Chapter 1

Basic Euclidean Geometry

This chapter is not intended to be a complete survey of basic Euclidean Geometry, but rather a review for those who have previously taken a geometry course. For a definitive account, see Euclid's *Elements.*

1.1 Triangles

A triangle is a (plane) figure bounded by three line segments. The most important result about triangles is that the sum of the angles of a triangle has measure equal to two right angles (or $180°$). This can be deduced from Fig. 1.1, where the line DAE is parallel to the line segment BC.

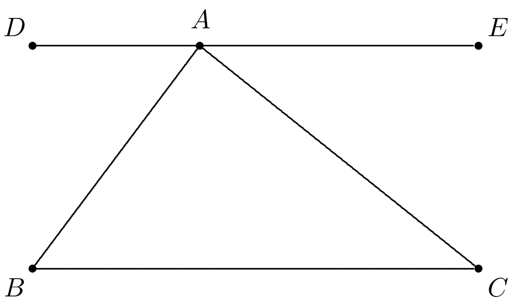


Fig. 1.1

If one of the angles of a triangle is a right angle, we call the the triangle a *right triangle*. For such triangles, we have *Pythagoras's Theorem* which states that the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides.

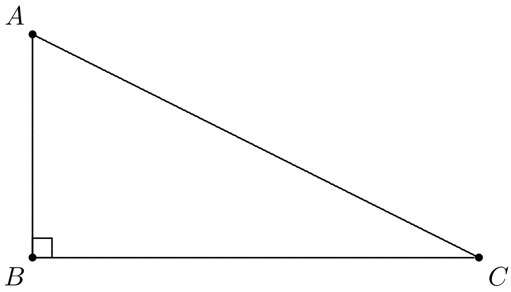


Fig. 1.2

In Fig. 1.2, we have that $AB^2 + BC^2 = CA^2$.

1.2 Similar Triangles

We start with two triangles ABC and $A'B'C'$. The definition of two triangles being similar can be given in one of two ways:
We say that the triangles ABC and $A'B'C'$ are *similar* if either

(1) the *sides* of the triangle are in proportion; that is, if

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'} \quad ;$$

or
(2) the *angles* of the triangles are equal; that is, if

$$\angle ABC = \angle A'B'C' \quad ,$$

$$\angle BCA = \angle B'C'A' \quad ,$$

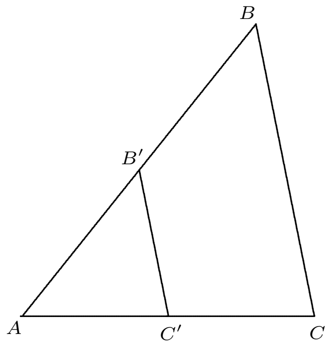
$$\text{and} \quad \angle CAB = \angle C'A'B' \quad .$$

It is easy to see that the two definitions are equivalent. Thus, showing either relationship gives the other.
We ask the question: what is sufficient to show that two triangles are similar? Do we have to show that all three angles are equal; do we have to show that all three ratios are equal?
In the case of *angles*, it is sufficient to show that *two* of the angles are equal, since this will automatically give that the third angles are equal. (Why?)

However, it is not sufficient to show that two of the ratios of the sides are equal. To see this, simply consider two isosceles triangles, ABC and $A'B'C'$ with $AB = BC$ and $A'B' = B'C'$.
Then, $\frac{AB}{A'B'} = \frac{BC}{B'C'}$, but $\triangle ABC$ and $\triangle A'B'C'$ will only be similar if $\angle ABC = \angle A'B'C'$.

Problem 1.1. *Let ABC be a triangle. Points B' and C' are chosen on AB and AC, respectively, such that $\frac{AB'}{B'B} = \frac{AC'}{C'C}$. Prove that triangles ABC and $AB'C'$ are similar.*

1.3 Congruent Triangles



Similar triangles can be of different sizes. (We see that in the diagram above.) If the ratio of the sides has value 1, then we say that the triangles are *congruent*. This means that the two triangles are *identical* in every way, although their orientation and position may differ.
There are several conditions that are sufficient for showing that two triangles are congruent. They are

1. Three sides equal **SSS**
2. Two sides and the included angle **SAS**
3. Two angles and the corresponding side **ASA**

In the third case, we may as well assume that the side is common to both angles; hence, the notation **ASA**.
We do not insist on the same orientation for congruence.

1.4 Quadrilaterals

A quadrilateral is a plane figure bounded by four line segments. In this section we look briefly at some particular quadrilaterals.

(1) **Trapezoid**

A trapezoid is a quadrilateral with one pair of (opposite) sides parallel. a *symmetric* trapezoid is a trapezoid where the non-parallel sides are equally inclined to the other sides. See Fig. 1.3.

