Chapter 1

Euclidean Barycentric Coordinates and the Classic Triangle Centers

In order to set the stage for the comparative introduction of barycentric calculus, we introduce in this Chapter Euclidean barycentric coordinates, employ them for the determination of several triangle centers, and exemplify their use for tetrahedron centers.

Unlike parallelograms and circles, triangles have many centers, four of which have already been known to the ancient Greeks. These four classic centers of the triangle are: the centroid, G, the orthocenter, H, the incenter, I, and the circumcenter O. Three of these, G, H, and O, are collinear, lying on the so called *Euler line*.

(1) The centroid, G, of a triangle is the point of concurrency of the triangle medians. The triangle centroid is also known as the triangle barycenter.
(2) The orthocenter, H, of a triangle is the point of concurrency of the triangle altitudes.
(3) The incenter, I, of a triangle is the point of concurrency of the triangle angle bisectors. Equivalently, it is the point on the interior of the triangle that is equidistant from the triangle three sides.
(4) The circumcenter, O, of a triangle is the point in the triangle plane equidistant from the three triangle vertices.

There are many other triangle centers. In fact, an on-line Encyclopedia of Triangle Centers that contains more that 3000 triangle centers is maintained by Clark Kimberling [Kimberling (web); Kimberling (1998)].

1.1 Points, Lines, Distance and Isometries

In the Cartesian model $\mathbb{R}^n$ of the n-dimensional Euclidean geometry, where n is any positive integer, we introduce a Cartesian coordinate system relative to which points of $\mathbb{R}^n$ are given by n-tuples, like $X = (x_1, x_2, \ldots, x_n)$ or $Y = (y_1, y_2, \ldots, y_n)$, etc., of real numbers. The point $\mathbf{0} = (0, 0, \ldots) \in \mathbb{R}^n$ is called the *origin* of $\mathbb{R}^n$. The Cartesian model $\mathbb{R}^n$ of the n-dimensional Euclidean geometry is a real inner product space [Marsden (1974)] with addition, subtraction, scalar multiplication and inner product given, respectively, by the equations

$$\begin{aligned}(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)\\ (x_1, x_2, \ldots, x_n) - (y_1, y_2, \ldots, y_n) &= (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)\\ r(x_1, x_2, \ldots, x_n) &= (rx_1, rx_2, \ldots, rx_n)\\ (x_1, x_2, \ldots, x_n)\cdot(y_1, y_2, \ldots, y_n) &= x_1y_1 + x_2y_2 + \ldots + x_ny_n\end{aligned} \tag{1.1}$$

for any real number $r \in \mathbb{R}$ and any points $X, Y \in \mathbb{R}^n$. Unless it is otherwise specifically stated, we shall always adopt the convention that $n \geq 2$. In the study of spheres and tetrahedra it is assumed that $n \geq 3$.

In our Cartesian model $\mathbb{R}^n$ of Euclidean geometry, it is convenient to define a line by the set of its points. Let $A, B \in \mathbb{R}^n$ be any two distinct points. The unique line L_{AB} that passes through these points is the set of all points

$$L_{AB} = A + (-A + B)t \tag{1.2}$$

for all $t \in \mathbb{R}$, that is, for all $-\infty < t < \infty$. Equation (1.2) is said to be the line representation in terms of points A and B. Obviously, the same line can be represented by any two distinct points that lie on the line.

The *norm* $\|X\|$ of $X \in \mathbb{R}^n$ is given by

$$\|X\|^2 = X\cdot X \tag{1.3}$$

satisfying the Cauchy – Schwartz inequality

$$|X\cdot Y| \leq \|X\|\|Y\| \tag{1.4}$$

and the triangle inequality

$$\|X + Y\| \leq \|X\| + \|Y\| \tag{1.5}$$

for all $X, Y \in \mathbb{R}^n$.

The distance $d(X, Y)$ between points $X, Y \in \mathbb{R}^n$ is given by the distance function

$$d(X, Y) = \| -X + Y\| \tag{1.6}$$

that obeys the triangle inequality

$$\| -X + Y\| + \| -Y + Z\| \geq \| -X + Z\| \tag{1.7}$$

or, equivalently,

$$d(X, Y) + d(Y, Z) \geq d(X, Z) \tag{1.8}$$

for all $X, Y, Z \in \mathbb{R}^n$.

A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is *isometric*, or an *isometry*, if it preserves distance, that is, if

$$d(f(X), f(Y)) = d(X, Y) \tag{1.9}$$

for all $X, Y \in \mathbb{R}^n$.

The set of all isometries of $\mathbb{R}^n$ forms a group that contains, as subgroups, the set of all translations of $\mathbb{R}^n$ and the set of all rotations of $\mathbb{R}^n$ about its origin. The group of all translations of $\mathbb{R}^n$ and all rotations of $\mathbb{R}^n$ about its origin, known as the Euclidean group of motions, plays an important role in Euclidean geometry. The formal definition of groups, therefore, follows.

Definition 1.1 (Groups). *A group is a pair $(G, +)$ of a nonempty set and a binary operation in the set, whose binary operation satisfies the following axioms. In G there is at least one element, 0, called a left identity, satisfying*

(G1) $\quad 0+a=a$

for all $a \in G$. There is an element $0 \in G$ satisfying Axiom (G1) such that for each $a \in G$ there is an element $-a \in G$, called a left inverse of a, satisfying

(G2) $\quad -a + a = 0.$

Moreover, the binary operation obeys the associative law

(G3) $\quad (a + b) + c = a + (b + c)$

for all $a, b, c \in G$.

Definition 1.2 (Commutative Groups). *A group $(G, +)$ is commutative if its binary operation obeys the commutative law*

(G6) $\quad a + b = b + a$

for all $a, b \in G$.

A natural extension of (commutative) groups into (gyrocommutative) gyrogroups, which is sensitive to the needs of exploring hyperbolic geometry, will be presented in Defs. 2.2 – 2.3, p. 73.

A translation $T_X A$ of a point A by a point X in $\mathbb{R}^n$, is given by

$$T_X A = X + A \tag{1.10}$$

for all $X, A \in \mathbb{R}^n$. Translation composition is given by point addition. Indeed,

$$T_X T_Y A = X + (Y + A) = (X + Y) + A = T_{X+Y} A \tag{1.11}$$

for all $X, Y, A \in \mathbb{R}^n$, thus obtaining the translation composition law

$$T_X T_Y = T_{X+Y} \tag{1.12}$$

for translations of $\mathbb{R}^n$. The set of all translations of $\mathbb{R}^n$, accordingly, forms a commutative group under translation composition.

Let $SO(n)$ be the *special orthogonal group* of order n, that is, the group of all $n \times n$ orthogonal matrices with determinant 1. A rotation R of a point $A \in \mathbb{R}^n$, denoted RA, is given by the matrix product RA^t of a matrix $R \in SO(n)$ and the transpose A^t of $A \in \mathbb{R}^n$. A rotation of $\mathbb{R}^n$ is a linear map of $\mathbb{R}^n$, so that it leaves the origin of $\mathbb{R}^n$ invariant. Rotation composition is given by matrix multiplication, so that the set of all rotations of $\mathbb{R}^n$ about its origin forms a noncommutative group under rotation composition.

Translations of $\mathbb{R}^n$ and rotations of $\mathbb{R}^n$ about its origin are isometries. The set of all translations of $\mathbb{R}^n$ and all rotations of $\mathbb{R}^n$ about its origin forms a group under transformation composition, known as the Euclidean group of motions. In group theory, this group of motions turns out to be the so called *semidirect product* of the group of translations and the group of rotations.

Following Klein's 1872 *Erlangen Program* [Mumford, Series and Wright (2002)][Greenberg (1993), p. 253], the geometric objects of a geometry are the invariants of the group of motions of the geometry so that, conversely, objects that are invariant under the group of motions of a geometry possess geometric significance. Accordingly, for instance, the distance between two points of $\mathbb{R}^n$ is geometrically significant in Euclidean geometry since it is invariant under the group of motions, translations and rotations, of the Euclidean geometry of $\mathbb{R}^n$.

1.2 Vectors, Angles and Triangles

Definition 1.3 (Equivalence Relations and Classes). *A relation on a nonempty set S is a subset R of $S \times S$, $R \subset S \times S$, written as $a \sim b$ if $(a, b) \in R$. A relation $\sim$ on a set S is*

(1) *Reflexive if $a \sim a$ for all $a \in S$.*
(2) *Symmetric if $a \sim b$ implies $b \sim a$ for all $a, b \in S$.*
(3) *Transitive if $a \sim b$ and $b \sim c$ imply $a \sim c$ for all $a, b, c \in S$.*

A relation is an equivalence relation if it is reflexive, symmetric and transitive.

An equivalence relation $\sim$ on a set S gives rise to equivalence classes. The equivalence class of $a \in S$ is the subset $\{x \in S : x \sim a\}$ of S of all the elements $x \in S$ that are related to a by the relation $\sim$.

Two equivalence classes in a set S with an equivalence relation $\sim$ are either equal or disjoint, and the union of all the equivalence classes in S equals S. Accordingly, we say that the equivalence classes of a set S with an equivalence relation form a *partition* of S.

Points of $\mathbb{R}^n$, denoted by capital italic letters A, B, P, Q, etc., give rise to vectors in $\mathbb{R}^n$, denoted by bold roman lowercase letters $\mathbf{u}, \mathbf{v}$, etc. Any two ordered points $P, Q \in \mathbb{R}^n$ give rise to a unique rooted vector $\mathbf{v} \in \mathbb{R}^n$, rooted at the point P. It has a tail at the point P and a head at the point Q, and it has the value $-P + Q$,

$$\mathbf{v} = -P + Q \tag{1.13}$$

The length of the rooted vector $\mathbf{v} = -P + Q$ is the distance between its tail, P, and its head, Q, given by the equation

$$\|\mathbf{v}\| = \|-P + Q\| \tag{1.14}$$

Two rooted vectors $-P+Q$ and $-R+S$ are equivalent if they have the same value, $-P + Q = -R + S$, that is,

$$-P+Q \quad \sim \quad -R+S \qquad \text{if and only if} \qquad -P+Q = -R+S \tag{1.15}$$

The relation $\sim$ in (1.15) between rooted vectors is reflexive, symmetric and transitive. Hence, it is an equivalence relation that gives rise to equivalence classes of rooted vectors. To liberate rooted vectors from their roots we define a *vector* to be an equivalence class of rooted vectors. The vector $-P + Q$ is thus a representative of all rooted vectors with value $-P + Q$.

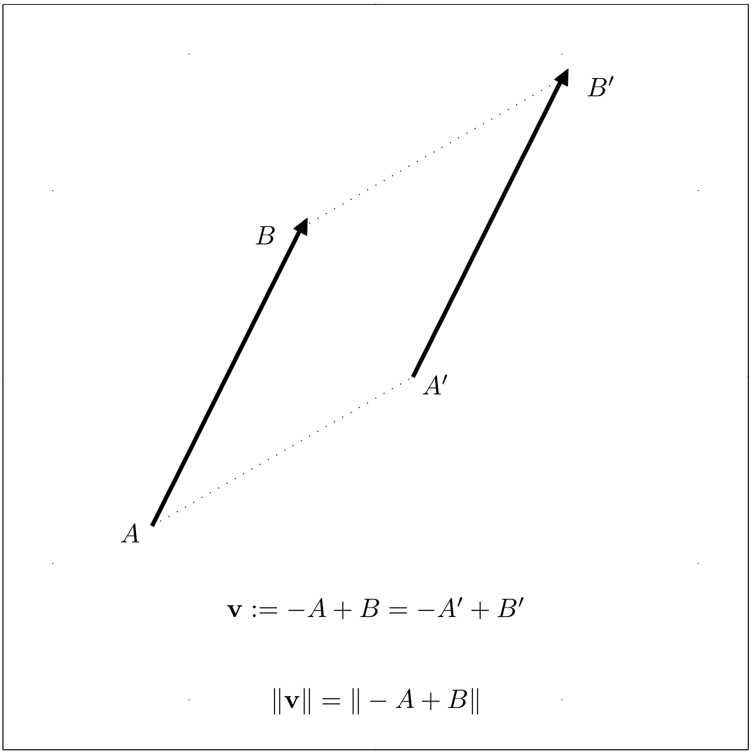


Fig. 1.1 The vectors $-A + B$ and $-A' + B'$ have equal values, that is, $-A + B = -A' + B'$, in a Euclidean space $\mathbb{R}^n$. As such, these two vectors are equivalent and, hence, indistinguishable in their vector space and its underlying Euclidean geometry. Two equivalent nonzero vectors in Euclidean geometry are parallel, and possess equal lengths, as shown here for $n = 2$. Vectors in hyperbolic geometry are called gyrovectors. For the hyperbolic geometric counterparts, see Fig. 2.2, p. 102, and Fig. 2.13, p. 144.

As an example, the two distinct rooted vectors $-A+B$ and $-A'+B'$ in Fig. 1.1 possess the same value so that, as vectors, they are indistinguishable.

Vectors add according to the parallelogram addition law. Hence, vectors in Euclidean geometry are equivalence classes of ordered pairs of points that add according to the parallelogram law.

A point $P \in \mathbb{R}^n$ is identified with the vector $-O + P$, O being the arbitrarily selected origin of the space $\mathbb{R}^n$. Hence, the algebra of vectors can be applied to points as well.

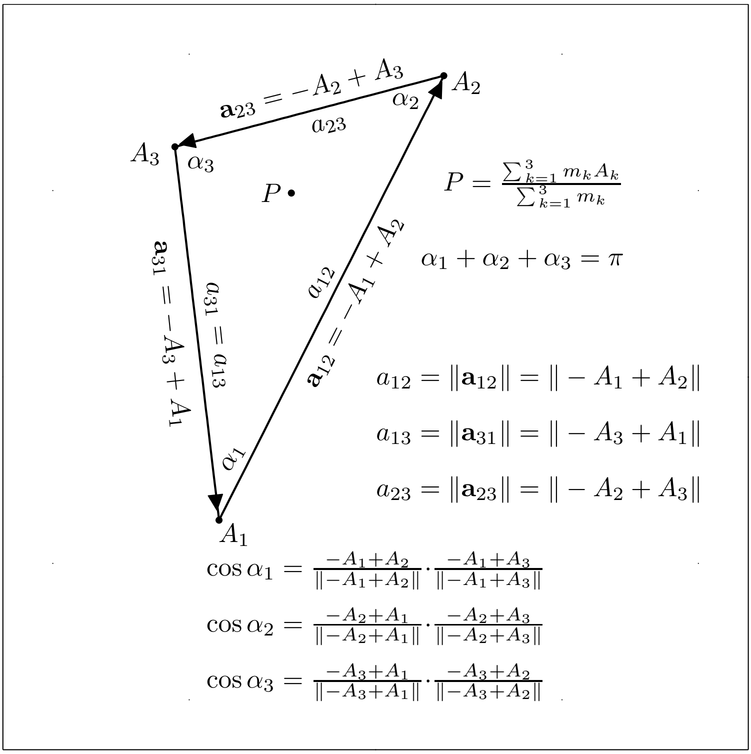


Fig. 1.2 A triangle $A_1A_2A_3$ in $\mathbb{R}^n$ is shown here for $n = 2$, along with its associated standard index notation. The triangle vertices, A_1, A_2 and A_3, are any non-collinear points of $\mathbb{R}^n$. Its sides are presented graphically as line segments that join the vertices. They form vectors, $\mathbf{a}_{ij}$, side-lengths, $a_{ij} = \|\mathbf{a}_{ij}\|$, $1 \le i, j \le 3$, and angles, α_k, $k = 1, 2, 3$. The triangle angle sum is π. The cosine function of the triangle angles is presented. The point P is a generic point in the triangle plane, with barycentric coordinates $(m_1 : m_1 : m_3)$ with respect to the triangle vertices.

Let $-A_1 + A_2$ and $-A_1 + A_3$ be two rooted vectors with a common tail A_1, Fig. 1.2. They include an angle $\alpha_1 = \angle A_2A_1A_3 = \angle A_3A_1A_2$, the measure of which is given by the equation

$$\cos\alpha_1 = \frac{-A_1 + A_2}{\| - A_1 + A_2\|} \cdot \frac{-A_1 + A_3}{\| - A_1 + A_3\|} \tag{1.16}$$

Accordingly, the angle α_1 in Fig. 1.2 has the radian measure

$$\alpha_1 = \cos^{-1} \frac{-A_1 + A_2}{\| - A_1 + A_2\|} \cdot \frac{-A_1 + A_3}{\| - A_1 + A_3\|} \tag{1.17}$$

The angle α_1 is invariant under translations. Indeed,

$$\begin{aligned}\cos\alpha_1' &= \frac{-(X+A_1)+(X+A_2)}{\|-(X+A_1)+(X+A_2)\|}\cdot\frac{-(X+A_1)+(X+A_3)}{\|-(X+A_1)+(X+A_3)\|}\\ &= \frac{-A_1+A_2}{\|-A_1+A_2\|}\cdot\frac{-A_1+A_3}{\|-A_1+A_3\|}\\ &= \cos\alpha_1\end{aligned} \tag{1.18}$$

for all $A_1, A_2, A_3, X \in \mathbb{R}^n$. Similarly, the angle α_1 is invariant under rotations of $\mathbb{R}^n$ about its origin. Indeed,

$$\begin{aligned}\cos\alpha_1'' &= \frac{-RA_1+RA_2}{\|-RA_1+RA_2\|}\cdot\frac{-RA_1+RA_3}{\|-RA_1+RA_3\|}\\ &= \frac{R(-A_1+A_2)}{\|R(-A_1+A_2)\|}\cdot\frac{R(-A_1+A_3)}{\|R(-A_1+A_3)\|}\\ &= \frac{-A_1+A_2}{\|-A_1+A_2\|}\cdot\frac{-A_1+A_3}{\|-A_1+A_3\|}\\ &= \cos\alpha_1\end{aligned} \tag{1.19}$$

for all $A_1, A_2, A_3 \in \mathbb{R}^n$ and $R \in SO(n)$, since rotations $R \in SO(n)$ are linear maps that preserve the inner product in $\mathbb{R}^n$.

Being invariant under the motions of $\mathbb{R}^n$, angles are geometric objects of the Euclidean geometry of $\mathbb{R}^n$. Triangle angle sum in Euclidean geometry is π. The standard index notation that we use with a triangle $A_1A_2A_3$ in $\mathbb{R}^n$, $n \geq 2$, is presented in Fig. 1.2 for $n = 2$. In our notation, triangle $A_1A_2A_3$, thus, has (i) three vertices, A_1, A_2 and A_3; (ii) three angles, α_1, α_2 and α_3; and (iii) three sides, which form the three vectors $\mathbf{a}_{12}$, $\mathbf{a}_{23}$ and $\mathbf{a}_{31}$; with respective (iv) three side-lengths a_{12}, a_{23} and a_{31}.

1.3 Euclidean Barycentric Coordinates

A barycenter in astronomy is the point between two objects where they balance each other. It is the center of gravity where two or more celestial bodies orbit each other. In 1827 Möbius published a book whose title, *Der Barycentrische Calcul*, translates as *The Barycentric Calculus*. The word *barycenter* means center of gravity, but the book is entirely geometrical and, hence, called by Jeremy Gray [Gray (1993)], *Möbius's Geometrical*

Mechanics. The 1827 Möbius book is best remembered for introducing a new system of coordinates, the *barycentric coordinates*. The historical contribution of Möbius' barycentric coordinates to vector analysis is described in [Crowe (1994), pp. 48–50].

The Möbius idea, for a triangle as an illustrative example, is to attach masses, m_1, m_2, m_3, respectively, to three non-collinear points, A_1, A_2, A_3, in the Euclidean plane $\mathbb{R}^2$, and consider their center of mass, or momentum, P, called *barycenter*, given by the equation

$$P = \frac{m_1 A_1 + m_2 A_2 + m_3 A_3}{m_1 + m_2 + m_3} \tag{1.20}$$

The barycentric coordinates of the point P in (1.20) in the plane of triangle $A_1 A_2 A_3$ relative to this triangle may be considered as weights, m_1, m_2, m_3, which if placed at vertices A_1, A_2, A_3, cause P to become the balance point for the plane. The point P turns out to be the center of mass when the points of $\mathbb{R}^2$ are viewed as position vectors, and the center of momentum when the points of $\mathbb{R}^2$ are viewed as relative velocity vectors.

