Chapter 7

Collisions and Scattering

7.1 Collinear, Elastic Collisions

A particle of mass m_1 and velocity $\vec{v}_1$ approaches a particle of mass m_2 moving with velocity $\vec{v}_2$; after the collision the velocities are $\vec{v}\,'_1$ and $\vec{v}\,'_2$, respectively, as shown.

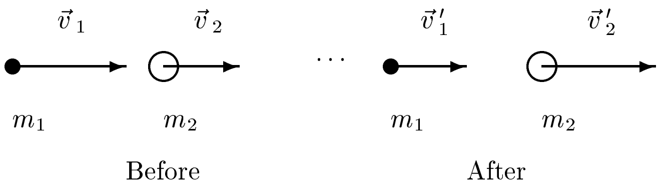

Figure 7.1: Collinear Collision

The total momentum is conserved for this isolated collinear collision:

$$m_1\, v_1 + m_2\, v_2 = m_1\, v_1' + m_2\, v_2' \tag{7.1}$$

Let us suppose that the kinetic energy of motion is also conserved (an Elastic Collision), so that

$$\frac{1}{2} m_1\, v_1^2 + \frac{1}{2}\, m_2\, v_2^2 = \frac{1}{2} m_1\, v_1'^2 + \frac{1}{2}\, m_2\, v_2'^2 \tag{7.2}$$

One might expect elastic collisions to occur between small elementary objects. In addition, they may occur when objects interact through their gravitational or electromagnetic fields and then separate.

We may cast these equations into the form

$$m_1\,(v_1^2 - v_1^{'2}) = m_2\,(v_2^{'2} - v_2^2)$$

$$m_1\,(v_1 - v_1') = m_2\,(v_2' - v_2)$$

Taking the ratio of these equations, we obtain

$$v_1 + v_1' = v_2' + v_2$$

$$\text{or} \qquad v_1 - v_2 = v_2' - v_1' \tag{7.3}$$

Thus, the relative speed of approach of the particles before the collision is equal to their relative speed of separation after the collision.

When a light object collides with a much heavier one ($m_1 \ll m_2$), the light object bounces off the heavy one. When the two objects are initially approaching one another ($v_1 > 0$ and $v_2 < 0$), the light object acquires speed, whereas the speed of the heavy object is virtually unchanged:

$$|v_1'| = |v_1| + 2|v_2|$$

This effect is the basis for the Gravitational Slingshot. A projectile makes a hyperbolic orbit around a much heavier body planet or asteroid), and thereby increases its speed by twice the orbital speed of the much heavier body. For example, the Cassini satellite, launched in 1997, received boosts from Venus (twice), the earth, and Jupiter, arriving near Saturn in 2004.

In an Inelastic Collision, a certain amount Q of mechanical energy is converted into heat. For this case the relation (7.1) remains valid, whereas Eq.(7.2) is replaced by

$$\frac{1}{2}\,m_1 v_1^2 + \frac{1}{2}\,m_2\,v_2^2 = Q + \frac{1}{2}\,m_1 v_1^{'2} + \frac{1}{2}\,m_2\,v_2^{'2}$$

Into this equation we make the substitution

$$v_2' = v_2 + \frac{m_1}{m_2}\,(v_1 - v_1')$$

which is obtained from Eq.(7.1), to obtain

$$m_1\,m_2\,(v_1 - v_1')\,(v_1 + v_1' - 2v_2) = 2m_2\,Q + m_1^2\,(v_1 - v_1')^2$$

For fixed incident velocities v_1, v_2, the maximum heat lost during the collision is obtained by differentiating this relation with respect to v_1', and setting

$$\frac{dQ}{dv_1'} = 0$$

We obtain

$$v_1' = \frac{m_1\,v_1 + m_2\,v_2}{m_1 + m_2} = v_2'$$

$$Q_{max} = \frac{1}{2}\frac{m_1\,m_2}{m_1 + m_2}(v_1 - v_2)^2$$

For this Maximally Inelastic Collision the particles stick together after the collision.

7.2 Classical Zeno Process

"Il n'est pas sûr que la Nature soit simple. Pouvons-nous sans danger faire comme si elle l'était?"

- Henri Poincaré

Let us consider a mass $m = m_0$ and initial speed $u = u_0$ to the left, which collides elastically and collinearly in sequence with the masses $m_1, \ldots, m_N$, initially at rest along the incident direction, as shown.[1]

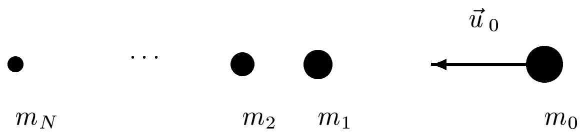

Figure 7.2: Zeno Process

Let us define the mass ratio

$$\mu_n = \frac{m_{n+1}}{m_n} \qquad \text{and} \qquad n = 0, 1, \ldots, N-1$$

where we require $\mu_n < 1$ for all n. After the collision the $(n+1)$-st ball goes to the left with velocity

$$u_{n+1} = \frac{2m_n}{m_n + m_{n+1}}\, u_n = \frac{2}{1+\mu_n}\, u_n = \qquad \text{where} \qquad n = 0, 1, \ldots, N-1$$

whereas the recoil velocity of the n-th ball after its second collision is

$$v_n = u_{n+1} - u_n = \frac{m_n - m_{n+1}}{m_n + m_{n+1}}\, u_n \qquad \text{where} \qquad n = 0, 1, \ldots, N-1$$

We obtain

$$v_n = \frac{1-\mu_n}{1+\mu_n}\, u_n = \frac{1-\mu_n}{1+\mu_n}\, \frac{2}{1-\mu_{n-1}}\, v_{n-1}$$

Note that the recoil velocity v_n is positive for all n. We may express the speed u_n in terms of the initial speed:

$$u_n = u_0 \prod_{j=0}^{n-1} \frac{2}{1+\mu_j}$$

[1]A thorough discussion of this type of sequential collision has been given by David Atkinson, "Losing energy in classical, relativistic and quantum mechanics". Studies in History and Philosophy of Modern Physics 38, (2007) 170-180.

We also wish to require that v_n be monotically non-decreasing with n, so that subsequent collisions between the masses do not occur after the N-th mass is struck. This requirement is equivalent to the conditions

$$\frac{1-\mu_n}{1+\mu_n} \geq \frac{1-\mu_{n-1}}{2} \tag{7.4}$$

for all n. The final particle, with mass m_N and final velocity u_N, is struck only once – after which it has kinetic energy T_N and momentum P_N, where

$$2T_N = m_N\,u_N^2 = m_0\,u_0^2 \prod_{n=0}^{N-1} \frac{4\mu_n}{(1+\mu_n)^2} = m_0\,u_0^2 \prod_{n=0}^{N-1} \left[1 - \left(\frac{1-\mu_n}{1+\mu_n}\right)^2\right]$$

$$P_N = m_N\,u_N = m_0\,u_0 \prod_{n=0}^{N-1} \frac{2\mu_n}{1+\mu_n} = m_0\,u_0 \prod_{n=0}^{N-1} \left[1 - \frac{1-\mu_n}{1+\mu_n}\right] \tag{7.5}$$

