Chapter 1

Spaces of Functions

In this chapter we present spaces of functions which play an essential role in the theory of distributions. They are called fundamental spaces or test function spaces. They are endowed with suitable topologies said to be canonical ones. Finally, density of certain spaces in others are given.

1.1 The space $\mathcal{C}(\Omega)$

We begin with a space which plays an important role in many domains of analysis. For an open subset Ω of $\mathbb{R}^n$, put

$$\mathcal{C}(\Omega) = \{f : \Omega \longrightarrow \mathbb{C} \ / \ f \text{ is continuous}\},$$

where $\mathbb{C}$ is the field of complex numbers. It is a complex vector space. Endow it with the topology of uniform convergence on compact sets, defined by the family $(p_K)_K$ of seminorms, where K runs over the collection of compact subsets of Ω, given by

$$p_K(f) = \sup_{x \in K} |f(x)| .$$

Then $(\mathcal{C}(\Omega), (p_K)_K)$ is a separated (i.e. Hausdorff) locally convex space ($l.c.s.$, in brief). An important property of this space is that it is metrizable. Indeed, it is sufficient to notice that its topology can be defined by the sequence $(p_{K_l})_l$ of seminorms, where $(K_l)_l$ is an exhaustive sequence of compact subsets of Ω. We do have the following essential result.

Proposition 1.1. *The space* $(\mathcal{C}(\Omega), (p_K)_K)$ *is a complete metrizable l.c.s., so a Fréchet space.*

Proof. Let $(f_p)_p$ be a Cauchy sequence in $\mathcal{C}(\Omega)$. Fix a compact subset K of Ω. One has

$$\forall \varepsilon > 0,\ \exists N_K : p, q \geq N_K \Longrightarrow \sup_{x\in K} |f_p(x) - f_q(x)| \leq \varepsilon,$$

so, for every $x \in K$, the sequence $(f_p(x))_p$ is Cauchy in $\mathbb{C}$ which is complete. Whence one gets a function f_K defined on K by $f_K(x) = \lim_{p\to+\infty} f_p(x)$. In fact, one has uniform convergence on K and hence f_K is continuous on K. The family $(f_K)_K$ of functions allows to obtain a function f defined on Ω by $f_{|K} = f_K$. This function f is continuous. Finally f is, by its construction, the limit of the sequence $(f_p)_p$ for the topology of $\mathcal{C}(\Omega)$.

□

1.2 The space $\mathcal{E}^m(\Omega)$

For $m \in \mathbb{N}^*$, put

$$\mathcal{C}^m(\Omega) = \{f : \Omega \longrightarrow \mathbb{C} \ / \ f \text{ is of class } \mathcal{C}^m\}.$$

It is a complex vector space. One endows it with a topology for which it is a Fréchet *l.c.s.*. The following notations are needed. If $j = (j_1, j_2, \ldots, j_n)$ is a multi-index, i.e., an n-tuple of positive integers, put

$$D^j = \left(\frac{\partial}{\partial x_1}\right)^{j_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{j_n} = \frac{\partial^{j_1}}{\partial x_1^{j_1}} \cdots \frac{\partial^{j_n}}{\partial x_n^{j_n}}$$

and call $|j| = j_1 + \cdots + j_n$ the length of j.

Consider on $\mathcal{C}^m(\Omega)$ the seminorms $p_{K,m}$, where K is any compact subset of Ω and

$$p_{K,m}(f) = \max_{|j|\leq m} \sup_{x\in K} \left|D^j f(x)\right|.$$

So it is endowed with a separated structure of a *l.c.s.*. We have the following.

Proposition 1.2. *The space $(\mathcal{C}^m(\Omega), (p_{K,m})_K)$ is a Fréchet space.*

Proof. For the metrizability use the same argument as for $\mathcal{C}(\Omega)$. Concerning completeness, let $(f_p)_p$ be a Cauchy sequence in $\mathcal{C}^m(\Omega)$. For any $|j| \leq m$ and any compact subset K of Ω, one has

$$\sup_{x\in K} \left|D^j f_p(x) - D^j f_q(x)\right| \underset{p,\,q\to+\infty}{\longrightarrow} 0.$$

So for any $|j| \leq m$, $(D^j f_p)_p$ is a Cauchy sequence in $\mathcal{C}(\Omega)$ which is complete. Hence $D^j f_p \underset{p\to+\infty}{\longrightarrow} g_j$ in $\mathcal{C}(\Omega)$. And in particular, for $|j| = 0$, one has $f_p \underset{p\to+\infty}{\longrightarrow} g_0 = f$ in $\mathcal{C}(\Omega)$. It remains now to see that $f \in \mathcal{C}^m(\Omega)$ and that $f_p \underset{p\to+\infty}{\longrightarrow} f$ in $\mathcal{C}^m(\Omega)$. For this, it is sufficient to show that $g_j = D^j f$, for any j with $|j| \leq m$. One argues by induction on $|j|$. For $|j| = 0$, we do have $g_0 = f$. Suppose the result is true for $|j| \leq r$ with $r \leq m$ and let $|j| = r+1$. Consider the smallest integer k such that $j_k \neq 0$. So

$$D^j f_p = \frac{\partial}{\partial x_k}(D^{j'} f_p) \quad \text{with} \quad |j'| = r.$$