Definition 1.4 **(Euclidean Pointwise Independence** – Hocking and Young [Hocking and Young (1988), pp. 195−200]). *A set S of N points $S = \{A_1, \dots, A_N\}$ in $\mathbb{R}^n$, $n \geq 2$, is pointwise independent if the $N - 1$ vectors $-A_1 + A_k$, $k = 2, \dots, N$, are linearly independent.*

The notion of pointwise independence proves useful in the following definition of Euclidean barycentric coordinates.

Definition 1.5 **(Euclidean Barycentric Coordinates).** *Let $S=\{A_1, \dots, A_N\}$ be a pointwise independent set of N points in $\mathbb{R}^n$. Then, the real numbers $m_1, \dots, m_N$, satisfying*

$$\sum_{k=1}^{N} m_k \neq 0 \tag{1.21}$$

are barycentric coordinates of a point $P \in \mathbb{R}^n$ with respect to the set S if

$$P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k} \tag{1.22}$$

Equation (1.22) *is said to be a barycentric coordinate representation of P with respect to the set $S = \{A_1, \dots, A_N\}$.*

Barycentric coordinates are homogeneous in the sense that the barycentric coordinates $(m_1, \dots, m_N)$ of the point P in (1.22) *are equivalent to* *the barycentric coordinates* $(\lambda m_1, \ldots, \lambda m_N)$ *for any real nonzero number* $\lambda \in \mathbb{R}$, $\lambda \neq 0$. *Since in barycentric coordinates only ratios of coordinates are relevant, the barycentric coordinates* $(m_1, \ldots, m_N)$ *are also written as* $(m_1 \colon \ldots \colon m_N)$ *so that*

$$(m_1 : m_2 : \ \ldots \ : m_N) = (\lambda m_1 : \lambda m_2 : \ \ldots \ : \lambda m_N) \tag{1.23}$$

for any real $\lambda \neq 0$.

Barycentric coordinates that are normalized by the condition

$$\sum_{k=1}^{N} m_k = 1 \tag{1.24}$$

are called special barycentric coordinates.

The point P *in* (1.22) *is said to be a barycentric combination of the points of the set* S, *possessing the barycentric coordinate representation* (1.22).

The barycentric combination (1.22) *is positive if all the coefficients* m_k, $k = 1, \ldots, N$, *are positive. The set of all positive barycentric combinations of the points of the set* S *is called the convex span of* S.

The constant

$$m_0 = \sum_{k=1}^{N} m_k \tag{1.25}$$

is called the constant of the point P *with respect to the set* S.

The pointwise independence of the set S in Def. 1.5 insures that the barycentric coordinate representation of a point with respect to the set S is unique.

Definition 1.6 (Euclidean Simplex). *The convex span (see Def. 1.5) of the pointwise independent set* $S = \{A_1, \ldots, A_N\}$ *of* $N \geq 2$ *points in* $\mathbb{R}^n$ *is an* $(N-1)$*-dimensional simplex, called an* $(N-1)$*-simplex and denoted* $A_1, \ldots, A_N$. *The points of* S *are the vertices of the simplex. The convex span of* $N-1$ *of the points of* S *is a face of the simplex, said to be the face opposite to the remaining vertex. The convex span of each two of the vertices is an edge of the simplex.*

Any two distinct points A, B of $\mathbb{R}^n$ are pointwise independent, and their convex span is the interior of the segment AB, which is a 1-simplex. Similarly, any three non-collinear points A, B, C of $\mathbb{R}^n$, $n \geq 2$, are pointwise independent, and their convex span is the interior of the triangle ABC, which is a 2-simplex, and the convex span of any four pointwise independent points A, B, C, D of $\mathbb{R}^n$, $n \geq 3$, is the interior of the tetrahedron $ABCD$, which is a 3-simplex.

1.4 Analogies with Classical Mechanics

Barycentric coordinate representations of points of the Euclidean space $\mathbb{R}^n$ with respect to the set $S = \{A_1, \ldots, A_N\}$ of vertices of a simplex in $\mathbb{R}^n$ admit a classical mechanical interpretation.

Guided by analogies with classical mechanics, the $(N-1)$-simplex of the N points of the pointwise independent set $S = \{A_1, A_2, \ldots, A_N\}$ along with barycentric coordinates $(m_1 : m_2 : \ \ldots \ : m_N)$ may be viewed as an isolated system $S = \{A_k, m_k, k = 1, \ldots, N\}$ of N noninteracting particles, where $m_k \in \mathbb{R}$ is the mass of the kth particle and $A_k \in \mathbb{R}^n$ is the velocity of the kth particle, $k = 1, \ldots, N$, relative to the arbitrarily selected origin $O = \mathbf{0} = (0, \ldots, 0)$ of the Newtonian velocity space $\mathbb{R}^n$. Each point of the Newtonian velocity space $\mathbb{R}^n$ represents a velocity of an inertial frame. In particular, the origin $O = \mathbf{0}$ of $\mathbb{R}^n$ represents the rest frame.

By analogy with classical mechanics, the point P in (1.22) is the velocity of the center of momentum (CM) frame of the particle system S relative to the rest frame. The CM frame of S, in turn, is an inertial reference frame relative to which the momentum, $\sum_{k=1}^{N} m_k A_k$, of the particle system S vanishes.

Finally, the constant m_0 in (1.25) of the point P with respect to the set S in (1.25) is viewed in the context of classical mechanics as the total mass of the particle system S.

Along these remarkable analogies between Euclidean geometry and classical mechanics, there is an important disanalogy. As opposed to classical mechanics, where masses are always positive, in Euclidean geometry the "masses" m_k, $k = 1, \ldots, N$, considered as barycentric coordinates of points, need not be positive.

The analogies with classical mechanics will help us in this book to form a bridge to hyperbolic geometry, where analogies with classical mechanics are replaced by corresponding analogies with relativistic mechanics. Thus, specifically, in our transition from Euclidean to hyperbolic geometry,

(1) the Euclidean space of Newtonian velocities is replaced by the Euclidean ball of Einsteinian velocities, that is, by the ball of all relativistically admissible velocities,

(2) the Newtonian velocity addition law, which is the ordinary vector addition in Euclidean space, is replaced by Einstein velocity addition law in the ball of relativistically admissible velocities, and
(3) the Newtonian mass is replaced by the relativistic mass, which is velocity dependent.

1.5 Barycentric Representations are Covariant

It is easy to see from (1.22) that barycentric coordinates are independent of the choice of the origin of their vector space, that is,

$$W + P = \frac{\sum_{k=1}^{N} m_k (W + A_k)}{\sum_{k=1}^{N} m_k} \tag{1.26}$$

for all $W \in \mathbb{R}^n$. The proof that (1.26) follows from (1.22) is immediate, owing to the result that scalar multiplication in vector spaces is distributive over vector addition.

It follows from (1.26) that the barycentric coordinate representation (1.22) of a point P is *covariant* with respect to translations of $\mathbb{R}^n$ since the point P and the points A_k, $k = 1, \ldots, N$, of its generating set $S = \{A_1, \ldots, A_N\}$ vary in (1.26) together under translations.

Let $R \in SO(n)$ be an element of the *special orthogonal group* $SO(n)$ of all $n \times n$ orthogonal matrices with determinant 1, which represent rotations of the space $\mathbb{R}^n$ about its origin. Since R is linear, it follows from (1.22) that

$$RP = \frac{\sum_{k=1}^{N} m_k R A_k}{\sum_{k=1}^{N} m_k} \tag{1.27}$$

for all $R \in SO(n)$.

It follows from (1.27) that the *barycentric coordinate representation*, (1.22), of a point P is covariant with respect to rotations of $\mathbb{R}^n$ since the point P and the points A_k, $k = 1, \ldots, N$, of its generating set S vary in (1.27) together under rotations.

The group of all translations and all rotations of $\mathbb{R}^n$ forms the group of motions of $\mathbb{R}^n$, which is the group of all direct isometries of $\mathbb{R}^n$ (that is, isometries preserving orientation) for the Euclidean distance function (1.6).

The point P in (1.22) is determined by the points A_k, $k = 1, \ldots, N$, of its generating set S. It is said to be covariant since the point P and the points of its generating set S vary together in $\mathbb{R}^n$ under the motions of $\mathbb{R}^n$.

The set of all points in $\mathbb{R}^n$ for which the barycentric coordinates with respect to S are all positive forms an open convex subset of $\mathbb{R}^n$, that is, the open *N-simplex* with the N vertices $A_1, \ldots, A_N$. The N-simplex with vertices $A_1, \ldots, A_N$, is denoted by $A_1 \ldots A_N$ so that, for instance, A_1A_2 is the open segment joining points A_1 with A_2 in $\mathbb{R}^n$, $n \geq 1$, and $A_1A_2A_3$ is the interior of the triangle with vertices A_1, A_2 and A_3 in $\mathbb{R}^n$, $n \geq 2$. If the positive number m_k is viewed as the mass of a massive object with Newtonian velocity $A_k \in \mathbb{R}^n$, $1 \leq k \leq N$, the point P in (1.22) turns out to be the center of momentum of the N masses m_k, $1 \leq k \leq N$. If, furthermore, all the masses are equal, the center of momentum is the *centroid* of the N-simplex.

As an application of the covariance of barycentric coordinate representations in (1.26) and for later reference we present the following lemma:

Lemma 1.7 *Let $A_1A_2A_3$ be a triangle in a Euclidean space $\mathbb{R}^n$, and let*

$$P = \frac{m_1A_1 + m_2A_2 + m_3A_3}{m_1 + m_2 + m_3} \tag{1.28}$$

be the barycentric coordinate representation of a point $P \in \mathbb{R}^n$ with respect to the set $\{A_1, A_2, A_3\}$ of the triangle vertices. Then,

$$\begin{aligned}
\|-A_1 + P\|^2 &= \frac{m_2^2a_{12}^2 + m_3^2a_{13}^2 + m_2m_3(\ \ a_{12}^2 + a_{13}^2 - a_{23}^2)}{(m_1 + m_2 + m_3)^2} \\
\|-A_2 + P\|^2 &= \frac{m_1^2a_{12}^2 + m_3^2a_{23}^2 + m_1m_3(\ \ a_{12}^2 - a_{13}^2 + a_{23}^2)}{(m_1 + m_2 + m_3)^2} \\
\|-A_3 + P\|^2 &= \frac{m_1^2a_{13}^2 + m_2^2a_{23}^2 + m_1m_2(-a_{12}^2 + a_{13}^2 + a_{23}^2)}{(m_1 + m_2 + m_3)^2}
\end{aligned} \tag{1.29}$$

Proof. By the covariance property (1.26) of barycentric coordinate representations and the standard triangle index notation in Fig. 1.2, p. 7, it follows from (1.28) that

$$\begin{aligned}
-A_1 + P &= \frac{m_1(-A_1 + A_1) + m_2(-A_1 + A_2) + m_3(-A_1 + A_3)}{m_1 + m_2 + m_3} \\
&= \frac{m_2\mathbf{a}_{12} + m_3\mathbf{a}_{13}}{m_1 + m_2 + m_3}
\end{aligned} \tag{1.30}$$

so that

$$\|-A_1 + P\|^2 = \frac{m_2^2a_{12}^2 + m_3^2a_{13}^2 + 2m_2m_3a_{12}a_{13}\cos\alpha_1}{(m_1 + m_2 + m_3)^2} \tag{1.31}$$

Applying the law of cosines to triangle $A_1A_2A_3$ and its angle α_1 in Fig. 1.2, p. 7, we have

$$2a_{12}a_{13}\cos\alpha_1 = a_{12}^2 + a_{13}^2 - a_{23}^2 \tag{1.32}$$

Eliminating $\cos\alpha_1$ between (1.31) and (1.32) we obtain the first equation in (1.29). The second and the third equations in (1.29) are obtained from the first by cyclic permutations of the triangle vertices. □

1.6 Vector Barycentric Representation

Two points $P, P' \in \mathbb{R}^n$ define a vector $\mathbf{v} = -P' + P$ in $\mathbb{R}^n$ with a tail P' and a head P. In the following theorem we show that if each of the points P and P' possesses a barycentric representation with respect to a pointwise independent set $S = \{A_1, \ldots, A_N\}$ of N points in $\mathbb{R}^n$, then the vector $\mathbf{v} = -P' + P$ possesses an induced representation with respect to the vectors $\mathbf{a}_{ij} = -A_j + A_i$, $i, j = 1, \ldots, N$, $i < j$, called a *vector barycentric representation.*

Theorem 1.8 (The Vector Barycentric Representation). *Let*

$$P = \frac{\sum_{i=1}^{N} m_i A_i}{\sum_{i=1}^{N} m_i} \tag{1.33}$$

and

$$P' = \frac{\sum_{j=1}^{N} m'_j A_j}{\sum_{j=1}^{N} m'_j} \tag{1.34}$$

be barycentric representations of two points $P, P' \in \mathbb{R}^n$ with respect to a pointwise independent set $S = \{A_1, \ldots, A_N\}$ of N points of $\mathbb{R}^n$. Then, the vector $\mathbf{v}$ formed by the point difference $\mathbf{v} = -P + P'$ possesses the vector barycentric representation

$$\mathbf{v} = -P + P' = \frac{\sum_{\substack{i,j=1\\ i<j}}^{N} (m_i m'_j - m'_i m_j)(-A_i + A_j)}{\sum_{k=1}^{N} m_i \sum_{j=1}^{N} m'_j} \tag{1.35}$$

Proof. The proof is given by the following chain of equations, which are numbered for subsequent explanation.

$$
\begin{aligned}
-P + P' &\overset{(1)}{=\!=} \frac{\sum_{j=1}^{N} m'_j(-P + A_j)}{\sum_{j=1}^{N} m'_j} = -\frac{\sum_{j=1}^{N} m'_j(-A_j + P)}{\sum_{j=1}^{N} m'_j} \\
&\overset{(2)}{=\!=} -\frac{\sum_{j=1}^{N} m'_j \left(-A_j + \frac{\sum_{i=1}^{N} m_i A_i}{\sum_{i=1}^{N} m_i}\right)}{\sum_{j}^{N} m'_j} \\
&\overset{(3)}{=\!=} -\frac{\sum_{j=1}^{N} m'_j \frac{\sum_{i=1}^{N} m_i(-A_j + A_i)}{\sum_{i=1}^{N} m_i}}{\sum_{j}^{N} m'_j} \\
&\overset{(4)}{=\!=} \frac{\sum_{j=1}^{N} m'_j \sum_{i=1}^{N} m_i(-A_i + A_j)}{\sum_{i=1}^{N} m_i \sum_{j=1}^{N} m'_j} \\
&\overset{(5)}{=\!=} \frac{\sum_{\substack{i,j=1\\i<j}}^{N} m_i m'_j(-A_i + A_j) + \sum_{\substack{i,j=1\\i>j}}^{N} m_i m'_j(-A_i + A_j)}{\sum_{i=1}^{N} m_i \sum_{j=1}^{N} m'_j} \\
&\overset{(6)}{=\!=} \frac{\sum_{\substack{i,j=1\\i<j}}^{N} m_i m'_j(-A_i + A_j) - \sum_{\substack{i,j=1\\i<j}}^{N} m_j m'_i(-A_i + A_j)}{\sum_{i=1}^{N} m_i \sum_{j}^{N} m'_j} \\
&\overset{(7)}{=\!=} \frac{\sum_{\substack{i,j=1\\i<j}}^{N} (m_i m'_j - m'_i m_j)(-A_i + A_j)}{\sum_{i=1}^{N} m_i \sum_{j=1}^{N} m'_j} \\
&\overset{(8)}{=\!=} \frac{\sum_{\substack{i,j=1\\i<j}}^{N} (m_i m'_j - m'_i m_j)\mathbf{a}_{ij}}{\sum_{i=1}^{N} m_i \sum_{j=1}^{N} m'_j}
\end{aligned}
\tag{1.36}
$$

Derivation of the numbered equalities in (1.36) follows:

(1) Follows from the barycentric representation (1.34) of P' along with the covariance property (1.26) of barycentric representations.
(2) Follows from (1) by a substitution of the barycentric representation (1.33) of P.

(3) Follows from (2) by the covariance property (1.26) of barycentric representations.
(4) Follows from (3) immediately.
(5) Follows from (4) straightforwardly, noting that the contribution of pairs (i,j) vanishes when $i=j$.
(6) Follows from (5) by interchanging the labels i and j of the two summation indexes in the argument of the second Σ on the numerator of (5).
(7) Follows from (6) immediately.
(8) The passage from (7) to (8) is merely a matter of notation that we introduce here for its importance in the book. In this notation, the vector $-A_i+A_j$ with tail A_i and head A_j is denoted by $\mathbf{a}_{ij}=-A_i+A_j$, and its magnitude is denoted by $a_{ij}=\|-A_i+A_j\|$.

□

Example 1.9 (A Vector Barycentric Representation). Let

$$I=\frac{a_{23}A_1+a_{13}A_2+a_{12}A_3}{a_{12}+a_{13}+a_{23}} \tag{1.37}$$

and

$$P=\frac{a_{13}A_2+a_{12}A_3}{a_{12}+a_{13}} \tag{1.38}$$

be barycentric representations of points $I,P\in\mathbb{R}^n$, where $a_{12},a_{13},a_{23}>0$, and $S=\{A_1,A_2,A_3\}$ is a pointwise independent set in $\mathbb{R}^n$, $n\geq 2$.

Then, in the barycentric coordinate notation in Theorem 1.8,

$$I=\frac{m_1A_1+m_2A_2+m_3A_3}{m_1+m_2+m_3} \tag{1.39}$$

and

$$P=\frac{m_1'A_1+m_2'A_2+m_3'A_3}{m_1'+m_2'+m_3'} \tag{1.40}$$

where

$$\begin{aligned} m_1&=a_{23}\\ m_2&=a_{13}\\ m_3&=a_{12} \end{aligned} \tag{1.41}$$

and

$$\begin{aligned} m_1' &= 0 \\ m_2' &= a_{13} \\ m_3' &= a_{12} \end{aligned} \tag{1.42}$$

Hence, by Identity (1.35) of Theorem 1.8, we have the vector barycentric representation

$$\begin{aligned} -I + P &= \frac{(m_1 m_2' - m_1' m_2)\mathbf{a}_{12} + (m_1 m_3' - m_1' m_3)\mathbf{a}_{13} + (m_2 m_3' - m_2' m_3)\mathbf{a}_{23}}{(m_1 + m_2 + m_3)(m_1' + m_2' + m_3')} \\ &= a_{23} \frac{a_{13}\mathbf{a}_{12} + a_{12}\mathbf{a}_{13}}{(a_{12} + a_{13})(a_{12} + a_{13} + a_{23})} \end{aligned} \tag{1.43}$$

1.7 Triangle Centroid

The triangle centroid is located at the intersection of the triangle medians, Fig. 1.3.

Let $A_1A_2A_3$ be a triangle with vertices A_1, A_2 and A_3 in a Euclidean n-space $\mathbb{R}^n$, and let G be the triangle centroid, as shown in Fig. 1.3 for $n = 2$. Then, G is given in terms of its barycentric coordinates $(m_1 : m_2 : m_3)$ with respect to the set $\{A_1, A_2, A_3\}$ by the equation

$$G = \frac{m_1 A_1 + m_2 A_2 + m_3 A_3}{m_1 + m_2 + m_3} \tag{1.44}$$

where the barycentric coordinates m_1, m_2 and m_3 of P_3 are to be determined in (1.50) below.

The midpoint of side A_1A_2 is given by

$$M_{A_1A_2} = \frac{A_1 + A_2}{2} \tag{1.45}$$

so that an equation of the line L_{123} through the points $M_{A_1A_2}$ and A_3 is

$$L_{123}(t_1) = A_3 + \Big(- A_3 + \frac{A_1 + A_2}{2} \Big) t_1 \tag{1.46}$$

with the line parameter $t_1 \in \mathbb{R}$.

