LECTURE 1

i) Definition of a group and examples

Definition. A group G is a set $S = \{g\}$ such that for any two elements $g_1, g_2 \in S$, a composition law $\circ$ called a product is defined. It has the properties:

1) $g_1, g_2 \in G \Longrightarrow g_1 \circ g_2 \in G$. We write $g_1 \circ g_2 = g_1 g_2$.
2) The product is associative : $g_1(g_2 g_3) = (g_1 g_2) g_3$.
3) G contains an identity e. That is, $ge = eg = g, \forall g \in G$.

It follows that the identity is unique. For if e' is another identity, then $ee' = e = e'$. e is sometimes written as 1, and if the composition law is addition (+), then it is written as 0.

4) For every $g \in G$ there is an inverse, $g^{-1} \in G$, such that $gg^{-1} = g^{-1}g = e$.

The inverse is unique. For if $e = g^{-1}g = \overline{g}g$, multiply on the right by $g^{-1} \Rightarrow g^{-1} = \overline{g}$.

Definition. The order of a group is the number of elements in it. It can be finite, countably infinite, or uncountably infinite.

Examples:

1) The permutation group S_n on n objects. The order is $n!$.
Take n objects, number them 1, 2, ..., n. Put them in n boxes in a

row. For $n = 4$, we could have : $\boxed{2\,|\,1\,|\,3\,|\,4}$

A permutation $s \in S_n$, written as

$$s = \begin{pmatrix} 1 & 2 \ldots & n \\ s_1 & s_2 \ldots & s_n \end{pmatrix}$$

sends object k to object s_k. Thus if

$$s = \begin{pmatrix} 1\,2\,3\,4 \\ 4\,1\,3\,2 \end{pmatrix},$$

then

$$s\ \boxed{2\,|\,1\,|\,3\,|\,4} = \boxed{1\,|\,4\,|\,3\,|\,2}\,.$$

The order of the columns of s is immaterial.

The group arises here from maps of an underlying set (the set being objects in boxes). It is a 'transformation group'. We now verify the group properties:

Closure :

$$\begin{pmatrix} 1 & 2 \ldots & n \\ t_1 & t_2 \ldots & t_n \end{pmatrix} \begin{pmatrix} 1 & 2 \ldots & n \\ s_1 & s_2 \ldots & s_n \end{pmatrix} = \begin{pmatrix} 1 & 2 \ldots & n \\ t_{s_1} & t_{s_2} \ldots & t_{s_n} \end{pmatrix} \in S_n.$$

Associativity :

$$\begin{pmatrix} 1 & 2 \ldots & n \\ u_1 & u_2 \ldots & u_n \end{pmatrix} \left[\begin{pmatrix} 1 & 2 \ldots & n \\ t_1 & t_2 \ldots & t_n \end{pmatrix} \begin{pmatrix} 1 & 2 \ldots & n \\ s_1 & s_2 \ldots & s_n \end{pmatrix} \right]$$

$$= \begin{pmatrix} 1 \ldots & n \\ u_1 \ldots & u_n \end{pmatrix} \begin{pmatrix} 1 \ldots & n \\ t_{s_1} \ldots & t_{s_n} \end{pmatrix} = \begin{pmatrix} 1 \ldots & n \\ u_{t_{s_1}} \ldots & u_{t_{s_n}} \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & 2 \ldots & n \\ u_1 & u_2 \ldots & u_n \end{pmatrix} \begin{pmatrix} 1 & 2 \ldots & n \\ t_1 & t_2 \ldots & t_n \end{pmatrix} \right] \begin{pmatrix} 1 & 2 \ldots & n \\ s_1 & s_2 \ldots & s_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 \ldots & n \\ u_{t_1} \ldots & u_{t_n} \end{pmatrix} \begin{pmatrix} 1 \ldots & n \\ s_1 \ldots & s_n \end{pmatrix} = \begin{pmatrix} 1 \ldots & n \\ u_{t_{s_1}} \ldots & u_{t_{s_n}} \end{pmatrix}.$$

