of the group $SU(2,\mathbb{C})$. As stated above, the subspace L_0 in the construction of a graded Lie algebra is an ordinary Lie algebra. It is natural therefore to take L_0 as the Lie algebra of $SU(2,\mathbb{C})$ with generators τ_1,τ_2,τ_3 and

$$\left[\tau_i,\tau_j\right] = -\epsilon_{ijk}\tau_k.$$

We define the product

$$\circ \;:\; L\times L \;\longrightarrow\; L$$

on the subspace L_0 as

$$\begin{aligned} \circ\;: L_0\times L_0 \;&\longrightarrow\; L_0, \\ (\tau_i,\tau_j) \;&\longmapsto\; \tau_i\circ\tau_j := \left[\tau_i,\tau_j\right] = -\epsilon_{ijk}\tau_k\in L_0. \end{aligned} \tag{2.7}$$

We denote the generators of the subspace L_1 by $Q_a, a=1,2,\ldots,N, N=\dim L_1$. We now have to define the product $\circ$ when multiplying any $\tau_i\in L_0$ by Q_a, and multiplying two $Q_a\in L_1$. In the case of the former we have

$$\circ \;:\; L_0\times L_1 \;\longrightarrow\; L_1.$$

Thus if we form the product of any $\tau_i\in L_0$ with a $Q_a\in L_1$, then according to Eq. (2.4) we must obtain an element of the subspace L_1. We define therefore:

$$\circ \;:\; (\tau_i,Q_a) \;\longmapsto\; \tau_i\circ Q_a = -(t_i)_{ab}Q_b\in L_1. \tag{2.8}$$

Here the $(t_i)_{ab}$ are coefficients which are restricted by the generalized Jacobi identity (2.6). In the present case we have

$$x_k\in L_k,\quad x_l\in L_l,\quad x_M\in L_m,$$

and

$$\tau_i\in L_0,\quad \tau_j\in L_0,\quad Q_a\in L_1,$$

and so the generalized Jacobi identity is

$$x_k\circ(x_l\circ x_m)(-1)^{km} + x_m\circ(x_k\circ x_l)(-1)^{ml} + x_l\circ(x_m\circ x_k)(-1)^{lk} = 0,$$

so that

$$\tau_i\circ(\tau_j\circ Q_a) + Q_a\circ(\tau_i\circ\tau_j) + \tau_j\circ(Q_a\circ\tau_i) = 0,$$

and, using Eqs. (2.7) and (2.8),

$$\tau_i\circ[-(t_j)_{ab}Q_b] + Q_a\circ[-\epsilon_{ijk}\tau_k] + \tau_j\circ[(t_i)_{ab}Q_b] = 0,$$

that is

$$-(t_j)_{ab}(\tau_i \circ Q_b) - \epsilon_{ijk}(Q_a \circ \tau_k) + (t_i)_{ab}(\tau_j \circ Q_b) = 0.$$

Again using Eq. (2.8),

$$(t_j)_{ab}(t_i)_{bc}Q_c - \epsilon_{ijk}(t_k)_{ab}Q_b - (t_i)_{ab}(t_j)_{bc}Q_c = 0,$$

so that

$$\big[(t_i)_{ab}(t_j)_{bc} - (t_j)_{ab}(t_i)_{bc}\big]Q_c = -\epsilon_{ijk}(t_k)_{ac}Q_c,$$

or since the Q_c's are independent:

$$\big[t_i, t_j\big]_{ac} = -\epsilon_{ijk}(t_k)_{ac}. \tag{2.9}$$

This relation states that the coefficients $(t_i)_{ab}$ of Eq. (2.8) constitute dim $L_1\times$ dim L_1-matrices which are representation matrices of the algebra L_0. If the dimension of L_1 is two, then $t_i = \tau_i$; if the dimension of L_1 is three then $(t_i)_{ab} = -\epsilon_{iab}$.

Finally we have to define the product $\circ$ on the subspace L_1. According to Eq. (2.4) we have for $Q_a, Q_b \in L_1$:

$$Q_a \circ Q_b \in L_{1+1} = L_2 \cong L_0.$$

According to Eq. (2.5) this product on the subspace L_1 has to be symmetric, *i.e.*

$$Q_a \circ Q_b = -(-1)^1 Q_b \circ Q_a = Q_b \circ Q_a.$$

Hence we define

$$\begin{aligned} \circ\ : L_1 \times L_1 &\longrightarrow L_0, \\ Q_a, Q_b &\longmapsto Q_a \circ Q_b = (h_i)_{ab}\tau_i, \end{aligned} \tag{2.10}$$

where $(h_i)_{ab} = (h_i)_{ba}$, and the matrices $h_i, i = 1, 2, 3$, are three symmetric dim $L_1 \times$ dim L_1 matrices. Again we make use of the generalized Jacobi identities to find restrictions for the matrices h_i. An arbitrary multiplicative constant factor can be absorbed in the Q's by a redefinition. Let

$$\tau_i \in L_0 \qquad \text{and} \qquad Q_a, Q_b \in L_1.$$

Then the generalized Jacobi identity reads for this case:

$$\tau_i \circ (Q_a \circ Q_b)(-1)^0 + Q_b \circ (\tau_i \circ Q_a)(-1)^1 + Q_a \circ (Q_b \circ \tau_i)(-1)^0 = 0,$$

so that

$$\tau_i \circ \big((h_j)_{ab}\tau_j\big) - Q_b \circ \big(-(t_i)_{ac}Q_c\big) + Q_a \circ \big((t_i)_{bc}Q_c\big) = 0,$$

or

$$(h_j)_{ab}\big(\tau_i \circ \tau_j\big) + (t_i)_{ac}\big(Q_b \circ Q_c\big) + (t_i)_{bc}\big(Q_a \circ Q_c\big) = 0.$$

Then:

$$-(h_j)_{ab}\epsilon_{ijk}\tau_k + (t_i)_{ac}(h_j)_{bc}\tau_j + (t_i)_{bc}(h_j)_{ac}\tau_j = 0.$$

Rewriting this in the form:

$$\big[(t_i)_{ac}(h_j)_{cb} + (t_i)_{bc}(h_j)_{ca}\big]\tau_j = \epsilon_{ijk}(h_j)_{ab}\tau_k = \epsilon_{ikj}(h_k)_{ab}\tau_j,$$

so that

$$\big[(t_i h_j)_{ab} + (t_i h_j)_{ba}\big]\tau_j = -\epsilon_{ijk}(h_k)_{ab}\tau_j,$$

and

$$t_i h_j + (t_i h_j)^\top = -\epsilon_{ijk}h_k. \tag{2.11}$$

If we take $N = \dim L_1 = 2$, then $t_i = \tau_i, i = 1,2,3$, and the most general form of h_i is given by[9]

$$(h_i)_{ab} = a_i\delta_{ab} + b_i(\tau_3)_{ab} + c_i(\tau_1)_{ab} = \begin{pmatrix} a_i + \frac{i}{2}b_i & \frac{i}{2}c_i \\ \frac{i}{2}c_i & a_i - \frac{i}{2}b_i \end{pmatrix}_{ab},$$

where $a_i, b_i, c_i \in \mathbb{C}$. The matrix τ_2 does not appear in this expansion of h_i, because, as stated earlier, $(h_i)_{ab}$ has to be symmetric in its indices a and b, and τ_2 is antisymmetric. The nine coefficients a_i, b_i and c_i with $i = 1,2,3$, have to be computed from Eq. (2.11), *i.e.* from the relation

$$\tau_i h_j + (\tau_i h_j)^\top = -\epsilon_{ijk}h_k, \quad i,j,k = 1,2,3.$$

Proposition 2.1:

The matrices h_i are given by

$$h_i = 2c_3\big(\tau_i\tau_2\big), \quad i = 1,2,3. \tag{2.12}$$

This result can be guessed, since h_i is a 2×2-matrix, we expect τ_i on the right hand side; to make this symmetric we must have $\tau_i\tau_2$; fixing the remaining overall multiplicative factor is a matter of convention.