Momentum and energy conservation correspond, respectively, to the relations

$$m_0\,u_0 = \sum_{n+0}^{N-1} m_n\,v_n + P_N$$

$$m_0\,u_0^2 = \sum_{n+0}^{N-1} m_n\,v_n^2 + 2T_N$$

Suppose that an infinite sequence of masses are prepared, for which the total mass

$$M = \sum_n m_n$$

is finite, and the collisions all take place within a finite time interval. It must be true that $m_n \to 0$ at large n. Furthermore, it follows from Eq.(7.5) that

$$m_n\,u_n^2 \leq m_0\,u_0^2$$

$$(m_n\,u_n)^2 \leq m_n\,(m_0\,u_0^2)$$

Thus, $m_n u_n$ approaches zero at large n, so that the total momentum is conserved. Remarkably, conservation of energy does not follow, as is seen with the following example:

$$m_n = \frac{2\,m}{(n+1)\,(n+2)} = 2\,m\left[\frac{1}{n+1} - \frac{1}{n+2}\right]$$

The total mass is

$$M = \sum_{n=0}^{\infty} m_n = 2m \sum_{n=0}^{\infty} \left[\frac{1}{n+1} - \frac{1}{n+2}\right] = 2\,m$$

In addition we have

$$\mu_n = \frac{m_{n+1}}{m_n} = \frac{n+1}{n+3}$$
$$u_{n+1} = \frac{2}{1+\mu_n} u_n = \frac{n+3}{n+2}\, u_n$$

As a consequence

$$u_n = \frac{n+3}{2}\, u_0 \qquad \text{and} \qquad v_n = u_{n+1} - u_n = \frac{u_0}{2}$$

Since all particles move away with the same final speed, there are no further collisions in this case. Note that

$$\lim_{N\to\infty} 2T_N = \lim_{N\to\infty} m_n\, u_N^2 = \lim_{N\to\infty} \frac{2m}{(N+1)(N+2)} \left(\frac{N+3}{2}\, u_0\right)^2 = \frac{1}{2} m\, u_0^2$$

The final momentum and energy are

$$P_\infty = \sum_{n=0}^{\infty} m_n\, v_n = \frac{1}{2} M\, u_0 = m\, u_0$$
$$2T_\infty = \sum_{n=0}^{\infty} m_n\, v_n^2 = \frac{1}{4} M\, u_0^2 = \frac{1}{2} m\, u_0^2$$

The total momentum is conserved, whereas half of the mechanical energy has disappeared in this Zeno Process.

This collision sequence is very puzzling from a physical point of view. One interpretation of the sequence is that, while the collisions between the individual masses are elastic, the system as a whole undergoes an inelastic collision, in which momentum is conserved but mechanical energy is lost. (See Problem 7.1.)

This choice of masses is not unique. For example, see Problem 7.2.

7.3 Non-collinear Elastic Collisions

First we analyze the non-collinear collision in the center of mass frame.

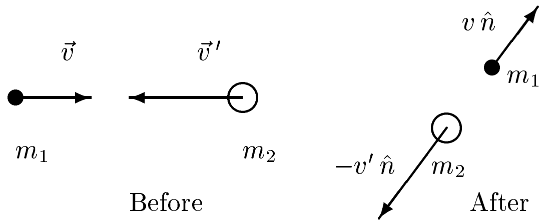

Figure 7.3: Elastic Collision in C of M Frame

The particles collide with equal and opposite momenta; $m_1\,v = m_2\,v'$, and after the collision they have the same speeds, but move in new directions $\pm\hat{n}$, where $\hat{n}$ is determined by the details of the collision (see Figure 7.3). Here are the velocities before and after the collision:

$$\begin{array}{llcll} m_1 : \text{Before} & v\,\hat{i} & \ldots & \text{After} & v\,\hat{n} \\ m_2 : \text{Before} & -(m_1/m_2)\,v\,\hat{i} & \cdots & \text{After} & -(m_1/m_2)\,v\,\hat{n} \end{array}$$

When viewed in the laboratory frame, in which the projectile m_1 approaches the target m_2 at rest, with velocity $\vec{v}_1 = v_1\hat{i}$, one must add the velocity of the center of mass:

$$\vec{v}_{cm} = \frac{m_1}{m_1 + m_2}\,\vec{v}_1$$

to these velocities. Because the velocity $\vec{v}_2$ is zero, one must have

$$\vec{v}_{cm} = \frac{m_1}{m_2}\,\vec{v}$$

$$\vec{v}_1 = \left(1 + \frac{m_1}{m_2}\right)\,\vec{v}$$

The velocities $\vec{v}\,'_1$ and $\vec{v}\,'_2$ are thus given by

$$\vec{v}\,'_1 = v\,\left(\frac{m_1}{m_2}\,\hat{i} + \hat{n}\right) = \frac{m_2}{m_1 + m_2}\,v_1\,\left(\frac{m_1}{m_2}\,\hat{i} + \hat{n}\right)$$

$$\vec{v}\,'_2 = \frac{m_1}{m_2}\,v\,\left(\hat{i} - \hat{n}\right) = \frac{m_1}{m_1 + m_2}\,v_1\,\left(\hat{i} - \hat{n}\right)$$

Note that the target mass m_1 always has a forward velocity component after the collision. Furthermore, when $m_1 \geq m_2$, the projectile m_2 always has a forward velocity component. Of special interest is the collision of equal masses $m_1 = m_2$, for which

$$\vec{v}\,'_1 \cdot \vec{v}\,'_2 = 0$$

so that the outgoing particles travel at an angle of 90° to one another.

7.4 Scattering from Hard Sphere

Let us consider the scattering of a freely moving point mass from a fixed hard sphere of radius R. The mass is moving toward the sphere at an impact parameter b with respect to the symmetry axis in the scattering plane, as shown:

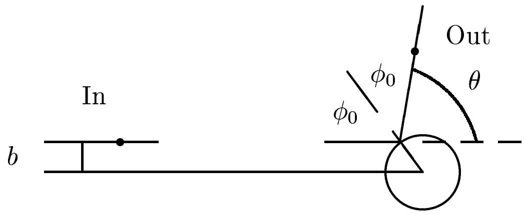

Figure 7.4: Scattering from a Hard Sphere

Note that the particle misses the sphere if $b > R$. Otherwise, it is reflected off the hard sphere, being scattered through an angle $\theta = \pi - 2\,\phi_0$, so that

$$b = R\,\sin\phi_0 = R\,\cos\frac{\theta}{2}$$

Particles with impact parameter b are scattered by an angle θ with respect to the incident direction. Those within an infinitesimal annulus of radius b and width db, with cross sectional area $d\sigma = 2\pi\,b\,db$, are scattered within an infinitesimal solid angle $d\Omega = 2\pi\,\sin\theta\,d\theta$. Thus

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \tag{7.6}$$

gives the Differential Scattering Cross Section

$$\frac{d\sigma}{d\Omega} = \frac{1}{\sin\theta}\left(R\,\cos\frac{\theta}{2}\right)\cdot\left(\frac{R}{2}\,\sin\frac{\theta}{2}\right) = \frac{R^2}{4}$$

Scattering from the hard sphere is isotropic, with equal probability for recoil in any direction. We integrate over the solid angle of the outgoing particle to obtain the Total Scattering Cross Section

$$\sigma_T = \pi\,R^2$$

The problem of scattering from two fixed spheres is considerably more complicated, since it depends upon the locations of each sphere in relation to the direction of the incident particle. Furthermore, for certain configurations the projectile may spend an infinite amount of time bouncing between the spheres. For an extended discussion of this matter see José and Salatan, Classical Dynamics: A Contemporary Approach, Section 4.1.3.