Identity:

$$e = \begin{pmatrix} 1 \ldots n \\ 1 \ldots n \end{pmatrix}.$$

Inverse:

$$\begin{pmatrix} 1 \dots & n \\ s_1 \dots & s_n \end{pmatrix}^{-1} = \begin{pmatrix} s_1 \dots & s_n \\ 1 \dots & n \end{pmatrix}.$$

S_n can also be thought of as linear transformations on a complex n-dimensional vector space $\mathbb{C}^n$. Let $e_1, e_2, \dots, e_n$ be a basis of this vector space. Define a family of $n!$ linear operators by

$$L\left[\begin{pmatrix} 1 & 2 \dots & n \\ s_1 & s_2 \dots & s_n \end{pmatrix}\right] \sum_i \xi_i e_i = \sum_i \xi_i e_{s_i}$$

where $\sum_i \xi_i e_i$ is a generic vector of $\mathbb{C}^n$. If s, t are permutations, then one sees easily that $L(s)L(t) = L(st)$. The set $\{L(s)\}$ is clearly a group with group multiplication as operator multiplication. It can also be thought of as S_n. [See Lecture 2.]

S_n appears in the discussion of i) Pauli exclusion principle in quantum mechanics and ii) some continuous groups and all finite groups.

2) The rotation group $O(3)$ in 3 dimensions.

Consider $\mathbb{R}^3$ = real three-dimensional vector space, i.e., $\mathbb{R}^3 = \{x = (x_1, x_2, x_3)\,|x_i \text{ real}\}$. The linear transformations $x \to Rx$, $x_i \to R_{ij}x_j$, which preserve the scalar product $(x,\ y) = \sum x_i y_i$ are such that $R^T R = 1$ (hence R is real and orthogonal), and $\det R^2 = 1$ or $\det R = 1$ or -1.

The set of all real, orthogonal matrices with det = +1 is the group $SO(3)$ of rotations *without* inversions. ("Det" is an abbreviation for determinant.) The set of all rotations (including reflections) is the group $O(3)$. The set of all elements with det = −1 does not form a group. (There is no identity, and the set is not closed under multiplication.)

Let

$$P = \begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \in O(3)$$

be "parity". We have $\det P = -1$ and $P^2 = e$. If $R \in O(3)$ had $\det R = -1$, write $R = PR'$ (where $R' = PR$). Here $R' \in O(3)$ and $\det R' = +1 \Rightarrow R' \in SO(3) \Rightarrow O(3) = SO(3) \cup PSO(3)$.

The corresponding groups in n dimensions are $SO(n)$, $O(n)$.

3) The group $SU(2) = \{g\}$. Here g is a 2×2 unitary matrix with $\det g = +1$. $SU(2)$ leaves $(z,\ z') = \sum_{i=1}^{2} z_i^* z_i'$ invariant. A similar definition holds for $SU(n)$ while $U(n) = \{g\}$, g being any $n \times n$ unitary matrix.

4) The Lorentz group $\mathcal{L} = \{\Lambda\}$. Λ is a real 4×4 matrix which leaves the form $(x,\ y) = \sum x_\mu y^\mu = x_0 y_0 - \sum_{i=1}^{3} x_i y_i$ invariant. If

$$\eta = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix},$$

then

$$(x,\ y) = x_i \eta_{ij} y_j$$

and

$$\Lambda^T g \Lambda = g.$$

5) The Poincaré group $\mathcal{P}$ consists of all Lorentz transformations and translations. An element of $\mathcal{P}$ is $(a,\ \Lambda)$ where $a = (a_0,\ a_1,\ a_2,\ a_4) \in \mathbb{R}^4$ and $\Lambda \in \mathcal{L}$. On a four-vector x,

$$(a,\ \Lambda)x = \Lambda x + a,$$

i.e. first Lorentz transform, and then translate. [This choice of order is only a convention.]