[9] Recall that the h_i are symmetric.

Proof: We consider the various cases separately.

The case $i = 1$: With

$$\tau_1 = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we get

$$\tau_1 h_j = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_j + \frac{i}{2}b_j & \frac{i}{2}c_j \\ \frac{i}{2}c_j & a_j - \frac{i}{2}b_j \end{pmatrix} = \frac{i}{2}\begin{pmatrix} \frac{i}{2}c_j & a_j - \frac{i}{2}b_j \\ a_j + \frac{i}{2}b_j & \frac{i}{2}c_j \end{pmatrix}.$$

This leads to:

$$\begin{aligned}
\tau_1 h_j + \left(\tau_1 h_j\right)^\top &= \frac{i}{2}\begin{pmatrix} \frac{i}{2}c_j & a_j - \frac{i}{2}b_j \\ a_j + \frac{i}{2}b_j & \frac{i}{2}c_j \end{pmatrix} + \frac{i}{2}\begin{pmatrix} \frac{i}{2}c_j & a_j + \frac{i}{2}b_j \\ a_j - \frac{i}{2}b_j & \frac{i}{2}c_j \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{2}c_j & ia_j \\ ia_j & -\frac{1}{2}c_j \end{pmatrix} \\
&\stackrel{!}{=} -\epsilon_{1jk}h_k \qquad\qquad (\text{ from Eq. (2.11)}) \\
&= -\epsilon_{1jk}\begin{pmatrix} a_k + \frac{i}{2}b_k & \frac{i}{2}c_k \\ \frac{i}{2}c_k & a_k - \frac{i}{2}b_k \end{pmatrix}.
\end{aligned}$$

Now consider the case $i = 1, j = 2$. In this case we obtain the matrix equation

$$\begin{pmatrix} -\frac{1}{2}c_2 & ia_2 \\ ia_2 & -\frac{1}{2}c_2 \end{pmatrix} = -\begin{pmatrix} a_3 + \frac{i}{2}b_3 & \frac{i}{2}c_3 \\ \frac{i}{2}c_3 & a_3 - \frac{i}{2}b_3 \end{pmatrix},$$

leading to

$$-\frac{1}{2}c_2 = -a_3 - \frac{i}{2}b_3, \quad ia_2 = -\frac{i}{2}c_3, \quad -\frac{1}{2}c_2 = -a_3 + \frac{i}{2}b_3,$$

so that

$$b_3 = 0, \quad a_3 = \frac{1}{2}c_2, \quad a_2 = -\frac{1}{2}c_3.$$

For the case $i = 1, j = 3$ we get

$$\begin{pmatrix} -\frac{1}{2}c_3 & ia_3 \\ ia_3 & -\frac{1}{2}c_3 \end{pmatrix} = \begin{pmatrix} a_2 + \frac{i}{2}b_2 & \frac{i}{2}c_2 \\ \frac{i}{2}c_2 & a_2 - \frac{i}{2}b_2 \end{pmatrix},$$

such that

$$-\frac{1}{2}c_3 = a_2 + \frac{i}{2}b_2, \quad ia_3 = \frac{i}{2}c_2, \quad -\frac{1}{2}c_3 = a_2 - \frac{i}{2}b_2.$$

That means:

$$b_2 = 0, \quad a_3 = \frac{1}{2}c_2, \quad a_2 = -\frac{1}{2}c_3.$$

Next we evaluate the case $i = 2$, using

$$\tau_2 = \frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

and obtain

$$\tau_2 h_j = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a_j + \frac{i}{2}b_j & \frac{i}{2}c_j \\ \frac{i}{2}c_j & a_j - \frac{i}{2}b_j \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{i}{2}c_j & a_j - \frac{i}{2}b_j \\ -a_j - \frac{i}{2}b_j & -\frac{i}{2}c_j \end{pmatrix}.$$

Then

$$\tau_2 h_j + (\tau_2 h_j)^\top = \frac{1}{2}\begin{pmatrix} ic_j & -ib_j \\ -ib_j & -ic_j \end{pmatrix} = \frac{i}{2}\begin{pmatrix} c_j & -b_j \\ -b_j & -c_j \end{pmatrix}.$$

Consider the case $i = 2, j = 1$: Equating the above to $-\epsilon_{213}h_3$ we obtain:

$$\frac{i}{2}\begin{pmatrix} c_1 & -b_1 \\ -b_1 & -c_1 \end{pmatrix} = \begin{pmatrix} a_3 + \frac{i}{2}b_3 & \frac{i}{2}c_3 \\ \frac{i}{2}c_3 & a_3 - \frac{i}{2}b_3 \end{pmatrix}.$$

Hence with $b_3 = 0$ we find

$$\frac{i}{2}c_1 = a_3 + \frac{i}{2}b_3 = a_3, \quad -\frac{i}{2}b_1 = \frac{i}{2}c_3, \quad -\frac{i}{2}c_1 = a_3 - \frac{i}{2}b_3 = a_3.$$

Thus $a_3 = 0, c_1 = 0, b_1 = -c_3$, and since $a_3 = 1/2\, c_2$, we have $c_2 = 0$.

Next we evaluate the case $i = 2, j = 3$:

$$\frac{i}{2}\begin{pmatrix} c_3 & -b_3 \\ -b_3 & -c_3 \end{pmatrix} = -\begin{pmatrix} a_1 + \frac{i}{2}b_1 & \frac{i}{2}c_1 \\ \frac{i}{2}c_1 & a_1 - \frac{i}{2}b_1 \end{pmatrix},$$

leading to:

$$\frac{i}{2}c_3 = -a_1 - \frac{i}{2}b_1, \quad -\frac{i}{2}b_3 = -\frac{i}{2}c_1, \quad -\frac{i}{2}c_3 = -a_1 + \frac{i}{2}b_1,$$

so that $b_3 = 0, c_1 = 0, a_1 = 0, b_1 = -c_3$. Hence, collecting all terms we have:

$$\begin{array}{lllllll} a_1 & = & 0, & b_1 & = & -c_3, & c_1 = 0, \\ a_2 & = & -\frac{1}{2}c_3, & b_2 & = & 0, & c_2 = 0, \\ a_3 & = & 0, & b_3 & = & 0, & c_3 \ \text{undetermined}, \end{array}$$

and therefore

$$h_1 = -\frac{i}{2}c_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2c_3\big(\tau_1\tau_2\big), \qquad h_2 = -\frac{1}{2}c_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2c_3\big(\tau_2\tau_2\big),$$
$$h_3 = \frac{i}{2}c_3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2c_3\big(\tau_3\tau_2\big),$$

i.e.

$$h_i = 2c_3\big(\tau_i\tau_2\big).$$

The constant c_3 can be absorbed in redefined generators Q_a as mentioned before.

We finally consider the *Jacobi identity* for three operators Q. In this case this relation is:

$$Q_a \circ \big(Q_b \circ Q_c\big)(-1) + Q_c \circ \big(Q_a \circ Q_b\big)(-1) + Q_b \circ \big(Q_c \circ Q_a\big)(-1) = 0,$$

and so

$$Q_a \circ \big((h_i)_{bc}\tau_i\big) + Q_c \circ \big((h_i)_{ab}\tau_i\big) + Q_b \circ \big((h_i)_{ca}\tau_i\big) = 0,$$

i.e.

$$(h_i)_{bc}\big(Q_a \circ \tau_i\big) + (h_i)_{ab}\big(Q_c \circ \tau_i\big) + (h_i)_{ca}\big(Q_b \circ \tau_i\big) = 0.$$

Using Eq. (2.8) with $t_i = \tau_i$,

$$(h_i)_{bc}(\tau_i)_{ad}Q_d + (h_i)_{ab}(\tau_i)_{cd}Q_d + (h_i)_{ca}(\tau_i)_{bd}Q_d = 0.$$

Using Eq. (2.12) and the fact that the Q's are independent of one another

$$\big(\tau_i\tau_2\big)_{bc}(\tau_i)_{ad} + \big(\tau_i\tau_2\big)_{ab}(\tau_i)_{cd} + \big(\tau_i\tau_2\big)_{ca}(\tau_i)_{bd} = 0.$$

This relation is valid, *i.e.* satisfied by the τ_i's, as can be checked by direct calculation.

