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COMPETITIVE OPTIMALITY OF LOGARITHMIC INVESTMENT*†

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Consider the two-person zero-sum game in which two investors are each allowed to invest in a market with stocks $(X_1, X_2, \ldots, X_m) \sim F$, where $X_i \geqslant 0$. Each investor has one unit of capital. The goal is to achieve more money than one's opponent. Allowable portfolio strategies are random investment policies $\underline{B} \in \mathbb{R}^m, \underline{B} \geqslant \underline{0}, E\sum_{i=1}^m \underline{B}_i = 1$. The payoff to player 1 for policy $\underline{B}_1$ vs. $\underline{B}_2$ is $P\{\underline{B}_1^t\underline{X} \geqslant \underline{B}_2^t\underline{X}\}$. The optimal policy is shown to be $\underline{B}^* = U\underline{b}^*$, where U is a random variable uniformly distributed on [0, 2], and $\underline{b}^*$ maximizes $E \ln \underline{b}^t\underline{X}$ over $\underline{b} \geqslant \underline{0}$, $\sum b_i = 1$.

Curiously, this competitively optimal investment policy $\underline{b}^*$ is the same policy that achieves the maximum possible growth rate of capital in repeated independent investments (Breiman (1961) and Kelly (1956)). Thus the immediate goal of outperforming another investor is perfectly compatible with maximizing the asymptotic rate of return.

1. Introduction. An investor is faced with a collection of stocks $(X_1, X_2, \ldots, X_m)$ drawn according to some known joint distribution function F. We shall assume that stock values X_i are nonnegative. A *portfolio* is a vector $\underline{b} = (b_1, \ldots, b_m)^t$, $b_i \geqslant 0, \sum b_i = 1$, with the interpretation that b_i is the proportion of capital allocated to stock i. The capital return S from investment portfolio $\underline{b}$ is

$$S = \underline{b}^t \underline{X} = \sum_{i=1}^{m} b_i X_i. \tag{1}$$

How should $\underline{b}$ be chosen? A currently accepted procedure is the efficient portfolio selection approach of Markowitz (1952, 1959). A portfolio $\underline{b}$ is said to be *efficient* if $(E\underline{b}^t\underline{X}, \operatorname{Var} \underline{b}^t\underline{X})$ is undominated. Criticisms of this approach are many. Only the first two moments are used in the analysis; there is no optimality of this procedure with respect to other obvious investment goals, and no choice procedure among the efficient portfolios is provided. (See Thorp (1971) and Samuelson (1969) for further comments.) Also, such a portfolio is not necessarily admissible [Hakansson (1971, p. 529), Thorp (1971, p. 20)] in the sense that it may be stochastically dominated by some other mixture S.

Another criterion for selecting $\underline{b}$, that of maximizing $E \ln S$, has been put forth by Kelly (1956) and Breiman (1961), and persuasively advocated by Thorp (1969, 1971, 1973). (Also see Latané (1959) and Williams (1936).) This portfolio is admissible, since it maximizes the expectation of a monotonic function of S. The resulting portfolio investment policy $\underline{b}^*$ has been demonstrated by Breiman to have the following properties:

P1. In repeated independent sequential investment, $\underline{b}^*$ maximizes $\liminf (1/n) \ln S_n$. Thus the asymptotic "interest rate" is maximized.

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P2. The time required to achieve a certain capital A is minimized by $\underline{b}^*$ (in a sense that can be made precise), in the limit as $A \to \infty$.

Yet $\underline{b}^*$ is not accepted in current economic practice. Perhaps one reason is that maximizing $E \ln S$ suggests that the investor has a logarithmic utility for money. However, the criticism of the choice of utility functions ignores the fact that maximizing $E \ln S$ is a consequence of the goals represented by properties P1 and P2, and has nothing to do with utility theory. (See Thorp (1971).) A thorough discussion of investment strategies and their relation to utility theory is developed in Arrow (1971).