To finish, apply successively the induction hypothesis and the following result (cf. [15]): If a sequence $(f_p)_p \subset \mathcal{C}^m(\Omega)$ converges to f in $\mathcal{C}(\Omega)$ and if the sequence $\left(\frac{\partial f_p}{\partial x_i}\right)_p$ converges to h_i in $\mathcal{C}(\Omega)$, then f is differentiable with respect to x_i and $h_i = \frac{\partial f}{\partial x_i}$. □

Remark 1.1. If we want the notation $\mathcal{C}^m(\Omega)$ to include the case $\mathcal{C}(\Omega)$, we should adopt the convention $\mathcal{C}^0(\Omega) = \mathcal{C}(\Omega)$. In the theory of distributions it is usual to write $\mathcal{E}^m(\Omega)$ instead of $\mathcal{C}^m(\Omega)$, with obviously $\mathcal{E}^0(\Omega) = \mathcal{C}(\Omega)$.

1.3 The space $\mathcal{E}(\Omega)$

Put

$$\mathcal{C}^\infty(\Omega) = \{f : \Omega \longrightarrow \mathbb{C} \;/\; f \text{ is of class } C^\infty\}.$$

It is a complex vector space. One endows it with a topology for which it is a Fréchet *l.c.s.*. In $\mathcal{C}^\infty(\Omega)$, consider the seminorms $p_{K,m}$, where m is any positive integer, K any compact subset of Ω, and

$$p_{K,m}(f) = \max_{|j|\leq m} \sup_{x\in K} \left|D^j f(x)\right|.$$

Depend on in comparison with the space $\mathcal{C}^m(\Omega)$, of the previous section, it is not only the compact set K that varies but also the positive integer m. The space $\mathcal{C}^\infty(\Omega)$ endowed with the family $(p_{K,m})_{K,m}$ of seminorms is a separated *l.c.s.* And we have the following:

Proposition 1.3. *The space* $(\mathcal{C}^\infty(\Omega), (p_{K,m})_{K,m})$ *is a Fréchet space.*

Proof. For the metrizability it is again the same argument as for $\mathcal{C}(\Omega)$. Concerning the completeness, let $(f_p)_p$ be a Cauchy sequence in $\mathcal{C}^\infty(\Omega)$. It is then so in each $\mathcal{C}^m(\Omega)$ in which it is convergent because these spaces are complete. So $f_p \underset{p\to+\infty}{\longrightarrow} h_m$ in $\mathcal{C}^m(\Omega)$, for every m. In fact h_m does not depend of m. Indeed if $m \neq m'$, one has in particular $f_p \underset{p\to+\infty}{\longrightarrow} h_m$ and $f_p \underset{p\to+\infty}{\longrightarrow} h_{m'}$ in $\mathcal{C}(\Omega)$. Therefore $h_m = h_{m'}$. If we put $f = h_0$, we then have $f_p \underset{p\to+\infty}{\longrightarrow} f$, in $\mathcal{C}^m(\Omega)$ for every m, i.e., $f_p \underset{p\to+\infty}{\longrightarrow} f$ in $\mathcal{C}^\infty(\Omega)$. □

Remark 1.2. In the theory of distributions, it is customary to write $\mathcal{E}(\Omega)$ instead of $\mathcal{C}^\infty(\Omega)$.

1.4 The space $\mathcal{D}_K^m(\Omega)$

Recall that the support of a function $f : \Omega \longrightarrow \mathbb{C}$, denoted by *supp* f, is defined as follows

$$supp\ f = \overline{\{x \in \Omega : f(x) \neq 0\}}.$$

$\bar{A}$ denotes the closure of a set A. If K is a compact subset of Ω and m a positive integer, put

$$\mathcal{D}_K^m(\Omega) = \{f : \Omega \longrightarrow \mathbb{C}\ /\ f \in \mathcal{C}^m(\Omega) \text{ and } supp\ f \subset K\}.$$

It is a complex vector space. One endows it with the topology induced by that one of $\mathcal{E}^m(\Omega)$. It can be defined by the single seminorm $p_{K,m}$ given by

$$p_{K,m}(f) = \max_{|j|\leq m} \sup_{x\in K} \left|\mathcal{D}^j f(x)\right|.$$

In fact it is a norm. The same is sometimes denoted by $\|.\|_{K,m}$. One easily checks that $\mathcal{D}_K^m(\Omega)$ is closed in $\mathcal{E}^m(\Omega)$. So we have the following result.

Proposition 1.4. *The space* $\left(\mathcal{D}_K^m(\Omega), \|.\|_{K,m}\right)$ *is a Banach space.*

Remark 1.3. $\mathcal{D}_K^0(\Omega)$ is denoted by $\mathcal{K}_K(\Omega)$; so

$$\mathcal{K}_K(\Omega) = \{f : \Omega \longrightarrow \mathbb{C}\ /\ f \in \mathcal{C}(\Omega) \text{ and } supp\ f \subset K\}.$$

1.5 The space $\mathcal{D}_K(\Omega)$

Let K be a compact subset of $\mathbb{R}^n$ and put

$$\mathcal{D}_K(\Omega) = \{f : \Omega \longrightarrow \mathbb{C}\ /\ f \in \mathcal{C}^\infty(\Omega) \text{ and } supp\ f \subset K\}.$$

It is a complex vector space. One endows it with the topology induced on it by $\mathcal{E}(\Omega)$. Recall that the latter is defined by the family $(p_{K,m})_m$ of seminorms, m running over the set of positive integers, with

$$p_{K,m}(f) = \max_{|j|\leq m} \sup_{x\in K} \left|D^j f(x)\right|.$$

Each one is in fact a norm. One easily checks that $\mathcal{D}_K(\Omega)$ is closed in $\mathcal{E}(\Omega)$. Hence we have the following result.

