Lecture 16

Quadratic Surd Expressions and Their Operations

Definitions

For an even positive integer n, by the notation $\sqrt[n]{a}$, where $a \geq 0$, we denote the non-negative real number x which satisfies the equation $x^n = a$. In particular, when $n = 2$, $\sqrt[2]{a}$ is called **square root of a**, and denoted by $\sqrt{a}$ usually.

For odd positive integer n and any real number a, by the notation $\sqrt[n]{a}$ we denote the real number x which satisfies the equation $x^n = a$.

An algebraic expression containing $\sqrt{a}$, where $a > 0$ is not a perfect square number, is called **quadratic surd expression**, like $1 - \sqrt{2}$, $\frac{1}{2-\sqrt{3}}$, etc.

Basic Operational Rules on $\sqrt{a}$

(I) $(\sqrt{a})^2 = a$, where $a \geq 0$.

(II) $\sqrt{a^2} = |a| = \begin{cases} a & \text{for } a > 0, \\ 0 & \text{for } a = 0, \\ -a & \text{for } a < 0. \end{cases}$

(III) $\sqrt{ab} = \sqrt{|a|} \cdot \sqrt{|b|}$ if $ab \geq 0$.

(IV) $\sqrt{\frac{a}{b}} = \frac{\sqrt{|a|}}{\sqrt{|b|}}$ if $ab \geq 0,\ b \neq 0$.

(V) $(\sqrt{a})^n = \sqrt{a^n}$ if $a \geq 0$.

(VI) $a\sqrt{c} + b\sqrt{c} = (a+b)\sqrt{c}$ if $c \geq 0$.

LECTURE NOTES ON MATHEMATICAL OLYMPIAD COURSES - For Junior Section (In 2 Volumes)

Rationalization of Denominators

(I) $\dfrac{1}{a\sqrt{b}+c\sqrt{d}} = \dfrac{a\sqrt{b}-c\sqrt{d}}{a^2b-c^2d}$, where a, b, c, d are rational numbers, $b, d \geq 0$ and $a^2b - c^2d \neq 0$.

(II) $\dfrac{1}{a\sqrt{b}-c\sqrt{d}} = \dfrac{a\sqrt{b}+c\sqrt{d}}{a^2b-c^2d}$, where a, b, c, d are rational numbers, $b, d \geq 0$ and $a^2b - c^2d \neq 0$.

In algebra, the expressions $A + B\sqrt{C}$ and $A - B\sqrt{C}$, where A, B, C are rational and $\sqrt{C}$ is irrational, are called **conjugate surd expressions**.

The investigation of surd forms is necessary and very important in algebra, since surd forms and irrational number have close relation. For example, all the numbers of the form $\sqrt{n}, n \in \mathbb{N}$ are irrational if the positive integer n is not a perfect square. In other words, the investigation of surd form expressions is the investigation of irrational numbers and their operations essentially.

Examples

Example 1. Simplify the expression $\dfrac{a}{a-2b}\sqrt{\dfrac{a^2-4ab+4b^2}{a(2b-a)}}$.

Solution Since $a - 2b \neq 0$, so

$$\frac{a^2-4ab+4b^2}{a(2b-a)} = \frac{(a-2b)^2}{a(2b-a)} > 0 \Rightarrow a(2b-a) > 0.$$

Therefore $\dfrac{a}{a-2b} < 0$ and $\dfrac{a}{2b-a} > 0$, so

$$\begin{aligned}
\frac{a}{a-2b}\sqrt{\frac{a^2-4ab+4b^2}{a}} &= -\left(\frac{a}{2b-a}\right)\sqrt{\frac{(2b-a)^2}{a(2b-a)}} \\
&= -\sqrt{\frac{a^2}{(2b-a)^2}\cdot\frac{(2b-a)}{a}} = -\sqrt{\frac{a}{2b-a}}.
\end{aligned}$$

Example 2. Given that $c > 1$ and

$$x = \frac{\sqrt{c+2}-\sqrt{c+1}}{\sqrt{c}-\sqrt{c-1}},\ y = \frac{\sqrt{c+2}-\sqrt{c+1}}{\sqrt{c+1}-\sqrt{c}},\ z = \frac{\sqrt{c}-\sqrt{c-1}}{\sqrt{c+2}-\sqrt{c+1}},$$

arrange x, y, z in ascending order.