In summary we have the following construction. The graded Lie algebra of $su(2, \mathbb{C})$ is:

$$\begin{array}{rcl} \tau_i \circ \tau_j = -\epsilon_{ijk}\tau_k, & \quad & L_0 \times L_0 \longrightarrow L_0, \\ \tau_i \circ Q_a = -(\tau_i)_{ab}Q_b, & \quad & L_0 \times L_1 \longrightarrow L_{0+1} = L_1, \\ Q_a \circ Q_b = \big(\tau_i\tau_2\big)_{ab}\tau_i, & \quad & L_1 \times L_1 \longrightarrow L_{1+1} = L_2 \cong L_0. \end{array}$$

The structure of the coefficients of the product in $L_0 \times L_1$ is determined by the generalized Jacobi identities. The only freedom one has is the choice of representation for these coefficients. The important point is that the coefficient matrix of $\tau_i \circ Q_a$ has to be a representation matrix of the algebra given by L_0. This result is an immediate consequence of the generalized Jacobi identity with one Q_a. Thus, the graded Lie algebra $su(2, \mathbb{C})$ has five elements: $\tau_i, i = 1, 2, 3$, and Q_a with $a = 1, 2$, and is written $OSp(1/2)$ ('one slash two') [108]. The dimension of the chosen representation then determines the dimension of the subspace L_1 and therefore the number of operators Q_a; $a = 1, 2, \ldots,$ dim L_1. Once the representation has been chosen, the coefficient matrices of the product $Q_a \circ Q_b$ may be found from the Jacobi identity with two Q's. For consistency these matrices have to satisfy the generalized Jacobi identity with three Q's; this, however, is not always the case — for example, an extension of $SU(2, \mathbb{C})$ with an n-dimensional L_1 $(n > 2)$ is not possible, since the generalized Jacobi identity for three Q's then requires the h_i to be zero.

2.4 $\mathbb{Z}_2$ Graded Lie Algebras

We now discuss $\mathbb{Z}_2$ graded Lie algebras in more detail. We recall first some general properties of $\mathbb{Z}_2$ graded Lie algebras.

Definition: $\mathbb{Z}_2$ graded Lie algebra

A linear algebra $L :=$ Span $\{X_\mu\}$ is given a $\mathbb{Z}_2$ grading if L is the direct sum of two subspaces L_0 and L_1, *i.e.* $L = L_0 \oplus L_1$, where we set:

$$\begin{aligned} L_0 &= \text{ Span } \big\{E_i\big\},\ i = 1, \ldots,\ \text{dim } L_0, \\ L_1 &= \text{ Span } \big\{Q_a\big\},\ a = 1, \ldots,\ \text{dim } L_1, \end{aligned}$$

on which a composition law:

$$\circ\ :\ L \times L\ \longrightarrow\ L$$

acts as follows:

$$L_0 \circ L_0 \subset L_0, \qquad L_0 \circ L_1 \subset L_1, \qquad L_1 \circ L_1 \subset L_0.$$

The pair $(L_0, \circ)$ establishes an ordinary Lie algebra.

Definition:

We assign to any $X_\mu \in L$ a *degree* $g \in \mathbb{Z}_2$, by defining:

$$g_\mu := g(X_\mu) = 0 \iff X_\mu \in L_0, \tag{2.13}$$
$$g_\mu := g(X_\mu) = 1 \iff X_\mu \in L_1. \tag{2.14}$$

We say the element $X_\mu \in L$ is *even*, if $g_\mu = 0$, and the element $X_\mu \in L$ is *odd*, if $g_\mu = 1$.

With these definitions of degrees of elements we see that the set of generators $E_i, i = 1, 2, 3, \ldots,$ dim L_0, which is a basis of L_0, consists of even elements, whereas the generators Q_a, which span the subspace L_1, are odd.

The subspace L_0 of the graded Lie algebra L, containing the even elements, is called the *bosonic sector*, the subspace L_1 is called the *fermionic sector*.

Definition:

We define the product $\circ$ on L by the assignment:

$$\circ \, : \, L \times L \longrightarrow L$$

where:

$$(X_\mu, X_\nu) \longmapsto X_\mu \circ X_\nu := X_\mu X_\nu - (-1)^{g_\mu g_\nu} X_\nu X_\mu. \tag{2.15}$$

We now consider this product separately on the two subspaces L_0 and L_1:

(i) $\circ \, : \, L_0 \times L_0 \longrightarrow L_0$
Let $E_i, E_j \in L_0$ and according to Eq. (2.13):

$$g(E_i) = g(E_j) = 0.$$

Then

$$E_i \circ E_j \overset{(2.15)}{=} E_i E_j - (-1)^{g_i g_j} E_j E_i = E_i E_j - E_j E_i = \big[E_i, E_j\big]. \tag{2.16}$$

With this construction we are consistent with the basic requirement that the product $\circ$ be antisymmetric on the subspace L_0 (see Eq. (2.5)).

(ii) $\circ \, : \, L_0 \times L_1 \longrightarrow L_1$
Let $E_j \in L_0, Q_a \in L_1$; then we get

$$g(E_i) = 0; \quad g(Q_a) = 1,$$

and we find

$$E_i \circ Q_a \overset{(2.15)}{=} E_i Q_a - (-1)^{g_i g_a} Q_a E_i = E_i Q_a - Q_a E_i = \big[E_i, Q_a\big]. \tag{2.17}$$

Again the product is antisymmetric on $L_0 \times L_1$, as demanded by Eq. (2.5).

(iii) $\circ \, : \, L_1 \times L_1 \longrightarrow L_0$
Let $Q_a, Q_b \in L_1$ and according to Eq. (2.14)

$$g(Q_a) = g(Q_b) = 1.$$

Then

$$Q_a \circ Q_b \overset{(2.15)}{=} Q_a Q_b - (-1)^{g_a g_b} Q_a Q_b = Q_a Q_b + Q_b Q_a = \{Q_a, Q_b\}. \tag{2.18}$$

From Eq. (2.18) we see that the product is symmetric on the subspace L_1.

We next introduce *generalized structure constants*. Since $X_\mu \circ X_\nu \in L$, the product $X_\mu \circ X_\nu$ must be a linear combination of the basis elements X_ω, *i.e.*

$$X_\mu \circ X_\nu = c_{\mu\nu}{}^{\omega} X_\omega, \tag{2.19}$$

where

$$c_{\mu\nu}{}^{\omega} = -(-1)^{g_\mu g_\nu} c_{\nu\mu}{}^{\omega} \tag{2.20}$$

are the *generalized structure constants* of the graded Lie algebra L. According to Eq. (2.20) the structure constants are antisymmetric for even-even and even-odd pairs of elements of L, and symmetric for odd-odd pairs. We now prove the relation (2.20).

In order to show Eq. (2.20) we start from Eq. (2.19), *i.e.*

$$X_\mu \circ X_\nu = c_{\mu\nu}{}^{\omega} X_\omega \qquad \text{and} \qquad X_\nu \circ X_\mu = c_{\nu\mu}{}^{\omega} X_\omega.$$

On the other hand using Eq. (2.15) we have

$$\begin{aligned} X_\mu \circ X_\nu &= X_\mu X_\nu - (-1)^{g_\mu g_\nu} X_\nu X_\mu, \\ X_\nu \circ X_\mu &= X_\nu X_\mu - (-1)^{g_\nu g_\mu} X_\mu X_\nu = X_\nu X_\mu - (-1)^{g_\mu g_\nu} X_\mu X_\nu, \end{aligned}$$

and again using Eq. (2.15)

$$\begin{aligned} (-1)^{g_\mu g_\nu} X_\nu \circ X_\mu &= (-1)^{g_\mu g_\nu} X_\nu X_\mu - \left[(-1)^{g_\mu g_\nu}\right]^2 X_\mu X_\nu \\ &= (-1)^{g_\mu g_\nu} X_\nu X_\mu - X_\mu X_\nu \\ &= -\left[X_\mu X_\nu - (-1)^{g_\mu g_\nu} X_\nu X_\mu\right] = -\left(X_\mu \circ X_\nu\right). \end{aligned}$$

Hence

$$X_\mu \circ X_\nu + (-1)^{g_\mu g_\nu} X_\nu \circ X_\mu = 0,$$

and with Eq. (2.19)

$$c_{\mu\nu}{}^{\omega} X_\omega + (-1)^{g_\mu g_\nu} c_{\nu\mu}{}^{\omega} X_\omega = 0, \quad \forall X_\omega \in L,$$

i.e.

$$c_{\mu\nu}{}^{\omega} = -(-1)^{g_\mu g_\nu} c_{\nu\mu}{}^{\omega},$$

since the X_ω are independent.

