Preface

Historically, Euclidean geometry became analytic with the appearance of Cartesian coordinates that followed the publication of René Descartes' (1596-1650) masterpiece in 1637, allowing algebra to be applied to Euclidean geometry. About 200 years later hyperbolic geometry was discovered following the publications of Nikolai Ivanovich Lobachevsky (1792-1856) in 1830 and János Bolyai (1802-1860) in 1832, and about 370 years later the hyperbolic geometry of Bolyai and Lobachevsky became analytic following the adaption of Cartesian coordinates for use in hyperbolic geometry in [Ungar (2001b); Ungar (2002); Ungar (2008a)], allowing novel nonassociative algebra to be applied to hyperbolic geometry.

The history of Vector Algebra dates back to the end of the Eighteenth century, considering complex numbers as the origin of vector algebra as we know today. Indeed, complex numbers are ordered pairs of real numbers with addition given by the parallelogram addition law. In the beginning of the nineteenth century there were attempts to extend this addition law into three dimensions leading Hamilton to the discovery of the quaternions in 1843. Quaternions, in turn, led to the notion of scalar multiplication in modern vector algebra. The key role in the creation of modern vector analysis as we know today, played by Willar Gibbs (1839–1903) and Oliver Heaviside (1850–1952), along with the contribution of Möbius' barycentric coordinates to vector analysis, is described in [Crowe (1994)].

The success of the use of vector algebra along with Cartesian coordinates in Euclidean geometry led Varičak to admit in 1924 [Varičak (1924)], for his chagrin, that the adaption of vector algebra for use in hyperbolic space was just not possible, as the renowned historian Scott Walter notes in [Walter (1999b), p. 121]. Fortunately however, along with the adaption of Cartesian coordinates for use in hyperbolic geometry, trigonometry and vector algebra have been adapted for use in hyperbolic geometry as well in [Ungar (2001b); Ungar (2002); Ungar (2008a)], leading to the adaption in this book of Möbius barycentric coordinates for use in hyperbolic geometry. As a result, powerful tools that are commonly available in the study of Euclidean geometry became available in the study of hyperbolic geometry as well, enabling one to explore hyperbolic geometry in novel ways.

The notion of Euclidean barycentric coordinates dates back to Möbius, 1827, when he published his book *Der Barycentrische Calcul* (*The Barycentric Calculus*). The word *barycentric* is derived from the Greek word *barys* (heavy), and refers to center of gravity. Barycentric calculus is a method of treating geometry by considering a point as the center of gravity of certain other points to which weights are ascribed. Hence, in particular, barycentric calculus provides excellent insight into triangle and tetrahedron centers. This unique book provides a comparative introduction to the fascinating and beautiful subject of triangle and tetrahedron centers in hyperbolic geometry along with analogies they share with familiar triangle and tetrahedron centers in Euclidean geometry. As such, the book uncovers magnificent unifying notions that Euclidean and hyperbolic triangle and tetrahedron centers share.

The hunt for Euclidean triangle centers is an old tradition in Euclidean geometry, resulting in a repertoire of more than three thousands triangle centers that are determined by their barycentric coordinate representations with respect to the vertices of their reference triangles. Several triangle and tetrahedron centers are presented in the book as an illustration of the use of Euclidean barycentric calculus in the determination of Euclidean triangle centers, and in order to set the stage for analogous determination of triangle and tetrahedron centers in the hyperbolic geometry of Bolyai and Lobachevsky.

The adaption of Cartesian coordinates, barycentric coordinates, trigonometry and vector algebra for use in various models of hyperbolic geometry naturally leads to the birth of *comparative analytic geometry* in this book, in which triangles and tetrahedra in three models of geometry are studied comparatively along with their comparative advantages, comparative features and comparative patterns. Indeed, the term "comparative analytic geometry" affirms the idea that the three models of geometry that are studied in this book are to be compared. These three models of analytic geometry are:

(1) The standard Cartesian model of n-dimensional Euclidean geometry. It is regulated by the associative-commutative algebra of vector spaces, and it possesses the comparative advantage of being relatively simple and familiar.
(2) The Cartesian-Beltrami-Klein model of n-dimensional hyperbolic geometry. It is regulated by the gyroassociative-gyrocommutative algebra of Einstein gyrovector spaces, and it possesses the comparative advantage that its hyperbolic geodetic lines, called gyrolines, coincide with Euclidean line segments. As a result, points of concurrency of gyrolines in this model of hyperbolic geometry can be determined by familiar methods of linear algebra.
(3) The Cartesian-Poincaré model of n-dimensional hyperbolic geometry. It is regulated by the gyroassociative-gyrocommutative algebra of Möbius gyrovector spaces, and it possesses the comparative advantage of being conformal so that, in particular, its hyperbolic circles, called gyrocircles, coincide with Euclidean circles (noting, however, that the center and gyrocenter of a given circle/gyrocircle need not coincide).

The idea of comparative study of the three models of geometry is revealed with particular brilliance in comparative features, one of which emerges from the result that barycentric coordinates that are expressed trigonometrically in the three models are model invariant.

Following the adaption of barycentric coordinates for use in hyperbolic geometry, this book heralds the birth of comparative analytic geometry, and provides the starting-point for the hunt for novel centers of hyperbolic triangles and hyperbolic tetrahedra.

Abraham A. Ungar
2010