Now

$$\begin{aligned}(a^{'},\ \Lambda^{'})(a,\ \Lambda)x &= (a^{'},\ \Lambda^{'})(\Lambda x + a) \\ &= (\Lambda^{'}(\Lambda x + a) + a^{'}) \\ &= (\Lambda^{'} a + a^{'},\ \Lambda^{'}\Lambda)x\end{aligned}$$

or

$$(a^{'},\ \Lambda^{'})(a,\ \Lambda) = (a^{'} + \Lambda^{'} a,\ \Lambda^{'}\Lambda).$$

The identity is $(0,\ 1)$ and the inverse of $(a,\ \Lambda)$ is

$$(a,\ \Lambda)^{-1} = (-\Lambda^{-1} a,\ \Lambda^{-1}).$$

The groups above appear concretely as *transformation groups* on underlying sets. In the associated *abstract* group, we just consider the group structure (and forget about their origin as transformation groups).

LECTURE 2

i) Mapping and functions for sets

Consider two sets E and F, which can be either distinct or the same.

A law f which assigns to each $x \in E$, a unique element $f(x) \in F$ is a function from E to F.

If to each $y \in F$, there is an $x \in E$ such that $y = f(x)$, then the map f is onto F.

If the image of E under f is not all of F, the map is into F.

If to each $y \in F$, $\exists$ only one $x \in E$ such that $f(x) = y$, the map is one-to-one (1-1). In this case,

$$f(x_2) = f(x_1) \Rightarrow x_2 = x_1.$$

The symbol f^{-1} is defined by: for $y \in F$, $f^{-1}(y)$ is the *set* in E such that each member of the set is mapped to y by f. f^{-1} is in general not a function unless f is 1-1.

ii) Isomorphisms and homomorphisms

Assume that the mapping is onto. For if the mapping were not onto, we can restrict attention to the image.

1) A map f of a group G onto a group G' is a homomorphism if it preserves the group laws, i.e.

$$f(g_1)f(g_2) = f(g_1g_2).$$

This rule implies the following :

(a) If $e \in G$, $e' \in G'$ are identities then $f(e) = e'$. For $f(e)f(g) = f(eg) = f(ge) = f(g)f(e) = f(g)$. Since f is an onto map, and e' is unique, $f(e) = e'$.

(b) $f(g^{-1}) = f(g)^{-1}$. For $f(g^{-1})f(g) = f(g^{-1}g) = e'$.

The set of all elements in G mapped to e' by f is called the *kernel* of the homomorphism f and denoted as Ker f.

2) If in 1), f is 1-1, then the homomorphism is an *isomorphism*. Then G and G' as groups are identified.

3) If $G' = G$, then replace "homo" by "endo", and "iso" by "auto" in the above.

Examples:

1) Let

$$s = \begin{pmatrix} 1 \ldots & n \\ s_1 \ldots & s_n \end{pmatrix} \in S_n.$$

Let $L(s)$ be the linear operator on $\mathbb{C}^n$ with basis $e_1, e_2, \ldots, e_n$ defined by

$$L(s)e_i = e_{s_i}.$$

Then $\{L(s)\}$ forms a group under multiplication, $L(s)L(t) = L(st)$. Call this group $S'_n \equiv \{L(s)\}_{s \in S_n}$. Then

$$L : S_n \longrightarrow S'_n, \ s \xrightarrow{L} L(s)$$

is an isomorphism.

2) Let g_0 be a fixed element in a group G. Consider

$$G \ni g \xrightarrow{f_{g_0}} g_0 g g_0^{-1} \in G.$$

This map is an automorphism, a so-called *inner* automorphism.

Proof :

$$1 - 1 : g_0 g g_0^{-1} = g_0 g' g_0^{-1} \Rightarrow g = g'$$

Onto : For arbitrary $g \in G$, $g_0^{-1} g g_0 \in G$ and this is mapped by f_{g_0} to g.

Homomorphism: $f_{g_0}(g_1) f_{g_0}(g_2) = (g_0 g_1 g_0^{-1})(g_0 g_2 g_0^{-1}) = g_0(g_1 g_2) g_0^{-1} = f_{g_0}(g_1 g_2)$.

iii) *SU*(2) and *SO*(3)

There is a 2-1 homomorphism R from $SU(2)$ onto $SO(3)$ with $\pm U \in SU(2)$ having the same image $R(U) \in SO(3)$. $SU(2)$ is called the *covering group* (in fact the *universal covering group*) of $SO(3)$.

