CHAPTER 2

Multivariate Copulae

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Though dating back to 1959 when the term “copulae” was coined, copula models only started their triumphal procession in the mid-1990s. Application of copulae was primarily restricted to the world of finance and insurance but now the copula concept has found its way into nearly all relevant statistical and mathematical literature where multivariate dependence structures are involved. Whereas the bivariate case was central in most of the publications and seems to be well-explored at present, there is still an ongoing and active debate on the construction of multivariate copula models. Apart from pair-copula constructions, which are the focus of this book and intensively discussed in the following chapters, this chapter briefly reviews both different copula classes and construction schemes of multivariate models.

2.1 Copulae

Loosely speaking, a *copula* incorporates the information on the dependence structure of $n > 1$ random variables $X_1, \ldots, X_n$. For reasons of simplicity, let us assume that the corresponding distribution functions $F_1, \ldots, F_n$ are *continuous* with the inverse functions $F_1^{-1}, \ldots, F_n^{-1}$ (details on discrete margins can be found, for instance, in Genest and Neslehova[20]). It follows from the probability integral transform that $U_i \equiv F_i(X_i)$ is uniformly distributed on $(0,1)$ for $i = 1, \ldots, n$. Conversely, $X_i = F_i^{-1}(U_i)$ for $i = 1, \ldots, n$. With this in mind,

$$\begin{aligned} P(X_1 \leq F_1^{-1}(x_1), \ldots, X_n \leq F_n^{-1}(x_n)) &= P(U_1 \leq x_1, \ldots, U_n \leq x_n) \\ &\equiv C(x_1, \ldots, x_n). \end{aligned}$$

Obviously, the function C is a distribution function with support on $[0,1]^n$ with uniform margins, a so-called *copula*.[a] Conversely, we obtain the following decomposition:

$$\begin{aligned} P(X_1 \leq x_1, \ldots, X_n \leq x_n) &= P(F_1(X_1) \leq F_1(x_1), \ldots, F_n(X_n) \leq F_n(x_n)) \\ &= C(F_1(x_1), \ldots, F_n(x_n)). \end{aligned}$$

Under the above assumptions, there is a one-to-one correspondence between the copula C and the distribution of $\boldsymbol{X} = (X_1, \ldots, X_n)'$, as stated in the fundamental theorem of Sklar.

Theorem 2.1 (Sklar[43]). *Given random variables $X_1, \ldots, X_n$ with continuous distribution functions $F_1, \ldots, F_n$ and joint distribution function F, there exists a unique copula C such that for all $\boldsymbol{x} = (x_1, \ldots, x_n)' \in \mathbb{R}^n$:*

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)). \tag{2.1}$$

Conversely, given any distribution functions $F_1, \ldots, F_n$ and copula C, F defined through Eq. (2.1) *is an n-variate distribution function with marginals $F_1, \ldots, F_n$.*

According to (2.1), the copula "couples" the marginal distributions to the joint distribution function F. Hence, Eq. (2.1) enables us to construct the joint distribution function F as follows: At the first stage, the marginal distribution $F_1, \ldots, F_n$ have to be specified, whereas, at the second stage,

[a] A formal definition of multivariate copulae is provided by Nelsen.[36]

the underlying copula model has to be selected. On the other hand, Eq. (2.1) can be re-written as follows:

$$F(F_1^{-1}(x_1), \ldots, F_n^{-1}(x_n)) = C(u_1, \ldots, u_n). \tag{2.2}$$

Equation (2.2) reveals how to extract the copula of a (given) multivariate distribution. Take, for instance, elliptical copulae which are discussed in the next subsection. We conclude this section with an example that contains simple but prominent copulae.

Example 2.1.

- *Independence copula*: Assume that the random variables $X_1, \ldots, X_n$ are independent. According to (2.1), the underlying ("independence") copula is given by

$$C^{\perp}(\mathbf{u}) \equiv C(u_1, \ldots, u_n) = u_1 \cdots u_n.$$

- *Copula bounds*: Every multivariate copula is bounded from above and below by the so-called Fréchet–Hoeffding bounds, i.e.,

$$\max\{u_1 + \cdots + u_n - (n-1), 0\} \le C(u_1, \ldots, u_n) \le \min\{u_1, \ldots, u_n\}.$$

Note that only the upper bound is a valid copula for $n > 2$.

By the end of this chapter, we will have looked at much more flexible, parametric copula classes and construction schemes for multivariate copulae, without claiming to be fully comprehensive. For a long time, both practitioners and theorists have relied solely on the multivariate Gaussian distribution and Gaussian copula, respectively, where the dependence structure is completely determined by pairwise correlations. More generally, elliptical copulae (see Section 2.2) still maintain many of their attractive properties. But while elliptical distributions are able to model moderate and/or heavy tails, they fail to capture asymmetric dependence structures. Among the classes of non-elliptical copulae, Archimedean copulae and its generalizations enjoy great popularity and are the subject of Section 2.3. Within this chapter, the focus is primarily on these two copula classes and on selected construction schemes of multivariate copulae published recently (e.g., Refs. 17, 30, 35). Beyond that, there exist a bundle of multivariate copulae which are excluded from this overview. To name only a few, we refer to multivariate extreme-value copulae (see, for instance, McNeil *et al.*[32] or Joe[24]), multivariate Farlie–Gumbel–Morgenstern copulae (see, for instance, Drouet and Kotz[12]) or multivariate Marshall–Olkin copulae (see, for instance, Joe[24]).