The line $L_{123}(t_1)$ contains one of the three medians of triangle $A_1A_2A_3$. Invoking cyclicity, equations of the lines L_{123}, L_{231} and L_{312}, which contain,

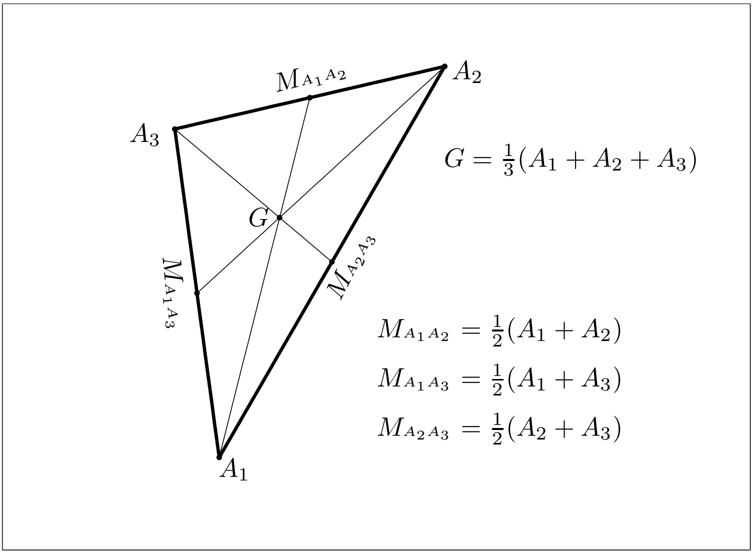


Fig. 1.3 The side midpoints M and the centroid G of triangle $A_1A_2A_3$ in a Euclidean plane $\mathbb{R}^2$.

respectively, the three triangle medians are obtained from (1.46) by index cyclic permutations,

$$\begin{aligned} L_{123}(t_1) &= \frac{t_1}{2}A_1 + \frac{t_1}{2}A_2 + (1-t_1)A_3 \\ L_{231}(t_2) &= \frac{t_2}{2}A_2 + \frac{t_2}{2}A_3 + (1-t_2)A_1 \\ L_{312}(t_3) &= \frac{t_3}{2}A_3 + \frac{t_3}{2}A_1 + (1-t_3)A_2 \end{aligned} \tag{1.47}$$

$t_1, t_2, t_3 \in \mathbb{R}$.

The triangle centroid G, Fig. 1.3, is the point of concurrency of the three lines in (1.47). This point of concurrency is determined by solving the equation $L_{123}(t_1) = L_{231}(t_2) = L_{312}(t_3)$ for the unknowns $t_1, t_2, t_3 \in \mathbb{R}$, obtaining $t_1 = t_2 = t_3 = 2/3$. Hence, G is given by the equation

$$G = \frac{A_1 + A_2 + A_3}{3} \tag{1.48}$$

Comparing (1.48) with (1.44) we find that the special barycentric coordinates (m_1, m_2, m_3) of G with respect to the set $\{A_1, A_2, A_3\}$ are given by

$$m_1 = m_2 = m_3 = \frac{1}{3} \tag{1.49}$$

Hence, convenient barycentric coordinates $(m_1 : m_2 : m_3)$ of G may be given by

$$(m_1 : m_2 : m_3) = (1 : 1 : 1) \tag{1.50}$$

as it is well-known in the literature; see, for instance, [Kimberling (web); Kimberling (1998)].

1.8 Triangle Altitude

Let $A_1A_2A_3$ be a triangle with vertices A_1, A_2, and A_3 in a Euclidean n-space $\mathbb{R}^n$, and let the point P_3 be the orthogonal projection of vertex A_3 onto its opposite side, A_1A_2 (or its extension), as shown in Fig. 1.4 for $n = 2$. Furthermore, let $(m_1 : m_2)$ be barycentric coordinates of P_3 with respect to the set $\{A_1, A_2\}$. Then, P_3 is given in terms of its barycentric coordinates $(m_1 : m_2)$ with respect to the set $\{A_1, A_2\}$ by the equation

$$P_3 = \frac{m_1A_1 + m_2A_2}{m_1 + m_2} \tag{1.51}$$

where the barycentric coordinates m_1 and m_2 of P_3 are to be determined in (1.61) below.

By the covariance (1.26) of barycentric coordinate representations with respect to translations we have, in particular, for $X = A_1$ and $X = A_2$,

$$\begin{aligned} \mathbf{p}_1 &= -A_1 + P_3 = \frac{m_2(-A_1 + A_2)}{m_1 + m_2} = \frac{m_2\mathbf{a}_{12}}{m_1 + m_2} \\ \mathbf{p}_2 &= -A_2 + P_3 = \frac{m_1(-A_2 + A_1)}{m_1 + m_2} = \frac{-m_1\mathbf{a}_{12}}{m_1 + m_2} \end{aligned} \tag{1.52}$$

As indicated in Fig. 1.2, p. 7, we use the notation

$$\mathbf{a}_{ij} = -A_i + A_j, \qquad a_{ij} = \|\mathbf{a}_{ij}\| \tag{1.53}$$

$i, j = 1, 2, 3$, $i \neq j$. Clearly, in general, $\mathbf{a}_{ij} \neq \mathbf{a}_{ji}$, but $a_{ij} = a_{ji}$.

We also use the notation

$$\begin{aligned} \mathbf{p}_1 &= -A_1 + P_3, \qquad p_1 = \|\mathbf{p}_1\| \\ \mathbf{p}_2 &= -A_2 + P_3, \qquad p_2 = \|\mathbf{p}_2\| \end{aligned} \tag{1.54}$$

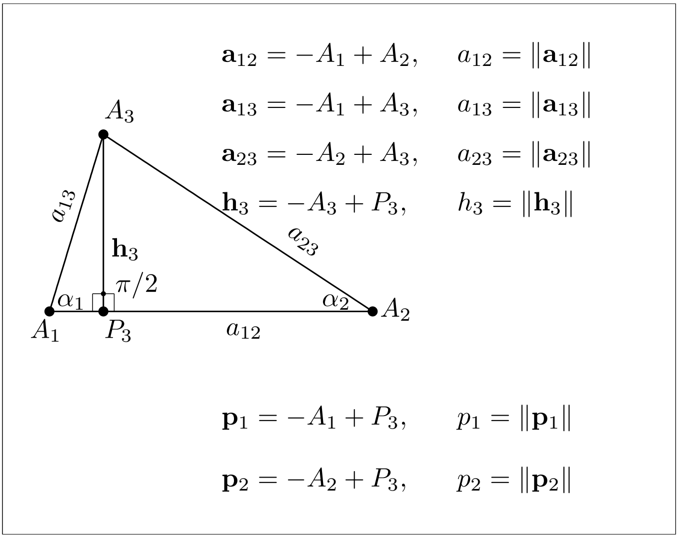


Fig. 1.4 Orthogonal projection, P_3, of a point, A_3, onto a segment, A_1A_2, in a Euclidean n-space $\mathbb{R}^n$. The segment A_3P_3 is the altitude $\mathbf{h}_3$ of triangle $A_1A_2A_3$ dropped perpendicularly from vertex A_3 to its foot P_3 on its base, which is side A_1A_2 of the triangle. Barycentric coordinates $\{m_1 : m_2\}$ of the point P_3 with respect to the set of points $\{A_1, A_2\}$, satisfying (1.51), are determined in (1.61).

and

$$\mathbf{h} = -A_3 + P_3, \qquad h = \|\mathbf{h}\| \tag{1.55}$$

In this notation, the vector equations (1.52) lead to the scalar equations

$$\begin{aligned} p_1 &= \frac{m_2 a_{12}}{m_1 + m_2} \\ p_2 &= \frac{m_1 a_{12}}{m_1 + m_2} \end{aligned} \tag{1.56}$$

The Pythagorean identity for the right-angled triangles $A_1P_3A_3$ and $A_2P_3A_3$ in Fig. 1.4 implies

$$h^2 = a_{13}^2 - p_1^2 = a_{23}^2 - p_2^2 \tag{1.57}$$

Hence, by (1.56)–(1.57),

$$a_{13}^2 - \frac{m_2^2 a_{12}^2}{(m_1 + m_2)^2} = a_{23}^2 - \frac{m_1^2 a_{12}^2}{(m_1 + m_2)^2} \tag{1.58}$$

Normalizing m_1 and m_2,

$$m_1 + m_2 = 1 \tag{1.59}$$

and solving (1.58)−(1.59) for m_1 and m_2, the special barycentric coordinates $\{m_1, m_2\}$ of the point P_3 with respect to the set $\{A_1, A_2\}$ are

$$\begin{aligned} m_1 &= \frac{a_{12}^2 - a_{13}^2 + a_{23}^2}{2a_{12}^2} \\ m_2 &= \frac{a_{12}^2 + a_{13}^2 - a_{23}^2}{2a_{12}^2} \end{aligned} \tag{1.60}$$

so that convenient barycentric coordinates $(m_1 : m_2)$ of P_3 with respect to the set $\{A_1, A_2\}$ may be given by

$$\begin{aligned} m_1 &= a_{12}^2 - a_{13}^2 + a_{23}^2 \\ m_2 &= a_{12}^2 + a_{13}^2 - a_{23}^2 \end{aligned} \tag{1.61}$$

Hence, by (1.51) and either (1.60) or (1.61) we have

$$P_3 = \frac{1}{2}\left(\frac{a_{12}^2 - a_{13}^2 + a_{23}^2}{a_{12}^2}\right) A_1 + \frac{1}{2}\left(\frac{a_{12}^2 + a_{13}^2 - a_{23}^2}{a_{12}^2}\right) A_2 \tag{1.62}$$

Following the law of sines,

$$\frac{a_{23}}{\sin\alpha_1} = \frac{a_{13}}{\sin\alpha_2} = \frac{a_{12}}{\sin\alpha_3} \tag{1.63}$$

for triangle $A_1A_2A_3$ in Fig. 1.5, (1.62) can be written in terms of the triangle angles as

$$P_3 = \frac{\sin^2\alpha_1 - \sin^2\alpha_2 + \sin^2\alpha_3}{2\sin^2\alpha_3} A_1 + \frac{-\sin^2\alpha_1 + \sin^2\alpha_2 + \sin^2\alpha_3}{2\sin^2\alpha_3} A_2 \tag{1.64}$$

Taking advantage of the triangle π condition

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi \tag{1.65}$$

that triangle angles obey, we have the trigonometric elegant identities

$$\begin{aligned} -\sin^2\alpha_1 + \sin^2\alpha_2 + \sin^2\alpha_3 &= 2\cos\alpha_1\sin\alpha_2\sin\alpha_3 \\ \sin^2\alpha_1 - \sin^2\alpha_2 + \sin^2\alpha_3 &= 2\sin\alpha_1\cos\alpha_2\sin\alpha_3 \end{aligned} \tag{1.66}$$

where $\alpha_3 = \pi - \alpha_1 - \alpha_2$.

Substituting (1.66) into (1.64), we have

$$P_3 = \frac{\sin\alpha_1\cos\alpha_2}{\sin(\alpha_1+\alpha_2)}A_1 + \frac{\cos\alpha_1\sin\alpha_2}{\sin(\alpha_1+\alpha_2)}A_2 \tag{1.67}$$

so that the special trigonometric barycentric coordinates (m_1, m_2) of P_3 with respect to the set $\{A_1, A_2\}$ is

$$(m_1, m_2) = \left(\frac{\sin\alpha_1\cos\alpha_2}{\sin(\alpha_1+\alpha_2)}, \frac{\cos\alpha_1\sin\alpha_2}{\sin(\alpha_1+\alpha_2)}\right) \tag{1.68}$$

and, accordingly, convenient trigonometric barycentric coordinates $(m_1' : m_2')$ of P_3 with respect to the set $\{A_1, A_2\}$ are

$$\begin{aligned}(m_1' : m_2') &= (\sin\alpha_1\cos\alpha_2 : \cos\alpha_1\sin\alpha_2)\\ &= (\tan\alpha_1 : \tan\alpha_2)\end{aligned} \tag{1.69}$$

The altitude $\mathbf{h}_3$ of triangle $A_1A_2A_3$ in Fig. 1.4 is the vector

$$\begin{aligned}\mathbf{h}_3 &= -A_3 + P_3\\ &= \frac{1}{2}\left(\frac{a_{12}^2 - a_{13}^2 + a_{23}^2}{a_{12}^2}\right)(-A_3 + A_1) + \frac{1}{2}\left(\frac{a_{12}^2 + a_{13}^2 - a_{23}^2}{a_{12}^2}\right)(-A_3 + A_2)\\ &= \frac{1}{2}\left(\frac{a_{12}^2 - a_{13}^2 + a_{23}^2}{a_{12}^2}\right)\mathbf{a}_{31} + \frac{1}{2}\left(\frac{a_{12}^2 + a_{13}^2 - a_{23}^2}{a_{12}^2}\right)\mathbf{a}_{31}\end{aligned} \tag{1.70}$$

as we see from (1.62) by employing the covariance property (1.26), p. 12, of barycentric coordinate representations. Note that $\mathbf{a}_{31} = -\mathbf{a}_{13}$, so that $a_{13} = \|\mathbf{a}_{31}\| = \|\mathbf{a}_{13}\| = a_{13}$, etc.

Noting the law of cosines,

$$a_{12}^2 = a_{13}^2 + a_{23}^2 - 2a_{13}a_{23}\cos\alpha_3 \tag{1.71}$$

in the notation of Fig. 1.4, we have

$$\begin{aligned}2\mathbf{a}_{31}\cdot\mathbf{a}_{32} &= 2(-A_3 + A_1)\cdot(-A_3 + A_2)\\ &= 2a_{13}a_{23}\cos\alpha_3\\ &= -a_{12}^2 + a_{13}^2 + a_{23}^2\end{aligned} \tag{1.72}$$

Hence, by (1.70) and (1.72), the squared length h_3^2 of altitude $\mathbf{h}_3$, Fig. 1.4, is given by

$$\begin{aligned} h_3^2 &= \|\mathbf{h}_3^2\| \\ &= \frac{1}{4}\left\{\left(\frac{a_{12}^2 - a_{13}^2 + a_{23}^2}{a_{12}^2}\right)a_{13}^2 + \left(\frac{a_{12}^2 + a_{13}^2 - a_{23}^2}{a_{12}^2}\right)a_{23}^2 \right. \\ &\quad \left. + \frac{a_{12}^2 - a_{13}^2 + a_{23}^2}{a_{12}^2}\frac{a_{12}^2 + a_{13}^2 - a_{23}^2}{a_{12}^2}(-a_{12}^2 + a_{13}^2 + a_{23}^2)\right\} \\ &= \frac{(a_{12} + a_{13} + a_{23})(-a_{12} + a_{13} + a_{23})(a_{12} - a_{13} + a_{23})(a_{12} + a_{13} - a_{23})}{4a_{12}^2} \\ &= \frac{F_2(a_{12}, a_{13}, a_{23})}{4a_{12}^2} \end{aligned} \tag{1.73}$$

Here $F_2(a_{12}, a_{13}, a_{23})$, given by

$$\begin{aligned} &F_2(a_{12}, a_{13}, a_{23}) \\ &= (a_{12} + a_{13} + a_{23})(-a_{12} + a_{13} + a_{23})(a_{12} - a_{13} + a_{23})(a_{12} + a_{13} - a_{23}) \\ &= -\begin{vmatrix} 0 & a_{12}^2 & a_{13}^2 & 1 \\ a_{21}^2 & 0 & a_{23}^2 & 1 \\ a_{31}^2 & a_{32}^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \end{aligned} \tag{1.74}$$

[Veljan (2000)] is a symmetric function of the triangle side lengths that possesses an elegant determinant representation, in terms of the so called *Cayley-Menger determinant.*

Equation (1.73) gives rise to Heron's formula of the triangle area in the following theorem:

Theorem 1.10 (Heron's Formula). *The area $|A_1A_2A_3|$ of triangle $A_1A_2A_3$ in a Euclidean space $\mathbb{R}^n$ is given by Heron's Formula*

$$\begin{aligned} |A_1A_2A_3| &= \tfrac{1}{2}a_{12}h_3 \\ &= \frac{1}{4}\sqrt{a_{12} + a_{13} + a_{23}}\sqrt{-a_{12} + a_{13} + a_{23}}\sqrt{a_{12} - a_{13} + a_{23}}\sqrt{a_{12} + a_{13} - a_{23}} \end{aligned} \tag{1.75}$$

Owing to their importance, we elevate the results in (1.62) and (1.73) to the status of theorems:

Theorem 1.11 (Point to Line Perpendicular Projection). *Let A_1 and A_2 be any two distinct points of a Euclidean space $\mathbb{R}^n$, and let $L_{A_1A_2}$ be the line passing through these points. Furthermore, let A_3 be any point of the space that does not lie on $L_{A_1A_2}$, as shown in Fig. 1.4. Then, in the notation of Fig. 1.4, the perpendicular projection of the point A_3 on the line $L_{A_1A_2}$ is the point P_3 on the line given by,* (1.62), (1.69),

$$\begin{aligned} P_3 &= \frac{1}{2}\left(\frac{a_{12}^2 - a_{13}^2 + a_{23}^2}{a_{12}^2}\right) A_1 + \frac{1}{2}\left(\frac{a_{12}^2 + a_{13}^2 - a_{23}^2}{a_{12}^2}\right) A_2 \\ &= \frac{\tan\alpha_1 A_1 + \tan\alpha_2 A_2}{\tan\alpha_1 + \tan\alpha_2} \end{aligned} \tag{1.76}$$

Theorem 1.12 (Point to Line Distance). *Let A_1 and A_2 be any two distinct points of a Euclidean space $\mathbb{R}^n$, and let $L_{A_1A_2}$ be the line passing through these points. Furthermore, let A_3 be any point of the space that does not lie on $L_{A_1A_2}$, as shown in Fig. 1.4. Then, in the notation of Fig. 1.4, the distance $h_3 = \|-A_3 + P_3\|$ between the point A_3 and the line $L_{A_1A_2}$ is given by the equation*

$$\begin{aligned} h_3^2 &= \frac{F_2(a_{12}, a_{13}, a_{23})}{4a_{12}^2} \\ &:= \frac{(a_{12}+a_{13}+a_{23})(-a_{12}+a_{13}+a_{23})(a_{12}-a_{13}+a_{23})(a_{12}+a_{13}-a_{23})}{4a_{12}^2} \\ &= \frac{2(a_{12}^2a_{13}^2 + a_{12}^2a_{23}^2 + a_{13}^2a_{23}^2) - (a_{12}^4 + a_{13}^4 + a_{23}^4)}{4a_{12}^2} \end{aligned} \tag{1.77}$$

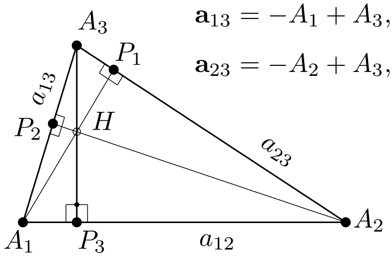
Following the result, (1.77), of Theorem 1.12 we have

$$a_{12}^2 h_3^2 = \frac{2(a_{12}^2a_{13}^2 + a_{12}^2a_{23}^2 + a_{13}^2a_{23}^2) - (a_{12}^4 + a_{13}^4 + a_{23}^4)}{4} \tag{1.78}$$

1.9 Triangle Orthocenter

The triangle orthocenter is located at the intersection of its altitudes, Fig. 1.5.

$$\mathbf{a}_{12} = -A_1 + A_2, \qquad a_{12} = \|\mathbf{a}_{12}\|$$
$$\mathbf{a}_{13} = -A_1 + A_3, \qquad a_{13} = \|\mathbf{a}_{13}\|$$
$$\mathbf{a}_{23} = -A_2 + A_3, \qquad a_{23} = \|\mathbf{a}_{23}\|$$



$$\alpha_1 = \angle A_2 A_1 A_3$$
$$\frac{a_{23}}{\sin\alpha_1} = \frac{a_{13}}{\sin\alpha_2} = \frac{a_{12}}{\sin\alpha_3} \qquad \alpha_2 = \angle A_1 A_2 A_3$$
$$\alpha_3 = \angle A_1 A_3 A_2$$

Fig. 1.5 The Triangle Orthocenter H. A triangle orthocenter is the point at which the three altitudes are concurrent. The standard triangle index notation along with its law of sines is presented.

Let P_1, P_2 and P_3 be the feet of the three altitudes of a triangle $A_1A_2A_3$ in a Euclidean n-space $\mathbb{R}^n$, shown in Fig. 1.5 for $n = 2$. The barycentric coordinate representation of the altitude feet with respect to the set $\{A_1, A_2, A_3\}$ are

$$\begin{aligned}
P_1 &= \frac{1}{2}\left(\frac{-a_{12}^2 + a_{13}^2 + a_{23}^2}{a_{23}^2}\right) A_2 + \frac{1}{2}\left(\frac{a_{12}^2 - a_{13}^2 + a_{23}^2}{a_{23}^2}\right) A_3 \\
P_2 &= \frac{1}{2}\left(\frac{-a_{12}^2 + a_{13}^2 + a_{23}^2}{a_{13}^2}\right) A_1 + \frac{1}{2}\left(\frac{a_{12}^2 + a_{13}^2 - a_{23}^2}{a_{13}^2}\right) A_3 \\
P_3 &= \frac{1}{2}\left(\frac{a_{12}^2 - a_{13}^2 + a_{23}^2}{a_{12}^2}\right) A_1 + \frac{1}{2}\left(\frac{a_{12}^2 + a_{13}^2 - a_{23}^2}{a_{12}^2}\right) A_2
\end{aligned} \tag{1.79}$$

The third equation in (1.79) is established in (1.62), and the first two equations in (1.79) are obtained from the third by vertex cyclic permutations.