We show the following [and hence the existence of the 2-1 homomorphism]:

1. There exists an R which maps U to a real 3×3 matrix $R(U)$ such that

$$R(U_1)R(U_2) = R(U_1U_2).$$

2. The map is into $SO(3)$. [i.e. $R(U)^T R(U) = 1$, $\det R(U) = +1$.]
3. The map is onto $SO(3)$.
4. $R(U_1) = R(U_2) \Rightarrow U_1 = \pm U_2$.

Proofs:

1) Let τ_1, τ_2, τ_3 be Pauli matrices : $\tau_i^\dagger = \tau_i$, $tr\ \tau_i = 0$. Any 2×2 matrix m is a linear combination of the unit matrix 1 and τ_i:

$$m = \alpha_0\ 1 + \alpha_i\ \tau_i.$$

If m is traceless and $m^\dagger = m$, then

$$\alpha_0 = 0$$

and

$$\alpha_i \tau_i = \alpha_i^* \tau_i\ \ or\ \ \alpha_i^* = \alpha_i.$$

Thus any traceless hermitian 2×2 matrix is a real linear combination of the τ_i's.

Now consider $U\ \tau_i\ U^\dagger = \tau_i'$ with $U \in SU(2)$. Then

$$Tr\ \tau_i' = 0,\ \tau_i'^\dagger = \tau_i' \Rightarrow U\ \tau_i\ U^\dagger = R_{ji}(U)\tau_j$$

where the real $R_{ji}(U)$ are uniquely defined from U by linear independence of the τ's. So $U \to R(U)$ is a map.

Thus for each U, we have a real 3×3 matrix $R(U)$.

For $V \in SU(2)$,

$$VU\tau_iU^\dagger V^\dagger = (VU)\tau_i(VU)^\dagger$$

or

$$V\{R_{ki}(U)\tau_k\}V^\dagger = R_{ji}(VU)\tau_j$$

or

$$R_{ki}(U)R_{jk}(V)\tau_j = R_{ji}(VU)\tau_j$$

or

$$R(V)R(U) = R(VU).$$

2) Using the identity $\tau_i\ \tau_j = \delta_{ij}\ 1 + i\epsilon_{ijk}\ \tau_k$, we have

$$\begin{aligned} U\tau_i\tau_j U^\dagger &= U\tau_i U^\dagger U\tau_j U^\dagger \\ &= R_{li}(U)\tau_l R_{mj}(U)\tau_m \\ &= R_{li}R_{mj}\{\delta_{lm} + i\epsilon_{lmn}\tau_n\} \\ &= (R^T R)_{ij}1 + i\{R_{li}R_{mj}\epsilon_{lmn}\tau_n\}. \end{aligned}$$

Also

$$\begin{aligned} U\tau_i\tau_j U^\dagger &= \delta_{ij}1 + i\epsilon_{ijk}R_{nk}\tau_n. \\ &\Rightarrow R^T R = 1 \ or \ R \in O(3) \end{aligned}$$

and

$$R_{li}R_{mj}\epsilon_{lmn} = \epsilon_{ijk}R_{nk}.$$

Now multiply by R_{ns} and use $R^T R = 1$. Then

$$R_{li}R_{mj}R_{ns}\epsilon_{lmn} = \epsilon_{ijs},$$

i.e. $\det R = 1$.

So

$$R \in SO(3).$$

LECTURE 3

i) $SU(2)$ and $SO(3)$ (continued)

We have shown that the image of $SU(2)$ under R is contained in $SO(3)$. We want to show the equality, i.e. $\{R(U)\} = SO(3)$.

3) To show that the map R is onto $SO(3)$:

Let $S \in SO(3)$. We know that $S = S_3(\gamma)S_2(\beta)S_3(\alpha)$ where $S_i(\phi) \equiv$ rotation by ϕ around the i^{th} axis. We show that $U_3(\alpha)$ exists such that $R[U_3(\alpha)] = S_3(\alpha)$. Similarly one finds $U_2(\beta)$, $U_3(\gamma)$. Finally because R is a homomorphism, it follows that $U_3(\gamma)U_2(\beta)U_3(\alpha) \xrightarrow{R} S$.