For a detailed introduction to copulae we refer the reader to the textbooks.[12,24,36] Application of copulae to finance can be found in

Refs. 7, 9, 32. Furthermore, overviews of copulae and some background theory are provided in Genest and Favre,[21] Embrechts *et al.*[14,15] or, from a more critical point of view, in Mikosch.[34]

2.2 Elliptical Copulae and Generalizations

2.2.1 *Elliptical copulae*

The class of *elliptical copulae* (EC) constitutes the prime example of implicit copulae stated in (2.2). EC are copulae associated with elliptical distributions[b] and are widely used in statistics and econometrics, especially in finance. Note that EC are not elliptical distributions themselves. EC have the virtue that they extend easily to arbitrary dimensions n and are rich in parameters, at least $n(n-1)/2$. However, radial asymmetries and asymmetric tail behavior cannot be captured within this class. Due to their implicit definition, explicit expressions for the copula are not available. Evaluating an elliptical copula requires the calculation of multiple integrals without closed-form solutions, which must be done numerically. Applications and limitations of EC are discussed in more detail by Frahm *et al.*,[19] whereas Hult and Lindskog[23] and Abdous *et al.*[1] deal with extremal dependence and tail dependence of elliptically contoured distributions. Within the elliptical class, both Gaussian and t-copulae play a predominant role.

Example 2.2 (Gaussian copula). Let $\mathbf{\Phi}^n_{\mathbf{R}}$ denote the standardized n-variate normal distribution with correlation matrix $\mathbf{R}$. Applying (2.2), the *Gaussian copula* is defined as follows:

$$C(\mathbf{u};\mathbf{R}) = \mathbf{\Phi}^n_{\mathbf{R}}(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_n)),$$

where $\mathbf{\Phi}^{-1}$ denotes the quantile function of a univariate standard normal distribution. Per construction, the Gaussian copula generates the standard Gaussian joint distribution if and only if the margins follow a standard normal distribution. The corresponding copula density is given by

$$c(\mathbf{u};\mathbf{R}) = \frac{\frac{1}{(2\pi)^{n/2}\sqrt{|\mathbf{R}|}}\exp(-0.5\boldsymbol{\zeta}'\boldsymbol{R}^{-1}\boldsymbol{\zeta})}{\prod_{j=1}^n \frac{1}{\sqrt{2\pi}}\exp(-0.5\zeta_j^2)} = \frac{\exp(-0.5\boldsymbol{\zeta}'(\boldsymbol{R}^{-1}-\mathbf{I}_n)\boldsymbol{\zeta})}{\sqrt{|\boldsymbol{R}|}},$$

with $\boldsymbol{\zeta} \equiv (\zeta_1,\ldots,\zeta_n)'$ and $\zeta_i = \Phi^{-1}(u_i)$ for $i = 1,\ldots,n$. Restricting to the bivariate case, a bivariate Gaussian variable admits no tail dependence

[b]A detailed treatment of elliptically contoured distribution is provided by Fang *et al.*[16]

(see, e.g., Ref. 15). Extensions to the Gaussian copula can be found in Andersen and Sidenius.[2]

Example 2.3 (Student-*t* copula). Let $\mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{\Sigma})$ and $R = \sqrt{\nu}/\sqrt{S}$ with $S \sim \chi^2(\nu)$, i.e. a chi-squared variable with ν degrees of freedom. Then the $\mathbb{R}^n$-valued random vector

$$\mathbf{Y} \equiv R\mathbf{Z} = (RZ_1, \dots, RZ_n)$$

has a t-distribution with ν degrees of freedom. If $\nu > 2$, $\mathrm{Cov}(\mathbf{Y}) = \frac{\nu}{\nu-2}\mathbf{\Sigma}$. Again, applying Sklar's theorem and defining $\boldsymbol{\rho} \equiv (\rho_{ij})_{1 \le i,j \le n}$ with $\rho_{ij} \equiv \Sigma_{ij}/\sqrt{\Sigma_{ii}\Sigma_{jj}}$, the implicit copula expression is given by

$$C_t(\mathbf{u}; \nu, \boldsymbol{\rho}) = \mathbf{t}^n_{\nu,\boldsymbol{\rho}}(t_\nu^{-1}(u), t_\nu^{-1}(v)),$$

where t_ν^{-1} denotes the inverse function of the classical univariate t-distribution. The associated density function of the t-copula is given by

$$c_t(\mathbf{u}; \boldsymbol{\rho}, \nu) = \frac{1}{\sqrt{|\boldsymbol{\rho}|}} \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)}\right)^n \frac{\prod_{j=1}^n \left(1 + \frac{t_\nu^{-1}(u_j)^2}{\nu}\right)^{\frac{\nu+1}{2}}}{\left(1 + \frac{\zeta' \boldsymbol{\rho}^{-1} \zeta}{\nu}\right)^{\frac{\nu+n}{2}}}. \tag{2.3}$$

Restricting again to the bivariate case, the t-copula has tail dependence coefficient

$$\lambda = \lambda_U = \lambda_L = 2t_{\nu+1}\left(-\frac{\sqrt{\nu+1}\sqrt{1-\rho}}{\sqrt{1-\rho}}\right) > 0,$$

provided that $\rho \ge -1$. Venter[44,45] deals with the estimation, application and limitations of the Student t-copula, whereas Kole *et al.*[28] perform stress testing under Student's t-dependence.

Still within the elliptical class, Mendes and Arslan[33] favor a generalized t-copula which allows for different degrees and types of linear and non-linear dependence. In particular, they derive expressions for its coefficients of upper and lower tail dependence and suggest applications in finance, including portfolio optimization and computation of measures of contagion.