Proposition 1.5. *The space* $(\mathcal{D}_K(\Omega), (p_{K,m})_m)$ *is a Fréchet space.*

Remark 1.4. If we want the notation $\mathcal{D}_K^m(\Omega)$ to include also the case $\mathcal{D}_K(\Omega)$, we are led to adopt the convention $\mathcal{D}_K^\infty(\Omega) = \mathcal{D}_K(\Omega)$. Henceforth, we will write $\mathcal{D}_K^m(\Omega)$, where $m \in \overline{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$.

1.6 The space $\mathcal{D}^m(\Omega)$

Given $m \in \overline{\mathbb{N}}$, put

$$\mathcal{D}^m(\Omega) = \{f : \Omega \longrightarrow \mathbb{C} \ / \ f \in \mathcal{C}^m(\Omega) \text{ and } \textit{supp}\, f \text{ is compact}\}.$$

We note that $\mathcal{D}^0(\Omega)$ is denoted by $\mathcal{K}(\Omega)$ and $\mathcal{D}^\infty(\Omega)$ by $\mathcal{D}(\Omega)$.

Here are some examples.

Example 1.1. For any $c > 0$, let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 0 & \text{if } |x| \geq 1, \\ \exp\left(\dfrac{c}{x^2-1}\right) & \text{if } |x| < 1. \end{cases}$$

Then $f \in \mathcal{D}(\mathbb{R})$. Obviously *supp* $f = [-1, 1]$. It is also clear that f admits derivatives of any order at points t such that $|t| \neq 1$. For $|t_0| = 1$, one has $\lim_{t\to t_0} f(t) = 0$ and $f(t_0) = 0$. So f is continuous at t_0. Moreover, one obtains by induction that for $|t| < 1$ and $k \in \mathbb{N}^*$,

$$f^{(k)}(t) = \frac{P_k(t)f(t)}{(t^2-1)^{2k}}$$

where P_k is a polynomial of degree $3k-2$. Finally considering the left and the right derivatives and the fact that $f^{(k)}(t) \underset{|t|\to 1}{\longrightarrow} 0$, one verifies that f admits also derivatives of any order at t_0 with $|t_0| = 1$.

Example 1.2. From Example 1.1, one obtains for any interval $[a, b]$, $a < b$, a function $g \in \mathcal{D}(\mathbb{R})$ such that *supp* $g = [a, b]$. Indeed, it is sufficient to take the composite of the function f with the function φ defined by

$$\varphi(t) = \frac{2}{b-a}t - \frac{b+a}{b-a}$$

from $[a, b]$ onto $[-1, 1]$. We have

$$g(t) = \begin{cases} 0 & \text{if} \quad t \notin]a, b[\,, \\ \exp\left(\dfrac{c\,(b-a)^2}{4\,(t-a)\,(t-b)}\right) & \text{if} \quad t \in]a, b[\,. \end{cases}$$

Example 1.3. The analogue of Example 1.1 in $\mathbb{R}^n$ is the following

$$h(x) = \begin{cases} 0 & \text{if} \quad \|x\| \geq 1, \\ \exp\left(\dfrac{c}{\|x\|^2 - 1}\right) & \text{if} \quad \|x\| < 1. \end{cases}$$

Clearly *supp* $h = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. Also h is obviously of class $\mathcal{C}^\infty$ at any point x such that $\|x\| \neq 1$. For $\|a\| = 1$, one has $\lim\limits_{x \to a} h(x) = 0 = h(a)$. Hence h is continuous at a. Again an induction argument as in Example 1.1 shows that the partial derivatives of any order exist and are continuous at a. So f is in $\mathcal{D}(\mathbb{R}^n)$.

Example 1.4. From Example 1.1, one obtains another function $g \in \mathcal{D}(\mathbb{R}^n)$ such that *supp* $g = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. Indeed, it suffices to take the composite of the function f with the function ψ defined by

$$\psi(x_1, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 = \|x\|^2$$

which is, of course, of class $\mathcal{C}^\infty$ from $\mathbb{R}^n$ to $\mathbb{R}$. Thus, we have

$$g(x) = \begin{cases} 0 & \text{if} \quad \|x\| \geq 1, \\ \exp\left(\dfrac{c}{\|x\|^4 - 1}\right) & \text{if} \quad \|x\| < 1. \end{cases}$$