We now consider the generalized Jacobi identities. For $X_\mu, X_\nu, X_\rho \in L$, these are defined by:

$$\begin{aligned}\left[X_\mu \circ \left(X_\nu \circ X_\rho\right)\right](-1)^{g_\mu g_\rho} &+ \left[X_\nu \circ \left(X_\rho \circ X_\mu\right)\right](-1)^{g_\nu g_\mu}\\ &+ \left[X_\rho \circ \left(X_\mu \circ X_\nu\right)\right](-1)^{g_\rho g_\nu} = 0. \qquad (2.21)\end{aligned}$$

We demonstrate that the map given in Eq. (2.15) obeys the generalized Jacobi identity (2.21):

$$\begin{aligned}
&\left[X_\mu \circ \left(X_\nu \circ X_\rho\right)\right](-1)^{g_\mu g_\rho} + \left[X_\rho \circ \left(X_\mu \circ X_\nu\right)\right](-1)^{g_\rho g_\nu}\\
&\qquad + \left[X_\nu \circ \left(X_\rho \circ X_\mu\right)\right](-1)^{g_\nu g_\mu}\\
&= \left[X_\mu\left(X_\nu \circ X_\rho\right) - (-1)^{g_\mu(g_\nu+g_\rho)}\left(X_\nu \circ X_\rho\right)X_\mu\right](-1)^{g_\mu g_\rho}\\
&\quad + \left[X_\rho\left(X_\mu \circ X_\nu\right) - (-1)^{g_\rho(g_\mu+g_\nu)}\left(X_\mu \circ X_\nu\right)X_\rho\right](-1)^{g_\rho g_\nu}\\
&\quad + \left[X_\nu\left(X_\rho \circ X_\mu\right) - (-1)^{g_\nu(g_\rho+g_\mu)}\left(X_\rho \circ X_\mu\right)X_\nu\right](-1)^{g_\nu g_\mu}\\
&= \Big[X_\mu\left(X_\nu X_\rho - (-1)^{g_\nu g_\rho} X_\rho X_\nu\right)\\
&\quad - (-1)^{g_\mu(g_\nu+g_\rho)}\left(X_\nu X_\rho - (-1)^{g_\nu g_\rho} X_\rho X_\nu\right)X_\mu\Big](-1)^{g_\mu g_\rho}\\
&\quad + \Big[X_\rho\left(X_\mu X_\nu - (-1)^{g_\mu g_\nu} X_\nu X_\mu\right)\\
&\quad - (-1)^{g_\rho(g_\mu+g_\nu)}\left(X_\mu X_\nu - (-1)^{g_\mu g_\nu} X_\nu X_\mu\right)X_\rho\Big](-1)^{g_\rho g_\nu}\\
&\quad + \Big[X_\nu\left(X_\rho X_\mu - (-1)^{g_\rho g_\mu} X_\mu X_\rho\right)\\
&\quad - (-1)^{g_\nu(g_\rho+g_\mu)}\left(X_\rho X_\mu - (-1)^{g_\rho g_\mu} X_\mu X_\rho\right)X_\nu\Big](-1)^{g_\nu g_\mu}\\
&= \Big[X_\mu X_\nu X_\rho - (-1)^{g_\nu g_\rho} X_\mu X_\rho X_\nu - (-1)^{g_\mu(g_\nu+g_\rho)} X_\nu X_\rho X_\mu\\
&\quad + (-1)^{g_\mu g_\nu+g_\mu g_\rho+g_\nu g_\rho} X_\rho X_\nu X_\mu\Big](-1)^{g_\mu g_\rho}\\
&\quad + \Big[X_\rho X_\mu X_\nu - (-1)^{g_\mu g_\nu} X_\rho X_\nu X_\mu - (-1)^{g_\rho g_\mu+g_\rho g_\nu} X_\mu X_\nu X_\rho\\
&\quad + (-1)^{g_\rho g_\mu+g_\rho g_\nu+g_\mu g_\nu} X_\nu X_\mu X_\rho\Big](-1)^{g_\rho g_\nu}\\
&\quad + \Big[X_\nu X_\rho X_\mu - (-1)^{g_\rho g_\mu} X_\nu X_\mu X_\rho - (-1)^{g_\nu g_\rho+g_\nu g_\mu} X_\rho X_\mu X_\nu\\
&\quad + (-1)^{g_\nu g_\rho+g_\nu g_\mu+g_\rho g_\mu} X_\mu X_\rho X_\nu\Big](-1)^{g_\nu g_\mu}
\end{aligned}$$

$$\begin{aligned}
&= X_\mu X_\nu X_\rho(-1)^{g_\mu g_\rho} - X_\mu X_\rho X_\nu(-1)^{g_\rho(g_\mu+g_\nu)} \\
&\quad - X_\nu X_\rho X_\mu(-1)^{g_\mu g_\nu} + X_\rho X_\nu X_\mu(-1)^{g_\nu(g_\mu+g_\rho)} \\
&\quad + X_\rho X_\mu X_\nu(-1)^{g_\rho g_\nu} - X_\rho X_\nu X_\mu(-1)^{g_\nu(g_\mu+g_\rho)} \\
&\quad - X_\mu X_\nu X_\rho(-1)^{g_\rho g_\mu} + X_\nu X_\mu X_\rho(-1)^{g_\mu(g_\rho+g_\nu)} \\
&\quad + X_\nu X_\rho X_\mu(-1)^{g_\nu g_\mu} - X_\nu X_\mu X_\rho(-1)^{g_\mu(g_\nu+g_\rho)} \\
&\quad - X_\rho X_\mu X_\nu(-1)^{g_\nu g_\rho} + X_\mu X_\rho X_\nu(-1)^{g_\rho(g_\mu+g_\nu)} \\
&= X_\mu X_\nu X_\rho\Big((-1)^{g_\mu g_\rho} - (-1)^{g_\rho g_\mu}\Big) \\
&\quad + X_\mu X_\rho X_\nu\Big((-1)^{g_\rho(g_\mu+g_\nu)} - (-1)^{g_\rho(g_\nu+g_\mu)}\Big) \\
&\quad + X_\rho X_\nu X_\mu\Big((-1)^{g_\nu(g_\mu+g_\rho)} - (-1)^{g_\nu(g_\mu+g_\rho)}\Big) \\
&\quad + X_\rho X_\mu X_\nu\Big((-1)^{g_\rho g_\nu} - (-1)^{g_\nu g_\rho}\Big) \\
&\quad + X_\nu X_\mu X_\rho\Big((-1)^{g_\mu(g_\rho+g_\nu)} - (-1)^{g_\mu(g_\rho+g_\nu)}\Big) \\
&\quad + X_\nu X_\rho X_\mu\Big((-1)^{g_\nu g_\mu} - (-1)^{g_\mu g_\nu}\Big) = 0.
\end{aligned}$$

Hence the composition law (2.15) is a product which obeys all conditions of the product of a graded Lie algebra as defined by Eqs. (2.4) to (2.6).

We now write out the generalized Jacobi identity for the four different possibilities.