Example 2.4 (GT-copula). Arslan[3] introduces a new family of multivariate generalized distributions as a scale mixture of a multivariate power exponential distribution (see Gómez *et al.*[22]) and an inverse generalized gamma distribution with a scale parameter, and shows that this family of distributions belongs to the family of elliptically contoured distributions that includes the multivariate normal distribution and the multivariate t-distribution as special or limiting cases. The corresponding copula

("GT-copula") is intensively discussed by Mendes and Arslan[33] who show that the bivariate copula density is given by

$$c(u_1, u_2; \rho, \nu, \beta) = \frac{K}{\sqrt{1-\rho^2}} \cdot \frac{\left[\frac{\nu}{2} + \left(\frac{\zeta_1^2+\zeta_2^2-2\rho\zeta_1\zeta_2}{1-\rho^2}\right)^{\beta}\right]^{-\frac{\nu}{2}-\frac{1}{\beta}}}{f(\zeta_1;\beta,\nu/2)f(\zeta_2;\beta,\nu/2)}, \tag{2.4}$$

with $K \equiv \frac{\beta\,\Gamma(n/2)q^q}{\pi^{n/2}B(q,n/2\beta)}$, $\zeta_i \equiv F_{GT}^{-1}(u_i)$ for $i = 1, 2$ and where the marginal density and distribution function of MGT marginals, respectively, are

$$f(x;\beta,q) = K \int_{x^2}^{\infty} \frac{(y-x^2)^{-1/2}}{(q+y^{\beta})^{q+1/\beta}}\,dy \tag{2.5}$$

and

$$F(x;\beta,q) = \frac{1}{2} + \int_{x^2}^{\infty} \frac{\arcsin(x/\sqrt{y})}{(q+y^{\beta})^{q+1/\beta}}\,dy. \tag{2.6}$$

Unfortunately, explicit formulae for the integrals in (2.5) and (2.6) are not available and numerical procedures are required in order to evaluate both copula and copula density.

Example 2.5 (Elliptical generalized hyperbolic (GH) copulae). Dating back to Barndorff-Nielsen,[4,5] both univariate and multivariate GH distributions have become very popular in the last decade, especially in finance (see, for instance, Prause[38]). This distribution family exhibits heavier tails than the Gaussian distribution but lighter ones than the t-distribution, both of which appear as limit cases. All moments of the GH distribution exist and the moment-generating function is available in closed form. Though multivariate GH distributions share the desirable characteristics of the univariate one (i.e., flexibility, semi-heavy tails), this distribution family possesses no parameter configuration for which the case of marginal independence can be modeled. Above that, the bivariate GH distribution is tail-independent (see, e.g., Schmidt[41]). In general, the multivariate version arises as a normal mean-variance mixture, i.e., as a multivariate normal distribution with (random) mean vector $\boldsymbol{\mu} + \boldsymbol{\beta}\tau\triangle$ and (random) covariance matrix $\tau\triangle$, where τ itself follows a univariate generalized inverse Gaussian distribution (see, e.g., Jørgensen[25]). The corresponding GH density is given by

$$f_n(\boldsymbol{x};\boldsymbol{\Theta}) = \frac{\left(\frac{\psi}{(\psi+\boldsymbol{\beta}\triangle\boldsymbol{\beta}')}\right)^{\lambda/2}\left(\frac{(\psi+\boldsymbol{\beta}\triangle\boldsymbol{\beta}')}{\chi}\right)^{n/4}}{(2\pi)^{n/2}\mathrm{K}_{\lambda}(\sqrt{\psi\chi})} \cdot \frac{\mathrm{K}_{\lambda-n/2}(\sqrt{(\psi+\boldsymbol{\beta}\triangle\boldsymbol{\beta}')(\chi+z)})}{(1+z/\chi)^{n/4-\lambda/2}e^{-\boldsymbol{\beta}'(\boldsymbol{x}-\boldsymbol{\mu})}}$$

with $z \equiv (\boldsymbol{x}-\boldsymbol{\mu})'\triangle^{-1}(\boldsymbol{x}-\boldsymbol{\mu})$, $\triangle$ being a positive definite matrix with determinant 1, parameter vector $\boldsymbol{\Theta} \equiv (\boldsymbol{\mu}, \chi, \boldsymbol{\beta}, \psi, \lambda, \triangle)'$ and where $\mathrm{K}_\lambda(x)$ denotes the modified Bessel function of the third kind. Ellipticity is achieved only if the asymmetry parameter vector $\boldsymbol{\beta}$ is set to zero. Despite the popularity of the GH distribution, the literature on the corresponding GH copula itself is relatively sparse (e.g., Schmidt[41,42] and Lentzas[29]). Under a slightly different parametrization (see McNeil *et al.*[32]) that has the property that mixing parameters remain invariant under linear affine transformations, Lentzas[29] derives the copula density of a GH distribution as follows:

$$c_{GH}(\boldsymbol{u}) = \frac{k \cdot K_{\lambda-\frac{n}{2}}\left(\sqrt{(\chi + (\boldsymbol{\zeta}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\zeta}-\boldsymbol{\mu}))(\psi + \boldsymbol{\gamma}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma})}\right)}{e^{-(\boldsymbol{\zeta}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}}((\chi + (\boldsymbol{\zeta}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\zeta}-\boldsymbol{\mu}))(\psi + \boldsymbol{\gamma}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}))^{\frac{n}{2}-\lambda}}$$
$$\times \left(\prod_{i=1}^{n} \frac{k_i \cdot K_{\lambda-\frac{1}{2}}\left\{\sqrt{\left(\chi + \frac{(\zeta_i-\mu_i)^2}{\Sigma_{ii}}\right)\left(\psi + \frac{\gamma_i^2}{\boldsymbol{\Sigma}_{ii}}\right)}\right\}}{e^{-\frac{\gamma_i(\zeta_i-\mu_i)}{\boldsymbol{\Sigma}_{ii}}} \sqrt{\left(\chi + \frac{(\zeta_i-\mu_i)^2}{\boldsymbol{\Sigma}_{ii}}\right)\left(\psi + \frac{\gamma_i^2}{\boldsymbol{\Sigma}_{ii}}\right)}^{\frac{1}{2}-\lambda}} \right)^{-1}$$