We are not interested in utility theory in this paper insofar as utility theory deals with the consistency of subjective preferences. We wish instead to emphasize the objective aspects of portfolio selection, i.e., properties of optimal portfolios that have some objective appeal. In particular we wish to add another goal to the list P1, P2, namely that of outperforming another investor (or even of outperforming oneself with respect to what one could have done). If we can show that all three goals are uniquely achieved by a given policy, we are on our way to making an objective case for utility-independent optimality of the stated portfolio.

An objection to P1 and P2 and, by extension, to $\underline{b}^*$, held by Samuelson (1967, 1969) and others, is that not all investors are interested in long term goals. Samuelson (1969, p. 245) writes, "Our analysis enables us to dispel a fallacy that has been borrowed into portfolio theory from information theory of the Shannon type. Associated with independent discoveries by J.B. Williams (1936), John Kelly (1956), and H.A. Latané (1959) is the notion that if one is investing for many periods, the proper behavior is to maximize the geometric mean of return rather than the arithmetic mean. I believe this to be incorrect (except in the Bernoulli logarithmic case where it happens to be correct for reasons quite distinct from the Williams-Kelly-Latané reasoning) . . . . It is a mistake to think that, just because a w^{**} decision ends up with almost-certain probability to be better than a w^* decision, this implies that w^{**} must yield a better expected value of utility."

Another possible objection is that $\underline{b}^*$ may be too conservative, since it optimizes a concave (risk averse) function of the return S. One interpretation of "too conservative" could be that $\underline{b}^*$ will be outperformed (i.e., $\underline{b}'\underline{X} > \underline{b}^{*\prime}\underline{X}$) with high probability by a more ambitious policy $\underline{b}$. Alternatively, too conservative might mean that with substantial probability $\underline{b}^*$ will be outperformed by a large factor (i.e., $\underline{b}'\underline{X} > c\underline{b}^{*\prime}\underline{X}$, for some constant $c > 1$) by a more risky policy $\underline{b}$. Thus a reasonable goal for an individual investor or a mutual fund would be good short term competitive performance.

With the above objections in mind, we are led to the analysis of one-stage investments. Consider the two-person zero-sum game in which two investors seek portfolio policies that are competitively best in the sense that at least half the time one achieves more capital than one's opponent. Surprisingly, the game theoretic optimal strategy will be shown to be $U\underline{b}^*$ where U is an independent uniform $[0, 2]$ random variable, and $\underline{b}^*$ is the same log optimal policy as before. Furthermore, among non-randomized strategies, $\underline{b}^*$ is shown to be competitively best in the sense that it will not be beaten by very much very often. Thus the alleged conservatism of $\underline{b}^*$ must be established on other grounds; and the short term value of $U\underline{b}^*$ is established competitively.

In the next section, we shall argue for the naturalness of the random variable U in the competitive investment game. Theorem 1, establishing $U\underline{b}^*$ as the solution of the game, will be proved in §3.

2. A game-theoretic digression.

Before proceeding, we establish the necessity for randomization in the competitive investment game.

Suppose 2 players each have 1 unit of capital. Their competitive positions are equal.

However, let us now suppose that player 2 has available to him any fair gamble ($EX = 1$, $X \geq 0$). By selecting the distribution of the gamble X judiciously, he can beat player 1 with probability $1 - \epsilon$. Simply let $P(X = 1/(1-\epsilon)) = 1 - \epsilon$, $P(X = 0) = \epsilon$. Then $P(X > 1) = 1 - \epsilon$. Therefore, player 1 must protect himself by randomizing his capital. This is a purely game theoretic maneuver and has nothing to do with maximizing investment return.