Now

$$S_3(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

To find $U_3(\alpha)$ such that $U_3(\alpha)\ \tau_i\ U_3(\alpha)^\dagger = S_3(\alpha)_{ji}\ \tau_j$, we first write out each term in this equation:

$$\begin{aligned} U_3(\alpha)\ \tau_1\ U_3(\alpha)^\dagger &= \cos\alpha\ \tau_1 - \sin\alpha\ \tau_2, \\ U_3(\alpha)\ \tau_2\ U_3(\alpha)^\dagger &= \sin\alpha\ \tau_1 + \cos\alpha\ \tau_2, \\ U_3(\alpha)\ \tau_2\ U_3(\alpha)^\dagger &= \ \tau_3. \end{aligned}$$

Try

$$\begin{aligned} U_3(\alpha) &= \mathrm{e}^{i\frac{\tau_3}{2}\alpha} \\ &= \cos\frac{\alpha}{2} + i\ \tau_3\ \sin\frac{\alpha}{2} \\ &= \begin{bmatrix} \mathrm{e}^{\frac{i\alpha}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{i\alpha}{2}} \end{bmatrix} \in SU(2). \end{aligned}$$

$$\begin{aligned} U_3(\alpha)^\dagger &= \mathrm{e}^{-i\frac{\tau_3}{2}\alpha} \\ &= \cos\frac{\alpha}{2} - i\ \tau_3\ \sin\frac{\alpha}{2}. \end{aligned}$$

Then

$$\begin{aligned} U_3(\alpha)\ \tau_2\ U_3(\alpha)^\dagger &= \left[\tau_1\ \cos\frac{\alpha}{2} - \tau_2\ \sin\frac{\alpha}{2}\right]\left[\cos\frac{\alpha}{2} - i\ \tau_3\ \sin\frac{\alpha}{2}\right] \\ &= \tau_1\ \left(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\right) + \tau_2\ \left(-2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\right) \\ &= \cos\alpha\ \tau_1 - \sin\alpha\ \tau_2. \end{aligned}$$

Similarly for the rest of the equations.

Finally we show that the map R is two-to-one.

4) To show that the map is $2 \to 1$:

(a) $R(U) = R(-U)$ so that $\pm U$ are mapped to the same $R(U)$.

(b) If $U\ \tau_i\ U^\dagger = V\ \tau_i\ V^\dagger$, then $U = \pm V$. For then,

$V^\dagger U\ \tau_i = \tau_i\ V^\dagger U$

or

$[V^\dagger U,\ \tau_i] = 0.$

But $[V^\dagger U,\ 1] = 0$, so $[V^\dagger U,\ M] = 0$ for any M. Consequently

$$V^\dagger U = \alpha_0\ 1.$$

Taking the determinant we have

$$\alpha_0^2 = 1$$

or

$$\alpha_0 = \pm 1.$$

This completes the proof.

ii) Subgroups

A subgroup H of a group G is a (non-empty) subset of G which itself forms a group with respect to the group composition defined on G.

Thus

$$h \in H \Rightarrow h^{-1} \in H.$$
$$h_1, h_2 \in H \Rightarrow h_1 h_2 \in H.$$
$$e \in H.$$

Examples:

1) S_3 has three distinct S_2 subgroups consisting of e and permutation of i and j alone. All are isomorphic.

2) $SO(3)$ consists of an infinite number of $SO(2)$ subgroups. A typical one will be all rotations about a fixed axis $\hat{n}$:

$$\{\mathrm{e}^{i\hat{n}\cdot\vec{J}\theta} | \hat{n} \text{ fixed, all } \theta\},$$

$$\vec{J} \equiv \text{spin one angular momentum operator.}$$

They are all isomorphic. The isomorphism map is:

$$\mathrm{e}^{i\hat{n}\cdot\vec{J}\theta} \rightarrow \mathrm{e}^{i\hat{m}\cdot\vec{J}\theta}.$$

3) Real orthogonal matrices are also unitary. So $O(n) \subset U(n)$, $SO(n) \subset SU(n)$. Also, $U(n) \supset U(n-1) \supset U(n-2) \supset \cdots \supset U(1)$.