Solution From

$$\begin{aligned} x &= \frac{\sqrt{c+2}-\sqrt{c+1}}{\sqrt{c}-\sqrt{c-1}} = \frac{[(\sqrt{c+2})^2-(\sqrt{c+1})^2](\sqrt{c}+\sqrt{c-1})}{(\sqrt{c+2}+\sqrt{c+1})[(\sqrt{c})^2-(\sqrt{c-1})^2]} \\ &= \frac{\sqrt{c}+\sqrt{c-1}}{\sqrt{c+2}+\sqrt{c+1}}, \\ y &= \frac{\sqrt{c+2}-\sqrt{c+1}}{\sqrt{c+1}-\sqrt{c}} = \frac{[(\sqrt{c+2})^2-(\sqrt{c+1})^2](\sqrt{c+1}+\sqrt{c})}{(\sqrt{c+2}+\sqrt{c+1})[(\sqrt{c+1})^2-(\sqrt{c})^2]} \\ &= \frac{\sqrt{c+1}+\sqrt{c}}{\sqrt{c+2}+\sqrt{c+1}}, \end{aligned}$$

it follows that $x < y$. Further,

$$\begin{aligned} z &= \frac{\sqrt{c}-\sqrt{c-1}}{\sqrt{c+2}-\sqrt{c+1}} = \frac{[(\sqrt{c})^2-(\sqrt{c-1})^2](\sqrt{c+2}+\sqrt{c+1})}{(\sqrt{c}+\sqrt{c-1})[(\sqrt{c+2})^2-(\sqrt{c+1})^2]} \\ &= \frac{\sqrt{c+2}+\sqrt{c+1}}{\sqrt{c}+\sqrt{c-1}}. \end{aligned}$$

Since $\sqrt{c}+\sqrt{c-1} < \sqrt{c+1}+\sqrt{c} < \sqrt{c+2}+\sqrt{c+1}$, thus $x < y < z$.

Example 3. (SSSMO/2003) Let x be a real number, and let

$$A = \frac{-1+3x}{1+x} - \frac{\sqrt{|x|-2}+\sqrt{2-|x|}}{|2-x|}.$$

Prove that A is an integer, and find the unit digit of A^{2003}.

Solution Since $|x|-2 \geq 0$ and $2-|x| \geq 0$ simultaneously implies $|x| = 2$, so $x = \pm 2$ only. Since the denominator $|x-2| \neq 0$, i.e. $x \neq 2$, so $x = -2$. Therefore $A = 7$. Then

$$7^{2003} = (7^4)^{500} \cdot 7^3 \equiv 243 \equiv 3 \pmod{10},$$

therefore the units digit of A is 3.

Example 4. Given $x = \frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}$, $y = \frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}$, find the value of $x^4 + y^4 + (x+y)^4$.

Solution Here an important technique is to express to $x^4 + y^4 + (x+y)^4$ by $x+y$ and xy instead of using the complicated expression of x and y. From

$$x = \frac{1}{7-3}(\sqrt{7}+\sqrt{3})^2 = \frac{1}{4}(10+2\sqrt{21}) = \frac{1}{2}(5+\sqrt{21}),$$
$$y = \frac{1}{7-3}(\sqrt{7}-\sqrt{3})^2 = \frac{1}{4}(10-2\sqrt{21}) = \frac{1}{2}(5-\sqrt{21}),$$

it follows that $x+y=5$ and $xy=1$. Therefore

$$\begin{aligned}&x^4+y^4+(x+y)^4\\&=(x^2+y^2)^2-2x^2y^2+5^4=[(x+y)^2-2(xy)]^2-2(xy)^2+625\\&=23^2-2+625=527+625=1152.\end{aligned}$$

Example 5. Simplify the expression $\dfrac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$ by rationalizing the denominator.