We now solve the following two-person zero-sum game. Let players 1 and 2 choose d.f.'s F and G, respectively, $\int x\,dF = \int y\,dG = 1$, $F(0^-) = G(0^-) = 0$. Assume $X \sim F$ and $Y \sim G$ are independently drawn. The freedom of choice we allow in the choice of F and G makes physical sense, since any capital distribution $F(x)$, $\int x\,dF(x) = 1$, $F(0^-) = 0$, is achievable from initial capital 1 by a sequential gambling scheme on fair coin tosses (Cover (1974)). The payoff to player 1 is

$$P(X \geq Y) = \int G\,dF. \tag{2}$$

LEMMA. *The value of this game is* $\frac{1}{2}$, *and the unique optimal strategies are*

$$F^*(t) = G^*(t) = \begin{cases} t/2, & 0 \leq t \leq 2, \\ 1, & t \geq 2. \end{cases} \tag{3}$$

PROOF. For F^* and for any G,

$$\begin{aligned} P(Y \geq X) = \int F^*\,dG &= \int_0^\infty \min\{t/2, 1\}\,dG(t) \\ &\leq \tfrac{1}{2} \int_0^\infty t\,dG(t) = \tfrac{1}{2}. \end{aligned} \tag{4}$$

Thus F^* achieves $\frac{1}{2}$ against any G.

Uniqueness of the optimal distribution $F^*(x)$ is proved by assuming $P(Y \geq X) \leq \frac{1}{2}$ for (i) Y uniform $[0, 2]$, (ii) Y a two point distribution at 0 and a point $c \in [1, 2]$, and (iii) Y a two point distribution at $c \in [0, 1]$ and 2. Then (i)$\Rightarrow F^*(2) = 1$; (ii)$\Rightarrow F^*(c) \leq c/2$, $1 \leq c \leq 2$; and (iii)$\Rightarrow F^*(c) \leq c/2$, $0 \leq c \leq 1$. Since $\int t\,dF^*(t) = 1$, we see $F^*(t) = t/2$, $0 \leq t \leq 2$. The proof of the uniqueness of G^* follows by symmetry.

We see that a gambler must exchange his unit capital for a r.v. U uniformly distributed on $[0, 2]$ in order to protect himself. We mention parenthetically that one way to achieve this on a sequence of fair coin flips is to divide the one unit of initial capital into piles of size $(\frac{1}{2})^i$, $i = 1, 2, \ldots$, then bet the i^{th} pile on the outcome of the i^{th} coin flip. Letting $\omega_1, \omega_2, \ldots$ be i.i.d. Bernoulli $(1/2)$ r.v.'s, we have the return

$$S = \sum_{i=1}^{\infty} 2\omega_i 2^{-i} = \omega_1.\omega_2\omega_3\omega_4 \ldots$$

in binary, which is clearly uniformly distributed on $[0, 2]$.

3. The competitive investment game. Let $\mathfrak{B}$ be the set of all r.v.'s $\underline{B} = (B_1, B_2, \ldots, B_m)'$, $\underline{B} \geq 0$, a.e., $E\sum_{i=1}^m B_i = 1$. Note that the random investment policy $\underline{B}$ can be achieved by first exchanging the 1 unit initial capital for a fair random return W drawn according to the distribution of $\sum_{i=1}^m B_i$. Observe that $W \geq 0$, $EW = 1$. Then W is distributed across the stocks according to the conditional joint distribution of $(B_1, B_2, \ldots, B_m)$ given $\sum_{i=1}^m B_i = W$. The latter distribution can be performed on paper. Happily, the allowed conditional randomization is not necessary in the game theoretic optimal policy in Theorem 1 below.

Let the investment vector $\underline{X} = (X_1, X_2, \ldots, X_m)'$ be a r.v. with known distribution function $F(\underline{x})$. We assume that $\underline{X} \geq 0$, a.e. To eliminate degeneracy, we also assume that

$$-\infty < \sup_{\underline{b}} E \ln \underline{b}' \underline{X} < \infty.$$

Consider the two-person zero-sum game in which players 1 and 2 choose $\underline{B}^{(1)} \in \mathfrak{B}, \underline{B}^{(2)} \in \mathfrak{B}$, and player 1 receives payoff

$$P\{\underline{B}^{(1)\prime}\underline{X} \geqslant \underline{B}^{(2)\prime}\underline{X}\}. \tag{5}$$

It is assumed that $\underline{B}^{(1)}$, $\underline{B}^{(2)}$ and $\underline{X}$ are jointly independent.

