# HIDDEN OSCILLATION IN DYNAMICAL SYSTEMS

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Effective methods of searching periodic oscillations of dynamical systems are described. Their applications to Hilbert's 16-th problem for quadratic systems and Aizerman's problem are considered. The synthesis of the method of harmonic linearization, the applied theory of bifurcations, and the numerical methods of computation of periodic oscillations are described.

*Keywords*: Hidden oscillation, Hilbert problem, Aizerman problem.

In the making and original development of the theory of nonlinear oscillations in the first half of twentieth century, particular attention has been given to analysis and synthesis of oscillating systems, for which a solution of the problem of existence of oscillating modes was not too difficult. The structure itself of many mechanical, electromechanical, and electronic systems was such that they have oscillating modes of operation, the existence of which was "almost obvious". Therefore the main attention of researchers was given to the analysis of forms and properties of these oscillations ("almost" harmonic, relaxation, synchronous, circular, orbitally stable, and so on).

In the 50-s of last century the attention of many scholar was concentrated on two famous problems: Hilbert's 16-th problem and Aizerman's problem, for which the proof of existence of periodic solutions was a nontrivial problem. And in studying these problems it was made much progress. It turns out that they have many similar features: while Hilbert has formulated, at first, the problem of searching periodic solutions for two-dimensional polynomial systems, in studying the Aizerman problem it was stated that differential equations of systems of automatic control, satisfying the generalized Routh–Hurwitz conditions, also can have periodic solutions. In addition for further investigations in this direction it also becomes actual a problem of searching periodic solutions of such differential equations.

These problems stimulate a great flow of investigations in the second half of twentieth century. While Hilbert's 16-th problem greatly encouraged



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a development of the theory of bifurcations and the theory of normal forms, the Aizerman problem encouraged a development of the theory of absolute stability.

Arnold writes [1]: "To estimate the number of limit cycles of quadratic vector fields on plane, A.N. Kolmogorov had distributed a few hundreds of such fields (with randomly chosen coefficients of quadratic expressions) among a few hundreds of students of Mechanics and Mathematics Faculty of MGU for a further mathematical practice. Each student had to find the number of limit cycles of his/her field. The result of this experiment was absolutely unexpected: not a single field had a limit cycle! It is known that a limit cycle persists under a small change of field coefficients. Therefore, the systems with one, two, three (and even, as has become known later, four) limit cycles form an open set in the space of coefficients and in this case for a random choice of polynomial coefficients, the probability of hitting in it is positive. The fact that this did not occur permits one to suggest that the above-mentioned probabilities are, evidently, small."

The result of this experiment shows also the necessity of development of goal-oriented methods (as analytic, as numerical ones) for the search of periodic oscillations, which would use all horsepower of modern computational technique. The present survey is devoted to description of the certain of such methods.

We shall consider the problem of Kolmogorov and clear up if there exist two-dimensional quadratic dynamical systems, for which the students taking part in the above-described practical work could find limit cycles. For this purpose we reduce an arbitrary quadratic system to special Lienard equation.

Consider a quadratic system

$$\begin{aligned} \dot{x} &= a_1x^2 + b_1xy + c_1y^2 + \alpha_1x + \beta_1y \\ \dot{y} &= a_2x^2 + b_2xy + c_2y^2 + \alpha_2x + \beta_2y, \end{aligned} \tag{1}$$

where $a_i, b_i, c_i, \alpha_i, \beta_i$ are real numbers.

*Proposition* 1. Without loss of generality, one can assume that $c_1 = 0$.

*Proposition 2.* In studying limit cycles, for $b_1 \neq 0$ system (1) with $c_1 = 0$ can be reduced to the Lienard equation

$$\dot{x} = y, \quad \dot{y} = -f(x)y - g(x), \tag{2}$$

where

$$f(x) = R(x)e^{p(x)} = R(x)|\beta_1 + b_1x|^q,$$

$$g(x) = P(x)e^{2p(x)} = P(x)|\beta_1 + b_1 x|^{2q}; \quad q = -c_2/b_1$$

$$R(x) = -\frac{(b_1 b_2 - 2a_1 c_2 + a_1 b_1)x^2}{(\beta_1 + b_1 x)^2} + \\ + \frac{(b_2\beta_1 + b_1\beta_2 - 2\alpha_1 c_2 + 2a_1\beta_1)x + \alpha_1\beta_1 + \beta_1\beta_2}{(\beta_1 + b_1 x)^2}$$

$$P(x) = -\left(\frac{a_2x^2 + \alpha_2 x}{\beta_1 + b_1 x} - \frac{(b_2 x + \beta_2)(a_1 x^2 + \alpha_1 x)}{(\beta_1 + b_1 x)^2} + \frac{c_2(a_1 x^2 + \alpha_1 x)^2}{(\beta_1 + b_1 x)^3}\right)$$

and for $b_1 = 0$ to equation (2) with the functions

$$f(x) = R(x)e^{qx}, \quad g(x) = P(x)e^{2qx}; \quad q = -c_2/\beta_1$$

For equation (2) with the smooth functions $f$ and $g$ it is well known the theorem of Lienard on the existence of limit cycle [2,3]. This theorem will be extended here to the case of discontinuous functions $f$ and $g$ and be applied then to the problem of Kolmogorov. We suppose that the functions $f(x)$ and $g(x)$ are differentiable on the interval $(a, +\infty)$ and for certain numbers $a < \nu_1 \leq x_0 \leq \nu_2$ the following conditions

