7. Heisenberg Equation of Motion

The *Heisenberg equation of motion* is given by

$$i\hbar\frac{d\hat{A}}{dt} = [\hat{A},\hat{H}](t) \tag{1}$$

where [,] denotes the *commutator*, i.e. $[X,Y] := XY - YX$. Let σ_x, σ_y, σ_z be the *Pauli spin matrices*

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

Let $\hat{H}$ be the Hamilton operator

$$\hat{H} = \hbar\omega\sigma_z\,. \tag{3}$$

We evaluate the time evolution of σ_x. Since

$$[\sigma_x,\hat{H}] = \hbar\omega[\sigma_x,\sigma_z] = -2i\hbar\omega\sigma_y \tag{4}$$

we obtain

$$\frac{d\sigma_x}{dt} = -2\omega\sigma_y(t). \tag{5}$$

Now we have to evaluate the time evolution of σ_y, i.e.,

$$i\hbar\frac{d\sigma_y}{dt} = [\sigma_y,\hat{H}](t). \tag{6}$$

Since

$$[\sigma_y,\hat{H}] = \hbar\omega[\sigma_y,\sigma_z] = 2i\hbar\omega\sigma_x \tag{7}$$

we find $d\sigma_y/dt = 2\omega\sigma_x(t)$. Thus we have the linear system of matrix differential equations

$$\frac{d\sigma_x}{dt} = -2\omega\sigma_y(t), \qquad \frac{d\sigma_y}{dt} = 2\omega\sigma_x(t). \tag{8}$$

The initial conditions are $\sigma_x(t=0) = \sigma_x$, $\sigma_y(t=0) = \sigma_y$. Then the solution of the initial value problem is given by

$$\sigma_x(t) = \sigma_x\cos(2\omega t) - \sigma_y\sin(2\omega t), \quad \sigma_y(t) = \sigma_y\cos(2\omega t) + \sigma_x\sin(2\omega t). \tag{9}$$

The solution of the Heisenberg equation of motion can also be given by

$$\sigma_x(t) = \exp(i\hat{H}t/\hbar)\sigma_x\exp(-i\hat{H}t/\hbar), \quad \sigma_y(t) = \exp(i\hat{H}t/\hbar)\sigma_y\exp(-i\hat{H}t/\hbar). \tag{10}$$

Since $\sigma_z^2 = I_2$ we have

$$\exp(-i\hat{H}t/\hbar) := \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{-i\hat{H}t}{\hbar} \right)^k = \begin{pmatrix} \exp(-i\omega t) & 0 \\ 0 & \exp(i\omega t) \end{pmatrix}. \quad (11)$$

The Heisenberg equation of motion

$$i\hbar \frac{d\sigma_x}{dt} = [\sigma_x, \hat{H}](t) = \hbar\omega[\sigma_x, \sigma_z](t) \quad (12)$$

can be brought into a dimensionless form when we set $\tau(t) = \omega t$, $\tilde{\sigma}_x(\tau(t)) = \sigma_x(t)$. Then we have

$$i\frac{d\tilde{\sigma}_x}{d\tau} = [\tilde{\sigma}_x, \tilde{\sigma}_z](\tau), \qquad i\frac{d\tilde{\sigma}_y}{d\tau} = [\tilde{\sigma}_y, \tilde{\sigma}_z](\tau). \quad (13)$$

In the program we calculate the time evolution of σ_x and $d\sigma_x/dt$, where $d\sigma_x/dt = -2\omega\sigma_y(t)$. We use the following notation: $d\sigma_x/dt \to$ `sxt`, $d^2\sigma_x/dt^2 \to$ `sxtt`.

```
// heisenberg.cpp

#include <iostream>
#include "symbolicc++.h"
using namespace std;

int main(void)
{
Symbolic sx("",2,2), sy("",2,2), sz("",2,2);
Symbolic i("i"), hb("hb"), om("omega");
sx = ((Symbolic(0),Symbolic(1)),(Symbolic(1),Symbolic(0)));
sy = ((Symbolic(0),         -i),(          i,Symbolic(0)));
sz = ((Symbolic(1), Symbolic(0)),(Symbolic(0),Symbolic(-1)));
Symbolic H = hb*om*sz;
Symbolic sxt = -i/hb*(sx*H-H*sx);     // commutator
Symbolic sxtt = -i/hb*(sxt*H-H*sxt);  // commutator
cout << "H = " << H << endl;
cout << "sxt = " << sxt << endl;
cout << "sxtt = " << sxtt[i*i==-1] << endl;
return 0;
}
```