THEOREM 1. *The solution for the competitive investment game is $\underline{B}^* = U\underline{b}^*$, where U is unif. on* [0, 2], *independent of $\underline{X}$, and $\underline{b}^*$ maximizes $E \ln \underline{b}'\underline{X}$. The value of the game is $\frac{1}{2}$.*

PROOF. The Kuhn Tucker Theorem (1951) implies that the $\underline{b}^*$ maximizing $E \ln \sum_{i=1}^{m} b_i X_i$ subject to the constraint $\sum b_i = 1$, $b_i \geqslant 0$, satisfies

$$E \frac{X_i}{\sum b_j^* X_j} \begin{cases} = \lambda, & b_i^* > 0, \\ \leqslant \lambda, & b_i^* = 0, \; i = 1, 2, \ldots, m, \end{cases} \tag{6}$$

where λ is chosen so that $\sum b_i^* = 1$. But we see $\lambda = 1$, since

$$\begin{aligned} \lambda = \sum b_i^* \lambda &= \sum b_i^* E X_i / \left(\sum b_j^* X_j\right) \\ &= E\left(\sum b_i^* X_i\right) / \left(\sum b_j^* X_j\right) = 1. \end{aligned} \tag{7}$$

We now investigate the payoff of $\underline{B}^* = U\underline{b}^*$ against any other investment policy $\underline{B} \in \mathfrak{B}$:

$$\begin{aligned} P\{\underline{B}'\underline{X} \geqslant \underline{B}^{*\prime}\underline{X}\} &= P\{\underline{B}'\underline{X} \geqslant U\underline{b}^{*\prime}\underline{X}\} \\ &= P\{U \leqslant (\underline{B}'\underline{X})/(b^{*\prime}\underline{X})\} \leqslant 1/2 E((\underline{B}'\underline{X})/(\underline{b}^{*\prime}\underline{X})) \\ &= 1/2 \sum_{i=1}^{m} E B_i E(X_i / (\sum b_j^* X_j)) \leqslant 1/2 \sum E B_i \lambda \\ &= \lambda / 2 E \sum B_i = \lambda/2 = 1/2. \end{aligned} \tag{8}$$

Thus $\underline{B}^* = U\underline{b}^*$ achieves the value of the game against any $\underline{B}$, and the proof is complete.

The above strategy $\underline{B}^*$ can be implemented by first exchanging the 1 unit initial capital for the fair gamble U, uniformly distributed over [0, 2], then distributing U on the investments according to the solution $\underline{b}^*$ maximizing $E \ln \underline{b}'\underline{X}$.

This result can be generalized to show that $\underline{B}^* = U\underline{b}^*$ will not be beaten by very much very often: 0

COROLLARY 1. $P\{\underline{B}'\underline{X} \geqslant cU\underline{b}^{*\prime}\underline{X}\} \leqslant \frac{1}{2}c$, *for all* $\underline{B} \in \mathfrak{B}$, $c > 0$.

PROOF. $P\{cU \leqslant \underline{B}'\underline{X}/\underline{b}^{*\prime}\underline{X}\} \leqslant (\frac{1}{2}c) E(\underline{B}'\underline{X}/\underline{b}^{*\prime}\underline{X})$, and the proof proceeds as in Theorem 1.

Dropping the randomization U increases this probability by at most a factor of 2:

COROLLARY 2.

$$P(\underline{B}'\underline{X} \geqslant c\underline{b}^{*\prime}\underline{X}) \leqslant 1/c, \quad \textit{for all } \underline{B} \in \mathfrak{B},\ c > 0. \tag{9}$$

PROOF. By Markov's inequality and (8),

$$P(\underline{B}'\underline{X} \geqslant c\underline{b}^{*\prime}\underline{X}) \leqslant (1/c) E(\underline{B}'\underline{X}/\underline{b}^{*\prime}\underline{X}) \leqslant 1/c.$$

REMARK 1. This is the best that can be attained by any nonrandomized strategy, as can be seen from the discussion at the beginning of §2.