4) G = the Poincaré group. The translations $\mathcal{T}_4 = \{(a, 1)\}_{a\in\mathbb{R}^4}$ form a subgroup of G. The Lorentz transformations $O(3, 1) = \{(0, \Lambda)\}$ form a subgroup of G.

iii) Cosets and invariant subgroups

Let H be a subgroup of a group G. Then the *left coset* of H with respect to an element $g \in G$ is the set $gH \equiv \{gh | h \in H\}$. (Similarly right cosets are given by $Hg = \{hg \mid h \in H\}$. The space of left (right) cosets is $\{gH\}_{g\in G}$ ($\{Hg\}_{g\in G}$).)

We will prove the following:

(1) $G = \cup_g gH$.

(2) $gH \cap g'H$ is either $gH = g'H$ or $\emptyset$(the null set).

(3) If $\overline{g} \in gH$, then $\overline{g}H = gH$.

Thus if we say that g_1 is equivalent to g_2, $g_1 \sim g_2$, when g_1 and g_2 are in the same left coset, then the symbol $\sim$ is an equivalence relation. That is, it is symmetric ($g_1 \sim g_2 \Rightarrow g_2 \sim g_1$), reflexive ($g \sim g$) and transitive ($g_1 \sim g_2,\ g_2 \sim g_3 \Rightarrow g_1 \sim g_3$). Also since left cosets are identical or totally disjoint, we can label them by picking one element $\overline{g}$ from each left coset gH. Then automatically, $\overline{g}H = gH$.

Similar statements hold for right cosets.

(1) is due to the fact that $e \in H$ and hence $g \in gH$.

(2) and (3) are proved by the following lemmas.

Lemma 1

g_1 and g_2 are in the same left coset iff ${g_1}^{-1}g_2 \in H$.

If: Let ${g_1}^{-1}g_2 = h \in H$. Then $g_2 = g_1 h$ or $g_2 H = g_1 hH = g_1 H$. ($hH = H$ because H is a group.)

Only if: Suppose g_1, $g_2 \in gH$. Then $\exists\ h_1, h_2 \in H$ such that $g_1 = gh_1$ and $g_2 = gh_2$. Consequently

$${g_1}^{-1}g_2 = {h_1}^{-1}h_2 \in H.$$

Lemma 2

For any two cosets $g_1H,\ g_2H$, either $g_1H = g_2H$ or $g_1H \cap g_2H = \emptyset$.

Proof:

Suppose g_1H and g_2H have an element g in common. Then $\exists\ h_1$, $h_2 \in H$ such that $g = g_1h_1 = g_2h_2$. This implies ${g_1}^{-1}g_2 = h_1{h_2}^{-1} \in H$ and (by Lemma 1) $g_1H = g_2H$.

Lemma 3

If $\overline{g} \in gH$, then $\overline{g}H = gH$.

Lemma 2 gives the result since both the cosets contain $\overline{g}$, e being in H.

A subgroup $H \subset G$ is an *invariant* or a *normal* subgroup if $gHg^{-1} = H$ for all $g \in G$.

Note that now each $g \in G$ induces an automorphism f_g of H where $f_g(h) = ghg^{-1}$.

LECTURE 4

i) Cosets and invariant subgroups (continued)

Examples:

1) Let V_n be an n-dimensional vector space. It is a group under vector addition. If V_m is an m-dimensional subspace of V_n, it is an invariant subgroup.

2) $G =$ the Poincaré group. It contains 4-dimensional translations $\mathcal{T}_4 = \{(a,\ 1)\}$ as an invariant subgroup.

3) $G = SO(3)$, $H =$ rotations around 3^{rd} axis $= \{R_3(\gamma)\}$. This H is not an invariant subgroup of G.