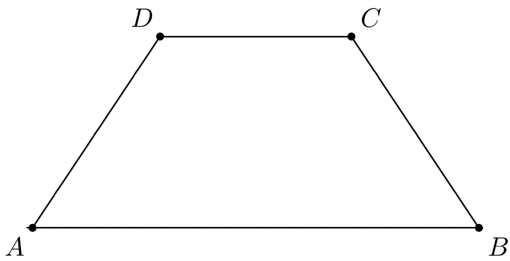


Fig. 1.3

In this example, we have both that $\angle ABC = \angle BAD$ and that $\angle ADC = \angle BCD$. We also see that the sum of the opposite angle of a symmetric trapezoid is equal to two right angles (π).

(2) **Parallelogram**

A parallelogram is a quadrilateral where the opposite sides are parallel. This gives the result that opposite angle are equal. See Fig. 1.4.

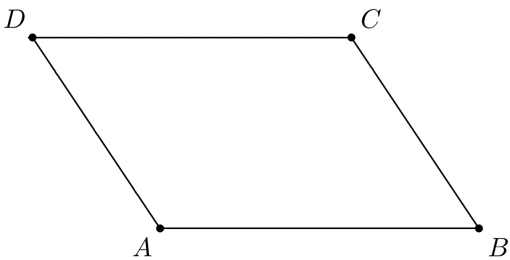


Fig. 1.4

(3) **Rhombus**
A rhombus is a parallelogram where all sides are equal.

(4) **Square**
A square is a particular quadrilateral where all sides are equal and all angles are equal to one right angle ($\pi/2$).
Alternatively, a square is a rhombus where an angle is a right angle (which gives all angles as right angles).

1.5 Polygons

A polygon is a plane figure bounded by line segments. Some special names are:

Sides	Name
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
n	Polygon

A basic result concerns the sum of the interior angles of a polygon. The value, $(2n-4)$ right angles, or $(n-2)\pi$, can be seen easily by triangulation: see Fig. 1.5.

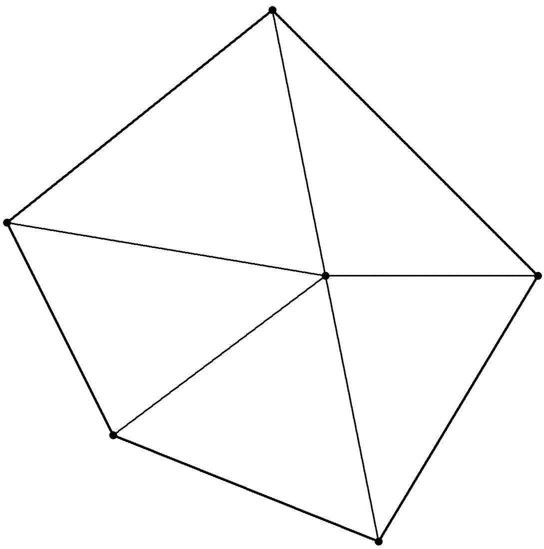


Fig. 1.5

Here we have n triangles. The total measure of the angles is then $n \times \pi$. From this, we must subtract the sum of the angles at the interior point, that is, 2π.
We shall be interested in a later chapter in *regular* polygons, and which can be constructed using straight edge and compasses.

1.6 Circles and Angles

In this section, we will recall a number of important results concerning angles, which arise naturally in the study of circles. The first concerns the size of an angle in a semicircle.

Theorem 1.1. *If AB is the diameter of a circle and C is any point on the circle distinct from A and B, then $\angle ACB = \pi/2$ (in radian measure).*

Proof.
Let O be the centre of the circle. to help, join OC, AC and CB. See Fig. 1.6.

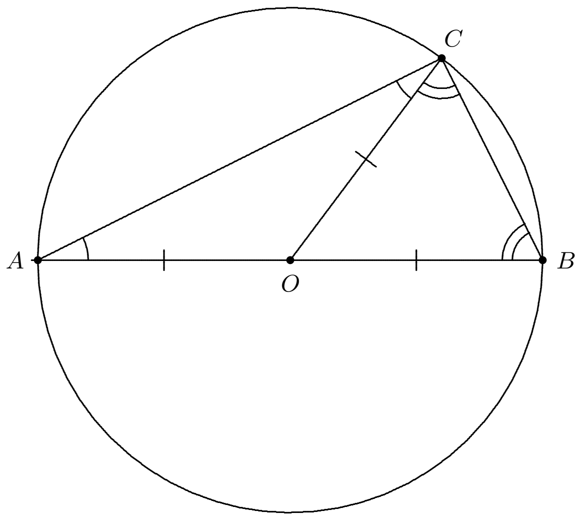


Fig. 1.6

Since $OA = OC$, we have $\angle OAC = \angle OCA$. Similarly, since $OC = OB$, we have $\angle OBC = \angle OCB$.
But we know that $\angle OAC + \angle OCA + \angle OCB + \angle OBC = \pi$ (sum of the interior angles of a triangle).

