Chapter 0

Introduction

This short chapter is an informal description of the main objects of the book — buildings and their Grassmannians. All precise definitions will be given in Chapter 2.

The simplest buildings are so-called *Coxeter complexes* — a class of simplicial complexes defined by Coxeter systems. Let W be a group and S be a set of generators for W such that each element of S is an involution. The pair (W, S) is a *Coxeter system* if the group W has the following presentation

$$\langle\ S\ :(ss')^{m(s,s')} = 1,\ (s, s') \in S \times S,\ m(s, s') < \infty\ \rangle,$$

where $m(s, s')$ is the order of ss'. The associated Coxeter complex $\Sigma(W, S)$ is the simplicial complex whose simplices can be identified with special subsets of type $w\langle X\rangle$ with $X \subset S$ and $w \in W$. Every Coxeter system can be uniquely (up to an isomorphism) reconstructed from its diagram. The diagram associated with (W, S) is the graph whose vertex set is S and $s, s' \in S$ are connected by $m(s, s') - 2$ edges. In the case when W is finite, we get a Dynkin diagram without directions. There is a complete description of all finite Coxeter systems.

Similarly, more complicated buildings can be obtained from the *Tits systems*. A Tits system (G, B, N, S) is a structure on a group G consisting of two subgroups B, N which span G and a set of generators S of the quotient group

$$W := N/(B \cap N);$$

note that the pair (B, N) is called a BN-*pair*. By one of the basic properties of the Tits systems, (W, S) is a Coxeter system. The second remarkable property is the fact that every subgroup containing B, such subgroups are called *special*, can be reconstructed from elements of W and B. Moreover, there is a natural one-to-one correspondence between special subgroups and subgroups of W generated by subsets of S. The building associated with the Tits system (G, B, N, S) is the simplicial complex $\Delta(G, B, N, S)$ whose simplices can be identified with special subsets of type gP, where $g \in G$ and P is a special subgroup. If $w \in W$ then for every $g, g' \in N$ belonging to w and every special subgroup P we have $gP = g'P$; denote this special subset by wP. All such special subsets form a subcomplex Σ isomorphic to $\Sigma(W, S)$. The left action of the group G on the building defines the family of subcomplexes $g\Sigma$, $g \in G$. These subcomplexes are isomorphic to $\Sigma(W, S)$ and called *apartments*.

Following [Tits (1974)], we define an abstract *building* as a simplicial complex Δ with a family of subcomplexes called *apartments* and satisfying the following axioms:

- all apartments are Coxeter complexes,
- for any two simplices of Δ there is an apartment containing both of them,
- a technical condition concerning the existence of "nice" isomorphisms between apartments.

By this definition, every Coxeter complex is a building with a unique apartment. So, there exists a Coxeter system (W, S) such that all apartments of Δ are isomorphic to $\Sigma(W, S)$. The type of the building Δ is defined by the diagram of (W, S). We restrict ourselves to so-called *thick* buildings only; in such buildings maximal simplices form a sufficiently wide class. Coxeter complexes do not satisfy this condition.

Let V be an $(n+1)$-dimensional vector space and $\Delta(V)$ be the flag complex of V, i.e., the simplicial complex consisting of all flags of V. For every base B of V the subcomplex Σ_B which consists of all flags formed by linear subspaces spanned by subsets of B is called the *apartment* of $\Delta(V)$ associated with the base B. Every Σ_B is isomorphic to the simplicial complex of the Coxeter system (W, S), where W is the group of all permutations on the set $\{1, \dots, n+1\}$ and S is formed by all transpositions $(i, i+1)$; the associated diagram is A_n. The simplicial complex $\Delta(V)$ together with the family of all such apartments is a building of type A_n. Note that this building can be obtained from the Tits system of the group $\mathrm{GL}(V)$.

Let Δ be a building and (W, S) be the associated Coxeter system. The vertex set of Δ can be naturally decomposed in $|S|$ disjoint subsets called *Grassmannians*. If the building is associated with a Tits system for a certain group G then the Grassmannians are the orbits of the left action of G on the vertex set. In the general case, this decomposition is related to the fact that the vertex set of Δ can be labeled by elements of S and this labeling is unique up to a permutation on S.