REMARK 2. Corollary 2 bears a strong resemblance to Markov's lemma, i.e., $Y \geqslant 0, EY = \mu \Rightarrow P(Y \geqslant c\mu) \leqslant 1/c, \forall c > 0$. This suggests that $\underline{b}^{*\prime}\underline{X}$ acts like the fixed amount of capital μ in Markov's lemma and that $\underline{b}^{*\prime}\underline{X}$ can be changed from a competitive standpoint only by *fair* randomization. Inequality (9) is true despite the fact that $E\underline{B}'\underline{X}$ may be greater than $E\underline{b}^{*\prime}\underline{X}$.

4. Example: The St. Petersburg paradox. In the St. Petersburg paradox, a gambler pays an entry fee c. He receives in return a random amount of capital X, where $P(X = 2^i) = 2^{-i}$, $i = 1, 2, \dots, \infty$. Note that $EX = \infty$.

Suppose that a gambler has total initial capital S_0. He is allowed to receive 1 unit of St. Petersburg investment for each c units that he pays as an entry fee. Let him invest the amount bS_0, $0 \leqslant b \leqslant 1$, and retain $(1 - b)S_0$ in cash. Thus his return S is given by $S = S_0((1 - b) + (b/c)X)$. In the framework of the previous sections, the investment vector is $\underline{X} = (X_1, X_2)' = (1, X/c)'$.

Let $b^* \in [0, 1]$ maximize $E \ln S$. We calculate

$$\frac{dE\ln S}{db} = E\frac{-1 + X/c}{(1 - b) + (b/c)X} = \sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^i \frac{(2^i/c - 1)}{2^i b/c + (1 - b)}. \tag{10}$$

Letting $b = 1$, we see that $dE\ln S/db = 1 - (c/3)$, which is $\geqslant 0$ for $c \leqslant 3$. Thus $b^* = 1$, for $0 \leqslant c \leqslant 3$. For $c > 3$, the solution b^* to (10) tends monotonically to zero as the entry fee $c \to \infty$. Finally it can be seen that b^* and $\max E \ln S$ are always strictly positive.

Investing a proportion of capital b^* guarantees that

(1) The investor is acting in accordance with an investment policy maximizing $\liminf (1/n)\ln S_n$, regardless of whether or not the other investment opportunities are of the St. Petersburg form; and

(2) the investor investing Ub^* is competitively optimal in the St. Petersburg game.

Moreover, we see that all entry fees c are "fair." However, the proportion b^* of total capital invested varies as a function of c. Also, b^* is independent of the total initial capital S_0.

Finally, if the investment fee is low enough, i.e., $0 \leqslant c \leqslant 3$, then $b^* = 1$ and all of the capital is invested. This results in $S_n \sim S_0(4/c)^n$ in the sense that $(1/n)\log_2 S_n \to 2 - \log_2 c$, for $0 \leqslant c \leqslant 3$.

5. Conclusions. It should now be clear that the investment policy $\underline{b}^*$ achieving $\max E \ln \underline{b}'\underline{X}$ has good short run as well as good long run properties. In addition, $\underline{b}^*$ is admissible in the sense that no other policy $\underline{b}$ stochastically dominates $\underline{b}^*$.

We wish to comment on the use of $U\underline{b}^*$ (as opposed to $\underline{b}^*$ alone) in practice. We have seen in §2 that the use of U is a purely game theoretic protection against competition and has nothing to do with increasing a player's capital. Thus we feel that $\underline{b}^*$ alone is sufficient to achieve all reasonable competitive investment goals, and we do not choose to advocate the additional randomization U.

Finally, it is tantalizing that $\underline{b}^*$ arises as the solution to such dissimilar problems as maximizing $\liminf(1/n)\ln S_n$ and maximizing $P\{\underline{B}_1'\underline{X} > \underline{B}_2'\underline{X}\}$. The underlying reason for this coincidence will be investigated.

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