Hence, $2\angle OCA + 2\angle OCB = \pi$, giving $\angle OCA + \angle OCB = \pi/2$. Therefore, $\angle ACB = \pi/2$ as desired.
Next, we consider the relationship between the angle subtended by an arc to a point on the circumference with the angle subtended by the same arc at the centre.

Theorem 1.2. *Let AB be a chord of a circle (with centre O) which is not a diameter, and let C be any point on the circle distinct from A and B.*

(1) If C is on the same side of AC as O, then $\angle AOB = 2\angle ACB$.
(2) If C is on the opposite side of AB from O, then $\angle AOB = 2\pi - 2\angle ACB$.

Proof.
Let C be on the same side of AB as O. To help, join AO, OB, AC and CB. First, assume that AC does not intersect the radius OB, and that BC does not intersect the radius OA as illustrated in Fig. 1.7.

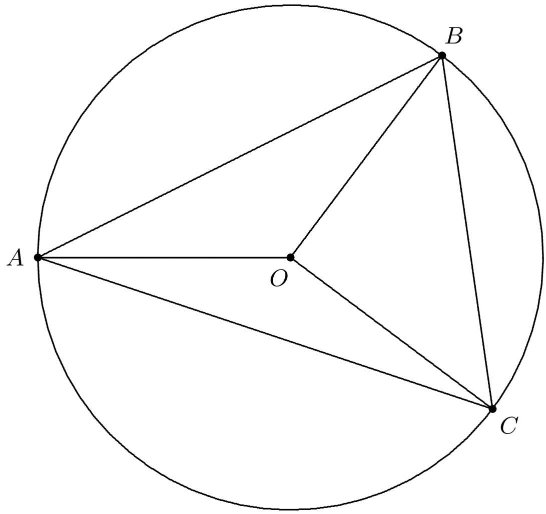


Fig. 1.7

Here, we have $\angle AOB = \pi - (\angle OAB + \angle OBA)$.
We also have $\angle OAC + \angle OCA + \angle OBC + \angle OCB = \pi - (\angle OAB + \angle OBA)$.
Hence, $\angle AOB = \angle OAC + \angle OCA + \angle OCB + \angle OBC$.
Since $OA = OC$, we have $\angle OAC = \angle OCA$, and since $OB = OC$, we have $\angle OBC = \angle OCB$.
Thus, $\angle OAB = 2(\angle OCA + \angle OCB) = 2\angle ACB$.

Next, assume that AC intersects the radius OB as illustrated below (the case where BC intersects OA is proved similarly, and is left as an exercise): see Fig. 1.8.

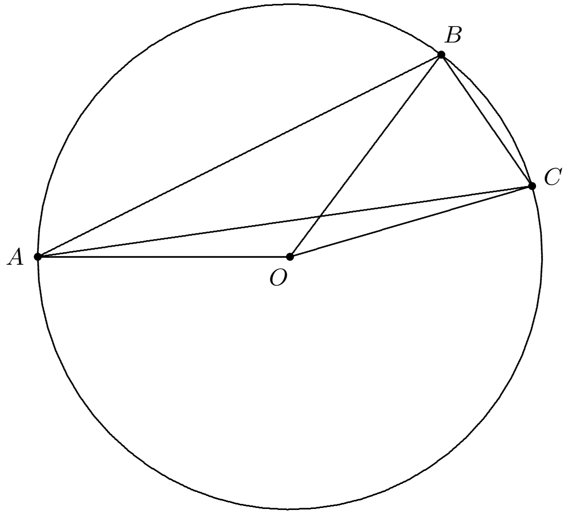


Fig. 1.8

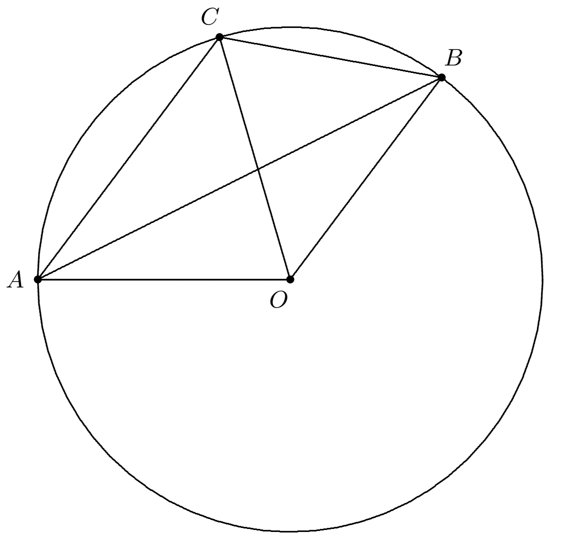


Fig. 1.9

Again, we have $\angle AOB = \pi - (\angle OAB + \angle OBA)$.