The equations of the lines that contain the altitudes of triangle $A_1A_2A_3$,

Fig. 1.5, are

$$\begin{aligned} L_{A_1P_1} &= A_1 + (-A_1 + P_1)t_1 \\ L_{A_2P_2} &= A_2 + (-A_2 + P_2)t_2 \\ L_{A_3P_3} &= A_3 + (-A_3 + P_3)t_3 \end{aligned} \tag{1.80}$$

for the three line parameters $-\infty < t_1, t_2, t_3 < \infty$, where the altitude feet P_1, P_2 and P_3 are given by (1.79).

In order to determine the point of concurrency H of the triangle altitudes, Fig. 1.5, if exists, we solve the vector equations

$$A_1 + (-A_1 + P_1)t_1 = A_2 + (-A_2 + P_2)t_2 = A_3 + (-A_3 + P_3)t_3 \tag{1.81}$$

for the three scalar unknowns t_1, t_2 and t_3. The solution turns out to be

$$\begin{aligned} t_1 &= \frac{2(-a_{12} - a_{13} + a_{23})}{D} a_{23} \\ t_2 &= \frac{2(-a_{12} + a_{13} - a_{23})}{D} a_{13} \\ t_3 &= \frac{2(\ \ a_{12} - a_{13} - a_{23})}{D} a_{12} \end{aligned} \tag{1.82}$$

where

$$D = a_{12}^2 + a_{13}^2 + a_{23}^2 - 2(a_{12}a_{13} + a_{12}a_{23} + a_{13}a_{23}) \tag{1.83}$$

Substituting the solution for t_1 (respectively, for t_2, t_3) in the first (respectively, second, third) equation in (1.80) we determine the orthocenter H of triangle $A_1A_2A_3$ in terms of barycentric coordinates,

$$H = \frac{m_1A_1 + m_2A_2 + m_3A_3}{m_1 + m_2 + m_3} \tag{1.84}$$

where convenient barycentric coordinates are

$$\begin{aligned} m_1 &= a_{23}^4 - (a_{12}^2 - a_{13}^2)^2 \\ m_2 &= a_{13}^4 - (a_{12}^2 - a_{23}^2)^2 \\ m_3 &= a_{12}^4 - (a_{23}^2 - a_{13}^2)^2 \end{aligned} \tag{1.85}$$

Following the law of sines (1.63), the barycentric coordinates of H in (1.84)–(1.85) can be written in terms of the triangle angles as

$$\begin{aligned} m_1 &= \frac{1 - \cos 2\alpha_1 - \cos 2\alpha_2 + \cos 2\alpha_3}{1 + \cos 2\alpha_1 - \cos 2\alpha_2 - \cos 2\alpha_3} \\ m_2 &= \frac{1 - \cos 2\alpha_1 - \cos 2\alpha_2 + \cos 2\alpha_3}{1 - \cos 2\alpha_1 + \cos 2\alpha_2 - \cos 2\alpha_3} \\ m_3 &= 1 \end{aligned} \tag{1.86}$$

Taking advantage of the relationship (1.65) between triangle angles, and employing trigonometric identities, (1.86) can be simplified, obtaining the elegant barycentric coordinates of the orthocenter H of triangle $A_1A_2A_3$ in terms of its angles,

$$\begin{aligned} m_1 &= \frac{\tan \alpha_1}{\tan \alpha_3} \\ m_2 &= \frac{\tan \alpha_2}{\tan \alpha_3} \\ m_3 &= 1 \end{aligned} \tag{1.87}$$

or equivalently, owing to the homogeneity of the barycentric coordinates,

$$\begin{aligned} m_1 &= \tan \alpha_1 \\ m_2 &= \tan \alpha_2 \\ m_3 &= \tan \alpha_3 \end{aligned} \tag{1.88}$$

Following (1.84) and (1.88), the orthocenter H of a triangle $A_1A_2A_3$ with vertices A_1, A_2 and A_3, and with corresponding angles α_1, α_2 and α_3, Fig. 1.5, is given in terms of its barycentric coordinates with respect to the set $\{A_1, A_2, A_3\}$ by the equation

$$H = \frac{\tan \alpha_1 A_1 + \tan \alpha_2 A_2 + \tan \alpha_3 A_3}{\tan \alpha_1 + \tan \alpha_2 + \tan \alpha_3} \tag{1.89}$$

1.10 Triangle Incenter

The incircle of a triangle is a circle lying inside the triangle, tangent to the triangle sides. The center, I, of the incircle is called the triangle incenter,

Fig. 1.8, p. 34. The triangle incenter is located at the intersection of the angle bisectors, Fig. 1.7, p. 30.

Let P_3 be a point on side A_1A_2 of triangle $A_1A_2A_3$ in a Euclidean n space $\mathbb{R}^n$ such that A_3P_3 is an angle bisector of angle $\angle A_1A_3A_2$, as shown in Fig. 1.6. Then, P_3 is given in terms of its barycentric coordinates (m_1, m_2) with respect to the set $\{A_1, A_2\}$ by the equation

$$P_3 = \frac{m_1A_1 + m_2A_2}{m_1 + m_2} \tag{1.90}$$

where the barycentric coordinates m_1 and m_2 of P_3 are to be determined in (1.98) – (1.99) below.

As in (1.52), by the covariance (1.26) of barycentric coordinate representations with respect to translations we have, in particular for $X = A_1$ and $X = A_2$,

$$\begin{aligned} \mathbf{p}_1 &= -A_1 + P_3 = \frac{m_2(-A_1 + A_2)}{m_1 + m_2} = \frac{m_2\mathbf{a}_{12}}{m_1 + m_2} \\ \mathbf{p}_2 &= -A_2 + P_3 = \frac{m_1(-A_2 + A_1)}{m_1 + m_2} = \frac{-m_1\mathbf{a}_{12}}{m_1 + m_2} \end{aligned} \tag{1.91}$$

As indicated in Fig. 1.6, we use the notation

$$\begin{aligned} \mathbf{a}_{12} &= -A_1 + A_2, & a_{12} &= \|\mathbf{a}_{12}\| \\ \mathbf{a}_{13} &= -A_1 + A_3, & a_{13} &= \|\mathbf{a}_{13}\| \\ \mathbf{a}_{23} &= -A_2 + A_3, & a_{23} &= \|\mathbf{a}_{23}\| \end{aligned} \tag{1.92}$$

and

$$\begin{aligned} \mathbf{p}_1 &= -A_1 + P_3, & p_1 &= \|\mathbf{p}_1\| \\ \mathbf{p}_2 &= -A_2 + P_3, & p_2 &= \|\mathbf{p}_2\| \end{aligned} \tag{1.93}$$

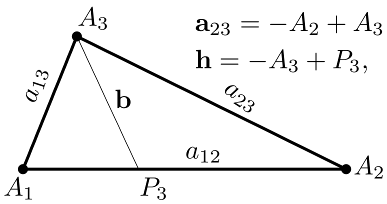
so that, by (1.91) – (1.93),

$$\begin{aligned} p_1 &= \frac{m_2a_{12}}{m_1 + m_2} \\ p_2 &= \frac{m_1a_{12}}{m_1 + m_2} \end{aligned} \tag{1.94}$$

implying

$$\frac{p_1}{p_2} = \frac{m_2}{m_1} \tag{1.95}$$

$$\begin{aligned} \mathbf{a}_{12} &= -A_1 + A_2, & a_{12} &= \|\mathbf{a}_{12}\| \\ \mathbf{a}_{13} &= -A_1 + A_3, & a_{13} &= \|\mathbf{a}_{13}\| \\ \mathbf{a}_{23} &= -A_2 + A_3, & a_{23} &= \|\mathbf{a}_{23}\| \\ \mathbf{h} &= -A_3 + P_3, & h &= \|\mathbf{h}\| \end{aligned}$$



$$\begin{aligned} \mathbf{p}_1 &= -A_1 + P_3, & p_1 &= \|\mathbf{p}_1\| \\ \mathbf{p}_2 &= -A_2 + P_3, & p_2 &= \|\mathbf{p}_2\| \end{aligned}$$

$$\alpha_1 = \angle A_2A_1A_3, \quad \alpha_2 = \angle A_1A_2A_3, \quad \alpha_3 = \angle A_1A_3A_2$$

$$\angle A_1A_3P_3 = \angle A_2A_3P_3$$

Fig. 1.6 Angle bisector, A_3P_3, of angle $\angle A_1A_3A_2$ in a Euclidean n-space $\mathbb{R}^n$, for $n = 2$. The segment A_3P_3 forms the angle bisector in triangle $A_1A_2A_3$, dropped from vertex A_3 to a point P_3 on its opposite side A_1A_2. Barycentric coordinates $\{m_1 : m_2\}$ of the point P_3 with respect to the set of points $\{A_1, A_2\}$, satisfying (1.90), are determined in (1.98) – (1.99).

By the *angle bisector theorem*, which follows immediately from the law of sines (1.63) and the equation $\sin\angle A_1P_3A_3 = \sin\angle A_2P_3A_3$ in Fig. 1.6, the angle bisector of an angle in a triangle divides the opposite side in the same ratio as the sides adjacent to the angle. Hence, in the notation of Fig. 1.6,

$$\frac{p_1}{p_2} = \frac{a_{13}}{a_{23}} \tag{1.96}$$

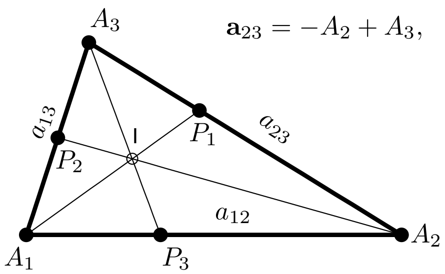
Hence, by (1.95) – (1.96), and by the law of sines (1.63),

$$\frac{m_2}{m_1} = \frac{a_{13}}{a_{23}} = \frac{\sin\alpha_2}{\sin\alpha_1} \tag{1.97}$$

so that barycentric coordinates of P_3 in (1.90) may be given by

$$\begin{aligned} m_1 &= a_{23} \\ m_2 &= a_{13} \end{aligned} \tag{1.98}$$

$$\mathbf{a}_{12} = -A_1 + A_2, \qquad a_{12} = \|\mathbf{a}_{12}\|$$
$$\mathbf{a}_{13} = -A_1 + A_3, \qquad a_{13} = \|\mathbf{a}_{13}\|$$
$$\mathbf{a}_{23} = -A_2 + A_3, \qquad a_{23} = \|\mathbf{a}_{23}\|$$



$$\angle A_2A_1P_1 = \angle A_3A_1P_1$$
$$\angle A_1A_2P_2 = \angle A_3A_2P_2$$
$$\angle A_1A_3P_3 = \angle A_2A_3P_3$$

$$\mathbf{p}_1 = -A_1 + P_1, \qquad p_1 = \|\mathbf{p}_1\|$$
$$\mathbf{p}_2 = -A_2 + P_2, \qquad p_2 = \|\mathbf{p}_2\|$$
$$\mathbf{p}_3 = -A_3 + P_3, \qquad p_3 = \|\mathbf{p}_3\|$$

$$\alpha_1 = \angle A_2A_1A_3, \quad \alpha_2 = \angle A_1A_2A_3, \quad \alpha_3 = \angle A_1A_3A_2$$

Fig. 1.7 The Triangle Incenter. The triangle angle bisectors are concurrent. The point of concurrency, I, is called the incenter of the triangle. Here $A_1A_2A_3$ is a triangle in a Euclidean n-space, $n = 2$, and the line A_kP_k is the angle bisector from vertex A_k to the intersection point P_k with the opposite side, $k = 1, 2, 3$.

or, equivalently, by

$$\begin{aligned} m_1 &= \sin\alpha_2 \\ m_2 &= \sin\alpha_1 \end{aligned} \tag{1.99}$$

Accordingly, P_3 in Fig. 1.7 is given in terms of its barycentric coordinates (m_1, m_2) with respect to the set $\{A_1, A_2\}$ by each of the two equations

$$\begin{aligned} P_3 &= \frac{a_{23}A_1 + a_{13}A_2}{a_{23} + a_{13}} \\ P_3 &= \frac{\sin\alpha_1 A_1 + \sin\alpha_2 A_2}{\sin\alpha_1 + \sin\alpha_2} \end{aligned} \tag{1.100}$$

The three bisector segments of triangle $A_1A_2A_3$ are A_1P_1, A_2P_2 and

A_3P_3, as shown in Fig. 1.7. It follows from (1.100) by vertex cyclic permutations that barycentric coordinates of their feet, P_1, P_2 and P_3, with respect to the set of the triangle vertices $\{A_1, A_2, A_3\}$ are given by

$$\begin{aligned} P_1 &= \frac{a_{13}A_2 + a_{12}A_3}{a_{13} + a_{12}} \\ P_2 &= \frac{a_{23}A_1 + a_{12}A_3}{a_{23} + a_{12}} \\ P_3 &= \frac{a_{23}A_1 + a_{13}A_2}{a_{23} + a_{13}} \end{aligned} \tag{1.101}$$

or, equivalently, by

$$\begin{aligned} P_1 &= \frac{\sin\alpha_2 A_2 + \sin\alpha_3 A_3}{\sin\alpha_2 + \sin\alpha_3} \\ P_2 &= \frac{\sin\alpha_1 A_1 + \sin\alpha_3 A_3}{\sin\alpha_1 + \sin\alpha_3} \\ P_3 &= \frac{\sin\alpha_1 A_1 + \sin\alpha_2 A_2}{\sin\alpha_1 + \sin\alpha_2} \end{aligned} \tag{1.102}$$

The equations of the lines that contain the angle bisectors of triangle $A_1A_2A_3$, Fig. 1.7, are

$$\begin{aligned} L_{A_1P_1} &= A_1 + (-A_1 + P_1)t_1 \\ L_{A_2P_2} &= A_2 + (-A_2 + P_2)t_2 \\ L_{A_3P_3} &= A_3 + (-A_3 + P_3)t_3 \end{aligned} \tag{1.103}$$

for the three line parameters $-\infty < t_1, t_2, t_3 < \infty$, where the angle bisector feet P_1, P_2 and P_3 are given by (1.101).

In order to determine the point of concurrency I of the triangle angle bisectors, Fig. 1.7, if exists, we solve the vector equations

$$A_1 + (-A_1 + P_1)t_1 = A_2 + (-A_2 + P_2)t_2 = A_3 + (-A_3 + P_3)t_3 \tag{1.104}$$

for the three scalar unknowns t_1, t_2 and t_3. The solution turn out to be

$$\begin{aligned} t_1 &= \frac{a_{12}+a_{13}}{a_{12}+a_{13}+a_{23}} \\ t_2 &= \frac{a_{12}+a_{23}}{a_{12}+a_{13}+a_{23}} \\ t_3 &= \frac{a_{13}+a_{23}}{a_{12}+a_{13}+a_{23}} \end{aligned} \tag{1.105}$$

Substituting the solution for t_1 (respectively, for t_2, t_3) in the first (respectively, second, third) equation in (1.103) we determine the incenter I of triangle $A_1A_2A_3$ in terms of barycentric coordinates,

$$I = \frac{m_1A_1 + m_2A_2 + m_3A_3}{m_1 + m_2 + m_3} \tag{1.106}$$

where the barycentric coordinates are

$$\begin{aligned} m_1 &= a_{23} \\ m_2 &= a_{13} \\ m_3 &= a_{12} \end{aligned} \tag{1.107}$$

or, equivalently by (1.97),

$$\begin{aligned} m_1 &= \sin\alpha_1 \\ m_2 &= \sin\alpha_2 \\ m_3 &= \sin\alpha_3 \end{aligned} \tag{1.108}$$

Following (1.106) and (1.107), the incenter I of a triangle $A_1A_2A_3$ with vertices A_1, A_2 and A_3, and with corresponding sidelengths a_{23}, a_{13} and a_{12}, Fig. 1.7, is given in terms of its barycentric coordinates with respect to the set $\{A_1, A_2, A_3\}$ by the equation

$$I = \frac{a_{23}A_1 + a_{13}A_2 + a_{12}A_3}{a_{23} + a_{13} + a_{12}} \tag{1.109}$$

Following (1.106) and (1.108), the incenter I of a triangle $A_1A_2A_3$ with vertices A_1, A_2 and A_3, and with corresponding angles α_1, α_2 and α_3, Fig. 1.7, is given in terms of its trigonometric barycentric coordinates with respect to the set $\{A_1, A_2, A_3\}$ by the equation

$$I = \frac{\sin\alpha_1 A_1 + \sin\alpha_2 A_2 + \sin\alpha_3 A_3}{\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3} \tag{1.110}$$

The sine of any triangle angle is positive. Hence, by convexity considerations, the incenter I of a triangle lies on the interior of the triangle.

1.11 Triangle Inradius

Let $A_1A_2A_3$ be a triangle with incenter I in a Euclidean space $\mathbb{R}^n$. Following (1.109) and the covariance property (1.26), p. 12, of barycentric representations, we have, in the notation of Fig. 1.8,

$$\begin{aligned} -A_1+I &= -A_1 + \frac{a_{23}A_1+a_{13}A_2+a_{12}A_3}{a_{23}+a_{13}+a_{12}} \\ &= \frac{a_{13}(-A_1+A_2)+a_{12}(-A_1+A_3)}{a_{23}+a_{13}+a_{12}} \\ &= \frac{a_{13}\mathbf{a}_{12}+a_{12}\mathbf{a}_{13}}{a_{23}+a_{13}+a_{12}} \end{aligned} \tag{1.111}$$

Hence,

$$\bar{a}_{13}^2 := \|-A_1+I\|^2 = \frac{2a_{12}^2a_{13}^2(1+\cos\alpha_1)}{(a_{12}+a_{13}+a_{23})^2} \tag{1.112}$$

noting that

$$(-A_1+A_2)\cdot(-A_1+A_3) = a_{12}a_{13}\cos\alpha_1 \tag{1.113}$$

By the law of cosines for triangle $A_1A_2A_3$,

$$2(1+\cos\alpha_1) = \frac{(a_{12}+a_{13})^2-a_{23}^2}{a_{12}a_{13}} \tag{1.114}$$

Hence, by (1.112)–(1.114),

$$\bar{a}_{13}^2 = \frac{a_{12}a_{13}}{(a_{12}+a_{13}+a_{23})^2}\{(a_{12}+a_{13})^2-a_{23}^2\} \tag{1.115}$$

Similarly,

$$\bar{a}_{23}^2 := \|-A_2+I\|^2 = \frac{2a_{12}^2a_{23}^2(1+\cos\alpha_2)}{(a_{12}+a_{13}+a_{23})^2} \tag{1.116}$$

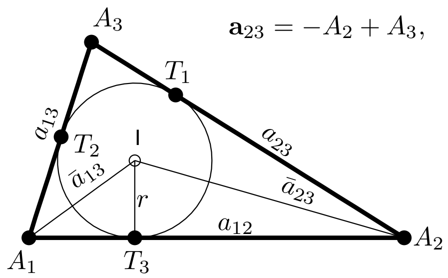
and hence

$$\bar{a}_{23}^2 = \frac{a_{12}a_{23}}{(a_{12}+a_{13}+a_{23})^2}\{(a_{12}+a_{23})^2-a_{13}^2\} \tag{1.117}$$

$$\mathbf{a}_{12} = -A_1 + A_2, \qquad a_{12} = \|\mathbf{a}_{12}\|$$

$$\mathbf{a}_{13} = -A_1 + A_3, \qquad a_{13} = \|\mathbf{a}_{13}\|$$

$$\mathbf{a}_{23} = -A_2 + A_3, \qquad a_{23} = \|\mathbf{a}_{23}\|$$



$$\alpha_1 = \angle A_3 A_1 A_2$$

$$\alpha_2 = \angle A_1 A_2 A_3$$

$$\alpha_3 = \angle A_2 A_3 A_1$$

$$\bar{\mathbf{a}}_{13} = -A_1 + I, \qquad \bar{a}_{13} = \| - A_1 + I\|$$

$$\bar{\mathbf{a}}_{23} = -A_2 + I, \qquad \bar{a}_{23} = \| - A_2 + I\|$$

Fig. 1.8 The Triangle Incircle, Incenter and Inradius. The triangle angle bisectors are concurrent. The point of concurrency, I, is called the incenter of the triangle. Here $A_1A_2A_3$ is a triangle in a Euclidean n-space, $\mathbb{R}^n$, and T_k is the point of tangency where the triangle incircle meets the triangle side opposite to vertex A_k, $k = 1, 2, 3$. The radius r of the triangle incircle, determined in (1.122), is called the triangle inradius.