7.5 Scattering by Central Potential

A particle of mass m and initial speed v_0 approaches a center of force with impact parameter b, subject to the repulsive potential $V(r)$. Let us describe its motion in the scattering plane in terms of polar coordinates (r, ϕ), as shown.

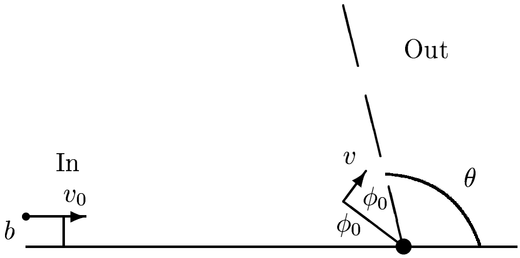

Figure 7.5: Scattering by Central Potential

The Lagrangian is

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r)$$

The energy and angular momentum are constants of the motion:

$$E = \frac{1}{2}m\,v_0^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + V(r)$$
$$\ell = m\,v_0\,b = m\,r^2\dot{\phi}$$

We use the second of these relations (angular momentum conservation) to write

$$\dot{\phi} = \frac{\ell}{m\,r^2} = \frac{v_0\,b}{r^2}$$
$$\dot{r} = \frac{dr}{d\phi}\,\dot{\phi} = \frac{dr}{d\phi}\,\frac{v_0\,b}{r^2}$$

We insert these formulas into the first relation (energy conservation) to obtain an equation for the trajectory $r(\phi)$:

$$\frac{1}{2}m\,v_0^2 = \frac{m\,v_0^2\,b^2}{2r^4}\left[\left(\frac{dr}{d\phi}\right)^2 + r^2\right] + V(r) \tag{7.7}$$

Next we insert the formula for the energy into this relation to obtain

$$y^2(r) \equiv r^2\left[1 - \frac{V(r)}{E}\right] = b^2\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\phi}\right)^2\right] \tag{7.8}$$

or

$$d\phi = \frac{dr}{r} \frac{b}{\sqrt{y^2(r) - b^2}}$$

We thus obtain an expression for the scattering angle θ:

$$\theta = \pi - 2\phi_0 = \pi - 2b \int_R^\infty \frac{dr}{r} \frac{1}{\sqrt{y^2(r) - b^2}} \tag{7.9}$$

where R is the distance of closest approach to the scattering center. Let us assume that $y(r)$ is monotonic, and change the integration variable from r to y:

$$\int_R^\infty \frac{dr}{r} \frac{1}{\sqrt{y^2(r) - b^2}} = \int_b^\infty \frac{dy}{\sqrt{y^2 - b^2}} \left(\frac{d}{dy} \log r(y) \right)$$

In addition, we use the identity

$$\pi = 2b \int_b^\infty \frac{dy}{y\sqrt{y^2 - b^2}} = 2\,b \int_b^\infty \frac{dy}{\sqrt{y^2 - b^2}} \left(\frac{d}{dy} \log y \right)$$

to obtain

$$\theta = 2b \int_b^\infty \frac{dy}{\sqrt{y^2 - b^2}} \left(\frac{d}{dy} \log \frac{y}{r(y)} \right) \tag{7.10}$$

We may perform the integration to determine the scattering angle at impact parameter b, and then use Eq.(7.6) to determine the scattering cross section.

Let us consider small angle scattering from the repulsive potential that has the asymptotic form

$$V(r) \to \frac{\kappa}{r^n}$$

at large r. It follows from Eq.(7.8) that, at large r,

$$\log \frac{y}{r} = \frac{1}{2} \log \left[1 - \frac{V(r)}{E} \right] \approx \frac{1}{2} \log \left[1 - \frac{\kappa}{E\,r^n} \right] \approx -\frac{\kappa}{2\,E\,r^n}$$

Since V approaches zero at large r, we may make the replacement $y = r$ on the right side to get this asymptotic formula from Eq.(7.10):

$$\theta = 2b \int_b^\infty \frac{dy}{\sqrt{y^2 - b^2}} \frac{d}{dy} \left(-\frac{\kappa}{2Ey^n} \right) = \frac{n\,\kappa\,b}{E} \int_b^\infty \frac{dy}{\sqrt{y^2 - b^2}} \frac{1}{y^{n+1}}$$

Let us make the substitution $y = b \sec\theta$ in the integral:

$$\int_b^\infty \frac{dy}{\sqrt{y^2 - b^2}} \frac{1}{y^{n+1}} = \frac{1}{b^{n+1}} \int_0^{\pi/2} d\theta\, \cos^n \theta = \frac{2^{n-1}}{b^{n+1}} \frac{\Gamma^2((n+1)/2)}{\Gamma(n+1)}$$

We thereby obtain

$$\theta = \frac{\kappa}{2E} \frac{\Gamma^2((n+1)/2)}{\Gamma(n)} \left(\frac{2}{b}\right)^n$$

Thus at large impact parameter b, or at small θ, we obtain

$$b = 2\left\{\frac{\kappa}{2E\theta} \frac{\Gamma^2((n+1)/2)}{\Gamma(n)}\right\}^{1/n} \tag{7.11}$$

and for θ small we use Eq.(7.6) to get

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta} \left|\frac{db}{d\theta}\right| \approx \left\{\frac{\kappa}{2E} \frac{\Gamma^2((n+1)/2)}{\Gamma(n)}\right\}^{2/n} \frac{4/n}{\theta^{2+2/n}}$$

In particular, for the repulsive Coulomb potential ($n = 1$) we obtain the following small angle asymptotic formula:

$$\frac{d\sigma}{d\Omega} \approx \frac{\kappa^2}{E^2\,\theta^4} \tag{7.12}$$

This scattering problem is solved in detail in the next section.

7.6 Rutherford Scattering

We next consider the scattering from the repulsive Coulomb potential

$$V(r) = \frac{\kappa}{r}$$

Let us express the trajectory equation (7.7) in terms of the inverse radius, $u = 1/r$:

$$\frac{1}{2}m\,v_0^2 = \frac{1}{2}m\,v_0^2\,b^2\left(\left(\frac{du}{d\phi}\right)^2 + u^2\right) + \kappa\,u \tag{7.13}$$

In particular, it follows that at $\phi = 0$, $r = \infty$, or $u = 0$, as well as $du/d\phi = 1/b$. Let us differentiate Eq.(7.13) to obtain

$$\frac{d^2u}{d\phi^2} + u + \frac{\kappa}{m\,v_0^2\,b^2} = 0$$

The following solution is consistent with the above conditions at $\phi = 0$:

$$\frac{1}{r(\phi)} = u(\phi) = -\frac{\kappa}{m\,v_0^2\,b^2}\,(1-\cos\phi) + \frac{1}{b}\,\sin\phi \tag{7.14}$$

The particle is repelled and slowed down by the repulsive force upon approach, traveling along a hyperbolic path. The value of ϕ_0 for the asymptotic direction of the particle is obtained by setting $u = 0$ in (7.14):

$$\frac{\sin\phi_0}{1-\cos\phi_0} = \cot\frac{\phi_0}{2} = \frac{\kappa}{m\,v_0^2\,b}$$

The angle of scattering is $\theta = \pi - 2\phi_0$, so that

$$b = \frac{\kappa}{m\,v_0^2}\,\cot\theta/2 \tag{7.15}$$

We then use Eq.(7.6) to determine the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \left(\frac{\kappa}{m\,v_0^2}\right)^2 \frac{1}{\sin\theta}\,\cot\theta/2\,\frac{\csc^2\theta/2}{2} = \left(\frac{\kappa}{4\,E}\right)^2 \frac{1}{\sin^4\theta/2}$$

This result is consistent with Eq.(7.12) at small angles.