with $\boldsymbol{\zeta} \equiv (\zeta_1, \ldots, \zeta_n)$, $\zeta_i \equiv F_{GH}^{-1}(u_i)$ and constants given by

$$k \equiv \frac{(\sqrt{\psi\chi})^{-\lambda}\psi^{\lambda}(\psi + \gamma'\boldsymbol{\Sigma}^{-1}\gamma)^{\frac{n}{2}-\lambda}}{(2\pi)^{\frac{n}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}K_\lambda(\sqrt{\psi\chi})}, \quad k_i \equiv \frac{(\sqrt{\psi\chi})^{-\lambda}\psi^{\lambda}\left(\psi + \frac{\gamma_i^2}{\boldsymbol{\Sigma}_{ii}}\right)^{\frac{1}{2}-\lambda}}{\sqrt{2\pi\boldsymbol{\Sigma}_{ii}}K_\lambda(\sqrt{\psi\chi})}.$$

Note that the hyperbolic quantile function has to be approximated numerically which complicates the evaluation of the GH copula. Lentzas[29] also deals with different estimation methods (ML estimation, rank correlation ML, Monte Carlo rank correlation ML, simulated GMM and based on the EM algorithm) for the unknown parameters of a GH copula.

2.2.2 *Generalized t-copulae*

The t-copula is often chosen when a multivariate model with extreme dependence is needed. However, the use of the standard t-copula is often criticized due to its restriction of having only a single parameter for the degrees of freedom that may limit its capability to model the tail dependence structure in a multivariate case. This motivates the next two examples, the grouped t-copula and the IT-copula.

Example 2.6 (Grouped *t*-copula). In order to increase the flexibility of the popular t-copula, Daul *et al.*[10] and Demarta and McNeil[11] introduce the grouped t-copula. Their aim is to describe the dependence among risk factors of different classes. For a given partition of $\{1,\ldots,n\}$ into m subsets of sizes $s_1,\ldots,s_m$ with $s_1+\cdots+s_m=n$,

$$\mathbf{Y} \equiv (R_1 Z_1,\ldots,R_1 Z_{s_1}, R_2 Z_{s_1+1},\ldots,R_2 Z_{s_1+s_2},\ldots,R_m Z_n)'.$$

The random vector $(Y_1,\ldots,Y_{s_1})'$ has s_1-dimensional t-distribution with ν_1 degrees of freedom and, for $k=1,\ldots,m-1$, $(Y_{s_1+\cdots+s_k+1},\ldots,Y_{s_1+\cdots+s_{k+1}})'$ has s_{k+1}-dimensional t-distribution with ν_{k+1} degrees of freedom. Finally, the grouped t-copula is the distribution function of the random vector

$$\mathbf{U} = (t_{\nu_1}(Y_1),\ldots,t_{\nu_1}(Y_{s_1}), t_{\nu_2}(Y_{s_1}+1),\ldots,t_{\nu_2}(Y_{s_1+s_2}),\ldots,t_{\nu_m}(Y_n))'$$

where again t_{ν_i} denotes the distribution function of a classical Student's t-distribution with ν_i degrees of freedom. Daul *et al.*[10] also show how to estimate the unknown parameters and give some application to credit risk modeling.

Example 2.7 (The IT-copula). Instead of grouping variables a priori in such a way that each group has a standard t-copula with its specific degrees of freedom parameter, both Luo and Shevchenko[31] and Barnett *et al.*[6] propose the so-called "individual" t-copula, or IT-copula, where each group boils down to one variable or risk factor only. Starting from the stochastic random vector $\mathbf{X} \equiv (R_1 Z_1,\ldots,R_n Z_n)'$ with R_i, Z_i as in Example 2.3, the IT-copula is defined as the cumulative distribution function of the random vector $\mathbf{U} \equiv (t_{\nu_1}(X_1),\ldots,t_{\nu_n}(X_n))'$. Clearly, both t-copula and grouped t-copula are special cases of this construction. Luo and Shevchenko[31] derive the corresponding explicit integral representation with $\overline{\nu} \equiv (\nu_1,\ldots,\nu_n)$

$$C(\mathbf{u};\overline{\nu},\mathbf{\Sigma}) = \int_0^1 \Phi_n(z_1(u_1,s),\ldots,z_n(u_n,s))\,ds \tag{2.7}$$

with $z_i(u_i,s) \equiv t_{\nu_i}(u_i)/G_{\nu_i}^{-1}(s)$, where $G_\nu^{-1}(x)$ corresponds to the distribution function of $\sqrt{\nu/S}$ for a χ^2_ν-variable S and Φ_n denotes the classical multivariate Gaussian distribution function. From (2.7), the density derives as

$$c(\mathbf{u};\overline{\nu},\mathbf{\Sigma}) = \frac{\int_0^1 \phi_n(z_1(u_1,s),\ldots,z_n(u_n,s)) \prod_{i=1}^n (G_{\nu_i}^{-1}(s))^{-1}\,ds}{\prod_{i=1}^n f_{\nu_i}(t_{\nu_i}^{-1}(u_i))}. \tag{2.8}$$