But this time, we have

$$(\angle OAB - \angle OAC) + (\angle OCB - \angle OCA) + (\angle OBA + \angle OBC) = \pi.$$

Thus, $\angle AOB = -\angle OAC + \angle OCB - \angle OCA + \angle OBC$.
As before, we have $\angle OAC = \angle OCA$ and $\angle OBC = \angle OCB$.
Hence, $\angle AOB = 2\angle OCB - 2\angle OCA = 2\angle ACB$.
Now, let C be on the opposite side of AB from the centre, and again, join AO, OB, AC, CB and OC. See Fig. 1.9.
Here, we have $\angle AOB = 2\pi - (\angle OAC + \angle OCA + \angle OCB + \angle OBC)$.
Since $OA = OC$, we have $\angle OAC = \angle OCA$, and since $OB = OC$, we have $\angle OBC = \angle OCB$.
Thus, $\angle AOB = 2\pi - 2\angle OCA - 2\angle OCB = 2\pi - 2\angle ACB$ as desired.
The next result, which follows immediately from theorem 1.2, will often be more important in applications.

Theorem 1.3. *If AB is a chord of a circle and C and D are two distinct points on the circle, distinct from A and B, both of which lie on the same side of AB, then $\angle ACB = \angle ADB$.*

Proof.
If AB is a diameter, then the result follows from theorem 1.1, since both $\angle ACB$ and $\angle ADB$ are right angles.

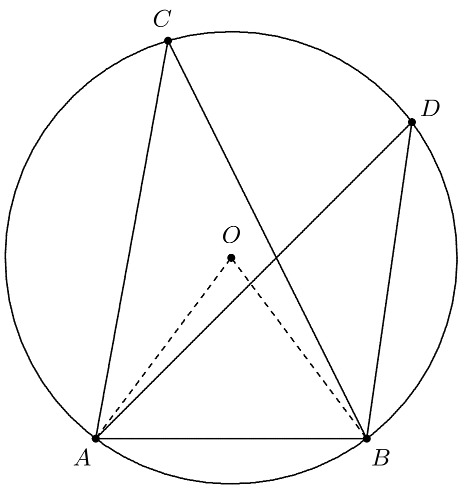


Fig. 1.10 The "Bow-Tie" Lemma

Assume, from now on, that AB is not a diameter. Let O be the centre of the circle.
If C and D are on the same side of AB as O, then theorem 1.2 (1) gives that $2\angle ACB = \angle AOB$ and $2\angle ADB = \angle AOB$, yielding that $\angle ACB = \angle ADB$.
If C and D are on the opposite side of AB from O, then theorem 1.2 (2) enables us also to conclude that $\angle ACB = \angle ADB$.
If C and D are on opposite sides of AB, we have the following:

Theorem 1.4. *If AB is a chord of a circle and C and D are two points on the circle, distinct for A and B, lying on opposite sides of AB, then $\angle ACB + \angle ADB = \pi$.*

Proof.
As in the previous result, if AB is a diameter, then
$$\angle ACB + \angle ADB = \pi/2 + \pi/2 = \pi.$$
Assume for the remainder of this proof that AB is not a diameter, and let O be the centre of the circle. Without loss of generality, assume that C and O lie on the same side of AB. Then, theorem 1.2 tells us that $\angle AOB = 2\angle ACB$ and that $\angle AOB = 2\pi - 2\angle ADB$.
Hence, $2\angle ACB = 2\pi - 2\angle ADB$, yielding that $\angle ACB + \angle ADB = \pi$.

Finally, we will make an interesting observation involving the angles between chords and tangents.

Theorem 1.5. *Let A, B and C be any three points on a circle, and let D be such that DA is tangent to the circle (at A) and that D is on the opposite side of line AB from C. Then $\angle DAB = \angle ACB$.*

Proof.
First note that if AB is a diameter of the circle, then $\angle ACB = \pi/2$ by theorem 1.1, and $\angle DAB = \pi/2$ since DA is a tangent. This, the result holds. Henceforth, we assume that AB is not a diameter.
Draw the diameter at A and call it AX. We consider two cases — either AX intersects the chord BC or it does not.
First, assume that AX intersects BC. Draw the chord BX. See Fig. 1.10. Note that $\angle ABX$ and $\angle DAX$ are both right angles. Also, by theorem 1.3, we have $\angle ACB = \angle AXB$.
Hence, $\angle DAB = \pi/2 - \angle BAX = \angle AXB = \angle ACB$, as desired.
Next, assume that AX does not intersect BC, and again, draw the chord BX. See Fig. 1.11.
As above, $\angle ABX$ and $\angle DAX$ are right angles.