(i) $X_\mu, X_\nu, X_\rho \in L_0$: We take

$$X_\mu = E_i, \quad X_\nu = E_j, \quad X_\rho = E_k,$$

and

$$(-1)^{g_\mu g_\rho} = (-1)^{g_\rho g_\nu} = (-1)^{g_\mu g_\nu} = +1.$$

Then Eq. (2.21) reads in the case of three generators of L_0:

$$\begin{aligned}
&\big[X_\mu \circ \big(X_\nu \circ X_\rho\big)\big](-1)^{g_\mu g_\rho} + \big[X_\rho \circ \big(X_\mu \circ X_\nu\big)\big](-1)^{g_\rho g_\nu} \\
&\qquad + \big[X_\nu \circ \big(X_\rho \circ X_\mu\big)\big](-1)^{g_\nu g_\mu} \\
&= E_i \circ \big(E_j \circ E_k\big) + E_k \circ \big(E_i \circ E_j\big) + E_j \circ \big(E_k \circ E_i\big) \\
&= E_i\big(E_j \circ E_k\big) - \big(E_j \circ E_k\big)E_i \\
&\quad + E_k\big(E_i \circ E_j\big) - \big(E_i \circ E_j\big)E_k \\
&\quad + E_j\big(E_k \circ E_i\big) - \big(E_k \circ E_i\big)E_j \\
&= E_i\big(E_jE_k - E_kE_j\big) - \big(E_jE_k - E_kE_j\big)E_i \\
&\quad + E_k\big(E_iE_j - E_jE_i\big) - \big(E_iE_j - E_jE_i\big)E_k \\
&\quad + E_j\big(E_kE_i - E_iE_k\big) - \big(E_kE_i - E_iE_k\big)E_j
\end{aligned}$$

$$
\begin{aligned}
&= E_i\big[E_j, E_k\big] - \big[E_j, E_k\big]E_i \\
&\quad + E_k\big[E_i, E_j\big] - \big[E_i, E_j\big]E_k \\
&\quad + E_j\big[E_k, E_i\big] - \big[E_k, E_i\big]E_j \\
&= \Big[E_i, \big[E_j, E_k\big]\Big] + \Big[E_k, \big[E_i, E_j\big]\Big] + \Big[E_j, \big[E_k, E_i\big]\Big] \\
&= 0.
\end{aligned}
$$

Hence the generalized Jacobi identity for three E's reduces to the ususal Jacobi identity for the Lie algebra L_0, *i.e.*

$$
\Big[E_i, \big[E_j, E_k\big]\Big] + \Big[E_k, \big[E_i, E_j\big]\Big] + \Big[E_j, \big[E_k, E_i\big]\Big] = 0. \tag{2.22}
$$

(ii) $X_\mu, X_\nu \in L_0, X_\rho \in L_1$: We take

$$
X_\mu = E_i, \quad X_\nu = E_j, \quad E_i, E_j \in L_0, \quad \text{and} \quad X_\rho = Q_a \in L_1.
$$

In this case Eq. (2.21) becomes

$$
\begin{aligned}
\big[X_\mu \circ \big(X_\nu \circ X_\rho\big)\big](-1)^{g_\mu g_\rho} &+ \big[X_\rho \circ \big(X_\mu \circ X_\nu\big)\big](-1)^{g_\rho g_\nu} \\
&+ \big[X_\nu \circ \big(X_\rho \circ X_\mu\big)\big](-1)^{g_\nu g_\mu} \\
&= E_i \circ \big(E_j \circ Q_a\big)(-1)^0 + Q_a \circ \big(E_i \circ E_j\big)(-1)^0 \\
&\quad + E_j \circ \big(Q_a \circ E_i\big)(-1)^0 \\
&= E_i\big(E_j \circ Q_a\big) - (-1)^{g_i(g_j+g_a)}\big(E_j \circ Q_a\big)E_i \\
&\quad + Q_a\big(E_i \circ E_j\big) - (-1)^{g_a(g_i+g_j)}\big(E_i \circ E_j\big)Q_a \\
&\quad + E_j\big(Q_a \circ E_i\big) - (-1)^{g_j(g_a+g_i)}\big(Q_a \circ E_i\big)E_j.
\end{aligned}
$$

Using again Eq. (2.15) we find

$$
\begin{aligned}
\big[X_\mu \circ \big(X_\nu \circ X_\rho\big)\big](-1)^{g_\mu g_\rho} &+ \big[X_\rho \circ \big(X_\mu \circ X_\nu\big)\big](-1)^{g_\rho g_\nu} \\
&+ \big[X_\nu \circ \big(X_\rho \circ X_\mu\big)\big](-1)^{g_\nu g_\mu} \\
&= E_i\big(E_jQ_a - (-1)^{g_jg_a}Q_aE_j\big) - \big(E_jQ_a - (-1)^{g_jg_a}Q_aE_j\big)E_i \\
&\quad + Q_a\big(E_iE_j - (-1)^{g_ig_j}E_jE_i\big) - \big(E_iE_j - (-1)^{g_ig_j}E_jE_i\big)Q_a \\
&\quad + E_j\big(Q_aE_i - (-1)^{g_ag_i}E_iQ_a\big) - \big(Q_aE_i - (-1)^{g_ag_i}E_iQ_a\big)E_j \\
&= E_i\big[E_j, Q_a\big] - \big[E_j, Q_a\big]E_i + Q_a\big[E_i, E_j\big] - \big[E_i, E_j\big]Q_a \\
&\quad + E_j\big[Q_a, E_i\big] - \big[Q_a, E_i\big]E_j \\
&= \Big[E_i, \big[E_j, Q_a\big]\Big] + \Big[Q_a, \big[E_i, E_j\big]\Big] + \Big[E_j, \big[Q_a, E_i\big]\Big] = 0.
\end{aligned}
$$

Hence the generalized Jacobi identity for two E's and one Q is:

$$
\Big[E_i, \big[E_j, Q_a\big]\Big] + \Big[Q_a, \big[E_i, E_j\big]\Big] + \Big[E_j, \big[Q_a, E_i\big]\Big] = 0. \tag{2.23}
$$

(iii) $X_\mu \in L_0, X_\nu, X_\rho \in L_1$: Here we set

$$X_\mu = E_i \in L_0, \quad X_\nu = Q_a \in L_1, \quad X_\rho = Q_b \in L_1.$$

Then Eq. (2.21) becomes:

$$\begin{aligned}
&\big[X_\mu \circ \big(X_\nu \circ X_\rho\big)\big](-1)^{g_\mu g_\rho} + \big[X_\rho \circ \big(X_\mu \circ X_\nu\big)\big](-1)^{g_\rho g_\nu} \\
&\qquad + \big[X_\nu \circ \big(X_\rho \circ X_\mu\big)\big](-1)^{g_\nu g_\mu} \\
&= E_i \circ \big(Q_a \circ Q_b\big)(-1)^{g_i g_b} + Q_b \circ \big(E_i \circ Q_a\big)(-1)^{g_b g_a} \\
&\quad + Q_a \circ \big(Q_b \circ E_i\big)(-1)^{g_a g_i} \\
&= E_i\big(Q_a \circ Q_b\big) - (-1)^{g_i(g_a+g_b)}\big(Q_a \circ Q_b\big)E_i \\
&\quad - Q_b\big(E_i \circ Q_a\big) + (-1)^{g_b(g_i+g_a)}\big(E_i \circ Q_a\big)Q_b \\
&\quad + Q_a\big(Q_b \circ E_i\big) - (-1)^{g_a(g_b+g_i)}\big(Q_b \circ E_i\big)Q_a \\
&= E_i\big(Q_aQ_b - (-1)^{g_a g_b}Q_bQ_a\big) - \big(Q_aQ_b - (-1)^{g_a g_b}Q_bQ_a\big)E_i \\
&\quad - Q_b\big(E_iQ_a - (-1)^{g_i g_a}Q_aE_i\big) - \big(E_iQ_a - (-1)^{g_i g_a}Q_aE_i\big)Q_b \\
&\quad + Q_a\big(Q_bE_i - (-1)^{g_i g_b}E_iQ_b\big) + \big(Q_bE_i - (-1)^{g_i g_b}E_iQ_b\big)Q_a \\
&= E_i\{Q_a, Q_b\} - \{Q_a, Q_b\}E_i - Q_b[E_i, Q_a] - [E_i, Q_a]Q_b \\
&\quad + Q_a[Q_b, E_i] + [Q_b, E_i]Q_a \\
&= \Big[E_i, \{Q_a, Q_b\}\Big] - \Big\{Q_b, [E_i, Q_a]\Big\} + \Big\{Q_a, [Q_b, E_i]\Big\} = 0.
\end{aligned}$$