Lemma

If H is an invariant subgroup of G, the left and right cosets are the same.

So in this case one can talk of cosets without specifying left or right and denote the space of cosets by

$$G/H = \{gH\} = \{Hg\}.$$

Theorem

If H is an invariant subgroup of G, G/H is a group, the factor group of G with respect to H. The multiplication rule is set multiplication of cosets. The identity is H.

For

$$gHg'H = gg'HH = gg'H = \text{another coset};$$
$$gHH = gH \text{ or } H \text{ is the identity};$$
$$(gH)^{-1} = g^{-1}H \text{ is the inverse since } g^{-1}HgH = H.$$

Examples:

1) $G = O(3)$; $H = SO(3)$. Because $\det(ghg^{-1}) = \det\ h = 1$, H is an invariant subgroup of G.
If $g \notin H$, $g = Ph$ where $h \in H$ and

$$P = \begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix}.$$

So $gH = PH$ and the cosets are H and PH. The group multiplication table becomes:

	H	PH
H	H	PH
PH	PH	H

This group is isomorphic to S_2.

2) $G =$ the Poincaré group, $H = T_4$. Since

$$T_4(a, \Lambda) = T_4(a, 1)(0, \Lambda) = T_4(0, \Lambda),$$

these cosets are in one-to-one correspondence with the elements of the Lorentz group $\{\Lambda\}$. Since

$$T_4(0, \Lambda)T_4(0, \Lambda') = T_4(0, \Lambda\Lambda'),$$

G/H is isomorphic to the Lorentz group.

3) $G = S_n$, $H = A_n$ = the alternating subgroup = the set of all even permutations. [Any permutation s can be written as $t_1 \ldots t_k$ where t_i is a transposition. If k is even (odd), then s is even (odd)(see Lecture 11).]

Clearly A_n is a group. Also since $a \in A_n$ implies sas^{-1} is even, A_n is an invariant subgroup.

If s is odd, then $s = t_0 a$ where t_0 is any transposition and a is even. Also

$$a \text{ even} \Rightarrow aA_n = A_n$$

while

$$s \text{ odd} \Rightarrow A_n = t_0 A_n.$$

So

$$S_n/A_n = S_2.$$

ii) Conjugate elements and classes

We say g_1 and g_2 are conjugate to each other if $\exists g_0 \in G$ such that

$$g_2 = g_0 g_1 g_0^{-1}.$$

Let us write $g_1 \sim g_2$ if such a g_0 exists in G. The relation $\sim$ is symmetric, reflexive, and transitive, i.e. a) $g_1 \sim g_2$ iff $g_2 \sim g_1$; b) $g \sim g$; c) $g_1 \sim g_2$, $g_2 \sim g_3 \Rightarrow g_1 \sim g_3$. So $\sim$ is an equivalence relation. The set of all such equivalent g's forms a class.

Examples:

1. The class containing $e \in G$ is $\{e\}$.
2. $G = SO(3)$. Consider a rotation by θ around the axis $\hat{n}$:

$$g = \mathrm{e}^{i\hat{n}\cdot\vec{j}\theta} = g\left(\hat{n}, \theta\right).$$

If R is a rotation, then

$$RJ_iR^{-1} = R_{ji}J_j.$$

So

$$\begin{aligned} RgR^{-1} &= \exp\left\{i\ \hat{n}_i RJ_iR^{-1}\theta\right\} \\ &= \exp\{i\ R_{ji}\hat{n}_i J_j\theta\}. \end{aligned}$$

So the class containing this g is that of rotations about every axis by the same angle θ.

iii) Simple groups

Definition. A group G is *simple* if it has no invariant subgroup besides the identity and itself.

Examples:

1. S_2 is simple.
2. $SO(3)$ and the group of proper orthochronous Lorentz transformations. $\mathcal{L}_+^{\uparrow}$ are simple. [We omit the proofs.]
3. $SU(2)$ is not simple as it has the invariant subgroup

$$Z_2 = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}.$$

4. $SU(n)$ is not simple as it has the invariant subgroup

$$Z_n = \left\{ \exp\left(i\frac{2\pi}{n}k \right) 1 \;\mid\; k = 0,\ 1,\ \ldots,\ n-1 \right\}.$$

5. The group $SO(2)$ is Abelian, so every discrete subgroup of $SO(2)$ is invariant. Therefore, $SO(2)$ is not simple. [See next lecture for the definition of Abelian groups.]