The vectors $\bar{\mathbf{a}}_{13}$ and $\bar{\mathbf{a}}_{23}$ along with their magnitudes $\bar{a}_{13}$ and $\bar{a}_{23}$ are shown in Fig. 1.8.

The tangency point T_3 where the incenter of triangle $A_1A_2A_3$ meets the triangle side A_1A_2 opposite to vertex A_3, Fig. 1.8, is the perpendicular projection of the incenter I on the line $L_{A_1A_2}$ that passes through the points A_1 and A_2. Hence, by the point to line perpendicular projection formula (1.76), p. 24,

$$T_3 = \frac{1}{2}\left(\frac{a_{12}^2 - \bar{a}_{13}^2 + \bar{a}_{23}^2}{a_{12}^2}\right) A_1 + \frac{1}{2}\left(\frac{a_{12}^2 + \bar{a}_{13}^2 - \bar{a}_{23}^2}{a_{12}^2}\right) A_2 \tag{1.118}$$

Substituting (1.115) and (1.117) into (1.118), we obtain

$$T_3 = \frac{a_{12} - a_{13} + a_{23}}{2a_{12}} A_1 + \frac{a_{12} + a_{13} - a_{23}}{2a_{12}} A_2 \tag{1.119}$$

Equation (1.119) gives rise to the following theorem:

Theorem 1.13 (The Incircle Tangency Points). *Let $A_1A_2A_3$ be a triangle in a Euclidean space $\mathbb{R}^n$, and let T_k be the point of tangency* *where the triangle incircle meets the side opposite to vertex* A_k, $k = 1, 2, 3$, *Fig. 1.8. Then, in the standard triangle notation, Figs. 1.7 – 1.8,*

$$\begin{aligned} T_1 &= \frac{-a_{12} + a_{13} + a_{23}}{2a_{23}} A_2 + \frac{a_{12} - a_{13} + a_{23}}{2a_{23}} A_3 \\ T_2 &= \frac{-a_{12} + a_{13} + a_{23}}{2a_{13}} A_1 + \frac{a_{12} + a_{13} - a_{23}}{2a_{13}} A_3 \\ T_3 &= \frac{a_{12} - a_{13} + a_{23}}{2a_{12}} A_1 + \frac{a_{12} + a_{13} - a_{23}}{2a_{12}} A_2 \end{aligned} \tag{1.120}$$

Proof. The third equation in (1.120) is established in (1.119). The first and the second equations in (1.120) are obtained from the first by vertex cyclic permutations. □

Applying the point to line distance formula (1.77), p. 24, to calculate the distance r between the point A_3 and the line $L_{A_1A_2}$ that contains the points A_1 and A_2, Fig. 1.8, we obtain the equation

$$r^2 = \frac{(a_{12} + \bar{a}_{13} + \bar{a}_{23})(-a_{12} + \bar{a}_{13} + \bar{a}_{23})(a_{12} - \bar{a}_{13} + \bar{a}_{23})(a_{12} + \bar{a}_{13} - \bar{a}_{23})}{4a_{12}^2} \tag{1.121}$$

Substituting (1.116) into (1.121), we obtain the following theorem:

Theorem 1.14 (The Triangle Inradius). *Let* $A_1A_2A_3$ *be a triangle in a Euclidean space* $\mathbb{R}^n$. *Then, in the standard triangle notation, Fig. 1.8, the triangle inradius* r *is given by the equation*

$$r = \sqrt{\frac{(p - a_{12})(p - a_{13})(p - a_{23})}{p}} \tag{1.122}$$

where p *is the triangle semiperimeter,*

$$p = \frac{a_{12} + a_{13} + a_{23}}{2} \tag{1.123}$$

Following Theorem 1.14 it is appropriate to present the well-known Heron's formula [Coxeter (1961)].

Theorem 1.15 (Heron's Formula). *Let* $A_1A_2A_3$ *be a triangle in a Euclidean space* $\mathbb{R}^n$. *Then, in the standard triangle notation, Fig. 1.2, the triangle area* $|A_1A_2A_3|$ *is given by Heron's formula,*

$$|A_1A_2A_3| = \sqrt{p(p - a_{12})(p - a_{13})(p - a_{23})} \tag{1.124}$$

or, equivalently,

$$|A_1A_2A_3|^2 = \frac{1}{16}F_2(a_{12}, a_{13}, a_{23}) \tag{1.125}$$

where $F_2(a_{12}, a_{13}, a_{23})$ *is the* 4×4 *Cayley-Menger determinant* (1.74), *p. 23.*

The determinant form (1.125) of Heron's formula possesses the comparative advantage of admitting a natural generalization to higher dimensions, as indicated in (1.194) – (1.195), p. 63.

Theorems 1.14 and 1.15 result in an elegant relationship between the triangle area $|A_1A_2A_3|$ and its inradius r,

$$r = \frac{|A_1A_2A_3|}{p} = \frac{2|A_1A_2A_3|}{a_{12} + a_{13} + a_{23}} \tag{1.126}$$

1.12 Triangle Circumcenter

The triangle circumcenter is located at the intersection of the perpendicular bisectors of its sides, Fig. 1.9. Accordingly, it is equidistant from the triangle vertices.

Let $A_1A_2A_3$ be a triangle with vertices A_1, A_2 and A_3 in a Euclidean n-space $\mathbb{R}^n$, and let O be the triangle circumcenter, as shown in Fig. 1.9. Then, O is given in terms of its barycentric coordinates $(m_1 : m_2 : m_3)$ with respect to the set $\{A_1, A_2, A_3\}$ by the equation

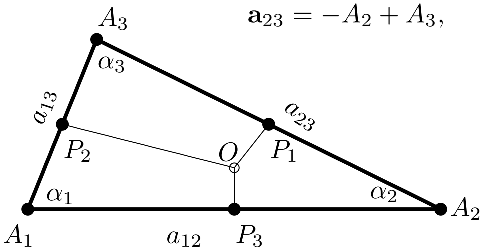
$$O = \frac{m_1A_1 + m_2A_2 + m_3A_3}{m_1 + m_2 + m_3} \tag{1.127}$$

where the barycentric coordinates m_1, m_2 and m_3 of P_3 are to be determined.

Applying Lemma 1.7, p. 13, to the point $P = O$ in (1.127) we obtain the equations

$$\begin{aligned}
\|-A_1 + O\|^2 &= \frac{m_2^2a_{12}^2 + m_3^2a_{13}^2 + m_2m_3(a_{12}^2 + a_{13}^2 - a_{23}^2)}{(m_1 + m_2 + m_3)^2} \\
\|-A_2 + O\|^2 &= \frac{m_1^2a_{12}^2 + m_3^2a_{23}^2 + m_1m_3(a_{12}^2 - a_{13}^2 + a_{23}^2)}{(m_1 + m_2 + m_3)^2} \\
\|-A_3 + O\|^2 &= \frac{m_1^2a_{13}^2 + m_2^2a_{23}^2 + m_1m_2(-a_{12}^2 + a_{13}^2 + a_{23}^2)}{(m_1 + m_2 + m_3)^2}
\end{aligned} \tag{1.128}$$

$$\mathbf{a}_{12} = -A_1 + A_2, \qquad a_{12} = \|\mathbf{a}_{12}\|$$
$$\mathbf{a}_{13} = -A_1 + A_3, \qquad a_{13} = \|\mathbf{a}_{13}\|$$
$$\mathbf{a}_{23} = -A_2 + A_3, \qquad a_{23} = \|\mathbf{a}_{23}\|$$



$$\mathbf{p}_1 = -A_1 + P_1, \qquad p_1 = \|\mathbf{p}_1\|$$
$$\mathbf{p}_2 = -A_2 + P_2, \qquad p_2 = \|\mathbf{p}_2\|$$
$$\mathbf{p}_3 = -A_3 + P_3, \qquad p_3 = \|\mathbf{p}_3\|$$

$$a_{23}^2 = a_{12}^2 + a_{13}^2 - 2a_{12}a_{13}\cos\alpha_1$$
$$a_{13}^2 = a_{12}^2 + a_{23}^2 - 2a_{12}a_{23}\cos\alpha_2$$
$$a_{12}^2 = a_{13}^2 + a_{23}^2 - 2a_{13}a_{23}\cos\alpha_3$$

Fig. 1.9 The Triangle Circumcenter is located at the intersection of its perpendicular bisectors. Accordingly, it is equidistant from the triangle vertices.

Equations (1.128) along with the triangle circumcenter condition, Fig. 1.9,

$$\begin{aligned} \|-A_1 + O\|^2 &= \|-A_2 + O\|^2 \\ \|-A_2 + O\|^2 &= \|-A_3 + O\|^2 \end{aligned} \tag{1.129}$$

and the normalization condition

$$m_1 + m_2 + m_3 = 1 \tag{1.130}$$

give the following system of three equations for the three unknowns m_1,

m_2 and m_3:

$$\begin{aligned}
m_2^2 a_{12}^2 + m_3^2 a_{13}^2 + m_2 m_3 (a_{12}^2 + a_{13}^2 - a_{23}^2) = \\
m_1^2 a_{12}^2 + m_3^2 a_{23}^2 + m_1 m_3 (\ \ a_{12}^2 - a_{13}^2 + a_{23}^2) \\
m_1^2 a_{12}^2 + m_3^2 a_{23}^2 + m_1 m_3 (a_{12}^2 - a_{13}^2 + a_{23}^2) = \\
m_1^2 a_{13}^2 + m_2^2 a_{23}^2 + m_1 m_2 (-a_{12}^2 + a_{13}^2 + a_{23}^2) \\
m_1 + m_2 + m_3 = 1
\end{aligned} \tag{1.131}$$

Substituting $m_3 = 1 - m_1 - m_2$ from the third equation in (1.131) into the first two equations in (1.131), and simplifying (the use of a computer system for algebra, like Mathematica or Maple, is recommended) we obtain two equations for the unknowns m_1 and m_2, each of which turns out to be linear in m_1 and quadratic in m_2. Eliminating m_2^2 between these two equations, we obtain the following single equation that relates m_1 to m_2 linearly:

$$a_{13}^2 - a_{23}^2 - m_1(a_{12}^2 + a_{13}^2 - a_{23}^2) + m_2(a_{12}^2 - a_{13}^2 + a_{23}^2) = 0 \tag{1.132}$$

A vertex cyclic permutation in (1.132) gives a second linear connection, between m_2 and m_3. A third linear connection, between m_1, m_2 and m_3, is provided by (1.130) thus obtaining the following system of three linear equations for the three unknowns m_1, m_2 and m_3:

$$\begin{aligned}
a_{13}^2 - a_{23}^2 - m_1(a_{12}^2 + a_{13}^2 - a_{23}^2) + m_2(\ \ a_{12}^2 - a_{13}^2 + a_{23}^2) = 0 \\
a_{12}^2 - a_{13}^2 - m_2(a_{12}^2 - a_{13}^2 + a_{23}^2) + m_3(-a_{12}^2 + a_{13}^2 + a_{23}^2) = 0 \\
m_1 + m_2 + m_3 = 1
\end{aligned} \tag{1.133}$$

The solution of the linear system (1.133) gives the special barycentric coordinates $\{m_1, m_2, m_3\}$ of the triangle circumcenter O:

$$\begin{aligned}
m_1 &= \frac{a_{23}^2(\ \ a_{12}^2 + a_{13}^2 - a_{23}^2)}{D} \\
m_2 &= \frac{a_{13}^2(\ \ a_{12}^2 - a_{13}^2 + a_{23}^2)}{D} \\
m_3 &= \frac{a_{12}^2(-a_{12}^2 + a_{13}^2 + a_{23}^2)}{D}
\end{aligned} \tag{1.134}$$

in terms of its side lengths, where D is given by

$$D = (a_{12}^2 + a_{13}^2 + a_{23}^2)(-a_{12}^2 + a_{13}^2 + a_{23}^2)(a_{12}^2 - a_{13}^2 + a_{23}^2)(a_{12}^2 + a_{13}^2 - a_{23}^2) \tag{1.135}$$

Finally, it follows from (1.134) that barycentric coordinates $\{m_1 : m_2 : m_3\}$ of the triangle circumcenter O are given by

$$\begin{aligned} m_1 &= a_{23}^2(a_{12}^2 + a_{13}^2 - a_{23}^2) \\ m_2 &= a_{13}^2(a_{12}^2 - a_{13}^2 + a_{23}^2) \\ m_3 &= a_{12}^2(-a_{12}^2 + a_{13}^2 + a_{23}^2) \end{aligned} \tag{1.136}$$

We now wish to find trigonometric barycentric coordinates for the triangle circumcenter, that is, barycentric coordinates that are expressed in terms of the triangle angles. Hence, we calculate m_1/m_3 and m_2/m_3 by means of (1.136) and the law of sines (1.63), p. 21, and employ the trigonometric identity $\sin^2\alpha = (1 - \cos 2\alpha)/2$, obtaining

$$\begin{aligned} \frac{m_1}{m_3} &= \frac{(1 + \cos 2\alpha_1 - \cos 2\alpha_2 - \cos 2\alpha_3)\sin^2\alpha_1}{(1 - \cos 2\alpha_1 - \cos 2\alpha_2 + \cos 2\alpha_3)\sin^2\alpha_3} \\ \frac{m_2}{m_3} &= \frac{(1 - \cos 2\alpha_1 + \cos 2\alpha_2 - \cos 2\alpha_3)\sin^2\alpha_2}{(1 - \cos 2\alpha_1 - \cos 2\alpha_2 + \cos 2\alpha_3)\sin^2\alpha_3} \end{aligned} \tag{1.137}$$

Hence, trigonometric barycentric coordinates $\{m_1 : m_2 : m_3\}$ for a triangle circumcenter are given by

$$\begin{aligned} m_1 &= (1 + \cos 2\alpha_1 - \cos 2\alpha_2 - \cos 2\alpha_3)\sin^2\alpha_1 \\ m_2 &= (1 - \cos 2\alpha_1 + \cos 2\alpha_2 - \cos 2\alpha_3)\sin^2\alpha_2 \\ m_3 &= (1 - \cos 2\alpha_1 - \cos 2\alpha_2 + \cos 2\alpha_3)\sin^2\alpha_3 \end{aligned} \tag{1.138}$$

Taking advantage of the relationship (1.65), p. 21, between triangle angles, and employing trigonometric identities, (1.138) can be simplified, obtaining the following elegant trigonometric barycentric coordinates of the circumcenter O of triangle $A_1A_2A_3$, Fig. 1.9, in terms of its angles,

$$\begin{aligned} m_1 &= \sin\alpha_1 \cos\alpha_1 \\ m_2 &= \sin\alpha_2 \cos\alpha_2 \\ m_3 &= \sin\alpha_3 \cos\alpha_3 \end{aligned} \tag{1.139}$$

For later reference we note that owing to the π-identity of triangles, (1.65), p. 21, the three equations in (1.139) are equivalent to the following three equations:

$$\begin{aligned} m_1 &= \sin\frac{-\alpha_1+\alpha_2+\alpha_3}{2}\sin\alpha_1 \\ m_2 &= \sin\frac{\alpha_1-\alpha_2+\alpha_3}{2}\sin\alpha_2 \\ m_3 &= \sin\frac{\alpha_1+\alpha_2-\alpha_3}{2}\sin\alpha_3 \end{aligned} \tag{1.140}$$

There is an important distinction between the elegant barycentric coordinates (1.140) and their simplified form (1.139). The former is free of the π-identity condition, while the latter embodies the π-identity. As a result, the validity of the latter is restricted to Euclidean geometry, where the π-identity holds. The former is also valid in Euclidean geometry but, unlike the latter, it survives unimpaired in hyperbolic geometry as well, where the π-identity does not hold.

Indeed, it will be found in (4.251), p. 248, that hyperbolic barycentric coordinates of the hyperbolic circumcenter of hyperbolic triangles in the Cartesian-Beltrami-Klein ball model of hyperbolic geometry are given by (1.140) as well.

1.13 Circumradius

The circumradius R of a triangle $A_1A_2A_3$ in a Euclidean space $\mathbb{R}^n$ is the radius of its circumcircle. Hence, in the notation of Fig. 1.10,

$$\begin{aligned} R^2 &= \|-A_1+O\|^2 \\ &= \|-A_2+O\|^2 \\ &= \|-A_3+O\|^2 \end{aligned} \tag{1.141}$$

where O is the triangle circumcenter.

The circumradius $R = \|-A_1+O\|$ is determined by successively substituting $\|-A_1+O\|^2$ from the first equation in (1.128), p. 36, and m_k,

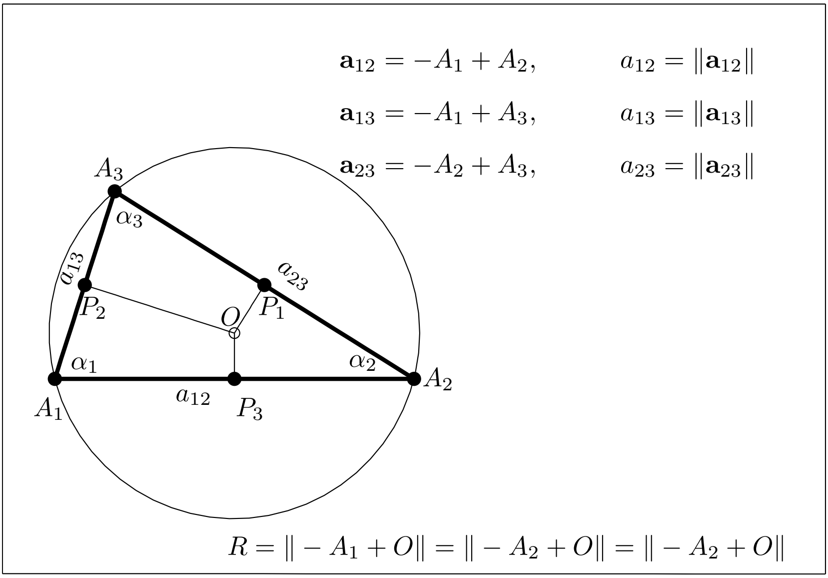


Fig. 1.10 The Circumcenter O and Circumradius R of a triangle $A_1A_2A_3$ in a Euclidean space $\mathbb{R}^n$.

$k = 1, 2, 3$, from (1.136), p. 39, into (1.141), obtaining

$$
\begin{aligned}
R^2 &= \frac{a_{12}^2 a_{13}^2 a_{23}^2}{16p(p-a_{12})(p-a_{13})(p-a_{23})} \\
&= \frac{a_{12}^2 a_{13}^2 a_{23}^2}{(a_{12}+a_{13}+a_{23})(-a_{12}+a_{13}+a_{23})(a_{12}-a_{13}+a_{23})(a_{12}+a_{13}-a_{23})} \\
&= \frac{a_{12}^2 a_{13}^2 a_{23}^2}{16|A_1A_2A_3|^2}
\end{aligned}
\tag{1.142}
$$

Hence, the triangle circumradius is given by

$$
R = \frac{a_{12}a_{13}a_{23}}{4\sqrt{p(p-a_{12})(p-a_{13})(p-a_{23}}} = \frac{a_{12}a_{13}a_{23}}{4|A_1A_2A_3|} = \frac{a_{12}a_{13}a_{23}}{4rp} \tag{1.143}
$$

where p and r are the triangle semiperimeter and inradius, and where $|A_1A_2A_3|$ is the triangle area given by Heron's formula (1.124), p. 35.