At the distance of closest approach, R, the speed of the particle is v. We may determine v and R from the conditions of energy and angular momentum conservation:

$$\ell = m\,v_0\,b = m\,v\,R$$
$$\frac{1}{2}m\,v_0^2 = \frac{1}{2}m\,v^2 + \frac{\kappa}{R}$$

We obtain

$$R^2 = b^2 + \frac{2\kappa\,R}{m\,v_0^2}$$
$$R = \frac{\kappa}{m\,v_0^2} + \sqrt{b^2 + \left(\frac{\kappa}{m\,v_0^2}\right)^2}$$
$$v = \frac{b}{R}\,v_0$$

The differential scattering cross section for the attractive Coulomb potential ($\kappa < 0$) is the same as that obtained in the repulsive case, although the scattering angle θ is always negative, because of the force is always attractive.

7.7 Cross Section for Repulsive $1/r^4$ Potential

Let us analyze scattering from the potential

$$V(r) = \frac{\kappa}{r^4}$$

We determine the function $y(r)$, given in Eq.(7.8):

$$y^2(r) = r^2\left[1 - \frac{\kappa}{E\,r^4}\right] = r^2 - \frac{\kappa}{E\,r^2}$$

and use formula (7.9) to express the scattering angle θ at a particular impact parameter b:

$$\theta = \pi - 2\,b \int_R^\infty \frac{dr}{\sqrt{r^4 - b^2\,r^2 - \kappa/E}}$$

Let us scale out the energy dependence by introducing these variables

$$r' = r\left(\frac{E}{\kappa}\right)^{1/4} \quad \text{and} \quad R' = R\left(\frac{E}{\kappa}\right)^{1/4} \quad \text{and} \quad b' = b\left(\frac{E}{\kappa}\right)^{1/4}$$

We obtain

$$\theta = \pi - 2\,b' \int_{R'}^\infty \frac{dr'}{\sqrt{r'^4 - b'^2\,r'^2 - 1}}$$

The minimum distance is R', where

$$R'^2 = \frac{b'^2}{2} + \sqrt{\frac{b'^4}{4} + 1}$$

and we obtain

$$\theta = \pi - 2b' \int_{R'}^\infty \frac{dr'}{\sqrt{(r'^2 - R'^2)\,(r'^2 + R'^2 - b'^2)}}$$

With the trigonometric substitution $r' = R' \sec\phi$, we obtain

$$\begin{aligned}
\theta &= \pi - \frac{2b'}{R'} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 + (1 - b'^2/R''2)\,\cos^2\phi}} \\
&= \pi - \frac{2b'}{\sqrt{2R'^2 - b'^2}} \int_0^{\pi/2} d\phi \left[1 - \frac{R'^2 - b'^2}{2R'^2 - b'^2}\,\sin^2\phi\right]^{-1/2} \\
&= \pi - \frac{2b'}{\sqrt{2R'^2 - b'^2}}\,K(k)
\end{aligned}$$

where $K(k)$ is the complete elliptic integral of the first kind, introduced in Section 2.5, with

$$k^2 = \frac{R'^2 - b'^2}{2R'^2 - b'^2} = \frac{1}{2}\left[1 - \frac{b'^2}{\sqrt{b'^4 + 4}}\right]$$

Note that $0 \le k^2 \le 1/2$. Equivalently,

$$b'^2 = \frac{1 - 2k^2}{k\sqrt{1 - k^2}}$$

Since

$$1 - 2k^2 = \frac{b'^2}{2R'^2 - b'^2} = \frac{b'^2}{\sqrt{b'^4 + 4}}$$

we obtain

$$\theta = \pi - 2\sqrt{1 - 2k^2}\,K(k) \tag{7.16}$$

A graph of b' versus θ (radians) is shown in Figure 7.6. Note that $\theta \to \pi$ at small b', whereas at large b', $\theta \to 3\pi/(4\,b'^4)$, in accord with Eq.(7.11).

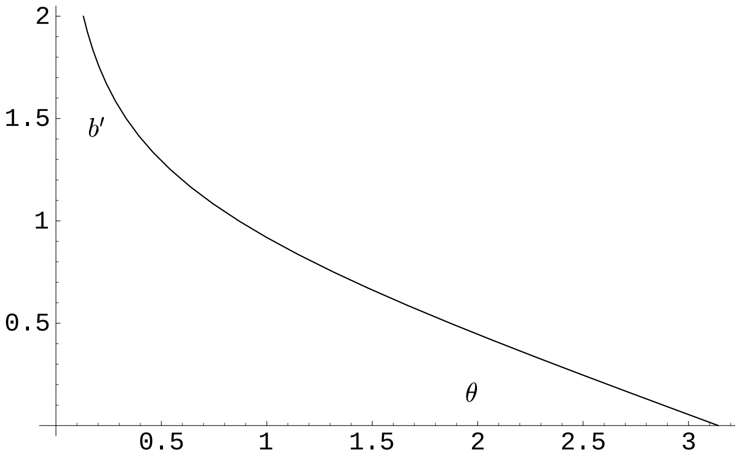

Figure 7.6: Scaled Impact Parameter b' versus Scattering Angle θ

Here is the Mathematica program that was used to obtain this graph:

```
b0 = 2.0;
theta[ksq_] := Pi -  2 Sqrt[1 - 2 ksq]  EllipticK[ksq] ;
ksq[b_] :=  (1  - b^2/Sqrt[b^4 + 4]) /2;
ParametricPlot[ {theta[ksq[b]], b}, {b, 0, b0} ];
```

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\sin\theta}\left|\frac{db^2}{d\theta}\right| = \frac{1}{2\sin\theta}\sqrt{\frac{\kappa}{E}}\left|\frac{db'^2}{d\theta}\right|$$

We employ the chain rule

$$\frac{db'^2}{d\theta} = \frac{db'^2}{dk}\frac{dk}{d\theta}$$

From relation (7.16) for $\theta(k)$, we obtain

$$\begin{aligned}\frac{d\theta}{dk} &= \frac{2}{\sqrt{1-2k^2}}\left\{2k\,K(k) - (1-2k^2)\,K'(k)\right\}\\ &= \frac{2}{\sqrt{k\,(1-2k^2)}}\left\{K(k) - \frac{1-2k^2}{1-k^2}\,E(k)\right\}\end{aligned}$$

and

$$\frac{db'^2}{dk} = -\frac{1}{k^2(1-k^2)^{3/2}}$$

Using the definition of the complete Elliptic Integrals of the First and Second Kind given in Problem 2.4, and an identity obtained there, one may show that

$$2k\,K(k) - (1-2k^2)\,\frac{dK}{dk} = \frac{1}{k}\left\{K(k) - \frac{1-2k^2}{1-k^2}\,E(k)\right\}$$

Note: this expression is positive, since $K(k) \geq E(k)$. Thus

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\sin\theta}\sqrt{\frac{\kappa}{E}}\,f(\theta)$$

$$f(\theta) = \frac{\sqrt{1-2k^2}}{k\,(1-k^2)^{3/2}}\,\frac{1}{|K(k) - (1-2k^2)\,E(k)/(1-k^2)|}$$

Since the quantity k is independent of the energy, and the dependence on the scattering angle θ is expressed in relation (7.16), the dependence of the differential cross section upon energy is consistent with the scaling result given in Problem 7.3. A graph of $f(\theta)$ versus θ (radians) is shown in Figure 7.7.