Obviously, the multivariate copula density involves an additional one-dimensional integration which makes fitting this copula more computationally demanding than fitting a standard t-copula. For details on model calibration and application to risk quantification, we refer to Luo and Shevchenko[31] and Barnett *et al.*[6]

2.3 Archimedean Copulae and Generalizations

2.3.1 *Classical Archimedean copulae*

Let φ: $[0,1] \to [0,\infty]$ be a continuous, strictly decreasing and convex function with $\varphi(1) = 0$, $\varphi(0) \leq \infty$ and let $\varphi^{[-1]}$ be the so-called pseudo-inverse of φ defined by

$$\varphi^{[-1]}(t) \equiv \begin{cases} \varphi^{-1}(t) & 0 \leq t \leq \varphi(0), \\ 0 & \varphi(0) \leq t \leq \infty\,. \end{cases}$$

It can be shown (see, e.g., Nelsen[36]) that

$$C(u_1, u_2) = \varphi^{[-1]}(\varphi(u_1) + \varphi(u_2))$$

defines a class of bivariate copulae, the so-called Archimedean copulae. The function φ is called the (additive) generator of the copula. Furthermore, if $\varphi(0) = \infty$, the pseudo-inverse describes an ordinary inverse function (i.e., $\varphi^{[-1]} = \varphi^{-1}$) and in this case φ is known as a strict generator.

Given a strict generator φ: $[0,1] \to [0,\infty]$, bivariate Archimedean copulae can be extended to the n-dimensional case. For every $n \geq 2$, the function C: $[0,1]^n \to [0,1]$ defined as

$$C(\mathbf{u}) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2) + \cdots + \varphi(u_n)) \tag{2.9}$$

is an n-dimensional Archimedean copula if and only if φ^{-1} is completely monotonic on $\mathbb{R}_+$, i.e., if $\varphi^{-1} \in \mathcal{L}_\infty$ with

$$\mathcal{L}_m \equiv \{\phi\colon \mathbb{R}_+ \to [0,1] | \phi(0) = 1,\, \phi(\infty) = 0,\, (-1)^k \phi^{(k)}(t) \geq 0\ ,\ k \leq m\}.$$

The Gumbel copula is derived from the generator $\varphi(t) = (-\ln t)^\theta, \theta \geq 1$ and the Clayton copula is generated by

$$\varphi(t) = \frac{1}{\theta}(t^{-\theta} - 1), \quad \theta > 0. \tag{2.10}$$

For an overview of further Archimedean copulae and the properties of the aforementioned ones, we refer the reader to the monographs of Nelsen[36] and Joe.[24]

2.3.2 *Non-exchangeable Archimedean copulae*

In order to increase flexibility and to allow for non-exchangeable dependence structures, several generalizations have emerged in the recent literature: A simple one — the so-called fully nested Archimedean (FNA) copulae — can be found in Joe[24] (p. 89), Whelan[46] and Savu and Trede,[39] and requires

$n-1$ generator functions $\varphi_1, \ldots, \varphi_{n-1}$ with $\varphi_1^{-1}, \ldots, \varphi_{n-1}^{-1} \in \mathcal{L}_\infty$ and $\varphi_{i+1} \circ \varphi_i^{-1}(t) = \varphi_{i+1}(\varphi_i^{-1}(t)) \in \mathcal{L}_\infty^*$ for

$$\mathcal{L}_n^* = \{\phi\colon \mathbb{R}_+ \to \mathbb{R}_+ | \phi(0) = 0,\, \phi(\infty) = \infty,\, (-1)^{k-1}\phi^{(k)}(t) \geq 0\ ,\ k \leq n\}.$$

The structure of FNA n-copulae is rather simple: One first couples u_1 and u_2, then the copula of u_1 and u_2 with u_3 to form a new copula, which is coupled afterwards with u_4 and so on. Hence, the FNA four-copula is of the form

$$C(\mathbf{u}) = \varphi_3^{-1}[\varphi_3(\varphi_2^{-1}[\varphi_2(\varphi_1^{-1}[\varphi_1(u_1) \,+\, \varphi_1(u_2)]) \,+\, \varphi_2(u_3)]) \,+\, \varphi_3(u_4)]. \tag{2.11}$$

Figure 2.1 illustrates one possible FNA copula for dimension $n = 5$.

Alternatively, mixing ordinary Archimedean and FNA copulae, partially nested Archimedean (PNA) copulae may be used. Again, for ease of notation, we focus on the four-variate case:

$$\begin{aligned} C(\boldsymbol{u}) \ &= \varphi^{-1}[\varphi\,(\varphi_{12}^{-1}\,[\varphi_{12}\,(u_1) + \varphi_{12}\,(u_2)]) \\ &\quad + \varphi\,(\varphi_{34}^{-1}\,[\varphi_{34}\,(u_3) + \varphi_{34}\,(u_4)])]. \end{aligned} \tag{2.12}$$

Note that $\varphi, \varphi_{12}, \varphi_{34}$ are generators with $\varphi^{-1}, \varphi_{12}^{-1}, \varphi_{34}^{-1} \in \mathcal{L}_\infty$ and $\varphi \circ \varphi_{12}^{-1}, \varphi \circ \varphi_{34}^{-1} \in \mathcal{L}_\infty^*$. Obviously, one first couples the pairs u_1, u_2 and u_3, u_4 with distinct generators. The resulting copula pair is then coupled using a third generator φ (which in turn might be coupled with an additional variable u_5 using a fourth generator ψ for an extension to the five-dimensional case). Another possible structure of a PNA copula is illustrated in Fig. 2.2.