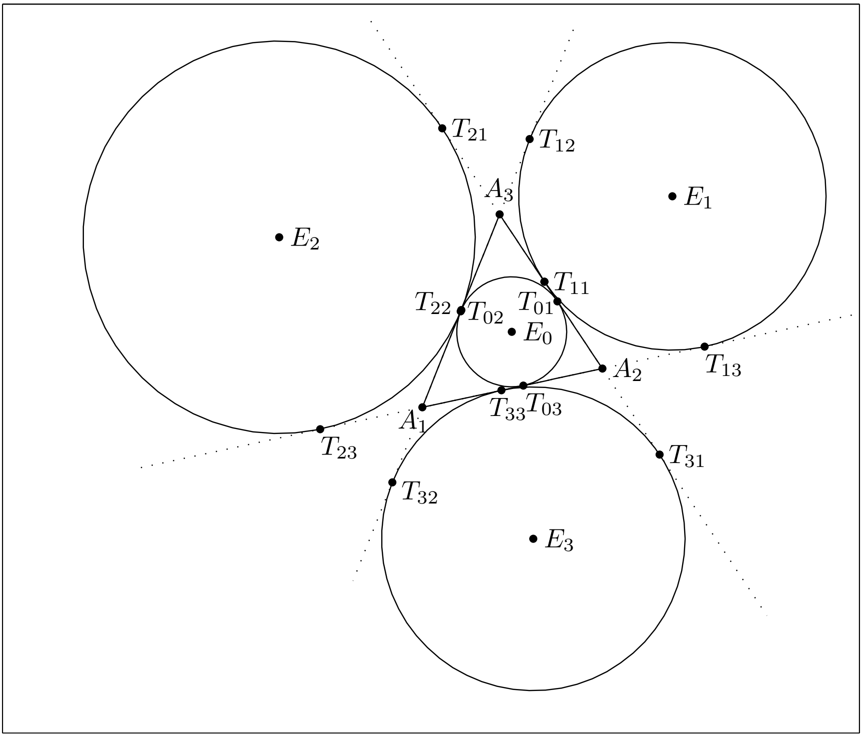


Fig. 1.11 The triangle incircle and excircles. Here T_{ij} is the tangency point where the in-excircle with center E_i, $i = 0, 1, 2, 3$, meets the triangle side, or its extension, opposite to vertex A_j, $j = 1, 2, 3$. The incircle points of tangency, T_{0j} are determined by Theorem 1.13, p. 34, and the excircle points of tangency, T_{ij}, $i = 1, 2, 3$, are determined by Theorem 1.17, p. 49. Trigonometric barycentric coordinate representations of the in-excircle tangency points T_{ij} are listed in (1.168), p. 50.

1.14 Triangle Incircle and Excircles

An incircle of a triangle is a circle lying inside the triangle, tangent to each of its sides, shown in Fig. 1.8, p. 34. The center and radius of the incircle of a triangle are called the triangle incenter and inradius. Similarly, an excircle of a triangle is a circle lying outside the triangle, tangent to one of its sides and tangent to the extensions of the other two. The centers and radii of the excircles of a triangle are called the triangle excenters and exradii. The incenter and excenters of a triangle, shown in Fig. 1.11, are equidistant from the triangles sides.

Let E be an incenter or an excenter of a triangle $A_1A_2A_3$, Fig. 1.11, in a Euclidean n space $\mathbb{R}^n$, and let

$$E = \frac{m_1A_1 + m_2A_2 + m_3A_3}{m_1 + m_2 + m_3} \tag{1.144}$$

be the barycentric coordinate representation of E with respect to the set $\{A_1, A_2, A_3\}$, where the barycentric coordinates m_k, $k = 1, 2, 3$, are to be determined in Theorem 1.16, p. 46.

Applying Lemma 1.7, p. 13, to the point $P = E$ in (1.144) we obtain the equations

$$\begin{aligned}
\|-A_1 + E\|^2 &= \frac{m_2^2a_{12}^2 + m_3^2a_{13}^2 + m_2m_3(a_{12}^2 + a_{13}^2 - a_{23}^2)}{(m_1 + m_2 + m_3)^2} \\
\|-A_2 + E\|^2 &= \frac{m_1^2a_{12}^2 + m_3^2a_{23}^2 + m_1m_3(a_{12}^2 - a_{13}^2 + a_{23}^2)}{(m_1 + m_2 + m_3)^2} \\
\|-A_3 + E\|^2 &= \frac{m_1^2a_{13}^2 + m_2^2a_{23}^2 + m_1m_2(-a_{12}^2 + a_{13}^2 + a_{23}^2)}{(m_1 + m_2 + m_3)^2}
\end{aligned} \tag{1.145}$$

Imposing the normalization condition

$$m_1 + m_2 + m_3 = 1 \tag{1.146}$$

in (1.144) – (1.145) is clearly convenient.

Let E represent each of the incenter and excenters E_k, $k = 0, 1, 2, 3$, of a triangle $A_1A_2A_3$ in a Euclidean n-space $\mathbb{R}^n$, shown in Fig. 1.11 for $n = 2$.

(1) The distance of E from the line $L_{A_1A_2}$ that passes through points A_1 and A_2, Fig. 1.11, is the altitude r_3 of triangle A_1A_2E drawn from base A_1A_2. Hence, by the point-line distance formula (1.77), p. 24, in Theorem 1.12, the distance r_3 between the point E and the line $L_{A_1A_2}$ is given by the equation

$$r_3^2 = \frac{(a_{12} + \bar{a}_{13} + \bar{a}_{23})(-a_{12} + \bar{a}_{13} + \bar{a}_{23})(a_{12} - \bar{a}_{13} + \bar{a}_{23})(a_{12} + \bar{a}_{13} - \bar{a}_{23})}{4a_{12}^2} \tag{1.147a}$$

where

$$\begin{aligned}
\bar{a}_{13}^2 &:= \|-A_1 + E\|^2 \\
\bar{a}_{23}^2 &:= \|-A_2 + E\|^2
\end{aligned} \tag{1.147b}$$

Substituting successively, $\bar{a}_{13}$ and $\bar{a}_{23}$ from (1.147b) and $\|-A_1+E\|$ and $\|-A_2+E\|$ from (1.145) into (1.147a) we obtain an equation of the form

$$r_3 = f_3(a_{12}, a_{13}, a_{23}, m_1, m_2, m_3) \tag{1.147c}$$

where r_3 is expressed as a function of the sides of triangle $A_1A_2A_3$ and the unknown barycentric coordinates of the point E in (1.144).

(2) The distance of E, Fig. 1.11, from the line $L_{A_1A_3}$ that passes through points A_1 and A_3, Fig. 1.11, is the altitude r_2 of triangle A_1A_3E drawn from base A_1A_3. Hence, by the point-line distance formula (1.77), p. 24, the distance r_2 between the point E and the line $L_{A_1A_3}$ is given by the equation

$$r_2^2 = \frac{(\bar{a}_{12}+a_{13}+\bar{a}_{23})(-\bar{a}_{12}+a_{13}+\bar{a}_{23})(\bar{a}_{12}-a_{13}+\bar{a}_{23})(\bar{a}_{12}+a_{13}-\bar{a}_{23})}{4a_{13}^2} \tag{1.148a}$$

where

$$\begin{aligned} \bar{a}_{12}^2 &:= \|-A_1+E\|^2 \\ \bar{a}_{23}^2 &:= \|-A_3+E\|^2 \end{aligned} \tag{1.148b}$$

Substituting successively, $\bar{a}_{12}$ and $\bar{a}_{23}$ from (1.148b) and $\|-A_1+E\|$ and $\|-A_3+E\|$ from (1.145) into (1.148a) we obtain an equation of the form

$$r_2 = f_2(a_{12}, a_{13}, a_{23}, m_1, m_2, m_3) \tag{1.148c}$$

where r_2 is expressed as a function of the sides of triangle $A_1A_2A_3$ and the unknown barycentric coordinates of the point E in (1.144).

(3) The distance of E from the line $L_{A_2A_3}$ that passes through points A_2 and A_3, Fig. 1.11, is the altitude r_1 of triangle A_2A_3E drawn from base A_2A_3. Hence, by the point-line distance formula (1.77), p. 24, the distance r_1 between the point E and the line $L_{A_2A_3}$ is given by the equation

$$r_1^2 = \frac{(\bar{a}_{12}+\bar{a}_{13}+a_{23})(-\bar{a}_{12}+\bar{a}_{13}+a_{23})(\bar{a}_{12}-\bar{a}_{13}+a_{23})(\bar{a}_{12}+\bar{a}_{13}-a_{23})}{4a_{23}^2} \tag{1.149a}$$

where

$$\begin{aligned} \bar{a}_{12}^2 &:= \| -A_2 + E \|^2 \\ \bar{a}_{13}^2 &:= \| -A_3 + E \|^2 \end{aligned} \tag{1.149b}$$

Substituting successively, $\bar{a}_{12}$ and $\bar{a}_{13}$ from (1.149b) and $\| -A_2 + E\|$ and $\| -A_3 + E\|$ from (1.145) into (1.149a) we obtain an equation of the form

$$r_1 = f_1(a_{12}, a_{13}, a_{23}, m_1, m_2, m_3) \tag{1.149c}$$

where r_1 is expressed as a function of the sides of triangle $A_1A_2A_3$ and the unknown barycentric coordinates of the point E in (1.144).

The condition that the point E, which represents each of the points E_k, $k = 0, 1, 2, 3$, Fig. 1.11, is equidistant from the sides of triangle $A_1A_2A_3$ is equivalent to the system of two equations

$$\begin{aligned} r_1 &= r_2 \\ r_1 &= r_3 \end{aligned} \tag{1.150}$$

These equations along with the normalization condition (1.146) form a system of three equations for the three unknowns m_1, m_2, m_3 in (1.144).

Solving the system (1.150) and, then, ignoring the normalization condition (1.146), we obtain

$$\begin{aligned} m_1^2 &= a_{23}^2 \\ m_2^2 &= a_{13}^2 \\ m_3^2 &= a_{12}^2 \end{aligned} \tag{1.151}$$

The equations in (1.151) present eight solutions for the triple (m_1, m_2, m_3). Owing to the homogeneity of barycentric coordinates, only four of the eight solutions are indistinguishable. These are:

$$\begin{aligned} (m_1 : m_2 : m_3) &= (\ \ a_{23} : \ \ a_{13} : \ \ a_{12}) \\ (m_1 : m_2 : m_3) &= (-a_{23} : \ \ a_{13} : \ \ a_{12}) \\ (m_1 : m_2 : m_3) &= (\ \ a_{23} : -a_{13} : \ \ a_{12}) \\ (m_1 : m_2 : m_3) &= (\ \ a_{23} : \ \ a_{13} : -a_{12}) \end{aligned} \tag{1.152}$$

Each of the four barycentric coordinate sets in (1.152) determines the barycentric coordinates of the point E in (1.144), which is equidistant from the sides of triangle $A_1A_2A_3$ in Fig. 1.11. Accordingly, the first equation in (1.152) gives the barycentric coordinates of the incenter, $E = E_0$, of triangle $A_1A_2A_3$, and the other equations in (1.152) give the barycentric coordinates of each of the excenters, $E = E_k$, $k = 1, 2, 3$, of the triangle.

By the law of sines (1.63), p. 21, and owing to their homogeneity, the barycentric coordinates in the first equation in (1.152) can be written as

$$\begin{aligned}(m_1 : m_2 : m_3) &= \left(\frac{a_{23}}{a_{12}} : \frac{a_{13}}{a_{12}} : 1\right) \\ &= \left(\frac{\sin\alpha_1}{\sin\alpha_3} : \frac{\sin\alpha_2}{\sin\alpha_3} : 1\right) \\ &= (\sin\alpha_1 : \sin\alpha_2 : \sin\alpha_3)\end{aligned} \tag{1.153}$$

Similarly, all the barycentric coordinates in (1.152) can be expressed trigonometrically in terms of the triangle angles.

Formalizing, we thus obtain the following theorem:

Theorem 1.16 (In-Excenters Barycentric Representations). *Let $A_1A_2A_3$ be a triangle with incenter E_0 and excenters E_k, $k = 1, 2, 3$, in a Euclidean space $\mathbb{R}^n$, Fig. 1.11. Then the barycentric coordinate representations of the triangle in-excenters E_k, $k = 0, 1, 2, 3$, are given by the equations*

$$\begin{aligned}E_0 &= \frac{a_{23}A_1 + a_{13}A_2 + a_{12}A_3}{a_{23} + a_{13} + a_{12}} \\ E_1 &= \frac{-a_{23}A_1 + a_{13}A_2 + a_{12}A_3}{-a_{23} + a_{13} + a_{12}} \\ E_2 &= \frac{a_{23}A_1 - a_{13}A_2 + a_{12}A_3}{a_{23} - a_{13} + a_{12}} \\ E_3 &= \frac{a_{23}A_1 + a_{13}A_2 - a_{12}A_3}{a_{23} + a_{13} - a_{12}}\end{aligned} \tag{1.154}$$

and their trigonometric barycentric coordinate representations are given by *the equations*

$$
\begin{aligned}
E_0 &= \frac{\sin\alpha_1 A_1 + \sin\alpha_2 A_2 + \sin\alpha_3 A_3}{\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3} \\
E_1 &= \frac{-\sin\alpha_1 A_1 + \sin\alpha_2 A_2 + \sin\alpha_3 A_3}{-\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3} \\
E_2 &= \frac{\sin\alpha_1 A_1 - \sin\alpha_2 A_2 + \sin\alpha_3 A_3}{\sin\alpha_1 - \sin\alpha_2 + \sin\alpha_3} \\
E_3 &= \frac{\sin\alpha_1 A_1 + \sin\alpha_2 A_2 - \sin\alpha_3 A_3}{\sin\alpha_1 + \sin\alpha_2 - \sin\alpha_3}
\end{aligned}
\tag{1.155}
$$

1.15 Excircle Tangency Points

Let $A_1A_2A_3$ be a triangle in a Euclidean space $\mathbb{R}^n$, with excircles centered at the points E_k, $k = 1, 2, 3$, and let the tangency point where the A_3-excircle meets the triangle side A_1A_2 be T_{33}, as shown in Fig. 1.11, p. 42. The tangency point T_{33} is the perpendicular projection of the point E_3 on the line $L_{A_1A_2}$ that passes through the points A_1 and A_2, Fig. 1.11. Hence, by the point to line perpendicular projection formula (1.76), p. 24, the point T_{33} possesses the barycentric coordinate representation

$$
T_{33} = \frac{1}{2}\left(\frac{a_{12}^2 - \bar{a}_{13}^2 + \bar{a}_{23}^2}{a_{12}^2}\right)A_1 + \frac{1}{2}\left(\frac{a_{12}^2 + \bar{a}_{13}^2 - \bar{a}_{23}^2}{a_{12}^2}\right)A_2 \tag{1.156}
$$

with respect to the set $S = \{A_1, A_2, A_3\}$ of the triangle vertices, where

$$
\begin{aligned}
\bar{a}_{13}^2 &:= \|-A_1 + E_3\|^2 \\
\bar{a}_{23}^2 &:= \|-A_2 + E_3\|^2
\end{aligned}
\tag{1.157}
$$

By Lemma 1.7, p. 13, we have

$$
\begin{aligned}
\|-A_1 + E_3\|^2 &= \frac{m_2^2 a_{12}^2 + m_3^2 a_{13}^2 + m_2 m_3(\ \ a_{12}^2 + a_{13}^2 - a_{23}^2)}{(m_1 + m_2 + m_3)^2} \\
\|-A_2 + E_3\|^2 &= \frac{m_1^2 a_{12}^2 + m_3^2 a_{23}^2 + m_1 m_3(\ \ a_{12}^2 - a_{13}^2 + a_{23}^2)}{(m_1 + m_2 + m_3)^2} \\
\|-A_3 + E_3\|^2 &= \frac{m_1^2 a_{13}^2 + m_2^2 a_{23}^2 + m_1 m_2(-a_{12}^2 + a_{13}^2 + a_{23}^2)}{(m_1 + m_2 + m_3)^2}
\end{aligned}
\tag{1.158}
$$

where m_k, $k = 1, 2, 3$, are the barycentric coordinates of E_3, given by

$$(m_1 : m_2 : m_3) = (a_{23} : a_{13} : -a_{12}) \tag{1.159}$$

as indicated in the fourth equation in (1.154).

Substituting successively, $\bar{a}_{12}^2$ and $\bar{a}_{13}^2$ from (1.157), $\| - A_1 + E_3\|^2$ and $\| - A_2 + E_3\|^2$ from (1.158), and m_k, $k = 1, 2, 3$, from (1.159) into (1.156) we obtain

$$T_{33} = \frac{1}{2}\left(\frac{a_{12} + a_{13} - a_{23}}{a_{12}}\right) A_1 + \frac{1}{2}\left(\frac{a_{12} - a_{13} + a_{23}}{a_{12}}\right) A_2 \tag{1.160}$$

Now let T_{32} be the tangency point where the A_3-excircle meets the extension of the triangle side A_1A_3, as shown in Fig. 1.11, p. 42. The tangency point T_{32} is the perpendicular projection of the point E_3 on the line $L_{A_1A_3}$ that passes through the points A_1 and A_3, Fig. 1.11. Hence, by the point to line perpendicular projection formula (1.76), p. 24, the point T_{32} possesses the barycentric coordinate representation

$$T_{32} = \frac{1}{2}\left(\frac{-\bar{a}_{12}^2 + a_{13}^2 + \bar{a}_{23}^2}{a_{13}^2}\right) A_1 + \frac{1}{2}\left(\frac{\bar{a}_{12}^2 + a_{13}^2 - \bar{a}_{23}^2}{a_{13}^2}\right) A_3 \tag{1.161}$$

with respect to the set $S = \{A_1, A_2, A_3\}$ of the triangle vertices, where

$$\begin{aligned} \bar{a}_{12}^2 &:= \| - A_1 + E_3\|^2 \\ \bar{a}_{23}^2 &:= \| - A_3 + E_3\|^2 \end{aligned} \tag{1.162}$$

Substituting successively, $\bar{a}_{12}^2$ and $\bar{a}_{23}^2$ from (1.162), $\| - A_1 + E_3\|^2$ and $\| - A_2 + E_3\|^2$ from (1.158), and m_k, $k = 1, 2, 3$, from (1.159) into (1.161) we obtain

$$T_{32} = \frac{1}{2}\left(\frac{a_{12} + a_{13} + a_{23}}{a_{13}}\right) A_1 + \frac{1}{2}\left(\frac{-a_{12} + a_{13} - a_{23}}{a_{13}}\right) A_3 \tag{1.163}$$

Finally, let T_{31} be the tangency point where the A_3-excircle meets the extension of the triangle side A_2A_3, as shown in Fig. 1.11, p. 42. Then,

$$T_{31} = \frac{1}{2}\left(\frac{a_{12} + a_{13} + a_{23}}{a_{23}}\right) A_2 + \frac{1}{2}\left(\frac{-a_{12} - a_{13} + a_{23}}{a_{23}}\right) A_3 \tag{1.164}$$

Here, (1.164) is obtained from (1.163) by interchanging the triangle vertices A_1 and A_2.

Equations (1.160) and (1.163)–(1.164) give rise to the following theorem:

Theorem 1.17 (Excircle Tangency Points). *Let $A_1A_2A_3$ be a triangle in a Euclidean space $\mathbb{R}^n$ and let T_{ij}, $i,j = 1,2,3$, be the points of tangency where the triangle A_i-excircle, $i = 1,2,3$, meets the side opposite to A_j, or its extension, of the triangle, Fig. 1.11, p. 42.*

Then, barycentric coordinate representations of the tangency points T_{ij} with respect to the pointwise independent set $S = \{A_1, A_2, A_3\}$ are given by the equations listed below.

$$\begin{aligned} T_{11} &= \frac{a_{12} - a_{13} + a_{23}}{2a_{23}} A_2 + \frac{-a_{12} + a_{13} + a_{23}}{2a_{23}} A_3 \\ T_{12} &= \frac{-a_{12} + a_{13} - a_{23}}{2a_{13}} A_1 + \frac{a_{12} + a_{13} + a_{23}}{2a_{13}} A_3 \\ T_{13} &= \frac{a_{12} - a_{13} - a_{23}}{2a_{12}} A_1 + \frac{a_{12} + a_{13} + a_{23}}{2a_{12}} A_2 \end{aligned} \tag{1.165a}$$

$$\begin{aligned} T_{21} &= \frac{-a_{12} - a_{13} + a_{23}}{2a_{23}} A_2 + \frac{a_{12} + a_{13} + a_{23}}{2a_{23}} A_3 \\ T_{22} &= \frac{a_{12} + a_{13} - a_{23}}{2a_{13}} A_1 + \frac{-a_{12} + a_{13} + a_{23}}{2a_{13}} A_3 \\ T_{23} &= \frac{a_{12} + a_{13} + a_{23}}{2a_{12}} A_1 + \frac{a_{12} - a_{13} - a_{23}}{2a_{12}} A_2 \end{aligned} \tag{1.165b}$$

$$\begin{aligned} T_{31} &= \frac{a_{12} + a_{13} + a_{23}}{2a_{23}} A_2 + \frac{-a_{12} - a_{13} + a_{23}}{2a_{23}} A_3 \\ T_{32} &= \frac{a_{12} + a_{13} + a_{23}}{2a_{13}} A_1 + \frac{-a_{12} + a_{13} - a_{23}}{2a_{13}} A_3 \\ T_{33} &= \frac{a_{12} + a_{13} - a_{23}}{2a_{12}} A_1 + \frac{a_{12} - a_{13} + a_{23}}{2a_{12}} A_2 \end{aligned} \tag{1.165c}$$

Proof. The proof of (1.165c) is established in (1.160) and (1.163)–(1.164). The proof of (1.165a)–(1.165b) follows from (1.165c) by invoking cyclicity, that is, by cyclic permutations of the triangle vertices. □

By the law of cosines for triangle $A_1A_2A_3$ in Theorem 1.17, with the triangle standard notation in Fig. 1.2, p. 7, and by the triangle π condition (1.65), p. 21, that triangle angles obey, we have the first equation in (1.166) below.

$$\begin{aligned}\frac{a_{12}-a_{13}+a_{23}}{2a_{23}} &= \frac{1}{2}\left(\frac{\sin\alpha_3}{\sin\alpha_1}-\frac{\sin\alpha_2}{\sin\alpha_1}+1\right)=\frac{\tan\frac{\alpha_3}{2}}{\tan\frac{\alpha_3}{2}+\tan\frac{\alpha_2}{2}}\\ \frac{-a_{12}+a_{13}+a_{23}}{2a_{23}} &= \frac{1}{2}\left(\frac{\sin\alpha_3}{\sin\alpha_1}+\frac{\sin\alpha_2}{\sin\alpha_1}+1\right)=\frac{\tan\frac{\alpha_2}{2}}{\tan\frac{\alpha_3}{2}+\tan\frac{\alpha_2}{2}}\end{aligned} \tag{1.166}$$

The second equation in (1.166) is obtained in a similar way.