1.7 The Topology of $\mathcal{D}^m(\Omega)$

One has $\mathcal{D}^m(\Omega) \subset \mathcal{E}^m(\Omega)$, $m \in \overline{\mathbb{N}}$. Of course $\mathcal{D}^m(\Omega)$ is a vector subspace of $\mathcal{E}^m(\Omega)$ but as a matter of fact, it is not complete for the topology induced by $\mathcal{E}^m(\Omega)$. To see this, take $\Omega = \mathbb{R}$ and $\varphi \in \mathcal{D}^m(\mathbb{R})$, $\varphi \geq 0$ and $\varphi \neq 0$ with *supp* $\varphi = [0, 1]$. Consider the sequence $(f_n)_n$ defined by

$$f_n(x) = \sum_{p=1}^{n} 2^{-p}\varphi(x - p).$$

It is Cauchy in $\mathcal{D}^m(\mathbb{R})$ for the topology induced by $\mathcal{E}^m(\mathbb{R})$. But its limit in $\mathcal{E}^m(\mathbb{R})$ does not have a compact support. Indeed take x_0 such that $\varphi(x_0) > 0$. Then for any positive integer k,

$$f(x_0 + k) = \sum_{p=1}^{+\infty} 2^{-p}\varphi\left[(x_0 + k) - p\right] \geq 2^{-k}\varphi(x_0) > 0.$$

We will then look for another *l.c.s.* topology τ on $\mathcal{D}^m(\Omega)$ such that $(\mathcal{D}^m(\Omega), \tau)$ is sequentially complete. Notice that

$$\mathcal{D}^m(\Omega) = \bigcup_K \mathcal{D}^m_K(\Omega),$$

where K runs over the collection of compact subsets of Ω and look for the topology τ so that $\tau \mid_{\mathcal{D}^m_K(\Omega)} = \tau_K$, where τ_K is the topology of $\mathcal{D}^m_K(\Omega)$. By definition $V \in \tau$ means that V is an open set for the topology τ. We are then led to consider

$$\mathcal{B} = \left\{W \subset \mathcal{D}^m(\Omega) : W \text{ is a disc and } W \cap \mathcal{D}^m_K(\Omega) \in \tau_K,\ \forall K\right\}.$$

In order to have $\mathcal{B}$ as a basis of 0-neighborhoods for τ, the following condition is necessary

$$\forall V,\ V \in \tau,\ V = \bigcup_{\varphi \in V} \left(\varphi + W_\varphi\right),\ W_\varphi \in \mathcal{B}.$$

We therefore consider the collection τ of all unions of subsets V of $\mathcal{D}^m(\Omega)$ of the form

$$\varphi + W,\ \varphi \in \mathcal{D}^m(\Omega),\ W \in \mathcal{B}.$$

Lemma 1.1. *Let $V \in \tau$. Then for any $\varphi \in V$, there is $W_\varphi \in \mathcal{B}$, such that $\varphi + W_\varphi \subset V$.*

Proof. If $\varphi \in V$, then there is $\psi \in \mathcal{D}^m(\Omega)$ and $W_\psi \in \mathcal{B}$ such that $\varphi \in \psi + W_\psi \subset V$. Let K be a compact subset of Ω such that $\varphi, \psi \in \mathcal{D}^m_K(\Omega)$. One has $\varphi - \psi \in \mathcal{D}^m_K(\Omega) \cap W_\psi$. So there is $U_\varphi \in \mathcal{V}_{\tau_K}(0)$ such that $\varphi - \psi + U_\varphi \subset \mathcal{D}^m_K(\Omega) \cap W_\psi$. Let $\alpha_\varphi > 0$ such that $\alpha_\varphi(\varphi - \psi) \subset U_\varphi$. So $(1+\alpha_\varphi)(\varphi-\psi) \in W_\psi$ or yet $\varphi-\psi \in (1-\delta_\varphi)W_\psi$, with $0 < \delta_\varphi < 1$. Therefore one has $\varphi - \psi + \delta_\varphi W_\psi \subset W_\psi$ with W_ψ convex. Whence $\varphi + \delta_\varphi W_\psi \subset V$, so that $\delta_\varphi W_\psi \in \mathcal{B}$. So we proved that

$$\tau = \{V \subset \mathcal{D}^m(\Omega) \;/\; \forall \varphi \in V,\ \exists W_\varphi \in \mathcal{B} : \varphi + W_\varphi \subset V\}.$$

This implies that τ is a topology on $\mathcal{D}^m(\Omega)$ for which $\mathcal{B}$ is a 0-basis of neighborhoods. □

Proposition 1.6. *The topology τ on $\mathcal{D}^m(\Omega)$ is locally convex. It is called the topology of $\mathcal{D}^m(\Omega)$.*

Proof. The origin admits a basis of absolutely convex neighborhoods. It is then sufficient to show that the addition and multiplication by scalars are continuous. The first assertion follows from the fact that for any $W \in \mathcal{B}$,

$$(\varphi + \frac{1}{2}W) + (\psi + \frac{1}{2}W) = \varphi + \psi + W;\ \forall \varphi,\ \psi \in \mathcal{D}^m(\Omega).$$

For the second, one writes

$$\alpha\varphi - \alpha_0\varphi_0 = \alpha(\varphi - \varphi_0) + (\alpha - \alpha_0)\varphi_0.$$

Take $\delta > 0$ such that $\delta\varphi_o \in \frac{1}{2}W$ and let c be such that

$$2(|\alpha_0| + \delta)\,|c| \leq 1.$$

Then $\alpha\varphi - \alpha_0\varphi_0 \in W$, whenever $|\alpha - \alpha_0| < \delta$ and $\varphi - \varphi_0 \in c\,W$. □

Remark 1.5.