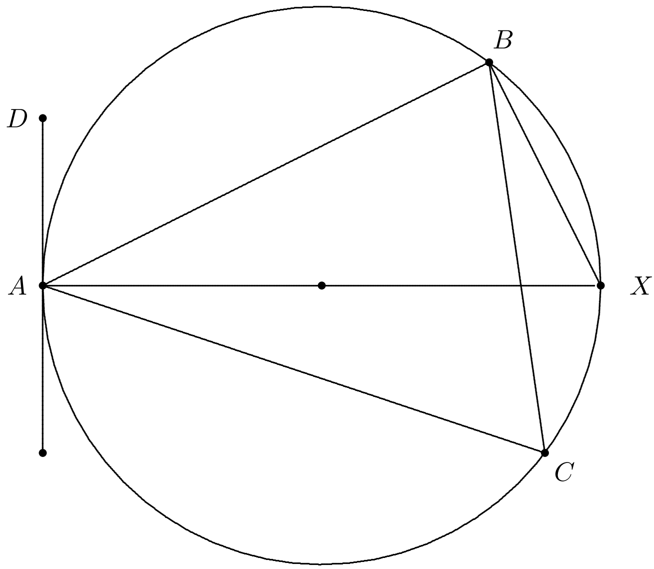


Fig. 1.11

In this case, $\angle ACB + \angle AXB = \pi$ by theorem 1.4. Hence,

$$\begin{aligned}\angle DAB = \pi/2 + \angle BAX \;&=\; \pi/2 + (\pi/2 - \angle AXB)\\ = \pi - (\pi - \angle ACB) \;&=\; \angle ACB.\end{aligned}$$

1.7 Cyclic Quadrilaterals

A quadrilateral whose vertices all lie on a circle is called a **cyclic quadrilateral**.

Now, any triangle has the property that a unique circle can be drawn through its vertices (for the demonstration, theorem 3.6 on page 55). Thus, to say that a quadrilateral is cyclic is really quite restrictive — in fact, we are demanding that D lie on the unique circle passing through A, B and C. Similarly, we could start with the unique circle through any other three of the four named points. Therefore, we should expect that cyclic quadrilaterals have some very particular properties.

One such property is clear — from the geometry of a circle, we see that each angle of a cyclic quadrilateral $ABCD$ must be less than π. See Fig. 1.12.

To facilitate terminology, we call any quadrilateral with this property **convex** — in fact, there is a more general definition of convex which applies

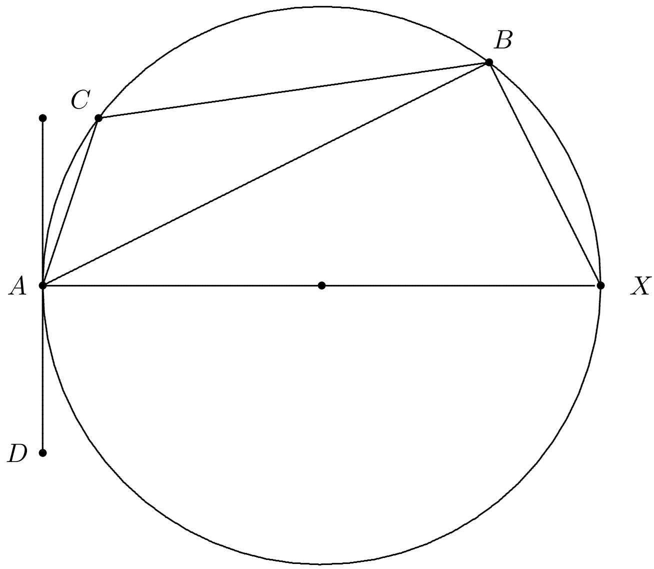


Fig. 1.12

to all polygons, but in the case of quadrilaterals, it is equivalent to ours.
Two other important properties of cyclic quadrilaterals follow immediately from other result proved earlier. In fact, theorems 1.3 and 1.4 can be re-stated as follows (we assume the vertices to be labelled in a clockwise order):

(1) If $ABCD$ is a cyclic quadrilateral, then $\angle ACB = \angle ADB$.
(2) If $ABCD$ is a cyclic quadrilateral, then $\angle ABC + \angle ADC = \pi$.