Hence, the Jacobi relation for one E and two Q's is:

$$\Big[E_i, \{Q_a, Q_b\}\Big] - \Big\{Q_b, [E_i, Q_a]\Big\} + \Big\{Q_a, [Q_b, E_i]\Big\} = 0. \tag{2.24}$$

(iv) $X_\mu, X_\nu, X_\rho \in L_1$: Here we set

$$X_\mu = Q_a \in L_1, \quad X_\nu = Q_b \in L_1, \quad X_\rho = Q_c \in L_1.$$

Then Eq. (2.21) becomes for this case of three Q's:

$$\begin{aligned}
&\big[X_\mu \circ \big(X_\nu \circ X_\rho\big)\big](-1)^{g_\mu g_\rho} + \big[X_\rho \circ \big(X_\mu \circ X_\nu\big)\big](-1)^{g_\rho g_\nu} \\
&\qquad + \big[X_\nu \circ \big(X_\rho \circ X_\mu\big)\big](-1)^{g_\nu g_\mu} \\
&= Q_a \circ \big(Q_b \circ Q_c\big)(-1)^{g_a g_c} + Q_c \circ \big(Q_a \circ Q_b\big)(-1)^{g_c g_b} \\
&\quad + Q_b \circ \big(Q_c \circ Q_a\big)(-1)^{g_b g_a} \\
&= -Q_a\big(Q_b \circ Q_c\big) + (-1)^{g_a(g_b+g_c)}\big(Q_b \circ Q_c\big)Q_a \\
&\quad - Q_c\big(Q_a \circ Q_b\big) + (-1)^{g_c(g_a+g_b)}\big(Q_a \circ Q_b\big)Q_c \\
&\quad - Q_b\big(Q_c \circ Q_a\big) + (-1)^{g_b(g_c+g_a)}\big(Q_c \circ Q_a\big)Q_b
\end{aligned}$$

$$\begin{aligned}
&= -Q_a\big(Q_bQ_c - (-1)^{g_bg_c}Q_cQ_b\big) + \big(Q_bQ_c - (-1)^{g_bg_c}Q_cQ_b\big)Q_a \\
&\quad - Q_c\big(Q_aQ_b - (-1)^{g_ag_b}Q_bQ_a\big) + \big(Q_aQ_b - (-1)^{g_ag_b}Q_bQ_a\big)Q_c \\
&\quad - Q_b\big(Q_cQ_a - (-1)^{g_cg_a}Q_aQ_c\big) + \big(Q_cQ_a - (-1)^{g_cg_a}Q_aQ_c\big)Q_b \\
&= -Q_a\big\{Q_b, Q_c\big\} + \big\{Q_b, Q_c\big\}Q_a \\
&\quad - Q_c\big\{Q_a, Q_b\big\} - \big\{Q_a, Q_b\big\}Q_c \\
&\quad - Q_b\big\{Q_c, Q_a\big\} + \big\{Q_c, Q_a\big\}Q_b \\
&= -\Big[Q_a, \big\{Q_b, Q_c\big\}\Big] - \Big[Q_c, \big\{Q_a, Q_b\big\}\Big] - \Big[Q_b, \big\{Q_c, Q_a\big\}\Big] \\
&= 0.
\end{aligned}$$

Hence, for three Q's the Jacobi identity is:

$$\Big[Q_a, \big\{Q_b, Q_c\big\}\Big] + \Big[Q_c, \big\{Q_a, Q_b\big\}\Big] + \Big[Q_b, \big\{Q_c, Q_a\big\}\Big] = 0. \tag{2.25}$$

Remark: It is important to observe that the commutators and anticommutators in Eqs. (2.22) to (2.25) are consistent with the composition law (2.4) for graded Lie Algebras. Consider for example the generalized Jacobi identity (2.24), *i.e.*

$$\Big[E_i, \big\{Q_a, Q_b\big\}\Big] - \Big\{Q_b, \big[E_i, Q_a\big]\Big\} + \Big\{Q_a, \big[Q_b, E_i\big]\Big\} = 0.$$

According to Eq. (2.4) we have $\{Q_a, Q_b\} \in L_0$; but on L_0 the product has to be antisymmetric, so an expression like $\big[E_i, \{Q_a, Q_b\}\big]$ is consistent with Eq. (2.6). Similarly $[E_i, Q_a] \in L_1$ and $\big\{Q_b, [E_i, Q_a]\big\}$ has the correct bracket structure.

In conclusion we introduce the so-called *generalized Killing form*. The generalized Killing form on L is defined by:[10]

$$b_{\mu\nu} = \sum_{\rho,\sigma} (-1)^{g_\sigma} c_{\mu\rho}{}^{\sigma} c_{\nu\sigma}{}^{\rho}, \tag{2.26}$$

where

$$\begin{gathered}
g_\sigma := g(X_\sigma), \qquad \big[X_\mu, X_\nu\big] = c_{\mu\nu}{}^{\sigma} X_\sigma, \\
X_\sigma \in L_0 \implies g_\sigma = 0, \quad X_\sigma \in L_1 \implies g_\sigma = 1.
\end{gathered}$$

Here, $b_{\mu\nu} = B(X_\mu, X_\nu)$ is a symmetric bilinear form when acting on the subspace L_0 and thus reduces to the usual Killing form for the Lie algebra

[10] See W. Miller, [72], p. 394, or A.S. Scarrino and P. Sorba, [104].

L_0. We can then apply *Cartan's criterion* [104]: The Lie algebra L_0 is semi-simple if and only if $B(X_\mu, X_\nu)|_{L_0 \times L_0}$ is nondegenerate, *i.e.*

$$\det\,(b_{\mu\nu}) \neq 0.$$

Furthermore, the Lie algebra L_0 is compact, if and only if the generalized Killing form restricted to L_0 is negative definite. Hence, a compact Lie algebra is always semi-simple. In addition, the generalized Killing form is antisymmetric when acting on the subspace L_1, *i.e.*

$$b_{ab} = B(Q_a, Q_b) = -B(Q_b, Q_a),$$

and

$$B(X_\mu, X_\nu)\Big|_{L_0 \times L_1} = 0.$$

2.5 Graded Matrices

An endomorphism

$$M \;:\; L \;\longrightarrow L, \qquad L = L_0 \oplus L_1$$

acting on a vector space L [104] can be represented by a *graded matrix* which has the following matrix structure. Let $\dim\, L_0 = n,\ \dim\, L_1 = m.$ Then

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{ End } (L) \tag{2.27}$$

is an $(n+m) \times (n+m)$-matrix where:

A is an $n \times n$ square matrix,

B is an $n \times m$ submatrix,

C is an $m \times n$ submatrix,

D is an $m \times m$ square matrix.