Hence, the barycentric coordinate representation of the tangency point T_{11} in (1.165a) can be written as the trigonometric barycentric coordinate representation

$$T_{11}=\frac{\tan\frac{\alpha_3}{2}A_2+\tan\frac{\alpha_2}{2}A_3}{\tan\frac{\alpha_3}{2}+\tan\frac{\alpha_2}{2}}=\frac{\cot\frac{\alpha_2}{2}A_2+\cot\frac{\alpha_2}{3}A_3}{\cot\frac{\alpha_2}{2}+\cot\frac{\alpha_2}{3}} \tag{1.167}$$

As in (1.167), all the barycentric coordinate representations of the tangency points in Theorem 1.13, p. 34, and Theorem 1.17, p. 49, shown in Fig. 1.11, can be written as trigonometric barycentric coordinate representations, obtaining the following theorem:

Theorem 1.18 (In-Excircle Tangency Points). *Let $A_1A_2A_3$ be a triangle in a Euclidean space $\mathbb{R}^n$ and let T_{ij}, $i=0,1,2,3$, $j=1,2,3$, be the points of tangency where the triangle in-excircle with center E_i meets the side opposite to A_j, or its extension, of the triangle, Fig. 1.11, p. 42.*

Then, trigonometric barycentric coordinate representations of the tangency points T_{ij} with respect to the pointwise independent set $S=\{A_1,A_2,A_3\}$, are given by the equations listed below.

$$\begin{aligned}T_{01}&=\frac{\tan\frac{\alpha_2}{2}A_2+\tan\frac{\alpha_3}{2}A_3}{\tan\frac{\alpha_2}{2}+\tan\frac{\alpha_3}{2}}\\ T_{02}&=\frac{\tan\frac{\alpha_1}{2}A_1+\tan\frac{\alpha_3}{2}A_3}{\tan\frac{\alpha_1}{2}+\tan\frac{\alpha_3}{2}}\\ T_{03}&=\frac{\tan\frac{\alpha_1}{2}A_1+\tan\frac{\alpha_2}{2}A_2}{\tan\frac{\alpha_1}{2}+\tan\frac{\alpha_2}{2}}\end{aligned} \tag{1.168a}$$

$$T_{11} = \frac{\cot \frac{\alpha_2}{2} A_2 + \cot \frac{\alpha_3}{2} A_3}{\cot \frac{\alpha_2}{2} + \cot \frac{\alpha_3}{2}}$$

$$T_{12} = \frac{\tan \frac{\alpha_1}{2} A_1 - \cot \frac{\alpha_3}{2} A_3}{\tan \frac{\alpha_1}{2} - \cot \frac{\alpha_3}{2}} \qquad (1.168b)$$

$$T_{13} = \frac{\tan \frac{\alpha_1}{2} A_1 - \cot \frac{\alpha_2}{2} A_2}{\tan \frac{\alpha_1}{2} - \cot \frac{\alpha_2}{2}}$$

$$T_{21} = \frac{\tan \frac{\alpha_2}{2} A_2 - \cot \frac{\alpha_3}{2} A_3}{\tan \frac{\alpha_2}{2} - \cot \frac{\alpha_3}{2}}$$

$$T_{22} = \frac{\cot \frac{\alpha_1}{2} A_1 + \cot \frac{\alpha_3}{2} A_3}{\cot \frac{\alpha_1}{2} + \cot \frac{\alpha_3}{2}} \qquad (1.168c)$$

$$T_{23} = \frac{\cot \frac{\alpha_1}{2} A_1 - \tan \frac{\alpha_2}{2} A_2}{\cot \frac{\alpha_1}{2} - \tan \frac{\alpha_2}{2}}$$

$$T_{31} = \frac{\cot \frac{\alpha_2}{2} A_2 - \tan \frac{\alpha_3}{2} A_3}{\cot \frac{\alpha_2}{2} - \tan \frac{\alpha_3}{2}}$$

$$T_{32} = \frac{\cot \frac{\alpha_1}{2} A_1 - \tan \frac{\alpha_3}{2} A_3}{\cot \frac{\alpha_1}{2} - \tan \frac{\alpha_3}{2}} \qquad (1.168d)$$

$$T_{33} = \frac{\cot \frac{\alpha_1}{2} A_1 + \cot \frac{\alpha_2}{2} A_2}{\cot \frac{\alpha_1}{2} + \cot \frac{\alpha_2}{2}}$$

Proof. The trigonometric barycentric coordinate representation of T_{11} in (1.168b) is established in (1.167). All the other trigonometric barycentric coordinate representations in the Theorem can be established in a similar way. □

Surprisingly, we will find in Theorem 5.2, p. 273, that the trigonometric barycentric coordinates in the trigonometric barycentric coordinate representations of tangency points in Theorem 1.18 survive unimpaired in the transition from Euclidean to hyperbolic geometry. This remarkable result indicates the comparative advantage of trigonometric barycentric coordinate representations in our comparative study of the transition from Euclidean to hyperbolic geometry.

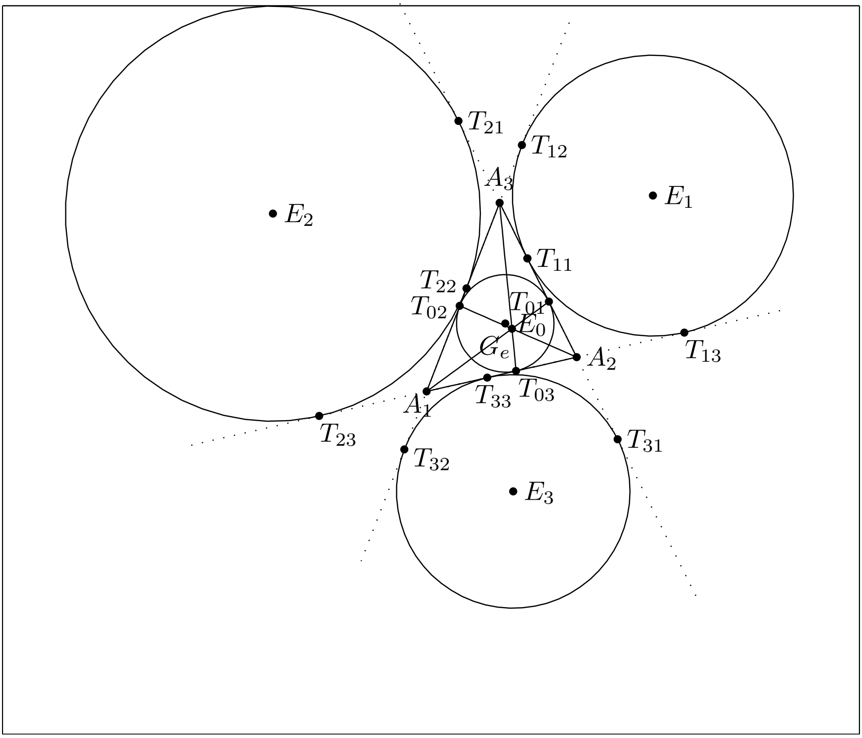


Fig. 1.12 Gergonne Point, G_e. In the notation of Fig. 1.11, p. 42, for the triangle in-excircle tangency points, the lines A_kT_{0k}, $k = 1, 2, 3$, are concurrent, and the resulting point of concurrency is the triangle Gergonne point G_e. Trigonometric barycentric coordinate representation of G_e is given by (1.169).

1.16 From Triangle Tangency Points to Triangle Centers

The triangle tangency points T_{ij}, $i = 0, 1, 2, 3$, $j = 1, 2, 3$, in Theorem 1.18, shown in Fig. 1.11, p. 42, give rise to the following three triangle centers:

(1) *Gergonne Point* G_e. In the notation of Fig. 1.11, the lines A_kT_{0k}, $k = 1, 2, 3$, are concurrent, Fig. 1.12. The resulting point of concurrency, called the triangle Gergonne point, G_e, possesses the trigonometric barycentric coordinate representation (see Exercise 9, p. 64)

$$G_e = \frac{\tan\frac{\alpha_1}{2}A_1 + \tan\frac{\alpha_2}{2}A_2 + \tan\frac{\alpha_3}{2}A_3}{\tan\frac{\alpha_1}{2} + \tan\frac{\alpha_2}{2} + \tan\frac{\alpha_3}{2}} \tag{1.169}$$

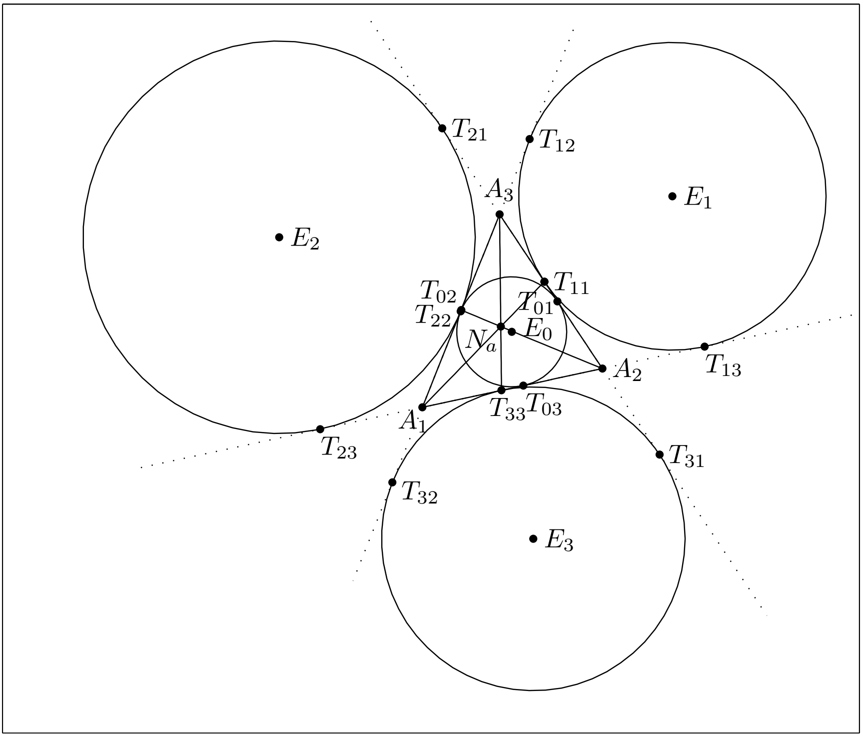


Fig. 1.13 Nagel Point, N_a. In the notation of Fig. 1.11, p. 42, for the triangle in-excircle tangency points, the lines A_kT_{kk}, $k = 1, 2, 3$, are concurrent, and the resulting point of concurrency is the triangle Nagel point. Trigonometric barycentric coordinate representation of N_a is given by (1.170).

with respect to the vertices of the reference triangle $A_1A_2A_3$.

(2) *Nagel Point* G_e. In the notation of Fig. 1.11, the lines A_kT_{kk}, $k = 1, 2, 3$, are concurrent, Fig. 1.13. The resulting point of concurrency, called the triangle Nagel point, N_a, possesses the trigonometric barycentric coordinate representation (see Exercise 10, p. 64)

$$N_a = \frac{\cot\frac{\alpha_1}{2}A_1 + \cot\frac{\alpha_2}{2}A_2 + \cot\frac{\alpha_3}{2}A_3}{\cot\frac{\alpha_1}{2} + \cot\frac{\alpha_2}{2} + \cot\frac{\alpha_3}{2}} \tag{1.170}$$

with respect to the vertices of the reference triangle $A_1A_2A_3$.

(3) *Point* P_u. In the notation of Fig. 1.11, the lines E_kT_{kk}, $k = 1, 2, 3$, are concurrent, Fig. 1.14. The resulting point of concurrency, called the triangle point P_u, possesses the trigonometric barycentric coordinate

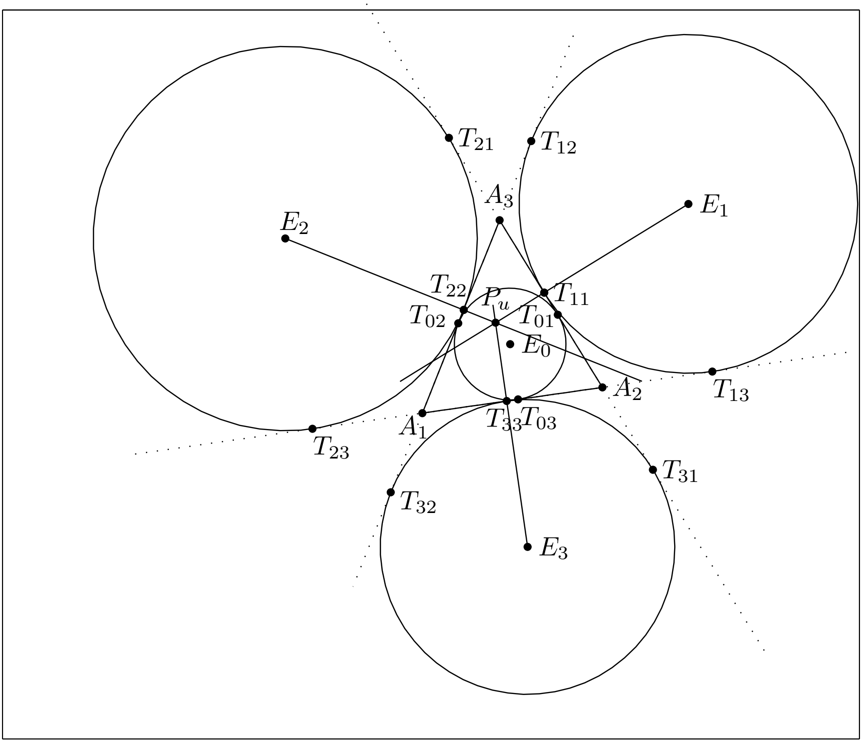


Fig. 1.14 The Triangle P_u Point. In the notation of Fig. 1.11, p. 42, for the triangle in-excircle tangency points, the lines E_kT_{kk}, $k = 1, 2, 3$, are concurrent, and the resulting point of concurrency is the triangle P_u point. Trigonometric barycentric coordinate representation of P_u is given by (1.171).

representation (see Exercise 11, p. 64)

$$P_u = \frac{m_1A_1 + m_2A_2 + m_3A_3}{m_1 + m_2 + m_3} \tag{1.171a}$$

where gyrotrigonometric gyrobarycentric coordinates $(m_1 : m_2 : m_3)$ of P_u, Fig. 1.14, in (1.171a) are

$$\begin{aligned} m_1 &= \sin\alpha_1(1 + \cos\alpha_1 - \cos\alpha_2 - \cos\alpha_3) \\ m_2 &= \sin\alpha_2(1 - \cos\alpha_1 + \cos\alpha_2 - \cos\alpha_3) \\ m_3 &= \sin\alpha_3(1 - \cos\alpha_1 - \cos\alpha_2 + \cos\alpha_3) \end{aligned} \tag{1.171b}$$

with respect to the vertices of the reference triangle $A_1A_2A_3$.

1.17 Triangle In-Exradii

The A_3-exradius r_3 of a triangle $A_1A_2A_3$ in a Euclidean space $\mathbb{R}^n$ is the distance from excenter E_3 to side A_1A_2 of the triangle, as shown in Fig. 1.11, p. 42. Applying the point to line distance formula (1.77), p. 24, to calculate the distance r_3 between the point E_3 and the line $L_{A_1A_2}$ that passes through the points A_1 and A_2, Figs. 1.11–1.14, we obtain the equation

$$r_3^2 = \frac{(a_{12} + \bar{a}_{13} + \bar{a}_{23})(-a_{12} + \bar{a}_{13} + \bar{a}_{23})(a_{12} - \bar{a}_{13} + \bar{a}_{23})(a_{12} + \bar{a}_{13} - \bar{a}_{23})}{4a_{12}^2} \tag{1.172}$$

where

$$\begin{aligned} \bar{a}_{13}^2 &:= \|-A_1 + E_3\|^2 \\ \bar{a}_{23}^2 &:= \|-A_2 + E_3\|^2 \end{aligned} \tag{1.173}$$

Substituting successively, $\bar{a}_{13}$ and $\bar{a}_{23}$ from (1.173) and $\|-A_1+E_3\|$ and $\|-A_2+E_3\|$ from (1.158), p. 47, along with the barycentric coordinates of E_3 in (1.159) into (1.172) we obtain the equation

$$r_3^2 = \frac{p(p-a_{13})(p-a_{23})}{p-a_{12}} \tag{1.174}$$

where p is the triangle semiperimeter (1.123), p. 35.

Equation (1.174) gives rise to the following theorem:

Theorem 1.19 (Triangle In-Exradii). *Let $A_1A_2A_3$ be a triangle in a Euclidean space $\mathbb{R}^n$ with inradius r_0 and exradii r_k, $k = 1, 2, 3$, and with a perimeter p. Then*

$$\begin{aligned} r_0^2 &= \frac{(p-a_{12})(p-a_{13})(p-a_{23})}{p} \\ r_1^2 &= \frac{p(p-a_{12})(p-a_{13})}{p-a_{23}} \\ r_2^2 &= \frac{p(p-a_{12})(p-a_{23})}{p-a_{13}} \\ r_3^2 &= \frac{p(p-a_{13})(p-a_{23})}{p-a_{12}} \end{aligned} \tag{1.175}$$

Proof. The first equation in (1.175) is the result of Theorem 1.14, p. 35. The fourth equation in (1.175) is established in (1.174). The second and the third equations in (1.175) follow from the fourth equation by cyclic permutations of the triangle vertices. □

By Heron's formula (1.15), p. 35, the equations in (1.175) for the in-exradii r_k, $k = 0, 1, 2, 3$, along with the equation in (1.143), p. 41, for the circumradius of triangle $A_1A_2A_3$ can be written as

$$\begin{aligned} r_0 &= \frac{2|A_1A_2A_3|}{a_{12}+a_{13}+a_{23}} \\ r_1 &= \frac{2|A_1A_2A_3|}{a_{12}+a_{13}-a_{23}} \\ r_2 &= \frac{2|A_1A_2A_3|}{a_{12}-a_{13}+a_{23}} \\ r_3 &= \frac{2|A_1A_2A_3|}{-a_{12}+a_{13}+a_{23}} \\ R &= \frac{a_{12}a_{13}a_{23}}{4|A_1A_2A_3|} \end{aligned} \tag{1.176}$$

implying

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r_0} \tag{1.177}$$

and

$$r_1 + r_2 + r_3 = 4R + r_0 \tag{1.178}$$

It can be shown that the in-exradii r_k, $k = 0, 1, 2, 3$, in (1.176) are related to the circumradius R of triangle $A_1A_2A_3$ by the equations

$$\begin{aligned} r_0 &= 4R\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\sin\frac{\alpha_3}{2} \\ r_1 &= 4R\sin\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}\cos\frac{\alpha_3}{2} \\ r_2 &= 4R\cos\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\cos\frac{\alpha_3}{2} \\ r_3 &= 4R\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}\sin\frac{\alpha_3}{2} \end{aligned} \tag{1.179}$$

or, equivalently, by the equations

$$
\begin{aligned}
r_0 &= \frac{1}{2}R\frac{\cos\frac{\alpha_1-\alpha_2-\alpha_3}{2}}{\cos\frac{\alpha_1}{2}}\frac{\cos\frac{-\alpha_1+\alpha_2-\alpha_3}{2}}{\cos\frac{\alpha_2}{2}}\frac{\cos\frac{-\alpha_1-\alpha_2+\alpha_3}{2}}{\cos\frac{\alpha_3}{2}}\\
r_1 &= \frac{1}{2}R\frac{\cos\frac{\alpha_1-\alpha_2-\alpha_3}{2}}{\cos\frac{\alpha_1}{2}}\frac{\cos\frac{-\alpha_1+\alpha_2-\alpha_3}{2}}{\sin\frac{\alpha_2}{2}}\frac{\cos\frac{-\alpha_1-\alpha_2+\alpha_3}{2}}{\sin\frac{\alpha_3}{2}}\\
r_2 &= \frac{1}{2}R\frac{\cos\frac{\alpha_1-\alpha_2-\alpha_3}{2}}{\sin\frac{\alpha_1}{2}}\frac{\cos\frac{-\alpha_1+\alpha_2-\alpha_3}{2}}{\cos\frac{\alpha_2}{2}}\frac{\cos\frac{-\alpha_1-\alpha_2+\alpha_3}{2}}{\sin\frac{\alpha_3}{2}}\\
r_3 &= \frac{1}{2}R\frac{\cos\frac{\alpha_1-\alpha_2-\alpha_3}{2}}{\sin\frac{\alpha_1}{2}}\frac{\cos\frac{-\alpha_1+\alpha_2-\alpha_3}{2}}{\sin\frac{\alpha_2}{2}}\frac{\cos\frac{-\alpha_1-\alpha_2+\alpha_3}{2}}{\cos\frac{\alpha_3}{2}}
\end{aligned}
\tag{1.180}
$$

Remarkably, the equations in (1.179) fail in hyperbolic geometry while the equations in (1.180) remain valid in hyperbolic geometry as well. Accordingly, we say that the equations in (1.179) embody the π-identity condition of triangles, (1.65), p. 21, which fails in hyperbolic geometry, while the equations in (1.180) are free of the π-identity condition.