Note that the first of these results also has three other possible conclusions, depending on which of BC, CD or DA is chosen as the chord in theorem 1.3. For example, $\angle DBA = \angle DCA$. As an exercise, the reader should list all possibilities.
Similarly to the second result, we also have $\angle BCD + \angle BAD = \pi$. But this could also be deduced from the fact that the sum of the interior angles of a quadrilateral is 2π.
It is important for subsequent material to note that the converses of these two results hold.

Theorem 1.6. *Suppose that $ABCD$ is a convex quadrilateral such that $\angle ACB = \angle ADB$. Then quadrilateral $ABCD$ is a cyclic quadrilateral.*

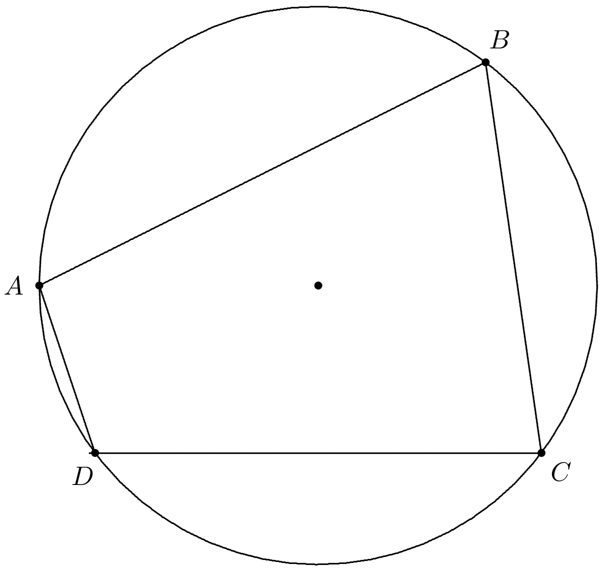


Fig. 1.13

Proof.
Draw the unique circle through A, B and C. We wish to prove that D lies on the circle. See Fig. 1.13.
Let E be the second point of intersection of the circle with BD (B being the first) and join AE. Since $ABCD$ is convex, the point E does exist and is on the same side of AC as is D. It follows that if $D \neq E$, then $\angle AEB \neq \angle ADB$. But, $ABCD$ is a cyclic quadrilateral, and thus, we have $\angle AEB = \angle ACB$. Since $\angle ACB = \angle ADB$ (as marked), this is a contradiction. Hence, $D = E$.

Theorem 1.7. *If $ABCD$ is a quadrilateral in which $\angle ABC + \angle ADC = \pi$, then $ABCD$ is cyclic.*

Proof.
Note that the given condition forces quadrilateral $ABCD$ to be convex. See Fig. 1.14.
As with theorem 1.6, draw the unique circle through A, B and C, and let E be the second point of intersection of the circle with BD. Join AE and AC. If $D \neq E$, then $\angle AEC \neq \angle ADC$. But $ABCE$ is cyclic, yielding that $\angle ABC + \angle AEC = \pi$. This implies that $\angle AEC = \angle ADC$, which is a contradiction.

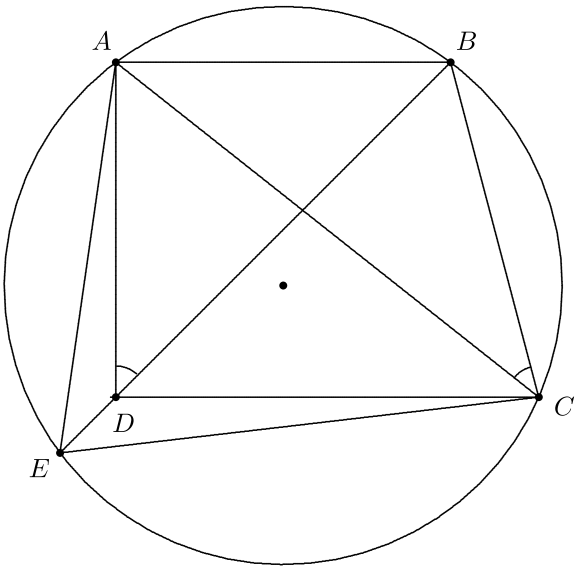


Fig. 1.14

1.8 Intersecting Chords

We consider a circle with two intersecting chords.
Suppose that AC and BD are two chords of a given circle which intersect at some point X inside the circle. See Fig. 1.15.
Here, we observe that theorem 1.3 implies that $\angle ABD = \angle ACD$, and thus, it follows that triangles ABX and DCX are similar. Hence, we have that $\frac{AX}{XD} = \frac{BX}{XC}$, or, equivalently,

$$AX \cdot XC \ = \ BX \cdot XD \ \ .$$

This result is known as the **Intersecting Chords Theorem**.
Of course, it can be immediately interpreted as a result about the diagonals of any cyclic quadrilateral.