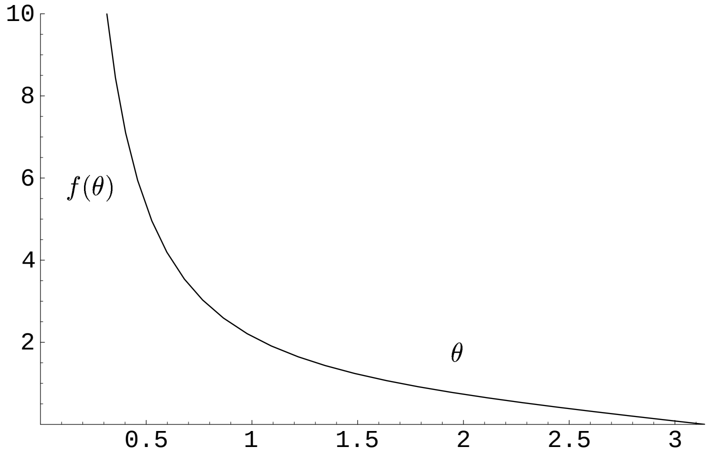

Figure 7.7: Scaled Cross Section $f(\theta)$ versus Scattering Angle θ

Here is the Mathematica program that was used to obtain the graph:

```
b0 = 1.6;
theta[k2_] := Pi -  2 Sqrt[1 - 2 k2]  EllipticK[k2] ;
ksq[b_] :=  (1  - b^2/Sqrt[b^4 + 4]) /2;
sigma[k2_] :=
```

```
    Sqrt[1 - 2 k2] / (Sqrt[k2] (1 - k2)^(3/2)) /
    (EllipticK[k2] - (1 - 2  k2)  EllipticE[k2] / ( 1 - k2));
ParametricPlot[{theta[ksq[b]], sigma[ksq[b]] }, {b,0,b0},
    PlotRange  -> { {0, Pi}, {0, 10}}]
```

7.8 Inverse Scattering Problem

To solve the inverse problem of obtaining the central potential $V(r)$ from scattering data, we employ relation (7.6) to determine the impact parameter b as a (monotonically decreasing) function of the scattering angle θ:

$$b\,\frac{db}{d\theta} = -\sin\theta\,\frac{d\sigma}{d\Omega}$$
$$b^2(\theta) = 2\int_\theta^\pi d\theta\,\sin\theta\,\frac{d\sigma}{d\Omega}$$

We then use $b(\theta)$ to determine the potential $V(r)$ by solving the inverse relation to (7.10) to get the function $y(r)$, defined by Eq.(7.8).

To invert (7.10) we introduce the function

$$\begin{aligned}T(y) &= \frac{1}{\pi}\int_y^\infty \frac{db}{\sqrt{b^2-y^2}}\,\theta(b)\\ &= \frac{1}{\pi}\int_y^\infty \frac{2b\,db}{\sqrt{b^2-y^2}}\left[\int_b^\infty \frac{dy}{\sqrt{u^2-b^2}}\left(\frac{d}{du}\log\frac{u}{r(u)}\right)\right]\\ &= \frac{1}{\pi}\int_y^\infty du\left(\frac{d}{du}\log\frac{u}{r(u)}\right)J(y,u)\end{aligned}$$

where

$$J(y,u) = \int_y^u \frac{2\,b\,db}{\sqrt{(b^2-y^2)\,(u^2-b^2)}} = \pi$$

The last step is demonstrated through the trigonometric substitution $b^2 = u^2\cos^2\rho + y^2\sin^2\rho$. We thus obtain

$$T(y) = \int_y^\infty du\left(\frac{d}{du}\,\log\frac{u}{r(u)}\right) = \log\frac{r(y)}{y}$$

or

$$r(y) = y\,e^{T(y)} \tag{7.17}$$

Note that Eq.(7.10) may be written as

$$\theta(b) = -2b\int_b^\infty \frac{dy\,T'(y)}{\sqrt{y^2-b^2}}$$

We have implicitly assumed that $b(\theta)$ is monotonically decreasing in θ, so that its inverse $\theta(b)$ exists and can be determined. In addition, we have assumed that the function $r(y)$ in Eq.(7.17) is monotonically increasing in y, so that we may determine $V(r)$. The procedure is viable when the corresponding function $V(r)$ is monotonic in r. Also, note that the potential is not determined from these data when r is less than the minimum distance of approach of the particle to the scattering center. In particular, we get the potential only when $V(r) < E$.[2]

Not every axially symmetric differential cross section corresponds to an appropriate potential $V(r)$, since the scattering data depend upon energy as well as scattering angle, and thus the potential determined from the inverse problem may be energy dependent, as well. The simple scaling relation of the differential cross section for r^{-n} potentials is considered in Problem 7.3.

As an example, let the differential cross section be

$$\frac{d\sigma}{d\Omega} = \frac{\pi^2\, a^2}{\sin\theta} \frac{\pi - \theta}{\theta^2\,(2\pi - \theta)^2} \tag{7.18}$$

What potential $V(r)$ produces these scattering data? First, let us calculate the impact parameter $b(\theta)$:

$$b^2(\theta) = 2\pi^2\, a^2 \int_\theta^\pi \frac{(\pi - \theta)\, d\theta}{\theta^2\,(2\pi - \theta)^2}$$

We change the integration variable to $y = \pi - \theta$:

$$\begin{aligned} b^2(\theta) &= 2\pi^2\, a^2 \int_0^{\pi-\theta} \frac{y\, dy}{(\pi^2 - y^2)^2} = \pi^2\, a^2 \int_0^{\pi-\theta} d\left\{\frac{1}{\pi^2 - y^2}\right\} \\ &= \pi^2\, a^2 \left\{\frac{1}{\pi^2 - (\pi - \theta)^2} - \frac{1}{\pi^2}\right\} = a^2 \left\{\frac{\pi^2}{\pi^2 - (\pi - \theta)^2} - 1\right\} \\ &= a^2\, \frac{(\pi - \theta)^2}{\pi^2 - (\pi - \theta)^2} \end{aligned}$$

Let us then solve for $\theta(b)$:

$$\begin{aligned} (\pi - \theta)^2 &= \pi^2\, \frac{b^2}{a^2 + b^2} \\ \theta &= \pi \left\{1 - \frac{b}{\sqrt{a^2 + b^2}}\right\} \end{aligned} \tag{7.19}$$

We next determine $T(y)$:

$$T(y) = \frac{1}{\pi} \int_y^\infty \frac{db}{\sqrt{b^2 - y^2}}\, \theta(b) = \int_y^\infty \frac{b\, db}{\sqrt{b^2 - y^2}} \left\{\frac{1}{b} - \frac{1}{\sqrt{a^2 + b^2}}\right\}$$

[2] An excellent discussion of the inverse problem appears in R. Newton Scattering Theory of Waves and Particles, Section 5.9.