(1) Notice that for any compact subset K of Ω one has

$$\forall V \in \tau,\ V \cap \mathcal{D}_K^m(\Omega) \in \tau_K.$$

Indeed if $\varphi \in V \cap \mathcal{D}_K^m(\Omega)$ then, by the construction of τ, there is $W_\varphi \in \mathcal{B}$ such that

$$(\varphi + W_\varphi) \cap \mathcal{D}_K^m(\Omega) \subset V \cap \mathcal{D}_K^m(\Omega).$$

Since $W_\varphi \cap \mathcal{D}_K^m(\Omega)$ is a 0-neighborhood in $\mathcal{D}_K^m(\Omega)$, the set $V \cap \mathcal{D}_K^m(\Omega)$ is a neighborhood of φ in $\mathcal{D}_K^m(\Omega)$. So $V \cap \mathcal{D}_K^m(\Omega) \in \tau_K$.

(2) Let W be a balanced and convex subset of $\mathcal{D}^m(\Omega)$. It is immediate, by (1), that W is in τ if and only if it is in $\mathcal{B}$.

We come now to fundamental properties of the topology τ.

Proposition 1.7. *For any compact subset K of Ω, one has $\tau_{|_{\mathcal{D}_K^m(\Omega)}} = \tau_K$.*

Proof. Let K be a given compact subset of Ω. By (1) of Remark 1.5, $\tau|_{\mathcal{D}_K^m(\Omega)}$ is coarser than τ_K. Conversely, for $U \in \tau_K$ let us show that there is $V \in \tau$ such that $U = V \cap \mathcal{D}_K^m(\Omega)$. We know that, for any $\varphi \in U$, there is N_φ and $\varepsilon > 0$ such that

$$B_{K,N_\varphi}(\varphi,\varepsilon) = \left\{\psi \in \mathcal{D}_K^m(\Omega) : p_{K,N_\varphi}(\psi - \varphi) < \varepsilon\right\} \subset U.$$

But

$$\begin{aligned} B_{K,N_\varphi}(\varphi,\varepsilon) &= \varphi + \left\{\chi \in \mathcal{D}_K^m(\Omega) : p_{K,N_\varphi}(\chi) < \varepsilon\right\} \\ &= \varphi + \left\{\chi \in \mathcal{D}_K^m(\Omega) : \max_{|j|\le N_\varphi} \sup_{x\in K} \left|\chi^{(j)}(x)\right| < \varepsilon\right\} \cap \mathcal{D}_K^m(\Omega) \\ &= \varphi + W_\varphi \cap \mathcal{D}_K^m(\Omega), \end{aligned}$$

where

$$W_\varphi = \left\{\chi \in \mathcal{D}^m(\Omega) : \max_{|j|\le N_\varphi} \sup_{x\in \Omega} \left|\chi^{(j)}(x)\right| < \varepsilon\right\}.$$

It is clear that $W_\varphi \in \mathcal{B}$. Then $V = \bigcup_{\varphi\in U} (\varphi + W_\varphi)$ answers the question. □

Proposition 1.8. *The space* $(\mathcal{D}^m(\Omega), \tau)$ *is sequentially complete.*

Proof. We will show that any Cauchy sequence in $\mathcal{D}^m(\Omega)$ is contained in some $\mathcal{D}_K^m(\Omega)$. It will be Cauchy there by the previous proposition. The conclusion then follows from the completeness of $\mathcal{D}_K^m(\Omega)$ and again from the previous proposition. Take $B = (f_n)_n$ a Cauchy sequence in $\mathcal{D}^m(\Omega)$ and let $(K_q)_q$ be an exhaustive sequence of compact subsets of Ω. If B were not contained in some $\mathcal{D}_K^m(\Omega)$, there would exist a sequence $(x_q)_q$ in Ω and a subsequence $(f_{n_q})_q$ of B, denoted by $(\varphi_q)_q$, such that $\varphi_q(x_q) \neq 0$ (Take $\varphi_q \notin \mathcal{D}_{K_q}^m(\Omega)$ and $x_q \notin K_q$). We will exhibit a 0-neighborhood W in $\mathcal{D}^m(\Omega)$ which does not absorb B. For that it suffices to find W such that $\varphi_q \notin qW$, for every q. Since the exhaustive sequence is increasing, any compact subset of Ω contains only a finite number of elements of the sequence $(x_q)_q$, for it is contained in some K_q. Given K a compact subset, let x_{q_i}, $i = 1, ..., l$, be the elements of the sequence that it contains. Put

$$W_{K,\varepsilon} = \{\psi \in \mathcal{D}_K^m(\Omega) : \ p_{K,0}(\psi) < \varepsilon\}.$$

For $\varphi_{q_i} \notin q_i W_{K,\varepsilon}$, it suffices that $\varepsilon \le \frac{1}{q_i}\left|\varphi_{q_i}(x_{q_i})\right|$. Take

$$W_K = \left\{\psi \in \mathcal{D}_K^m(\Omega) : \ p_{K,0}(\psi) < \frac{1}{q_i}\left|\varphi_{q_i}(x_{q_i})\right|, \quad i = 1, \ldots, l\right\}.$$