1) $g(x) < 0, \ \forall x \in (a, x_0); \ g(x) > 0, \forall x \in (x_0, +\infty)$

$$\lim_{x \to a} G(x) = \lim_{x \to +\infty} G(x) = +\infty; G(x) = \int_{x_0}^{x} g(z)dz$$

2) $f(x) > 0, \forall x \in (a, \nu_1) \cup (\nu_2, +\infty); \ F_1(\nu_2) \geq 0; F_k(x) =$

$$= \int_{\nu_k}^{x} f(z)dz, \ k = 1, 2$$

are satisfied.

*Theorem* 1. [4] Let Conditions 1 and 2 be satisfied and the point $x = x_0, y = 0$ is unstable Lyapunov focal equilibrium. Then system (2) has a limit cycle.

Theorem 1 provides Kolmogorov's problem solving. The parameters of system (1), discriminated by Theorem 1, correspond to the limit cycles, which can easily be computed by using modern packets. Thus, for example, for $a_1 = b_1 = \beta_1 = 1$, $c_1 = \alpha_1 = 0$, $b_2 = 0$, $c_2 = 3/4$, $\beta_2 = 1$, $\alpha_2 = -2$ by Theorem 1 the limit cycle of system (1) exists and can be found. It is shown in Fig. 1

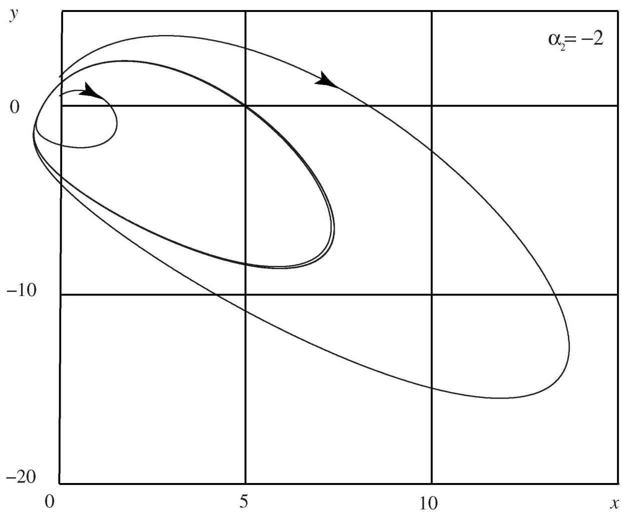


Fig. 1.

Consider a system

$$\begin{aligned} \dot{x}_1 &= -\omega_0 x_2 + b_1\varphi_0(x_1 + c^* x_3) \\ \dot{x}_2 &= \omega_0 x_1 + b_2\varphi_0(x_1 + c^* x_3) \\ \dot{x}_3 &= Ax_3 + b\varphi_0(x_1 + c^* x_3). \end{aligned} \tag{3}$$

Here $A$ is a constant $((n-2)\times(n-2))$-matrix, all eigenvalues of which have negative real arts, $b$ and $c$ are constant $(n-2)$-vectors, $b_1$ and $b_2$ are certain numbers:

$$\varphi_0(\sigma) = \begin{cases} \mu\sigma, & \forall\,\sigma \in (-\varepsilon, \varepsilon) \\ M\varepsilon^3, & \forall\,\sigma > \varepsilon \\ -M\varepsilon^3, & \forall\,\sigma < -\varepsilon \end{cases},$$

where $\mu, M$ are certain positive numbers, $\varepsilon$ is a small positive parameter.

Let us write now a transfer function of system (3):

$$W(p) = \frac{\alpha p + \beta}{p^2 + \omega_0^2} + c^*(A - pI)^{-1}b.$$

Here $\alpha = -b_1, \beta = b_2\omega_0$.

*Theorem* 2. [5] If the inequalities $\alpha > 0$ and $\mu\beta r^* q > \alpha\omega_0^2$ are satisfied, then system (3) has periodic solutions such that

$$x_1(0) = O(\varepsilon^2), \quad x_3(0) = O(\varepsilon^2), \quad x_2(0) = -\sqrt{\frac{\mu(\mu\beta r^* q - \alpha\omega_0^2)}{3\omega_0 M\alpha}} + O(\varepsilon).$$

In a number of cases this theorem provides Aizerman's problem solving.

## References


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