Third, copula C from (2.12) is also an example of a so-called hierarchical Archimedean (HA) copula. Borrowing the notation of Savu and Trede,[39] the basic idea of this approach is to build a hierarchy of Archimedean copulae with L different levels, indexed by $l = 1, \ldots, L$. At each level l, there are n_l distinct objects, indexed by $j = 1, \ldots, n_l$. In a first step (i.e., in level 1), the

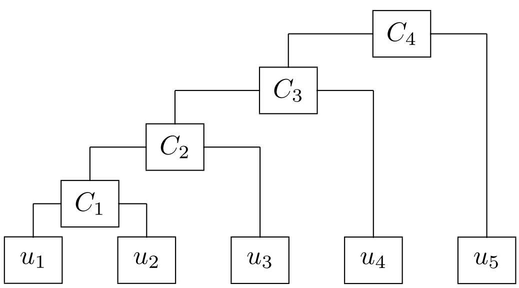

Figure 2.1. FNA copula for $n = 5$.

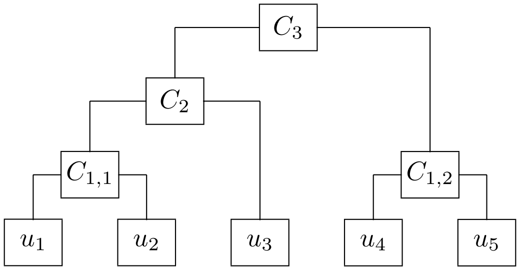

Figure 2.2. PNA copula for $n = 5$.

variables $u_1, \ldots, u_n$ are grouped into n_1 ordinary multivariate Archimedean copulae:

$$C_{1,j}(\mathbf{u}_{1,j}) = \varphi_{1,j}^{-1}\left(\sum_{\mathbf{u}_{1,j}} \varphi_{1,j}(\mathbf{u}_{1,j})\right), \quad j = 1, \ldots, n_1$$

with (possibly different) generators $\varphi_{1,j}$ and where $\mathbf{u}_{1,j}$ denotes the set of elements of $u_1, \ldots, u_n$ belonging to $C_{1,j}$. All copulae of the first level are again grouped into copulae at level $l = 2$. These copulae $C_{2,j}$ with generator function $\varphi_{2,j}$, $j = 1, \ldots, n_2$ are generalized Archimedean copulae, whose dependence structure is only of partial exchangeability and consists of copulae from the previous level (as elements), denoted by

$$C_{2,j}(\mathbf{C}_{2,j}) = \varphi_{2,j}^{-1}\left(\sum_{\mathbf{C}_{2,j}} \varphi_{2,j}(\mathbf{C}_{2,j})\right),$$

where $\mathbf{C}_{2,j}$ represents the set of all copulae from level $l = 1$ entering copula $C_{2,j}$. This procedure continues until only a single hierarchical Archimedean copula $C_{L,1}$ is achieved at level L. In order to ensure that $C_{L,1}$ is a proper copula, we have to proclaim that $\varphi_{l,j}^{-1} \in \mathcal{L}_\infty$ for $l = 1, \ldots, L$ and $j = 1, \ldots, n_l$, and that $\varphi_{l+1,i} \circ \varphi_{l,j}^{-1} \in \mathcal{L}_\infty^*$ for all $l = 1, \ldots, L$ and $j = 1, \ldots, n_l$, $i = 1, \ldots, n_{l+1}$ such that $C_{l,j} \in \mathbf{C}_{l+1,i}$. Moreover, a hierarchy is established if the number of copulae decreases at each level, if the top level contains only a single object and if at each level the dimensions of the copulae add up to n. Figure 2.3 displays the possible construction of a five-dimensional HA copula.

2.3.3 *Generalized multiplicative Archimedean copulae*

In this section, we focus on methods recently proposed by Morillas[35] and Liebscher.[30] Both approaches are based on a second functional representation of Archimedean copulae via so-called multiplicative generators (see

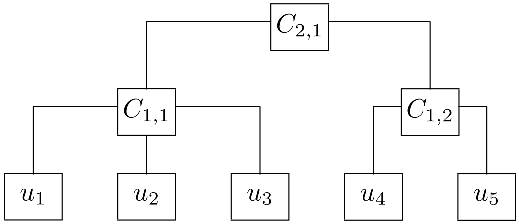

Figure 2.3. HA copula for $n = 5$.

Nelsen[36]). Setting $\vartheta(t) \equiv \exp(-\varphi(t))$ and $\vartheta^{[-1]}(t) \equiv \varphi^{[-1]}(-\ln t)$, Eq. (2.9) can be rewritten as

$$C(u_1, \ldots, u_n) = \vartheta^{[-1]}(\vartheta(u_1) \cdot \vartheta(u_2) \cdot \cdots \cdot \vartheta(u_n)). \tag{2.13}$$

The function ϑ is called a multiplicative generator of C. Due to the relationship between φ and ϑ, the function ϑ: $[0, 1] \to [0, 1]$ is continuous, strictly increasing and concave with $\vartheta(1) = 1$ and $\vartheta^{[-1]}(t) = 0$ if $0 \leq t \leq \vartheta(0)$ and $\vartheta^{[-1]}(t) = \vartheta^{-1}(t)$ if $\vartheta(0) \leq t \leq 1$.