1.18 A Step Toward the Comparative Study

In this chapter we have introduced the notion of Möbius barycentric coordinates in the Cartesian model $\mathbb{R}^n$ of Euclidean geometry and used it for the determination of several barycentric coordinate representations, including those of the four classical triangle centers, with respect to the vertices of reference triangles. Using the standard notation for a triangle $A_1A_2A_3$, Fig. 1.2, p. 7, we expressed barycentric coordinates $\{m_1 : m_2 : m_3\}$ (i) in terms of triangle side-lengths, a_{12}, a_{13}, a_{23}, and (ii) in terms of triangle angles $\alpha_1, \alpha_2, \alpha_3$. The resulting trigonometric barycentric coordinates, in which barycentric coordinates are expressed in terms of the angles of the reference triangle will prove useful in the extension of our study from Euclidean to hyperbolic geometry. Indeed, studying triangle centers comparatively, we will find in Tables 4.1 − 4.2, p. 254, that trigonometric barycentric coordinates that do not embody the π-identity condition of triangles, (1.65), p. 21, survive unimpaired the transition from Euclidean to hyperbolic geometry.

As a first step in the comparative study of triangle centers, a table of the trigonometric barycentric coordinates of the four classic triangle centers is presented in Table 1.1.

Table 1.1 Trigonometric barycentric coordinates of the classical triangle centers and the triangle altitude foot.

Center	Symbol	Trigonometric Barycentric Coordinates
Centroid	G, (1.50), p. 19 Fig. 1.3, p. 18	$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
Orthocenter	H, (1.88), p. 27 Fig. 1.5, p. 25	$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} \tan\alpha_1 \\ \tan\alpha_2 \\ \tan\alpha_3 \end{pmatrix}$
Incenter	I, (1.108), p. 32 Fig. 1.7, p. 30	$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} \sin\alpha_1 \\ \sin\alpha_2 \\ \sin\alpha_3 \end{pmatrix}$
Circumcenter	O, (1.140), p. 40 Fig. 1.9, p. 37	$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} \sin\frac{-\alpha_1+\alpha_2+\alpha_3}{2}\sin\alpha_1 \\ \sin\frac{\alpha_1-\alpha_2+\alpha_3}{2}\sin\alpha_2 \\ \sin\frac{\alpha_1+\alpha_2-\alpha_3}{2}\sin\alpha_3 \end{pmatrix} = \begin{pmatrix} \sin\alpha_1\cos\alpha_1 \\ \sin\alpha_2\cos\alpha_2 \\ \sin\alpha_3\cos\alpha_3 \end{pmatrix}$
Altitude Foot	P_3, (1.76), p. 24 Fig. 1.4, p. 20	$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} \tan\alpha_1 \\ \tan\alpha_2 \\ 0 \end{pmatrix}$

The comparative study of triangle centers, initiated here in Table 1.1, will be completed in Tables 4.1 − 4.2, p. 254.

1.19 Tetrahedron Altitude

Let $A_1A_2A_3A_4$ be a tetrahedron with vertices A_1, A_2, A_3 and A_4 in a Euclidean n-space $\mathbb{R}^n$, and let the point P_4 be the orthogonal projection of vertex A_4 onto its opposite face, $A_1A_2A_3$ (or its extension), as shown in Fig. 1.15 for $n = 3$. Furthermore, let (m_1, m_2, m_3) be barycentric coordinates of P_4 with respect to the set $\{A_1, A_2, A_3\}$. Then, P_4 is given in terms of its barycentric coordinates (m_1, m_2, m_3) with respect to the set $\{A_1, A_2, A_3\}$ by the equation

$$P_4 = \frac{m_1A_1 + m_2A_2 + m_3A_3}{m_1 + m_2 + m_3} \tag{1.181}$$

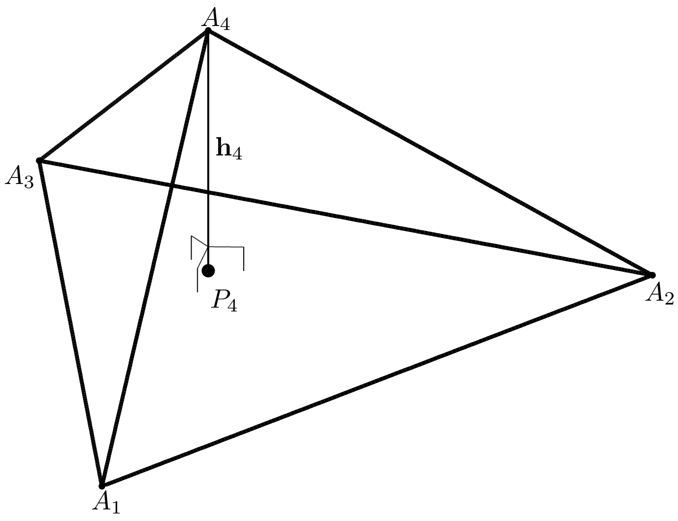


Fig. 1.15 The foot P_4 of the altitude $\mathbf{h}_4$ drawn from vertex A_4 of a tetrahedron $A_1A_2A_3$ in a Euclidean 3-space $\mathbb{R}^3$. The special (that is, normalized) barycentric coordinates (m_1, m_2, m_3) of P_4 with respect to the set $\{A_1A_2A_3\}$ are determined in (1.188), p. 61.

where the barycentric coordinates m_1, m_2 and m_3 of P_4, Fig. 1.15, are to be determined in (1.188) below.

By the covariance (1.26), p. 12, of barycentric coordinate representations with respect to translations we have, in particular, for $X = A_k$, $k = 1, 2, 3, 4$,

$$
\begin{aligned}
\mathbf{p}_1 &:= -A_1 + P_4 = \frac{m_2(-A_1 + A_2) + m_3(-A_1 + A_3)}{m_1 + m_2 + m_3} = \frac{m_2\mathbf{a}_{12} + m_3\mathbf{a}_{13}}{m_1 + m_2 + m_3} \\
\mathbf{p}_2 &:= -A_2 + P_4 = \frac{m_1(-A_2 + A_1) + m_3(-A_2 + A_3)}{m_1 + m_2 + m_3} = \frac{m_1\mathbf{a}_{21} + m_3\mathbf{a}_{23}}{m_1 + m_2 + m_3} \\
\mathbf{p}_3 &:= -A_3 + P_4 = \frac{m_1(-A_3 + A_1) + m_2(-A_3 + A_2)}{m_1 + m_2 + m_3} = \frac{m_1\mathbf{a}_{31} + m_2\mathbf{a}_{32}}{m_1 + m_2 + m_3} \\
\mathbf{h}_4 &:= -A_4 + P_4 = \frac{m_1(-A_4 + A_1) + m_2(-A_4 + A_2) + m_3(-A_4 + A_3)}{m_1 + m_2 + m_3} \\
&= \frac{m_1\mathbf{a}_{41} + m_2\mathbf{a}_{42} + m_3\mathbf{a}_{43}}{m_1 + m_2 + m_3}
\end{aligned}
\tag{1.182}
$$

where we use the notation

$$\mathbf{a}_{ij} = -A_i + A_j\,, \qquad a_{ij} = \|\mathbf{a}_{ij}\| \tag{1.183}$$

for $i, j = 1, 2, 3$, noting that $a_{ij} = a_{ji}$.

Let α_{4k} be the angle with vertex A_k, $k = 1, 2, 3$, of triangle $A_1A_2A_3$ that, in turn, forms the face opposite to vertex A_4 of tetrahedron $A_1A_2A_3A_4$, Fig. 1.15.

Then, the squared norms $p_k^2 = \|\mathbf{p}_k\|^2$, $k = 1, 2, 3$ are obtained from (1.182), in (1.184) below, by the law of cosines.

$$\begin{aligned} p_1^2 &= \frac{1}{m_0^2}(m_2^2a_{12}^2 + m_3^2a_{13}^2 + 2m_2m_3a_{12}a_{13}\cos\alpha_{41}) \\ &= \frac{1}{m_0^2}\{m_2^2a_{12}^2 + m_3^2a_{13}^2 + m_2m_3(a_{12}^2 + a_{13}^2 - a_{23}^2)\} \end{aligned} \tag{1.184a}$$

$$\begin{aligned} p_2^2 &= \frac{1}{m_0^2}(m_1^2a_{12}^2 + m_3^2a_{23}^2 + 2m_1m_3a_{12}a_{23}\cos\alpha_{42}) \\ &= \frac{1}{m_0^2}\{m_1^2a_{12}^2 + m_3^2a_{23}^2 + m_1m_3(a_{12}^2 - a_{13}^2 + a_{23}^2)\} \end{aligned} \tag{1.184b}$$

$$\begin{aligned} p_3^2 &= \frac{1}{m_0^2}(m_1^2a_{13}^2 + m_2^2a_{23}^2 + 2m_1m_2a_{13}a_{23}\cos\alpha_{43}) \\ &= \frac{1}{m_0^2}\{m_1^2a_{13}^2 + m_2^2a_{23}^2 + m_1m_2(-a_{12}^2 + a_{13}^2 + a_{23}^2)\} \end{aligned} \tag{1.184c}$$

where

$$m_0 = m_1 + m_2 + m_3 \tag{1.184d}$$

The condition that the tetrahedron altitude $\mathbf{h}$ is perpendicular to the vectors $\mathbf{p}_k$, $k = 1, 2, 3$, along with the Pythagorean theorem, imply

$$h_4^2 = a_{14}^2 - p_1^2 = a_{24}^2 - p_2^2 = a_{34}^2 - p_3^2 \tag{1.185}$$

thus obtaining the system of two equations

$$\begin{aligned} p_1^2 - a_{14}^2 &= p_2^2 - a_{24}^2 \\ p_1^2 - a_{14}^2 &= p_3^2 - a_{34}^2 \end{aligned} \tag{1.186}$$

for the three unknowns m_1, m_2 and m_3, Substituting p_k^2, $k = 1, 2, 3$, from (1.184) into (1.186) along with the normalization condition

$$m_3 = 1 - m_1 - m_2 \tag{1.187}$$

the system (1.186) reduces to a system of two linear equations for the two unknowns m_1 and m_2. Solving the resulting linear system for m_1 and m_2, and calculating m_3 from (1.187) we obtain the unique special barycentric coordinates (m_1, m_2, m_3) of the point P_4 in (1.181), p. 58,

$$\begin{aligned} m_1 = \frac{1}{D}\{&a_{24}^2(-a_{12}^2 + a_{13}^2 + a_{23}^2) \\ &+a_{34}^2(\ \ a_{12}^2 - a_{13}^2 + a_{23}^2) \\ &+a_{23}^2(\ \ a_{12}^2 + a_{13}^2 - a_{23}^2) - 2a_{14}^2a_{23}^2\} \end{aligned} \tag{1.188a}$$

$$\begin{aligned} m_2 = \frac{1}{D}\{&a_{14}^2(-a_{12}^2 + a_{13}^2 + a_{23}^2) \\ &+a_{13}^2(\ \ a_{12}^2 - a_{13}^2 + a_{23}^2) \\ &+a_{34}^2(\ \ a_{12}^2 + a_{13}^2 - a_{23}^2) - 2a_{13}^2a_{24}^2\} \end{aligned} \tag{1.188b}$$

$$\begin{aligned} m_3 = \frac{1}{D}\{&a_{12}^2(-a_{12}^2 + a_{13}^2 + a_{23}^2) \\ &+a_{14}^2(\ \ a_{12}^2 - a_{13}^2 + a_{23}^2) \\ &+a_{24}^2(\ \ a_{12}^2 + a_{13}^2 - a_{23}^2) - 2a_{12}^2a_{34}^2\} \end{aligned} \tag{1.188c}$$

where D is given by the equation

$$\begin{aligned} D &= 2(a_{12}^2a_{13}^2 + a_{12}^2a_{23}^2 + a_{13}^2a_{23}^2) - (a_{12}^4 + a_{13}^4 + a_{23}^4) \\ &= (a_{12} + a_{13} + a_{23})(-a_{12} + a_{13} + a_{23})(a_{12} - a_{13} + a_{23})(a_{12} + a_{13} - a_{23}) \\ &= 16|A_1A_2A_3|^2 \end{aligned} \tag{1.188d}$$

The third equation in (1.188d) follows from (1.75), p. 23, where, by Heron's formula, $|A_1A_2A_3|$ is the area of triangle $A_1A_2A_3$.

Convenient barycentric coordinates for the point P_4 can be obtained from (1.188) by omitting the nonzero common factor $1/D$.

Formalizing the main result of this section we obtain the following theorem:

Theorem 1.20 (Point to Plane Perpendicular Projection). *Let A_1, A_2 and A_3 be any three pointwise independent points of a Euclidean space $\mathbb{R}^n$, $n \geq 3$, and let $\pi_{A_1A_2A_3}$ be the plane passing through these points. Furthermore, let A_4 be any point of the space that does not lie on $\pi_{A_1A_2A_3}$, as shown in Fig. 1.15, p. 59. Then, the perpendicular projection P_4 of the point A_4 on the plane $\pi_{A_1A_2A_3}$ is given by,* (1.181),

$$P_4 = m_1A_1 + m_2A_2 + m_3A_3 \tag{1.189}$$

where the special barycentric coordinates m_1, m_2 and m_3 are given by (1.188)*, satisfying the normalization condition*

$$m_1 + m_2 + m_3 = 1 \tag{1.190}$$

1.20 Tetrahedron Altitude Length

By the last equation in (1.182), p. 59, the squared length of the tetrahedron altitude $\mathbf{h}_4$ is given by the equation

$$\begin{aligned} h_4^2 &= (m_1\mathbf{a}_{41} + m_2\mathbf{a}_{42} + m_3\mathbf{a}_{43})^2 \\ &= m_1^2a_{41}^2 + m_2^2a_{42}^2 + m_3^2a_{43}^2 \\ &+ 2m_1m_2a_{14}a_{24}\cos\alpha_{34} \\ &+ 2m_1m_3a_{14}a_{34}\cos\alpha_{24} \\ &+ 2m_2m_3a_{24}a_{34}\cos\alpha_{14} \end{aligned} \tag{1.191}$$

Hence, by the law of cosines,

$$\begin{aligned} h_4^2 = m_1^2a_{14}^2 &+ m_2^2a_{24}^2 + m_3^2a_{34}^2 \\ &+ m_1m_2(-a_{12}^2 + a_{14}^2 + a_{24}^2) \\ &+ m_1m_3(-a_{13}^2 + a_{14}^2 + a_{34}^2) \\ &+ m_2m_3(-a_{23}^2 + a_{24}^2 + a_{34}^2) \end{aligned} \tag{1.192}$$

We now substitute the special barycentric coordinates m_1, m_2 and m_3

from (1.188), p. 61, into (1.192) obtaining

$$h_4^2 = \frac{1}{2}\frac{F_3(a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34})}{F_2(a_{12}, a_{13}, a_{23})} \tag{1.193}$$

where F_2 is given by its 4×4 determinant representation in (1.74), p. 23, and where F_3 is given by the similar, 5×5 determinant representation

$$F_3(a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}) = \begin{vmatrix} 0 & a_{12}^2 & a_{13}^2 & a_{14}^2 & 1 \\ a_{21}^2 & 0 & a_{23}^2 & a_{24}^2 & 1 \\ a_{31}^2 & a_{32}^2 & 0 & a_{34}^2 & 1 \\ a_{41}^2 & a_{42}^2 & a_{43}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} \tag{1.194}$$

[Veljan (2000)] known as the 5×5 *Cayley-Menger determinant.*

We can now calculate the volume $|A_1A_2A_3A_4|$ of tetrahedron $A_1A_2A_3A_4$. By (1.125), p. 36, and (1.193)–(1.194),

$$|A_1A_2A_3A_4|^2 = \frac{1}{3^2}|A_1A_2A_3|^2 h_4^2 = \frac{1}{288}F_3(a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}) \tag{1.195}$$

1.21 Exercises

(1) Show that the pointwise independence of the set S in Def. 1.5, p. 9, insures that the barycentric coordinate representation of a point with respect to the set S is unique.
(2) Prove the trigonometric identities in (1.66), p. 21.
(3) Employ the π-identity of triangles, (1.65), p. 21, to simplify the barycentric coordinates $(m_1 : m_2 : m_3)$ in (1.138), p. 39, into the barycentric coordinates $(m_1 : m_2 : m_3)$ in (1.139).
(4) Employ the π-identity of triangles, (1.65), p. 21, to simplify the barycentric coordinates $(m_1 : m_2 : m_3)$ in (1.140), p. 39, into the barycentric coordinates $(m_1 : m_2 : m_3)$ in (1.139).
(5) Solve the vector equations in (1.81), p. 26, for the three scalar unknowns t_1, t_2 and t_3. Substitute the solution in (1.80) and, hence, obtain the equation in (1.84) for the point of concurrency, H, of the three lines in (1.80).
(6) Derive the circumradius R of a triangle $A_1A_2A_3$ in (1.142), p. 41 by successively substituting $\|-A_1+O\|^2$ from the first equation in (1.128), p. 36, and m_k, $k = 1, 2, 3$, from (1.136), p. 39, into (1.141).

(7) Derive the equations in (1.151), p. 45, from the equations in (1.150).
(8) Verify the trigonometric identities in (1.166), p. 50, under the triangle π condition (1.65), p. 21.
(9) Verify the trigonometric barycentric coordinate representation (1.169), p. 52, of the triangle Gergonne point G_e with respect to the vertices of its reference triangle in a Euclidean space $\mathbb{R}^n$. See also Exercise 2, p. 283.
(10) Verify the trigonometric barycentric coordinate representation (1.170), p. 53, of the triangle Nagel point N_a with respect to the vertices of its reference triangle in a Euclidean space $\mathbb{R}^n$. See also Exercise 3, p. 284.
(11) Verify the trigonometric barycentric coordinate representation (1.171), p. 54, of the triangle P_u Point with respect to the vertices of its reference triangle in a Euclidean space $\mathbb{R}^n$. See also Exercise 4, p. 284.
(12) Verify the equations in (1.179) and (1.180), p. 57.
(13) Substitute p_k^2, $k = 1, 2, 3$, from (1.184), p. 60, into (1.186) along with the normalization condition (1.187) to obtain a linear system of two equations for m_1 and m_2. Then solve the resulting linear system and calculate m_3 from the normalization condition and, hence, obtain the solution (m_1, m_2, m_3) in (1.188).
(14) Substitute the special barycentric coordinates m_1, m_2 and m_3 from (1.188), p. 61, into (1.192), p. 62, to obtain h_4^2 in (1.193), p. 63.