Problem 1.2. *In equilateral triangle ABC of side length 2, suppose that M and N are the mid-points of AB and AC, respectively. The triangle is inscribed in a circle. The line segment MN is extended to meet the circle at P. Determine the length of the line segment NP.* (Solution on page 157.)

Problem 1.3. *Prove the converse of the Intersecting Chords Theorem. That is, prove that if $ABCD$ is a convex quadrilateral, if X is the point of*

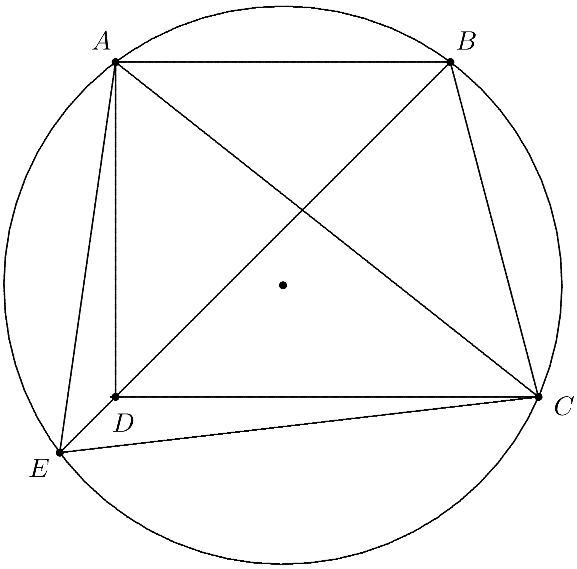


Fig. 1.15

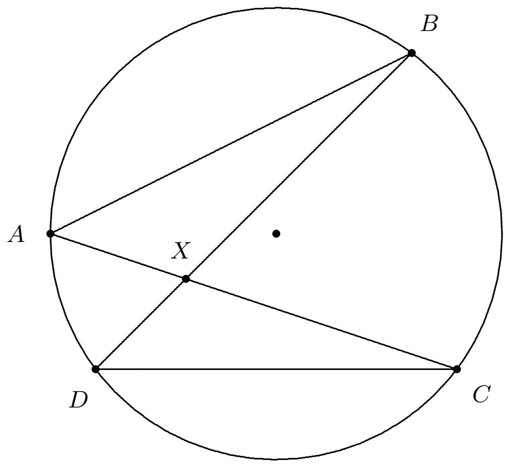


Fig. 1.16

intersection of the diagonals AC *and* BD, *and if* $AX \cdot XC = DX \cdot XB$, *then* $ABCD$ *is a cyclic quadrilateral.*

1.9 Inversion

The use of inversion can be very useful in solving some problems. We give the basic ideas here.
We work in the Euclidean plane, with one ideal point at infinity ∞.
Roughly speaking, an inversion is a transformation of the plane that generalizes the idea of reflection in a line.

1.9.1 *Reflection in a line*

In reflection in a line ℓ, a point X is mapped to a point X' that is the same distance from the line ℓ as is X, but is in the opposite half plane to X.

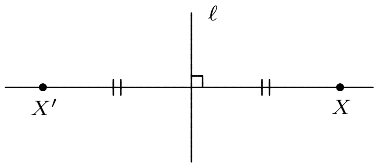


Fig. 1.17

1.9.2 *Inversion in a circle*

Generalize this notion of reflection by replacing the line ℓ by a circle Γ.

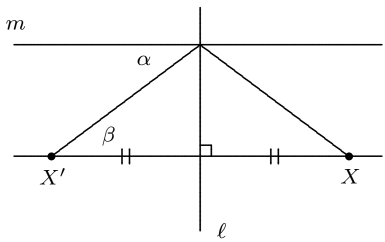


Fig. 1.18

Suppose that $m \| XX'$, meeting ℓ at P.
Reflecting $\angle PX'X$ gives $\angle PXX'$, which are equal.
Since $m \| XX'$, we have $\alpha = \beta$, so that $\angle PXX' = \alpha = \beta$.
Now, let your imagination expand so that ℓ is an infinitely large circle, with m lying on the radius of this circle through the point P. We now view is thus:

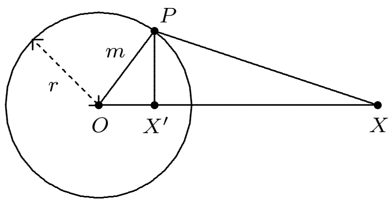


Fig. 1.19

We have $\triangle POX'$ similar to $\triangle XOP$, giving $\frac{OX'}{OP} = \frac{OP}{OX}$, so that $OX \cdot OX' = OP^2 = r^2$.
Now, suppose that O is a fixed point (called the CENTRE OF INVERSION) and that c is a fixed positive number (called the RADIUS OF INVERSION).