Equation (2.13) can also be expressed using the independence copula $C^{\perp}(\mathbf{u}) = \prod_{i=1}^{n} u_i$:

$$C(u_1, \ldots, u_n) = \vartheta^{[-1]}(C^{\perp}(\vartheta(u_1), \ldots, \vartheta(u_n))).$$

Morillas[35] substitutes $C^{\perp}$ by an arbitrary n-copula C in order to obtain

$$C_{\vartheta}(u_1, \ldots, u_n) = \vartheta^{[-1]}(C(\vartheta(u_1), \vartheta(u_2), \ldots, \vartheta(u_n))) \tag{2.14}$$

and proves that C_{ϑ} is an n-copula if $\vartheta^{[-1]}$ is *absolutely monotonic of order n* on $[0, 1]$, i.e. if $\vartheta^{[-1]}(t)$ satisfies $(\vartheta^{[-1]})^{(k)}(t) = \frac{d^k \vartheta^{[-1]}(t)}{dt^k} \geq 0$ for $k = 1, 2, \ldots, n$ and $t \in (0, 1)$.

Examples of generator functions are stated in Morillas.[35] Notice that not every generator given there is absolutely monotonic for arbitrary $n > 1$. As one can easily verify, the generator $\vartheta(t) = t^r/(2 - t^r), r \in (0, 1/3]$ (see Table 1, no. 9 in Morillas[35]) has no absolutely monotonic pseudo-inverse of order $n \geq 3$ because the third derivative of $\vartheta^{[-1]}$ becomes negative. Hence, this generator is suitable only for a construction of generalized bivariate copulae. For the basic properties of such Morillas copulae we refer to Morillas.[35]

Another way of generalizing Archimedean copulae is the method proposed by Liebscher.[30] He introduces the following copula representation:

$$C(u_1, \ldots, u_n) = \Psi\left(\frac{1}{m} \sum_{j=1}^{m} \psi_{j1}(u_1) \cdot \psi_{j2}(u_2) \cdot \cdots \cdot \psi_{jn}(u_n)\right), \tag{2.15}$$

where Ψ and ψ_{jk}: $[0,1] \to [0,1]$ are functions satisfying the following conditions: First, it is assumed that $\Psi^{(n)}$ exists with $\Psi^{(k)}(u) \geq 0$ for $k = 1, 2, \ldots, n$ and $u \in [0,1]$, and that $\Psi(0) = 0$. Second, ψ_{jk} is assumed to be differentiable and monotone increasing with $\psi_{jk}(0) = 0$ and $\psi_{jk}(1) = 1$ for all k, j. Third, Liebscher's construction requires that

$$\Psi\left(\frac{1}{m}\sum_{j=1}^{m}\psi_{jk}(v)\right) = v \qquad \text{for } k = 1, 2, \ldots, n \text{ and } v \in [0,1].$$

The three conditions guarantee that C as defined in (2.15) is actually a copula.

It is easily seen that the approaches of Morillas and Liebscher coincide for $m = 1$, $\vartheta^{[-1]} = \Psi$ in (2.15) and $C_\vartheta = C^\perp$ in (2.14).

Liebscher[30] also states a general method for deriving appropriate functions ψ_{jk}. Let h_{jk}: $[0,1] \to [0,1], j = 1, \ldots, m, k = 1, \ldots, n$ be a differentiable and bijective function such that $h'_{jk}(u) > 0$ for $u \in (0,1)$, $h_{jk}(0) = 0$, $h_{jk}(1) = 1$ and $m \cdot u = \sum_{j=1}^{m} h_{jk}(u)$, $u \in [0,1]$, $k = 1, \ldots, n$. Let $\psi = \Psi^{-1}$ be the differentiable inverse function of Ψ. An appropriate choice is setting $\psi_{jk}(u) = h_{jk}(\psi(u))$, since $\psi'_{jk}(u) = h'_{jk}(\psi(u)) \cdot \psi'(u) > 0$ for $j = 1, \ldots, m$ and $u \in [0,1]$.

2.3.4 *Koehler–Symanowski copulae*

Just like Archimedean copulae, Koehler–Symanowski (KS) copulae admit closed-form representations. Although KS copulae are not Archimedean in general, the (Archimedean) Clayton copula with generator function given in (2.10) is included as a KS copula under certain parameter restrictions. More generally, Koehler and Symanowski[27] introduce a multivariate distribution as follows: For the index set $V = \{1, 2, \ldots, n\}$, let $\mathcal{V}$ denote the power set of V and $\mathcal{I} \equiv \{I \in \mathcal{V} \text{ with } |I| \geq 2\}$. Let further $\mathbf{X}$ denote an n-dimensional random vector with univariate marginal distributions $F_i(x_i), i \in V$. For all subsets $I \in \mathcal{I}$, let $\alpha_I \in \mathbb{R}_0^+$ and $\alpha_i \in \mathbb{R}_0^+$ for all $i \in V$ such that $\alpha_{i+} = \alpha_i + \sum_{I \in \mathcal{I}} \alpha_I > 0$ for $i \in I$. Then the common distribution function F is defined by

$$F(\boldsymbol{x}) = \frac{\prod_{i\in V} F_i(x_i)}{\prod_{I\in\mathcal{I}}\left[\sum_{i\in I}\prod_{j\in I, j\neq i} F_j(x_j)^{\alpha_{j+}} - (|I|-1)\prod_{i\in I} F_i(x_i)^{\alpha_{i+}}\right]^{\alpha_I}}.$$

The terms $K_I = \sum_{i\in I}\prod_{j\in I, j\neq i} F_j(x_j)^{\alpha_{j+}} - (|I|-1)\prod_{i\in I} F_i(x_i)^{\alpha_{i+}}$ are called association terms. Moreover, Koehler and Symanowski[27] showed that the joint density function exists if the marginal density functions f_i exist for all $i \in V$. Due to the design of the Koehler–Symanowski (KS) distribution, the corresponding copula has a similar functional form: setting $u_i = F_i(x_i)$ for all $i \in V$, the KS copula is

$$C(u_1,\ldots,u_d) = \frac{\prod_{i\in V} u_i}{\prod_{I\in\mathcal{I}}\left[\sum_{i\in I}\prod_{j\in I, j\neq i} u_j^{\alpha_{j+}} - (|I|-1)\prod_{i\in I} u_i^{\alpha_{i+}}\right]^{\alpha_I}}\,.$$