A matrix with such a structure is also called a *supermatrix.* Since L is the direct sum of L_0 and L_1, *i.e.* $L = L_0 \oplus L_1$, a vector $v \in L$ has the structure

$$v = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \quad v_0 \in L_0, \quad v_1 \in L_1, \tag{2.28}$$

with $g(v_0) = 0,\ g(v_1) = 1$ according to Eqs. (2.13) and (2.14). Here v_0 is an element of the bosonic sector of L, *i.e.* L_0, and v_1 is an element of the fermionic sector of L, *i.e.* L_1. Introducing the following index notation

$$\begin{aligned}
A &= (A_{ij}), && i,j = 1,2,\dots,n,\\
B &= (B_{ia}), && i = 1,2,\dots,n;\ a = 1,2,\dots,m,\\
C &= (C_{ai}), && a = 1,2,\dots,m;\ i = 1,2,\dots,n,\\
D &= (D_{ab}), && a,b = 1,2,\dots,m,\\
v_0 &= (v_{0i}), && i = 1,2,\dots,n,\\
v_1 &= (v_{1a}), && a = 1,2,\dots,m,
\end{aligned} \tag{2.29}$$

we obtain (with $w, z = 1,2,\dots,m+n$):

$$M_{wz}v_z = \begin{pmatrix} A_{ij} & B_{ia} \\ C_{bj} & D_{ba} \end{pmatrix}\begin{pmatrix} v_{0j} \\ v_{1a} \end{pmatrix} = \begin{pmatrix} A_{ij}v_{0j} + B_{ia}v_{1a} \\ C_{bj}v_{0j} + D_{ba}v_{1a} \end{pmatrix} \overset{\text{def.}}{=} \begin{pmatrix} v'_{0i} \\ v'_{1b} \end{pmatrix}.$$

Thus:

$$v'_0 = Av_0 + Bv_1 \ \in L_0, \qquad v'_1 = Cv_0 + Dv_1 \ \in L_1.$$

Since $v'_0 \in L_0$, we must have $g(v'_0) = 0$. But $g(v_0) = 0$ and $g(v_1) = 1$. Now since $Av_0 \in L_0$ and $Bv_1 \in L_0$, we must have $g(Av_0) = 0$ and $g(Bv_1) = 0 \bmod 2$. Hence

$$g(Av_0) = g(A) + g(v_0)$$

implies

$$g(A) = 0,$$

and

$$g(Bv_1) = g(B) + g(v_1)$$

implies

$$g(B) = 1.$$

Similarly since $v'_1 \in L_1$, we have $g(v'_1) = 1$. But $g(v_0) = 0,\ g(v_1) = 1$, so that

$$g(C) = 1 \qquad \text{and} \qquad g(D) = 0.$$

This means that A and D must be even submatrices with degree $g(A) = g(D) = 0$, and B and C have to be odd with degree $g(B) = g(C) = 1$. It follows, the elements of B and C are anticommuting variables and therefore behave as Grassmann numbers, *i.e.*

$$B_{ia}C_{bj} = -C_{bj}B_{ia}. \tag{2.30}$$

From this relation we obtain the *transposition rule*

$$BC = -\left(C^\top B^\top\right)^\top \tag{2.31}$$

by setting in Eq. (2.30) $a = b = 1, \ldots, m$, so that

$$\left(BC\right)_{ij} = -C_{aj}B_{ia} = -\left(C^\top B^\top\right)_{ji}.$$

Definition:

The *supertrace* of a graded matrix M is defined by:

$$\text{STr } M = \text{Tr } A - \text{Tr } D, \tag{2.32}$$

where A and D are submatrices as in Eq. (2.27).

Proposition 2.2:

Let M_1, M_2 be two graded matrices. The supertrace has the property:

$$\text{STr } (M_1 M_2) = \text{STr } (M_2 M_1). \tag{2.33}$$

Proof: We consider two graded matrices

$$M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}.$$

Then:

$$\begin{aligned} \text{STr } (M_1 M_2) &= \text{STr } \left[\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}\right] \\ &= \text{STr } \begin{pmatrix} A_1A_2 + B_1C_2 & A_1B_2 + B_1D_2 \\ C_1A_2 + D_1C_2 & C_1B_2 + D_1D_2 \end{pmatrix}. \end{aligned}$$

Using Eqs. (2.30) and (2.32) this becomes

$$\begin{aligned} \text{STr } (M_1M_2) &= \text{Tr } \left(A_1A_2 + B_1C_2\right) - \text{Tr } \left(C_1B_2 + D_1D_2\right) \\ &= \text{Tr } \left(A_1A_2\right) + \text{Tr } \left(B_1C_2\right) - \text{Tr } \left(C_1B_2\right) - \text{Tr } \left(D_1D_2\right) \\ &= \text{Tr } \left(A_2A_1\right) - \text{Tr } \left(C_2B_1\right) + \text{Tr } \left(B_2C_1\right) - \text{Tr } \left(D_2D_1\right) \\ &= \text{Tr } \left(A_2A_1 + B_2C_1\right) - \text{Tr } \left(C_2B_1 + D_2D_1\right) \\ &= \text{STr } \begin{pmatrix} A_2A_1 + B_2C_1 & A_2B_1 + B_2D_1 \\ C_2A_1 + D_2C_1 & C_2B_1 + D_2D_1 \end{pmatrix} \\ &= \text{STr } \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \\ &= \text{STr } (M_2M_1). \end{aligned}$$

We observe that the supertrace is defined in such a way that it is cyclic for graded matrices paralleling the analogous property of the trace for ordinary matrices.

Definition:

> The determinant of the supermatrix M is the *superdeterminant* Sdet M, defined by
>
> $$\text{Sdet } M := \exp\{\text{STr } \ln M\}. \tag{2.34}$$

If

$$M = \exp X, \tag{2.35}$$

we have, of course,

$$\text{Sdet } M = \exp\{\text{STr } X\}. \tag{2.36}$$

The superdeterminant is defined in analogy to the well-known result

$$\det M = \exp\{ \text{ Tr } \ln M\}$$

for ordinary square matrices M. The latter is easily verified by setting $M = 1 + L$. Then:

$$\text{Tr } \ln(1+L) = \text{ Tr } \Big(L - \frac{1}{2}L^2 + \frac{1}{3}L^3 - \cdots \Big).$$

If U is a matrix with

$$\text{Tr } (L) = \text{ Tr } \big(ULU^{-1}\big) = \text{ Tr } (\Lambda) = \sum_i \lambda_i,$$

where Λ is diagonal, then:

$$\begin{aligned}
\text{Tr } \ln(1+L) &= \sum_i \lambda_i - \frac{1}{2}\sum_i \lambda_i^2 + \frac{1}{3}\sum_i \lambda_i^3 - \cdots \\
&= \sum_i \ln\big(1+\lambda_i\big) \\
&= \ln \prod_i \big(1+\lambda_i\big) \\
&= \ln\det\big(1+\Lambda\big) \\
&= \ln\det\big(1+L\big),
\end{aligned}$$

so that

$$\text{Tr } \ln M = \ln\det M.$$

It is a natural consequence of Eq. (2.34) to define unimodular graded matrices M by:

$$\text{Sdet } M = 1,$$

and

$$\text{STr } X = 0.$$

Proposition 2.3:

Let M_1 and M_2 be two graded matrices. Then

$$\text{Sdet } \big(M_1\, M_2\big) = \text{Sdet } \big(M_1\big) \cdot \text{Sdet } \big(M_2\big). \tag{2.37}$$

Proof: Using the definition of the superdeterminant, *i.e.* Eq. (2.24), we write the left hand side of Eq. (2.37) as

$$\text{Sdet } \big(M_1 M_2\big) = \exp\{\text{STr } \ln M_1 M_2\}.$$

We now define two matrices:

$$A := \ln M_1, \quad B := \ln M_2,$$

and use the *Baker–Campbell–Hausdorff* formula[11]

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[B,A]]+\cdots} = M_1 M_2.$$

Taking the logarithm of both sides we obtain:

$$\ln\big(M_1 M_2\big) = \ln\big(e^A e^B\big) = A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[B,A]]+\cdots .$$

Next we take the supertrace of both sides so that

$$\begin{aligned} \text{STr } \ln\big(M_1 M_2\big) &= \text{STr } A + \text{STr } B + \frac{1}{2}\text{STr } \big[A,B\big] + \cdots \\ &= \text{STr } \ln M_1 + \text{STr } \ln M_2 + \frac{1}{2}\text{STr } \big(AB - BA\big) + \cdots \\ &= \text{STr } \ln M_1 + \text{STr } \ln M_2. \end{aligned}$$

All commutator terms vanish because the supertrace obeys Eq. (2.33). Hence

$$\begin{aligned} \text{Sdet } \big(M_1 M_2\big) &= \exp\big\{\text{STr } \ln\big(M_1 M_2\big)\big\} \\ &= \exp\big\{\text{STr } \ln M_1 + \text{STr } \ln M_2\big\} \\ &= \exp\big\{\text{STr } \ln M_1\big\} \cdot \exp\big\{\text{STr } \ln M_2\big\} \\ &= \text{Sdet } \big(M_1\big) \cdot \text{Sdet } \big(M_2\big) \end{aligned}$$

in view of Eq. (2.29).