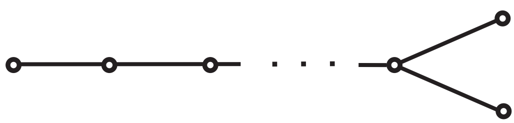
Let $\mathcal{G}$ be a Grassmannian of Δ. The intersection of $\mathcal{G}$ with an apartment of Δ is called an *apartment* of the Grassmannian $\mathcal{G}$. Two distinct elements $a, b \in \mathcal{G}$ are said to be *adjacent* if there exists a simplex $P \in \Delta$ such that $P \cup \{a\}$ and $P \cup \{b\}$ are maximal simplices. In this case, the subset formed by all $c \in \mathcal{G}$ such that $P \cup \{c\}$ is a maximal simplex will be called the *line* joining a and b. So, we get a structure known as a *partial linear space* or a *point-line geometry*, i.e., a set of points together with a family of subsets called lines and satisfying some simple axioms. This partial linear space is said to be the *Grassmann space* associated with $\mathcal{G}$.

The term "Grassmannian" is motivated by the following example. Let V be an $(n+1)$-dimensional vector space. The Grassmannians of the building $\Delta(V)$ are the usual Grassmannians $\mathcal{G}_k(V)$, $k \in \{1, \dots, n\}$, formed by all k-dimensional linear subspaces of V. The Grassmann spaces corresponding to $\mathcal{G}_1(V)$ and $\mathcal{G}_n(V)$ are the projective space associated with V and the dual projective space, respectively. In particular, any two distinct elements of these Grassmannians are adjacent. The Grassmann space of $\mathcal{G}_k(V)$, $1 < k < n$, is more complicated. It contains non-adjacent elements. Note that our adjacency relation coincides with the classical adjacency relation introduced in [Chow (1949)]: two elements of $\mathcal{G}_k(V)$ are adjacent if and only if their intersection is $(k-1)$-dimensional.

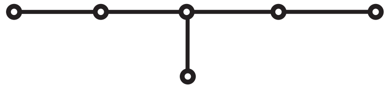
Recall that a building is spherical if the associated Coxeter system is finite, and it is irreducible if the diagram is connected. *Irreducible thick spherical buildings* of rank ≥ 3 were classified in [Tits (1974)]. There are precisely the following seven types of such buildings:

A_n

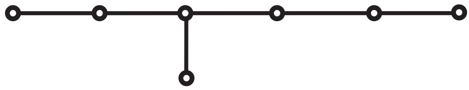
$\mathsf{B}_n = \mathsf{C}_n$



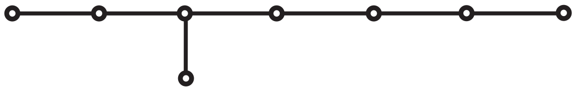
D_n



E_6



E_7



E_8

F_4

The first three types are called *classical*, the remaining four are known as *exceptional*. Every thick building of type A_n $(n \geq 3)$ is isomorphic to the flag complex of a certain $(n+1)$-dimensional vector space. All thick buildings of types C_n and D_n can be obtained from *polar spaces*, see Chapter 4. Exceptional buildings are related with so-called *metasymplectic* and *parapolar spaces*, see [Cohen (1995)].

In the present book we will consider Grassmannians associated with buildings of classical types only. Some information concerning Grassmannians of exceptional buildings can be found in [Cohen (1995)].

Investigation of building Grassmannians goes back to [Chow (1949); Dieudonné 2 (1951)] (see also Chapter III in [Dieudonné (1971)]) and is continued in [Cameron (1982)]. Currently, there are several research directions:

- axiomatic characterizations of Grassmann spaces, a survey can be found in [Cohen (1995)];
- embeddings in projective spaces, hyperplanes and generalized rank, see [Cooperstein (2003)] for a survey;
- subspaces of Grassmann spaces [Cooperstein, Kasikova and Shult (2005); Cooperstein (2005, 2007)];
- apartment properties [Blok and Brouwer (1998); Cooperstein and Shult (1997); Cooperstein, Kasikova and Shult (2005); Pankov 3 (2007)];

- characterizations of geometrical transformations of Grassmannians under "mild hypotheses" [Havlicek (1995); Huang (1998, 2000, 2001); Huang and Kreuzer (1995); Kreuzer (1998); Pankov, Prażmowski and Żynel (2006)] and the author's papers refereed in the book.

We describe all apartments preserving mappings of Grassmannians associated with buildings of classical types and collineations (isomorphisms) of the corresponding Grassmann spaces. Actually, the Fundamental Theorem of Projective Geometry and classical Chow's theorems [Chow (1949)] are partial cases of our results. The methods are based on deep structural properties of apartments (connections between the adjacency relation and apartments). Roughly speaking, we work in the latter two directions mentioned above. Also, we establish similar results for some geometric constructions non-related with buildings, for example, Grassmannians of infinite-dimensional vector spaces and the sets of conjugate linear involutions. One of them joins Chow's theorem with Dieudonné–Rickart's classification of automorphisms of the linear group [Dieudonné 1 (1951); Rickart (1950)].