We make the replacement

$$\frac{1}{b} - \frac{1}{\sqrt{a^2 + b^2}} = \int_0^a \frac{x\,dx}{(b^2 + x^2)^{3/2}}$$

and change the order of integration to obtain

$$T(y) = \int_0^a dx\, x\, K(x, y)$$
$$K(x, y) = \int_y^\infty \frac{b\,db}{\sqrt{b^2 - y^2}} \frac{1}{(b^2 + x^2)^{3/2}}$$

Next we make the substitution $b = y \cos\phi$

$$K(x, y) = y \int_0^{\pi/2} d\phi\, \cos\phi \frac{1}{(x^2 + y^2 - x^2 \sin^2\phi)^{3/2}}$$

Then the replacement $u = \sin\phi\, x/\sqrt{x^2 + y^2}$:

$$K(x, y) = \frac{y}{x\,(x^2 + y^2)} \int_0^{x/\sqrt{x^2+y^2}} \frac{du}{(1 - u^2)^{3/2}}$$

Let us then substitute $u = \sin\rho$:

$$K(x, y) = \frac{y}{x\,(x^2 + y^2)} \int_0^{\sin^{-1}(x/\sqrt{x^2+y^2})} d\rho\, \sec^2\rho$$
$$= \frac{y}{x\,(x^2 + y^2)} \tan\left(\sin^{-1}(x/\sqrt{x^2 + y^2})\right) = \frac{1}{x^2 + y^2}$$

Thus

$$T(y) = \int_0^a \frac{x\,dx}{x^2 + y^2} = \frac{1}{2} \log\left(1 + \frac{a^2}{y^2}\right)$$

We thus obtain

$$r(y) = y\, e^{T(y)} = y \sqrt{1 + \frac{a^2}{y^2}} = \sqrt{a^2 + y^2}$$

Equivalently; $y = \sqrt{r^2 - a^2}$, so that

$$\sqrt{1 - \frac{V(r)}{E}} = \frac{y}{r} = \sqrt{1 - \frac{a^2}{r^2}}$$

Consequently,

$$V(r) = \frac{E\,a^2}{r^2}$$

The cross section (7.18) leads to the repulsive inverse square potential given above. For the attractive inverse square potential

$$V(r) = -\frac{\alpha}{r^2}$$

it follows from energy and angular momentum conservation that

$$\frac{1}{2}m\,\dot{r}^2 + \frac{m\,v_0^2\,b^2}{2\,r^2} - \frac{\alpha}{r^2} = E = \frac{1}{2}m\,v_0^2$$

As a consequence, if $b < b_{min} = \sqrt{\alpha/E}$, the mass will disappear into the singular region at small r. We may thus calculate its Total Capture Cross Section

$$\sigma_c = \pi\,b_{min}^2 = \frac{\pi\,\alpha}{E}$$

For this attractive potential we may obtain the scattering angle from Eq.(7.9)

$$\theta = \pi - 2\,b\int_R^\infty \frac{dr}{r}\,\frac{1}{\sqrt{y^2 - b^2}}$$

where

$$y = r\,\sqrt{1 - V(r)/E} = \sqrt{r^2 + \frac{\alpha}{E}}$$

Thus

$$\theta = \pi - 2\,b\int_R^\infty \frac{dr}{r}\,\frac{1}{\sqrt{r^2 - R^2}}$$

where $R = \sqrt{b^2 - \alpha/E}$. We make the substitution $r = R\,\sec\rho$ to obtain

$$\theta = \pi - \frac{2\,b}{R}\int_0^{\pi/2} d\rho = \pi\left(1 - \frac{b}{R}\right)$$

Equivalently,

$$b^2 = \frac{\alpha}{E}\,\frac{(\pi - \theta)^2}{(\pi - \theta)^2 - \pi^2}$$

The scattering angle θ is negative, because of the attractive force. However, the differential cross section has the same form for the two cases.

7.9 Exercises

Problem 7.1 Totally Inelastic Collision with elastic Zeno Balls

For certain rather special Zeno collision sequences, all masses proceed after the second round of collisions in "lock step" with each other at the same speed v, and subsequent collisions cannot occur. Use the relations

$$u_{n+1} = \frac{2}{1+\mu_n}\, u_n \qquad \text{and} \qquad v = u_{n+1} - u_n \tag{7.20}$$

Determine the permissible mass ratios μ_n for such collisions to occur. First eliminate the variable u_{n+1} from Eq.(7.20):

$$u_n = \frac{1+\mu_n}{1-\mu_n}\, v = \alpha_n\, v \qquad \text{where} \qquad \alpha_n = \frac{1+\mu_n}{1-\mu_n}$$

Then show that

$$\alpha_{n+1} = \alpha_n + 1 \qquad \text{or} \qquad \alpha_n = \lambda + n$$

where the parameter λ is arbitrary. Show that the mass ratios, the individual masses, and total mass of the struck balls (initially all at rest) are given by

$$\mu_n = \frac{\lambda + n - 1}{\lambda + n + 1}$$

$$\frac{m_n}{m_0} = \prod_{k=0}^{n-1} \mu_k = \frac{\lambda\,(\lambda - 1)}{(\lambda + n)\,(\lambda + n - 1)}$$

$$\frac{M}{m_0} = \sum_{n=1}^{\infty} \frac{m_n}{m_0} = \lambda - 1$$

Show that the velocities after the first round of collisions are

$$u_n = (\lambda + n)\, v \qquad \text{where} \qquad \lambda = \frac{u_0}{v} = \frac{\beta_0}{v}$$

$$u_n = \beta_0 + n\, v \quad = \quad \left(1 + \frac{n}{\lambda}\right) \beta_0$$

The initial momentum P_i and energy T_i are

$$P_i = m_0\, u_0 \qquad \text{and} \qquad 2T_i = m_0\, u_0^2$$

Show that their final values are

$$P_f = \sum_{n=0}^{\infty} m_n\, v = m_0\, u_0 \qquad \text{and} \qquad 2T_f = \sum_{n=0}^{\infty} m_n\, v^2 = \frac{m_0\, u_0^2}{\lambda}$$

Thus, momentum is conserved for this collision, whereas mechanical energy is lost. Why?

One may gain insight into the nature of this collisional process by comparing it to a simple totally inelastic collision, in which a mass m_0 moves with velocity u_0 toward a mass M that is initially at rest. After the collision the masses stick together and move in the same direction with speed v. Momentum is conserved in this collision, so that

$$m_0\, u_0 = (M + m_0), v = \lambda\, m_0\, v$$

Note that $\lambda v = u_0 = 1$, just as in the case of the colliding balls. Furthermore, the same amount of mechanical energy has been dissipated as for the colliding balls:

$$T_{lost} = \frac{1}{2}\,\frac{m_0^2}{m_0 + M}\,u_0^2 = \frac{m_0}{m_0 + M}\,T_{in}$$

The latter collision of a projectile (m_0, u_0) with a macroscopic target $(M, 0)$ has the same outcome as the former collisional sequence, in that all masses move with speed v after the collision. Indeed, the former collision sequence can be considered as a microscopic rendition of the latter. Namely, if the mass M is partitioned into separate masses $(m_1, m_2, \ldots)$, the individual masses will not be separated from one another (nor from the projectile) after the collision – even if there is no "long range attraction" to provide cohesion of the large mass M. This decomposition represents a "natural cleavage" of the mass M to describe inelastic collision with the projectile.

This "maximally inelastic collision" produced the greatest loss of mechanical energy that can occur when a mass m_0 moving at velocity u_0 collides with a mass M, which is initially at rest. The kinetic energy of the center of mass

$$T_{cm} = \frac{m_0}{m_0 + M}\,T_{in}$$

cannot change during the collision(s), and for this case the final relative kinetic energy is zero.