Solution

$$\begin{aligned}&\frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}=1-\frac{2\sqrt{5}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}\\&=1-\frac{2\sqrt{5}(\sqrt{2}+\sqrt{3}-\sqrt{5})}{(\sqrt{2}+\sqrt{3})^2-5}=1-\frac{2(\sqrt{10}+\sqrt{15}-5)}{2\sqrt{2}\cdot\sqrt{3}}\\&=1-\frac{\sqrt{10}+\sqrt{15}-5}{\sqrt{6}}=1-\frac{\sqrt{60}+\sqrt{90}-5\sqrt{6}}{6}\\&=1-\frac{\sqrt{15}}{3}-\frac{\sqrt{10}}{2}+\frac{5\sqrt{6}}{6}.\end{aligned}$$

Example 6. Simplify $S=\sqrt{x^2+2x+1}-\sqrt{x^2+4x+4}+\sqrt{x^2-6x+9}$.

Solution From $S=\sqrt{x^2+2x+1}-\sqrt{x^2+4x+4}+\sqrt{x^2-6x+9}=|x+1|-|x+2|+|x-3|$, there are four possible cases as follows:

(i) When $x\le -2$, then $S=-(x+1)+(x+2)-(x-3)=-x+4$.

(ii) When $-2<x\le -1$, then $S=-(x+1)-(x+2)-(x-3)=-3x$.

(iii) When $-1<x\le 3$, then $S=(x+1)-(x+2)-(x-3)=-x+2$.

(iv) When $3<x$, then $S=(x+1)-(x+2)+(x-3)=x-4$.

Example 7. (SSSMO/2002/Q12) Evaluate

$(\sqrt{10}+\sqrt{11}+\sqrt{12})(\sqrt{10}+\sqrt{11}-\sqrt{12})(\sqrt{10}-\sqrt{11}+\sqrt{12})(\sqrt{10}-\sqrt{11}-\sqrt{12})$.

Solution Let $A=(\sqrt{10}+\sqrt{11}+\sqrt{12})(\sqrt{10}+\sqrt{11}-\sqrt{12})(\sqrt{10}-\sqrt{11}+\sqrt{12})(\sqrt{10}-\sqrt{11}-\sqrt{12})$. Then

$$\begin{aligned}A&=[(\sqrt{10}+\sqrt{11})^2-(\sqrt{12})^2][(\sqrt{10}-\sqrt{11})^2-(\sqrt{12})^2]\\&=(9+2\sqrt{10}\cdot\sqrt{11})(9-2\sqrt{10}\cdot\sqrt{11})=81-440=-359.\end{aligned}$$

Example 8. Evaluate $N = \dfrac{\sqrt{15}+\sqrt{35}+\sqrt{21}+5}{\sqrt{3}+2\sqrt{5}+\sqrt{7}}$.

Solution $N = \dfrac{(\sqrt{15}+\sqrt{21})+(\sqrt{35}+5)}{(\sqrt{3}+\sqrt{5})+(\sqrt{5}+\sqrt{7})} = \dfrac{(\sqrt{3}+\sqrt{5})(\sqrt{5}+\sqrt{7})}{(\sqrt{3}+\sqrt{5})+(\sqrt{5}+\sqrt{7})}$

$$\begin{aligned}\Longrightarrow \frac{1}{N} &= \frac{(\sqrt{3}+\sqrt{5})+(\sqrt{5}+\sqrt{7})}{(\sqrt{3}+\sqrt{5})(\sqrt{5}+\sqrt{7})} = \frac{1}{\sqrt{5}+\sqrt{7}} + \frac{1}{\sqrt{3}+\sqrt{5}} \\ &= \frac{1}{2}(\sqrt{7}-\sqrt{5}) + \frac{1}{2}(\sqrt{5}-\sqrt{3}) = \frac{1}{2}(\sqrt{7}-\sqrt{3}).\end{aligned}$$

$$\therefore N = \frac{2}{\sqrt{7}-\sqrt{3}} = \frac{2(\sqrt{7}+\sqrt{3})}{4} = \frac{\sqrt{7}+\sqrt{3}}{2}.$$

Example 9. (Training question for National Team of Canada) Simplify
$P = \dfrac{1}{2\sqrt{1}+\sqrt{2}} + \dfrac{1}{3\sqrt{2}+2\sqrt{3}} + \cdots + \dfrac{1}{100\sqrt{99}+99\sqrt{100}}$.