Notice that

$$W_K = \left\{ \psi \in \mathcal{D}_K^m(\Omega) : \; p_{K,o}(\psi) < \frac{1}{q} \left| \varphi_q(x_q) \right| ; \quad \forall q \right\}$$
$$= \left\{ \psi \in \mathcal{D}^m(\Omega) : \; \sup_{x \in \Omega} |\psi(x)| < \frac{1}{q} \left| \varphi_q(x_q) \right| ; \quad \forall_q \right\} \cap \mathcal{D}_K^m(\Omega).$$

Then

$$W = \left\{ \psi \in \mathcal{D}^m(\Omega) : \; \sup_{x \in \Omega} |\psi(x)| < \frac{1}{q} \left| \varphi_q(x_q) \right| ; \quad \forall_q \right\}$$

is the neighborhood looked for. □

Remark 1.6. The previous presentation is for the convenience of the reader. Actually (cf. [1], E.V.T. II 29, proposition 5), $\mathcal{B}$ is a fundamental system of neighborhoods of zero, in $\mathcal{D}^m(\Omega)$, for a locally convex topology τ. The latter is nothing else than the collection of unions of subsets V of $\mathcal{D}^m(\Omega)$ of the form

$$\varphi + W, \; \varphi \in \mathcal{D}^m(\Omega), \; W \in \mathcal{B}.$$

Also τ induces on each $\mathcal{D}_K^m(\Omega)$ its own topology (cf. [1], E.V.T. II 35, proposition 9).

As a consequence, we have the following useful result.

Proposition 1.9. *A sequence $(f_n)_n$ in $\mathcal{D}^m(\Omega)$ converges to an element f of $\mathcal{D}^m(\Omega)$ if and only if*

(1) there is a compact subset K of Ω such that supp $f \subset K$ and supp $f_n \subset K$, for every n.
(2) $(f_n)_n$ converges to f in $\mathcal{D}_K^m(\Omega)$.

Here is by the way a very useful fundamental property of the topology of $\mathcal{D}^m(\Omega)$.

Proposition 1.10. *A mapping f from $\mathcal{D}^m(\Omega)$ into any topological space is continuous if and only if its restriction f_K to every $\mathcal{D}_K^m(\Omega)$ is continuous.*

Remark 1.7. Since the spaces $\mathcal{D}_K^m(\Omega)$ are metrizable, the continuity of a mapping f from $\mathcal{D}^m(\Omega)$ into any topological space can be expressed via sequences.

1.8 Density results

One has

$$\mathcal{D}(\Omega) \subset \mathcal{D}^m(\Omega) \subset \mathcal{E}^m(\Omega), \text{ for every } m \in \mathbb{N}.$$

We will show that

(1) $\mathcal{D}(\Omega)$ is dense in $\mathcal{D}^m(\Omega)$, for every $m \in \mathbb{N}$.
(2) $\mathcal{D}^m(\Omega)$ is dense in $\mathcal{E}^m(\Omega)$, for every $m \in \mathbb{N}$.

For the first result, we will use the so-called regularization method.

Definition 1.1. A sequence $(\theta_j)_{j\in\mathbb{N}}$ in $\mathcal{D}(\mathbb{R}^n)$ is said to be regularizing if it satisfies the following conditions

(1) $\theta_j(x) \geq 0$, for every $x \in \mathbb{R}^n$.
(2) $supp\ \theta_j \subset \overline{B}(0, \varepsilon_j)$, with $(\varepsilon_j)_j$ tending to 0.
(3) $\displaystyle\int_{\mathbb{R}^n} \theta_j(x)dx = 1$.

Such sequences do always exist. Indeed it suffices to take $\varepsilon_j = (j+1)^{-1}$, $\theta_j(x) = (j+1)^n \theta\left[(j+1)x\right]$, where θ is in $\mathcal{D}(\mathbb{R}^n)$ with $supp\ \theta \subset \overline{B}(0,1)$ and $\displaystyle\int_{\mathbb{R}^n} \theta(x)dx = 1$.

Let $(\theta_j)_j$ be a regularizing sequence and $f \in \mathcal{E}^m(\mathbb{R}^n)$, $m \in \overline{\mathbb{N}}$. The functions $f_j = f * \theta_j$ the convolutions of f with the θ_j's are said to be the regularizations of f. They are indeed of class C^∞. Notice that

$$f_j(x) - f(x) = \int_{\|y\| \leq \varepsilon_j} \left[f(x-y) - f(x)\right] \theta_j(y)dy.$$

Due to this formula, the functions f_j play an essential role in density results.