Problem 7.2 Additional Zeno Ball Problem

Consider the Zeno collision for the mass sequence

$$m_n = m_0\,\frac{24}{(n+1)\,(n+2)\,(n+3)\,(n+4)}$$

Show for this case that $m_n \to 0$, and that the total mass is $M = 4/3\, m_0$. Show that

$$\mu_n = \frac{n+1}{n+5} \to 1 - \frac{4}{n}$$

at large n, and then obtain

$$u_n = \frac{(n+3)\,(n+4)}{12}\,v_0$$
$$v_n = u_{n+1} - u_n = \frac{n+4}{6}\,v_0$$

Show that further collisions do not occur, since v_n is positive and monotonically increasing in n. Finally, show that the limiting energy of the n-th ball is

$$T_n = m_0\,v_0^2\,\frac{(n+3)\,(n+4)}{12(n+1)\,(n+2)} \to \frac{1}{12}\,m_0\,v_0^2$$

Thus, one-sixth of the initial energy is lost, where as five-sixths of it resides in the outgoing masses.

Problem 7.3 Scaling of the Cross Section for the $1/r^n$ Potential

Show that for the potential

$$V(r) = \frac{\kappa}{r^n}$$

the differential scattering cross section obeys the scaling relation

$$\frac{d\sigma}{d\Omega} = \left[\frac{\kappa}{E}\right]^{2/n} f(\theta)$$

This scaling relation greatly simplifies the determination of the cross sections for this class of potentials.

Problem 7.4 Square Well Potential

Consider the scattering of a particle of mass m which is incident upon the scattering center with potential

$$V(r) = \begin{bmatrix} 0 & r > a \\ -v_0 & r < a \end{bmatrix}$$

Note: The particle experiences a radial impulse at $r = a$. The parameter n is defined as

$$n = \sqrt{1 + \frac{v_0}{E}}$$

Show that the impact parameter b for scattering angle θ satisfies

$$b^2(\theta) = a^2\frac{n^2\,\sin^2\theta/2}{1 - 2n\cos\theta/2 + n^2}$$

Show that for $\cos\theta/2 > 1/n$ the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{n^2\,a^2}{4\cos\theta/2}\,\frac{(n\cos\theta/2 - 1)\,(n - \cos\theta/2)}{(1 - 2n\cos\theta/2 + n^2)^2}$$

Problem 7.5 Gravitational Capture

A projectile of mass m is fired with energy E at a solid spherically symmetric body of total mass M and radius R. Show that the total cross section for collision with the surface of the body is

$$\sigma_T = \pi R^2 \left(1 + \frac{GMm}{RE}\right)$$

Note that, in the limit of high speeds, this reduces to the geometrical cross section πR^2.

Problem 7.6 Capture Cross Section for attractive r^{-4} Potential

For the potential

$$V(r) = -\frac{\alpha}{r^4}$$

determine the minimum impact parameter for which particles of energy E are scattered, rather than absorbed. Show that the absorption cross section is

$$\sigma_a = 2\pi\sqrt{\frac{\alpha}{E}}$$

Problem 7.7 Inverse Scattering Problem

The differential cross section at a particular energy E is given as

$$\frac{d\sigma}{d\Omega} = \frac{\kappa^2}{\sin\theta}\,\frac{\pi - \theta}{\theta^3}$$

Show that, at large r,

$$V(r) \to \frac{\kappa}{r}$$

Show that the impact parameter $b(\theta)$ is

$$b(\theta) = a\,\frac{\pi - \theta}{\theta} \qquad \text{where} \qquad a^2 = \frac{\kappa^2}{\pi}$$

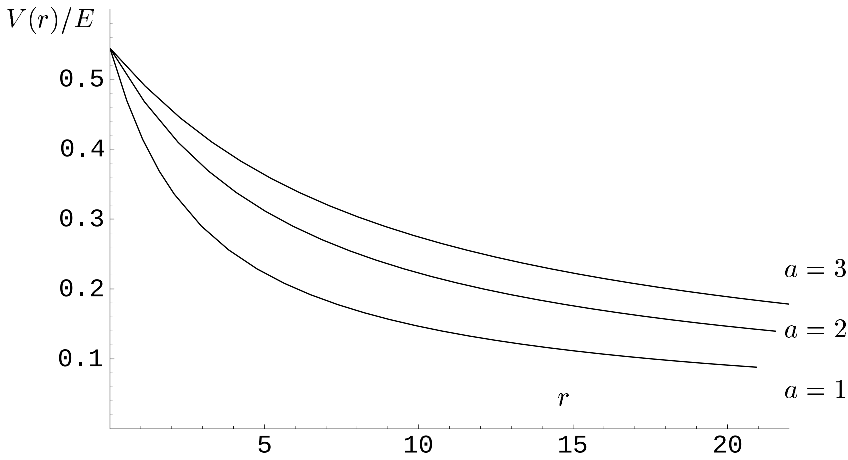

Figure 7.8: Scaled Potential $V(r)/E$ versus r for $a = 1, 2, 3$

Establish that the function $T(y)$ is given by

$$T(y) = \begin{cases} a/[2\sqrt{a^2 - y^2}]\,\tan^{-1}\sqrt{(a-y)/(a+y)} & y < a \\ 1/4 & y = a \\ a/[4\sqrt{y^2 - a^2}]\,\log\left([\sqrt{y+a} + \sqrt{y-a}]/[\sqrt{y+a} - \sqrt{y-a}]\right) & y > a \end{cases}$$

Determine the potential $V(r)/E$ numerically. The graphical solution is given for $a = 1, 2, 3$ in Figure 7.8.

Here is the Mathematica program that was used to obtain these results:

```
a1 = 1.0; a2 = 2; a3 = 3; amax = 20;
t[y_, a_] :=
  If[y >  a,
    Log[( Sqrt[y + a] + Sqrt[y - a])/
         (Sqrt[y + a] - Sqrt[y - a])] a /(4 Sqrt[y^2 - a^2]),
    ArcTan[Sqrt[(a - y)/(a + y) ]] a / (2 Sqrt[a^2 - y^2]),];
r[y_, tt_] := y  Exp[tt];
v[y_, rr_] := 1 - y^2 / rr^2;
ParametricPlot[{{r[y, t[y, a1]] , v[y, r[y, t[y, a1]] ] },
      {r[y, t[y, a2]] , v[y, r[y, t[y, a2]]]},
      {r[y, t[y, a3]] , v[y, r[y, t[y, a3]]]}} ,{y,0,amax},
        PlotRange -> {{0, 22}, {0, .6}}];
```

Problem 7.8 Cross Section

For the repulsive potential

$$V(r) = \frac{\kappa}{r} + \frac{\alpha}{r^2}$$

show that the relation between impact parameter b and scattering angle θ is

$$\theta = \pi - \frac{2b}{\sqrt{b^2 + \alpha/E}} \tan^{-1} \frac{\sqrt{b^2 + \alpha/E}}{\kappa/2E}$$

Show that for $\alpha = 0$ (pure Newtonian potential) this result reduces to Eq.(7.15), whereas for $\kappa = 0$ (pure inverse square potential) we obtain Eq.(7.19). Note that, for the mixed case, the scattering angle does not scale with energy.

Determine the differental cross section numerically as a function of the scattering angle for the case $\kappa = 3$, $\alpha = 3$, and $E = 1$. The graph for this case appears as Figure 7.9.