Solution For each positive integer n,

$$\begin{aligned}\frac{1}{(n+1)\sqrt{n}+n\sqrt{n+1}} &= \frac{1}{\sqrt{n(n+1)}(\sqrt{n+1}+\sqrt{n})} = \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n(n+1)}} \\ &= \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}},\end{aligned}$$

hence

$$\begin{aligned}P &= \left(1-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right) + \cdots + \left(\frac{1}{\sqrt{99}}-\frac{1}{\sqrt{100}}\right) \\ &= 1-\frac{1}{\sqrt{100}} = 1-\frac{1}{10} = \frac{9}{10}.\end{aligned}$$

Testing Questions (A)

1. If $x < 2$, then $|\sqrt{(x-2)^2}+\sqrt{(3-x)^2}|$ is equal to
   (A) $5-2x$ (B) $2x-5$ (C) 2 (D) 3.

2. Simplify $\dfrac{1+\sqrt{2}+\sqrt{3}}{1-\sqrt{2}+\sqrt{3}}$ by rationalizing the denominator.

3. Simplify the expression $\dfrac{x^2-4x+3+(x+1)\sqrt{x^2-9}}{x^2+4x+3+(x-1)\sqrt{x^2-9}}$, where $x > 3$.

4. Simplify $\dfrac{2+3\sqrt{3}+\sqrt{5}}{(2+\sqrt{3})(2\sqrt{3}+\sqrt{5})}$.

5. Evaluate

$$(\sqrt{5}+\sqrt{6}+\sqrt{7})(\sqrt{5}+\sqrt{6}-\sqrt{7})(\sqrt{5}-\sqrt{6}+\sqrt{7})(-\sqrt{5}+\sqrt{6}+\sqrt{7}).$$

6. (SSSMO(J)/1999) Suppose that $a=\sqrt{6}-2$ and $b=2\sqrt{2}-\sqrt{6}$. Then
   (A) $a>b$, (B) $a=b$, (C) $a<b$, (D) $b=\sqrt{2}a$, (E) $a=\sqrt{2}b$.

7. Arrange the three values $a=\sqrt{27}-\sqrt{26}, b=\sqrt{28}-\sqrt{27}, c=\sqrt{29}-\sqrt{28}$ in ascending order.

8. The number of integers x which satisfies the inequality $\dfrac{3}{1+\sqrt{3}}<x<\dfrac{3}{\sqrt{5}-\sqrt{3}}$ is
   (A) 2, (B) 3, (C) 4, (D) 5, (E) 6.

9. Calculate the value of $\dfrac{1}{1-\sqrt[4]{5}}+\dfrac{1}{1+\sqrt[4]{5}}+\dfrac{2}{1+\sqrt{5}}$.

10. Given $a>b>c>d>0$, and $U=\sqrt{ab}+\sqrt{cd},\ V=\sqrt{ac}+\sqrt{bd},\ W=\sqrt{ad}+\sqrt{bc}$. Use "$<$" to connect U, V, W.

Testing Questions (B)

1. (CHINA/1993) Find the units digit of the expression

$$x=\left(\frac{-2a}{4+a}-\frac{\sqrt{|a|-3}+\sqrt{3-|a|}}{3-a}\right)^{1993}.$$

2. (CHNMOL/1993) Simplify $\sqrt[3]{3}\left(\sqrt[3]{\dfrac{4}{9}}-\sqrt[3]{\dfrac{2}{9}}+\sqrt[3]{\dfrac{1}{9}}\right)^{-1}$.

3. (CHINA/1998) Evaluate $\sqrt{\dfrac{1998\times1999\times2000\times2001+1}{4}}$.

4. Given $a=\sqrt[3]{4}+\sqrt[3]{2}+1$, find the value of $\dfrac{3}{a}+\dfrac{3}{a^2}+\dfrac{1}{a^3}$.

5. Given that the decimal part of $M=(\sqrt{13}+\sqrt{11})^6$ is P, find the value of $M(1-P)$.