Theorem 1.1 (Regularization theorem). *Let $(\theta_j)_j$ be a regularizing sequence and X any one of the spaces $\mathcal{D}^m(\mathbb{R}^n)$, $\mathcal{K}(\mathbb{R}^n)$, $\mathcal{E}^m(\mathbb{R}^n)$, or $\mathcal{L}^p(\mathbb{R}^n)$. Then for every $f \in X$, one has $f = \lim_{j\to+\infty} f_j$ in X.*

Proof. Notice first that for every $j \in \mathbb{N}$, f_j is of class C^∞ on $\mathbb{R}^n$ and that

$$supp\ f_j \subset supp\ f + supp\ \theta_j.$$

(1) $X = \mathcal{C}(\mathbb{R}^n) = \mathcal{E}^0(\mathbb{R}^n)$. Let $f \in X$ and K be any compact subset of $\mathbb{R}^n$. One has

$$\sup_{x\in K} |f_j(x) - f(x)| \leq \int_{\|y\|\leq\varepsilon_j} \sup_{x\in K} |f(x-y) - f(x)|\, \theta_j(y)dy$$
$$\leq \sup_{\substack{x\in K \\ \|y\|\leq\varepsilon_j}} |f(x-y) - f(x)|\,.$$

Notice that if $x \in K$ and $y \in \overline{B}(0,\varepsilon_j)$, then

$$x - y \in K + \overline{B}(0,\varepsilon_j) \text{ and } x \in K + \overline{B}(0,\varepsilon_j).$$

We can suppose $\varepsilon_j \leq 1$ and therefore

$$K + \overline{B}(0,\varepsilon_j) \subset K + \overline{B}(0,1), \text{ for every } j.$$

The uniform continuity of f on the compact set $K_1 = K + \overline{B}(0,1)$ allows the conclusion.

(2) $X = \mathcal{E}^m(\mathbb{R}^n) = \mathcal{C}^m(\mathbb{R}^n)$. Let $f \in \mathcal{E}^m(\mathbb{R}^n)$. For any multi-index l with $|l| \leq m$, one has the following derivation formula

$$D^l f_j = \left(D^l f\right) * \theta_j = \theta_j * D^l f.$$

The result follows then from (1).

(3) $X = \mathcal{K}(\mathbb{R}^n) = \mathcal{D}^0(\mathbb{R}^n)$. Let $f \in \mathcal{K}(\mathbb{R}^n)$. With $\varepsilon_j \leq 1$, the f_j's and f have their supports in the compact subset $K = supp\ f + \overline{B}(0,1)$; then apply (1).

(4) $X = \mathcal{D}^m(\mathbb{R}^n)$. As in (2), using (3) instead of (1).

(5) Let $f \in \mathcal{L}^p(\mathbb{R}^n)$ with $1 \leq p < +\infty$. One has

$$f_j \in \mathcal{L}^p(\mathbb{R}^n) \text{ and } \|f_j\|_p \leq \|\theta_j\|_1 \|f\|_p = \|f\|_p$$

since $\|\theta_j\|_1 = 1$, for every $j \in \mathbb{N}$. But $\mathcal{K}(\mathbb{R}^n)$ is dense in $\mathcal{L}^p(\mathbb{R}^n)$. Hence for any $\varepsilon > 0$, there is $\varphi \in \mathcal{K}(\mathbb{R}^n)$ such that $\|f - \varphi\|_p \leq \varepsilon$. Then $\|f_j - \varphi_j\|_p \leq \|f - \varphi\|_p \leq \varepsilon$ and so

$$\|f - f_j\|_p \leq \|f - \varphi\|_p + \|\varphi - \varphi_j\|_p + \|\varphi_j - f_j\|_p$$
$$\leq 2\varepsilon + \|\varphi - \varphi_j\|_p\,.$$

Now when $j \to +\infty$, the sequence $(\varphi_j)_j$ converges to φ in $\mathcal{K}(\mathbb{R}^n)$, by (3); hence $\|\varphi - \varphi_j\|_p$ converges to 0. Whence $\|\varphi - \varphi_j\|_p \leq \varepsilon$, for j large enough, so that $\|f - f_j\|_p \leq 3\varepsilon$, for j large enough.

□

Theorem 1.2. *$\mathcal{D}(\Omega)$ is dense in $\mathcal{D}^m(\Omega)$, for every $m \in \mathbb{N}$.*

Proof. If $\Omega = \mathbb{R}^n$, the result follows from the regularization theorem. If $\Omega \neq \mathbb{R}^n$, let $f \in \mathcal{D}^m(\Omega)$ with support K. Extending f by 0 outside of Ω, one obtains a function $g \in \mathcal{D}^m(\mathbb{R}^n)$ the restriction of which to Ω is equal to f. Putting $g_j = g * \theta_j$, one has

$$supp\ g_j \subset K + \overline{B}(0, \varepsilon_j).$$

To have $supp\ g_j \subset \Omega$, just take $\varepsilon_j < d(K, \Omega^c)$. Supposing $(\varepsilon_j)_j$ decreasing, one has $K + \overline{B}(0, \varepsilon_j) \subset K + \overline{B}(0, \varepsilon_1)$. The g_j's and g have their supports in the compact subset $K_1 = K + \overline{B}(0, \varepsilon_1)$ of Ω. Consider the restrictions φ_j of g_j to Ω. They are in $\mathcal{D}^m(\Omega)$. Moreover for every multi-index l with $|l| \leq m$, one has

$$\sup_{x \in K_1} \left| D^l f(x) - D^l \varphi_j(x) \right| = \sup_{x \in K_1} \left| D^l g(x) - D^l g_j(x) \right|.$$

Finally, by the regularization theorem, the right-hand side tends to 0. □

We will show that $\mathcal{D}^m(\Omega)$ is dense in $\mathcal{E}^m(\Omega)$, for every $m \in \overline{\mathbb{N}}$. For any $f \in \mathcal{E}^m(\Omega)$, we have to find functions with compact supports approaching f. Multiplication by characteristic functions of compact subsets does not preserve derivability (in fact, even continuity). It is the technique called of truncating that solves the question.