[11] See e.g. W. Miller, [72], p. 161.

Proposition 2.4:

The superdeterminant (2.34) can be expressed in terms of ordinary determinants by means of the following formulas. Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a supermatrix; then

$$\begin{aligned} \text{Sdet } M &= \frac{\det(A - BD^{-1}C)}{\det D} && (2.38) \\ &= \frac{\det A}{\det(D - CA^{-1}B)}. && (2.39) \end{aligned}$$

Proof: In order to verify Eq. (2.38), we decompose the matrix M in the following form:

$$M = \begin{pmatrix} E & F \\ 0_{m\times n} & \mathbb{1}_{m\times m} \end{pmatrix} \begin{pmatrix} \mathbb{1}_{n\times n} & 0_{n\times m} \\ G & H \end{pmatrix} = M_1 M_2 = \begin{pmatrix} E + FG & FH \\ G & H \end{pmatrix}.$$

Comparing this decomposition with the standard form above, we obtain:

$$E + FG = A, \quad FH = B, \quad C = G, \quad D = H,$$

or

$$E = A - BD^{-1}C, \quad F = BD^{-1}, \quad G = C, \quad H = D.$$

Then

$$\begin{aligned} \text{Sdet } M &= \text{Sdet } \big(M_1 M_2\big) \\ &\overset{(2.37)}{=} \text{Sdet } M_1 \cdot \text{Sdet } M_2 \\ &= \exp\{\text{STr } \ln M_1\} \cdot \exp\{\text{STr } \ln M_2\}. \end{aligned}$$

For the particular form of M_1 under consideration

$$\text{STr } \ln M_1 = \text{Tr } \ln E,$$

as can be shown with the help of the power series expansion of the logarithm. Thus setting $M_1 = 1 + L$, so that

$$L = \begin{pmatrix} E - 1 & F \\ 0 & 0 \end{pmatrix}, \qquad L^2 = \begin{pmatrix} (E-1)^2 & (E-1)F \\ 0 & 0 \end{pmatrix},$$
$$L^3 = \begin{pmatrix} (E-1)^3 & (E-1)^2 F \\ 0 & 0 \end{pmatrix}, \ldots,$$

we have

$$\begin{aligned}
\mathrm{STr}\ \ln M_1 &= \mathrm{STr}\ \ln\bigl(1+L\bigr) \\
&= \mathrm{STr}\ \Bigl[L - \frac{1}{2}L^2 + \frac{1}{3}L^3 - \ldots\Bigr] \\
&= \mathrm{STr}\ \left[\begin{pmatrix} E-1 & F \\ 0 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} (E-1)^2 & (E-1)F \\ 0 & 0 \end{pmatrix} + \cdots\right] \\
&= \mathrm{Tr}\ \bigl[E-1\bigr] - \frac{1}{2}\mathrm{Tr}\ \bigl[(E-1)^2\bigr] + \frac{1}{3}\mathrm{Tr}\ \bigl[(E-1)^3\bigr] - \cdots \\
&= \mathrm{Tr}\ \bigl[\ln E\bigr].
\end{aligned}$$

Inserting this in our expression for Sdet M, we obtain:

$$\begin{aligned}
\mathrm{Sdet}\ M &= \exp\bigl\{\mathrm{Tr}\ \bigl[\ln E\bigr]\bigr\}\cdot\exp\bigl\{-\mathrm{Tr}\ \bigl[\ln H\bigr]\bigr\} \\
&= \det E\cdot(\det H)^{-1} \\
&= \frac{\det\bigl(A - BD^{-1}C\bigr)}{\det D}.
\end{aligned}$$

Equation (2.39) can be verified by using the decomposition

$$M = \begin{pmatrix} \mathbb{1}_{n\times n} & 0 \\ P & Q \end{pmatrix}\begin{pmatrix} R & S \\ 0 & \mathbb{1}_{m\times m} \end{pmatrix} = \begin{pmatrix} R & S \\ PR & PS+Q \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Hence

$$P = CA^{-1}, \quad Q = D - CA^{-1}B, \quad R = A, \quad B = S.$$

Then as above

$$\mathrm{Sdet}\ M = \det R\cdot(\det Q)^{-1} = \frac{\det A}{\det\bigl(D - CA^{-1}B\bigr)}.$$

This completes the proof of Proposition 2.4.

Definition:

Supertransposition of a supermatrix M is defined by

$$M^{S\top} = \begin{pmatrix} A^\top & -C^\top \\ B^\top & D^\top \end{pmatrix}. \tag{2.40}$$

Proposition 2.5:

Supertransposition as defined in Eq. (2.40) is constructed in such a way to mimic the ordinary law of transposition, *i.e.*

$$\big(M_1M_2\big)^{S\top} = M_2^{S\top}M_1^{S\top}. \tag{2.41}$$

Proof: We start with:

$$\begin{aligned}
M_2^{S\top}M_1^{S\top} &= \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}^{S\top}\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}^{S\top} \\
&\overset{(2.40)}{=} \begin{pmatrix} A_2^\top & -C_2^\top \\ B_2^\top & D_2^\top \end{pmatrix}\begin{pmatrix} A_1^\top & -C_1^\top \\ B_1^\top & D_1^\top \end{pmatrix} \\
&= \begin{pmatrix} A_2^\top A_1^\top - C_2^\top B_1^\top & -A_2^\top C_1^\top - C_2^\top D_1^\top \\ B_2^\top A_1^\top + D_2^\top B_1^\top & -B_2^\top C_1^\top + D_2^\top D_1^\top \end{pmatrix} \\
&\overset{(2.31)}{=} \begin{pmatrix} (A_1A_2)^\top + (B_1C_2)^\top & -(C_1A_2)^\top - (D_1C_2)^\top \\ (A_1B_2)^\top + (B_1D_2)^\top & (C_1B_2)^\top + (D_1D_2)^\top \end{pmatrix} \\
&= \begin{pmatrix} (A_1A_2 + B_1C_2)^\top & -(C_1A_2 + D_1C_2)^\top \\ (A_1B_2 + B_1D_2)^\top & (C_1B_2 + D_1D_2)^\top \end{pmatrix} \\
&\overset{(2.40)}{=} \begin{pmatrix} A_1A_2 + B_1C_2 & A_1B_2 + B_1D_2 \\ C_1A_2 + D_1C_2 & C_1B_2 + D_1D_2 \end{pmatrix}^{S\top} \\
&= \left[\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}\right]^{S\top} \quad = \quad \big(M_1M_2\big)^{S\top}.
\end{aligned}$$

Finally, we prove the following proposition.

Proposition 2.6:

Let M be a supermatrix. Then

$$\mathrm{Sdet}\,\big(M^{S\top}\big) = \mathrm{Sdet}\,\big(M\big). \tag{2.42}$$

Proof: Using Eqs. (2.38), (2.30) and again Eq. (2.38), we find:

$$\begin{aligned}
\mathrm{Sdet}\,\big(M^{S\top}\big) &= \mathrm{Sdet}\,\begin{pmatrix} A^\top & -C^\top \\ B^\top & D^\top \end{pmatrix} \\
&= \frac{\det\big(A^\top + C^\top (D^\top)^{-1}B^\top\big)}{\det(D^\top)} \\
&= \frac{\det\big(A^\top - (BD^{-1}C)^\top\big)}{\det D} \\
&= \frac{\det\big[\big(A - BD^{-1}C\big)^\top\big]}{\det D} \\
&= \frac{\det\big(A - BD^{-1}C\big)}{\det D} = \mathrm{Sdet}\,\big(M\big).
\end{aligned}$$