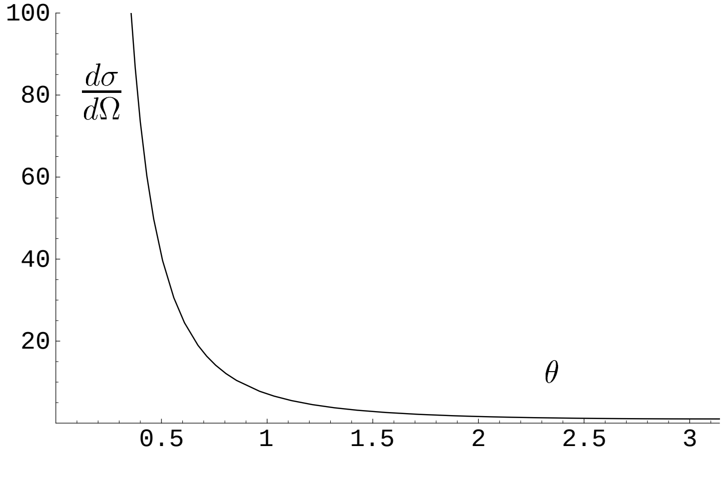

Figure 7.9: $d\sigma/d\Omega$ versus θ for the potential $V(r) = 3/r + 3/r^2$

Here is the Mathematica program that was used to obtain these results:

```
kappa0 = 3.0; alpha0 = 3.0; b0 = 100;
theta[b_, alpha_, kappa_] :=
  Pi - 2 b/Sqrt[b^2 + alpha] ArcTan[2 Sqrt[b^2 + alpha]/kappa];
thpr[b_, alpha_, kappa_] :=
```

```
  2 alpha/(b^2 + alpha)^(3/2) ArcTan[2 Sqrt[b^2 + alpha]/kappa]
   +  2 b^2 / (b^2 + alpha) /(1 + 2  Sqrt[b^2 + alpha]/ kappa);
sigma[b_, alpha_, kappa_] :=
  b / Sin[theta[b, alpha, kappa]] / thpr[b, alpha, kappa];
ParametricPlot[{theta[b,alpha0,kappa0],sigma[b,alpha0,kappa0]},
 {b, 0, b0}, PlotRange -> {{0, Pi}, {0, 100}} ];
```

Problem 7.9 Time Delay

The time delay in the scattering process can also be measured.

- Show that the time difference between the scattering path and the undeflected path at impact parameter b is

$$\Delta T(b) = \frac{2}{v_0}\int_0^{\phi_0} d\phi \left[\frac{r^2}{b} - \frac{b}{\sin^2\phi}\right]$$

  where ϕ_0 is the angle at the distance of closest approach.
- Using the trajectory (7.14), show that the time delay for Rutherford scattering

$$\Delta T(b) = \frac{2b}{v_0}\int_0^{\phi_0} d\phi \left[\frac{1}{[\sin\phi - 2\,\kappa\,(1-\cos\phi)/(E\,b)\,]^2} - \frac{1}{\sin^2\phi}\right]$$

  is infinite. This result is an indication that the undeflected path is approached very slowly at asymptotic distances.
- For the potential

$$V(r) = \frac{\alpha}{r^2}$$

  show that the trajectory at impact parameter b is

$$r(b) = \frac{\sqrt{b^2+\alpha/E}}{\sin(\phi\sqrt{1+\alpha/(E\,b^2)})}$$

  and that the time delay is

$$\Delta T(b) = \frac{2b}{v_0}\cot\left[\frac{\pi/2}{\sqrt{1+\alpha/(E\,b^2)}}\right]$$

  Note that this time delay becomes larger at smaller impact parameters.

Problem 7.10 Rutherford Scattering

Consider Coulomb scattering of 10 MeV alpha particles ($z = 2$) on a fixed gold foil of thickness one μm (10^{-6} m). determine the probability of scattering for θ greater than 90°. Note: For gold, $Z = 79$, $\rho = 1.9 \times 10^4\,\mathrm{kg/m^3}$, and $m_{Au} = 197\,M_p \approx 184\,\mathrm{MeV/c^2}$. Also,

$$\kappa = z\,Z\,\frac{e^2}{4\pi\,\epsilon_0} \qquad \text{where} \qquad \frac{e^2}{4\pi\,\epsilon_0} = 1.44\,\mathrm{MeV\ fm}$$

- Show that, if the alpha particle is scattered by an angle θ greater than 90°, it must come to within 65 fm of the gold nucleus.
- Show that the foil has about 6×10^{22} nuclei for every square meter of surface area.
- Show that the fraction of scattered alpha particles striking the foil at normal incidence that are scattered at 90° or greater is $f = 8 \times 10^{-4}$, neglecting shadowing.
- Show that, for randomly placed Gold nuclei, shadowing may be included through the replacement

$$f \to 1 - e^{-f}$$

Problem 7.11 Scattering from Hard Ellipsoid of Revolution

An ellipsoid of revolution, centered at the origin and symmetric about the ρ axis, is described in cylindrical coordinates by the equation

$$\frac{\rho^2}{\rho_0^2} + \frac{z^2}{z_0^2} = 1$$

A particle approaches this ellipsoid in the (horizontal) $+\rho$ direction at impact parameter $b = z$. The particle scatters elastically off the surface of the ellipsoid. Show that the direction of the particle after scattering is

$$\hat{f} = \cos\theta\,\hat{i} + \sin\theta\,\hat{k}$$

where

$$\tan\frac{\theta}{2} = \frac{z_0\sqrt{z_0^2 - b^2}}{b\,\rho_0}$$

and that

$$b^2(\theta) = \frac{z_0^4}{z_0^2 + \rho^2\,\tan^2\theta/2}$$

Show that the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{z_0^4 \rho_0^2}{4(z_0^2\, \cos^2\theta/2 + \rho_0^2\, \sin^2\theta/2)^2}$$

and that the total scattering cross section is

$$\sigma_t = \pi z_0^2$$

Problem 7.12 Rocket Approaching Planet

A small rocket of mass m and initial speed v_0 approaches a spherical planet of mass M and radius R at an impact parameter b. Show that the rocket strikes the planet for

$$b \le b_0 = R\sqrt{1 + \frac{2\,G\,M}{v_0^2\,R}}$$

Show that, when the rocket comes in with just above the impact parameter for impact, it is scattered through an angle θ, where

$$b_0 = \frac{G\,M}{v_0^2}\,\cot\theta/2$$

Note: apply Eq.(7.15). For all larger impact parameters, the scattering angle is less than this value.

Problem 7.13 Scattering by Inverse Sixth Power Potential

For the central potential

$$V(r) = \lambda\, r^{-6}$$

with $\lambda > 0$, show that the scattering angle θ for a particle of impact parameter b and incident energy E is

$$\theta = \pi - 2b\int_R^\infty dr\,\frac{r}{[r^2 - b^2 r^4 - \lambda/E]^{1/2}}$$

where the distance of closest approach R satisfies the equation

$$R^6 = b^2 R^4 + \lambda/E$$

Note that $R > b$. Show that

$$\theta = \pi - 2b\int_0^\infty \frac{dy}{(y^2+\alpha)\,(y^2+\beta)}$$

where

$$\alpha + \beta = 3R^2 - b^2 \qquad \alpha\,\beta = R^2\,(3R^2 - b^2)$$

Making the choice $\alpha > \beta$, show that

$$\theta = \pi - \frac{2\,b\,\beta}{\alpha} K[\sqrt{1 - \beta^2/\alpha^2}]$$

where K is the complete elliptic integral of the first kind.