Definition 1.1. P and Q are INVERSES with respect to O with radius c if

$$OP.OQ \;=\; c^2 \;\; .$$

Definition 1.2. The circle centre O and radius c is called the CIRCLE OF INVERSION.

Theorem 1.8. *The circle of inversion is invariant under inversion.*

Theorem 1.9. *The inverse of a line through the centre of inversion is that line. (But it is not an invariant.)*

The proofs of these theorems are left to the reader.

Theorem 1.10. *The inverse of a line, not through the centre of inversion, is a circle passing through the centre of inversion.*

Proof.
Let O be the centre of inversion and PQ be the line, such that $OP \perp PQ$.
Let Γ be the circle of inversion.
Let P' and Q' be the inverses of P and Q, respectively.
Then $OP.OP' \;=\; OQ.OQ'$, giving

$$\frac{OP}{OQ} \;=\; \frac{OP'}{OQ'} \;\; .$$

Thus, triangles $\triangle OPQ$ and $\triangle OQ'P'$ are similar.
Since $\angle QPO = 90°$, we have that $\angle P'Q'O = 90°$, giving that O, P' and Q' lie on a circle of diameter OP'.

THE CONVERSE IS ALSO TRUE.

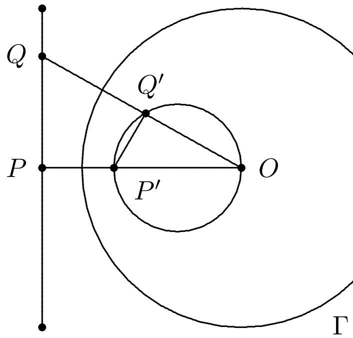


Fig. 1.20 Theorem 1.10

Theorem 1.11. *The inverse of a circle not passing through the centre of inversion in a circle (not passing through the centre of inversion).*

Proof.

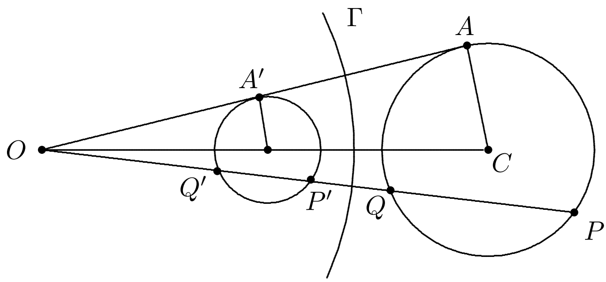


Fig. 1.21 Theorem 1.11

Let O be the centre of inversion and let the given circle have centre C. Then $OP.OP' = OQ.OQ' = c^2$.
Also, note that $OP.OQ = OA^2 = k^2$, where OA is the tangent from O to the given circle.
Thus,

$$c^4 = (OP.OP') \,.\, (OQ.OQ') = (OP'.OQ') \,.\, (OP.OQ) = (OP'.OQ') \,.\, k^2 \,,$$

whence,

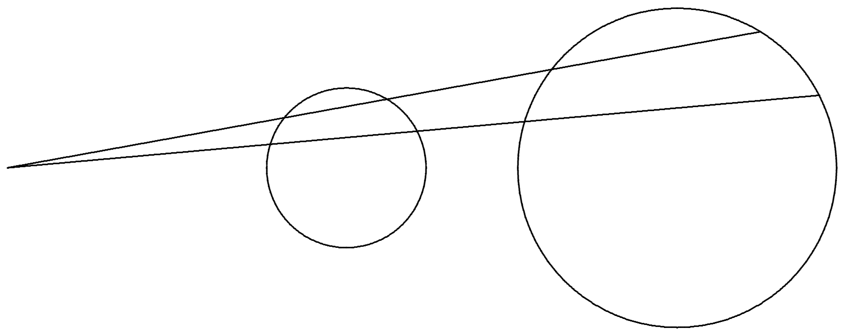
$$OP'.OQ' = \frac{k^2}{c^4} = (OA')^2 \,.$$

This means that the image is a circle!

Now, here is a result for you to try to obtain yourself!

Problem 1.4. *The measure of the angle between two intersecting circles is invariant under inversion*

Here is a useful figure.



Problem 1.5. *Two points, A and B are distinct and not collinear with the centre O of the circle of inversion. The images of the two points are A' and B', respectively.*
Prove that triangles OAB and $OB'A'$ are similar. Note that their orientations are reversed.

Problem 1.6. *A straight line passing through the centre of the circle of inversion maps onto itself.*