Definition 1.2. A truncating sequence on Ω, associated with an exhaustive sequence $(K_j)_j$ of compact subsets of Ω, is any sequence $(\psi_j)_j \subset \mathcal{D}(\Omega)$ of functions such that

(1) $0 \leq \psi_j \leq 1$,
(2) $\psi_j = 1$ on K_j,
(3) $supp\ \psi_j \subset \overset{o}{K}_{j+1}$.

Such sequences do exist. Indeed the separation lemma of Urysohn (cf. [15]) gives, for every j, a function $\psi_j \in \mathcal{D}(\overset{o}{K}_{j+1})$ such that $0 \leq \psi_j \leq 1$, $\psi_j = 1$ on K_j and $supp\ \psi_j \subset \overset{o}{K}_{j+1}$. We extend each ψ_j to Ω by 0 outside of $\overset{o}{K}_{j+1}$.

Theorem 1.3. *$\mathcal{D}^m(\Omega)$ is dense in $\mathcal{E}^m(\Omega)$, for every $m \in \overline{\mathbb{N}}$.*

Proof. Let $f \in \mathcal{E}^m(\Omega)$ and $(\psi_j)_j$ be a truncating sequence on Ω. The sequence $(\psi_j f)_j$ is in $\mathcal{D}^m(\Omega)$. Moreover, on any compact subset K one has $\psi_j f = f$, for j large enough. So the sequence $(\psi_j f)_j$ converges uniformly to f in any compact subset; and similarly $D^l(\psi_j f)$ to $D^l f$ for every multi-index l with $|l| \leq m$. □

Combining the two previous density theorems, one obtains the following result.

Theorem 1.4. *$\mathcal{D}(\Omega)$ is dense in $\mathcal{E}^m(\Omega)$, for every $m \in \overline{\mathbb{N}}$.*

Spaces of integrable functions play an important role in the theory of distributions. Notice that

$$\mathcal{D}(\Omega) \subset \mathcal{D}^m(\Omega) \subset \mathcal{K}(\Omega) \subset \mathcal{L}^p(\Omega),\ 1 \leq p < +\infty.$$

So a density result is in order. We know that $\mathcal{K}(\Omega) = \mathcal{D}^0(\Omega)$ is dense in $L^p(\Omega)$, $1 \leq p < +\infty$ and that $\mathcal{D}(\Omega)$ is dense in $\mathcal{K}(\Omega)$.

Proposition 1.11. *$\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$, $1 \leq p < +\infty$.*

We finish this chapter by the following recapitulating diagram:

$$\begin{array}{ccccccc} \mathcal{D}(\Omega) & \subset & \mathcal{D}^m(\Omega) & \subset & \mathcal{K}(\Omega) & \subset & L^p(\Omega) \\ \bigcap & & \bigcap & & \bigcap & & \bigcap \\ \mathcal{E}(\Omega) & \subset & \mathcal{E}^m(\Omega) & \subset & \mathcal{C}(\Omega) & \subset & L^p_{loc}(\Omega) \end{array}$$

where $1 \leq p \leq +\infty$.

The space $\mathcal{L}^1_{loc}(\Omega)$ (of locally integrable functions on Ω) contains all spaces of functions considered in this chapter. We will see that it plays a particular role in the theory of distributions.

The very basic spaces of Distribution Theory (fundamental spaces or test function spaces) are fully described. Their topologies are given in detail. These spaces often appear in several branches of mathematics. So this chapter may be useful for people who are not necessarily working in distribution theory. The notations are introduced step by step and remarks indicate the specific ones to the theory. Observe that Section 1.7 on the topology of $\mathcal{D}^m(\Omega)$ has to be carefully read, it is essential for the rest of the book. The chapter ends with density results which are frequently used in the sequel.

Among the useful inclusions one has

$$\mathcal{D}(\Omega) \subset \mathcal{D}^m(\Omega) \subset \mathcal{E}^m(\Omega), \text{ for every } m \in \mathbb{N}.$$

One shows that $\mathcal{D}(\Omega)$ is dense in $\mathcal{D}^m(\Omega)$ using the regularization method (see Theorem 1.1, p. 11). To obtain the density of $\mathcal{D}^m(\Omega)$ in $\mathcal{E}^m(\Omega)$, one needs the truncating method; by combining the two methods one gets the density of the first space in the third. Notice that all spaces of functions, considered in this chapter are subspaces of $\mathcal{L}^1_{loc}(\Omega)$. The latter actually plays a special role in distribution theory.

The subject matter of the chapter is contained in all books dealing with distributions. In our presentation, there are no long preliminaries on locally convex spaces nor on inductive limits as for instance in [15]. We have tried to make the proofs free of "tricks" and non necessary generalities. For more readings, see [3], [4], [5], [6], [7], [8], [12], [14], [15], and [16].