Preface

This textbook aims at introducing the reader to number theory.

Number theory (or higher arithmetic) was considered the "Queen of Mathematics" by Carl Friedrich Gauss (1777-1855). He wrote in particular: "The higher arithmetic presents us with an inexhaustible storehouse of interesting truths – of truths, too, which are not isolated but stand in the closest relation to one another, and between which, with every successive advance of the science, we continually discover new and sometimes wholly unexpected points of contact."

One hundred years later, Louis Mordell (1872-1952) added: "The theory of numbers is unrivalled for the number and variety of its results and for the beauty and wealth of its demonstrations. The higher arithmetic seems to include most of the romance of mathematics."

A great part of the beauty, charm and romance of number theory seems to result, as already observed by Gauss, from the contrast between the simplicity of many statements and the difficulty we encounter in proving them. Overcoming these difficulties has been often the historical source of deep and fruitful theories, which have shown themselves powerful and efficient in many other branches of mathematics.

For instance, its is known from ancient Babylonia (1900-1600 b.c.) that the equation $x^2 + y^2 = z^2$ has solutions for integers *x*, *y*, *z*. For example $3^2 + 4^2 = 5^2$. However, not until the seventieth century did Pierre de Fermat (1601-1665) prove that $x^4 + y^4 = z^4$ has no solution for non-zero integers *x*, *y*, *z*.

In a most famous conjecture, he even asserted that, in fact, the equation $x^n + y^n = z^n$ has no solution for non-zero integers x, y, z and $n \geq 3$.

Despite of the simplicity of its statement, *Fermat's last theorem* has shown incredibly difficult to prove. It was not until the end of the twentieth century that a complete proof was given by Andrew Wiles. This proof goes far beyond the scope of this book and rests on deep and complicated theories. We will content ourselves with proving Fermat's last theorem in the cases $n = 3, 4, 5$. However, even studying these "elementary" cases will convince the reader of the difficulty of the general problem.

Thus, this book aims at introducing the reader to the charm and beauty of number theory. It is intended for graduate and advanced undergraduate students, as well as for professional mathematicians, maths teachers, and mathematically educated laymen.

The prerequisites are an undergraduate course in algebra (groups, rings, fields and vector spaces) and calculus of one or two variables. A few parts of the book will make use of basic knowledge in complex analysis.

Evidently, it was not possible to cover, in one single book, all subjects in number theory. We have focused our attention on *diophantine problems*, named after Diophantus of Alexandria (around 250 b.c.), who wrote the first known book devoted exclusively to number theory. Diophantine problems can be roughly divided into two categories:

- Diophantine equations, that is equations whose unknowns are integers. This was the main subject of Diophantus book.
- Irrationality and transcendence problems, which go back to the irrationality of $\sqrt{2}$ (Pythagorean school, around 500 b.c.). The main tool for proving irrationality results consists in using *diophantine approximations*, that is approximations of real numbers by rational numbers.

From this entry point, the book offers the reader to discover many fundamental and instructive subjects in number theory, deliberately from an elementary point of view:

- Expansions of real numbers in series, infinite products and continued fractions
- Representations of integers as sums of squares
- Arithmetical functions (with an insight into prime number theory)
- Algebraic theory of numbers
- Diophantine equations
- Irrationality and transcendence methods…

The book is completely self-contained. Some proofs have been left to the reader as solved exercises. The reason for this is to promote a more "active" reading and also, sometimes, to highlight the main ideas of selected proofs or parts of the text.

However, a greater part of the 209 solved exercises consists in applications and further study of the concepts developed in the theory. They will hopefully help the reader understanding and mastering the powerful tools forged by mathematicians, all along the centuries, in order to handle and solve, in "honor of the human mind" (Carl Gustav Jacobi, 1804-1851), number-theoretical problems.

I am grateful to Michel Garcia for his kind assistance in checking thoroughly the first and second french editions of this book and pointing out a number of misprints and mathematical mistakes. Many thanks also to Jordan Taren, who helped me improving this translation into english by correcting a number of grammatical mistakes.

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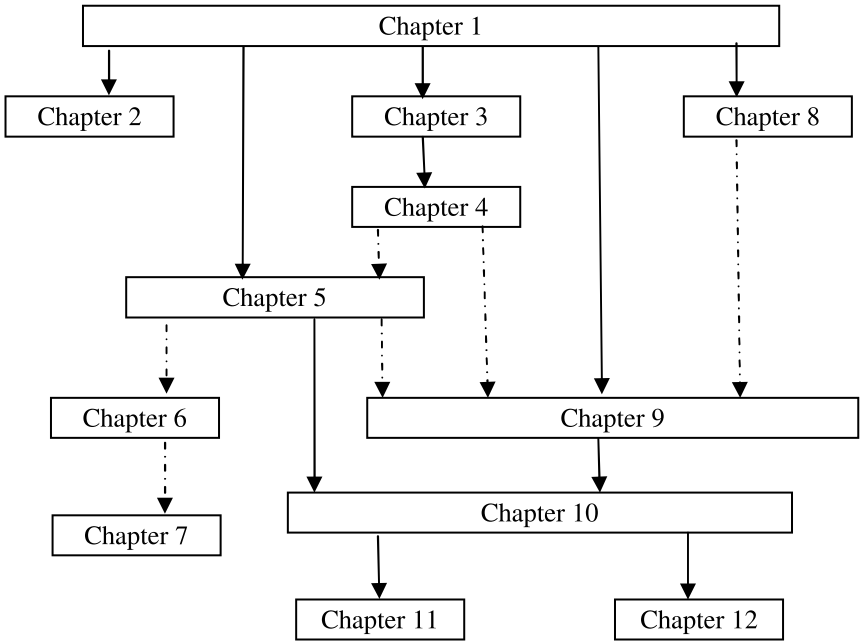
Structure of the book

An arrow linking chapter p to chapter q means that it is necessary to have read chapter p before studying chapter q. If the arrow is dotted, some parts of chapter q use results from chapter p, but studying chapter p is not necessary for the understanding of chapter q.

If there is no arrow between two chapters, these chapters are independent from each other.



NUMBER THEORY - An Elementary Introduction Through Diophantine Problems