In contrast to the cumulative distribution function, the functional representation of the density is quite complicated due to complex factors with additive components. Koehler and Symanowski[27] gave an explicit formula for the special case of a so-called KS(2) distribution (see also Caputo[8]), where all parameters α_I are set equal to zero for $|I| > 2$. The corresponding copula is termed a KS(2) copula. Assuming that $\alpha_{ij} \equiv \alpha_{ji} \geq 0$ for all $(i,j) \in V \times V$ and $\alpha_{i+} = \alpha_{i1} + \alpha_{i2} + \cdots + \alpha_{in} > 0$ for all $i \in V$, the KS(2) copula simplifies to

$$C(u_1, u_2, \ldots, u_n) = \prod_{i=1}^{n} u_i \prod\prod_{i<j} K_{ij}^{-\alpha_{ij}} \tag{2.16}$$

with $K_{ij} \equiv u_i^{1/\alpha_{i+}} + u_j^{1/\alpha_{j+}} - u_i^{1/\alpha_{i+}} u_j^{1/\alpha_{j+}} = K_{ji}$.

Palmitesta and Provasi[37] apply the KS(2) copula to financial return data. They also argue that this copula has the ability to model complex dependence structures among subsets of marginal distribution but they do not present any goodness-of-fit measure or comparison with other copulae. In contrast, Fischer *et al.*[18] show that the goodness-of-fit can be improved considerably if a four-dimensional association term is included as well.

2.4 Combinations of Arbitrary Copulae into a New One

Morillas' construction scheme in (2.14) can be seen as a distortion of a single but arbitrary copula. Similarly, one might be interested in constructing a new copula C from d given copulae $C_1, \ldots, C_d$ in order to increase flexibility and/or introduce asymmetry. A simple way is to consider linear combinations, where the weights sum up to one, i.e.,

$$C(\mathbf{u}) \equiv \alpha_1 C_1(\mathbf{u}) + \cdots + \alpha_d C_d(\mathbf{u}) \quad \text{with } \alpha_1 + \cdots + \alpha_d = 1. \tag{2.17}$$

Putting things differently, the copula in (2.17) results from a weighted arithmetic mean of $C_1, \ldots, C_d$. Klein *et al.*,[26] more generally, deal with conditions on the copulae such that the weighted Hölder mean of two copulae is again a copula. Recently, Liebscher[30] has discussed products of n-copulae of the form

$$C(u_1, \ldots, u_n) = \prod_{j=1}^{d} C_j(g_{j1}(u_1), \ldots, g_{jn}(u_n))$$

with a set of $d \cdot n$ admissible functions $g_{11}, \ldots, g_{1n}, \ldots, g_{d1}, \ldots, g_{dn}$, each of which, being bijective, monotonously increasing or identically equal to 1, satisfy

$$\prod_{j=1}^{d} g_{ji}(v) = v, \quad i = 1, \ldots, n. \tag{2.18}$$

Note that (2.18) reduces to $g_{1i}(v) = v$ for $d = 1$ and $i = 1, \ldots, n$, and C is recovered. In accordance with Liebscher,[30] the possible choices are

$$g_{ji}(v) \equiv v^{\theta_{ji}} \text{ with } \theta_{ji} > 0 \quad \text{and} \quad \sum_{j=1}^{d} \theta_{ji} = 1 \text{ for } i = 1, \ldots, n$$

or, for $\theta > 0$ and $\alpha \in (0, 1)$,

$$g_{1i}(v) \equiv f(v), \quad g_{2i}(v) \equiv \frac{v}{f(v)}, \quad f(v) = \left(\frac{1 - e^{-\theta_i v}}{1 - e^{-\theta_i}}\right)^{\alpha}.$$

Finally, Fischer and Köck[17] develop a construction scheme which includes both Morillas copulae in (2.14) and Liebscher copulae in (2.15) as special cases. The key idea of Morillas[35] is to replace the independence copula (which is implicitly assumed within the multiplicative Archimedean framework) with an arbitrary copula C and to prove that the new function is a copula, too. Taking a closer look at (2.15), one might be tempted to replace the product with an arbitrary n-copula in order to extend Liebscher's proposal. Assuming that Ψ is absolutely monotonic of order d and ψ_{ij} is differentiable and monotone increasing with $\psi_{ij}(0) = 0$, $\psi_{ij}(1) = 1$ and that $\Psi(\frac{1}{m}\sum_{j=1}^{m} \psi_{jk}(v)) = v$, and $C_1, \ldots, C_m$ are arbitrary copulae with existing copula densities, Fischer and Köck[17] showed that

$$C(u_1, \ldots, u_n) = \Psi\left(\frac{1}{m}\sum_{j=1}^{m} C_j(\psi_{j1}(u_1), \ldots, \psi_{jn}(u_n))\right), \quad m \geq 1, \;\; n \geq 2$$

is again a copula.

2.5 Summary

Whereas copulae seem to be well-explored in the bivariate case, there are several open issues in the multivariate setting. In particular, the construction of multivariate copula models which allow us to rebuild various types of dependencies and admit closed-form representations (at least for the copula density) in order to perform fast and easy parameter estimation is a challenging task. Within this chapter, we reviewed both popular copula classes and different construction schemes which emerged in the previous literature. Apart from pair-copula constructions, which are the focus of this book, special emphasis was put on elliptical copulae and selected generalizations as well as on generalized Archimedean copulae.

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