6. The group $SL(2,\mathbb{C})$ is $\{g\}$ where the element g is a 2×2 complex matrix with det $g = +1$. It is not simple because the subgroup

$$Z_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is invariant.

LECTURE 5

i) Abelian and semi-simple groups

Definition. A group $G = \{g_\alpha\}$ is Abelian if $g_\alpha g_\beta = g_\beta g_\alpha$, $\forall\ g_\alpha,\ g_\beta$.

Examples:

1) Translations in $\mathbb{R}^n$ are Abelian.
2) $SO(2)$ is Abelian.

Definition. A group G is *semi-simple* if it has no Abelian invariant subgroup besides the identity and itself.

Examples:

1) Any discrete subgroup of $SO(2)$ is Abelian and invariant. So $SO(2)$ is not semi-simple.
2) $SU(n)$ is not semi-simple since

$$Z_n = \left\{ \mathrm{e}^{2\pi ik/n}\ 1 \mid k = 0,\ 1,\ \ldots,\ n-1 \right\}$$

is an Abelian invariant subgroup.
3) The Poincaré group is not semi-simple since translations form an Abelian invariant subgroup.

ii) Representations of a group

Given a group $G = \{g\}$, a representation $\Gamma = \{T(g)\}$ of G is a set of linear operators on a vector space which forms a group under the usual product rule of linear operators such that

$$g \xrightarrow{T} T(g)$$

is a homomorphism. The vector space V on which Γ acts is the carrier or support space of the representation.

We will freely interchange the use of linear operators with matrices in view of their well-known correspondence.

The representation is called $faithful$ if T is an isomorphism.

Examples:

1) $SO(3)$ is a representation of $SU(2)$ which is not faithful. However, $SU(2)$ is not a representation of $SO(3)$. The spin j representation of $SU(2)$ is $\Gamma^{(j)} = \{D^{(j)}\}$ where $D^{(j)}$ are rotation matrices for angular momentum $j = 0, 1/2, 1, \ldots$. For j half an odd integer, the representation is faithful for $SU(2)$, but not so for j an integer. For j an integer, they are representations of $SO(3)$ and when $j \neq 0$, they are faithful representations of $SO(3)$. [For proof, see Lectures 3 and 25.]

2)

$$P = \begin{pmatrix} 1\,2\,3\,4 \\ 2\,3\,4\,1 \end{pmatrix},$$

P^2, P^3, $P^4 = e$ form a group. Now the fourth roots of unity are $e^{2\pi ik/4}$, $k = 0,\ 1,\ 2,\ 3$. Define

$$T(P^k) = e^{2\pi ik/4}1.$$

$T(P^k)$ is a linear operator on the one-dimensional complex vector space $\mathbb{C}^1 = \{z\}$. $\Gamma = \{T(P^k)\}$ is a faithful representation of the above group.

3) A four-dimensional representation of $SO(3)$ is given by

$$SO(3) \ni R \rightarrow \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}.$$

A $3n$-dimensional representation is given by

$$R \rightarrow \begin{bmatrix} R & 0 & . & . & 0 \\ 0 & R & . & . & . \\ . & . & R & . & . \\ . & . & . & . & . \\ 0 & . & . & . & R \end{bmatrix}.$$

4) Suppose S is a fixed 3×3 non-singular matrix and $R \in SO(3)$. Then the map $R \to SRS^{-1}$ gives a representation $\{SRS^{-1}\}$ of $SO(3)$.

iii) The regular representations

Consider the set V of all functions from G to $\mathbb{C}$:

$$G \ni g \xrightarrow{f \in V} f(g) \in \mathbb{C}.$$

Define for α, β complex numbers and f_1, $f_2 \in V$ the function $\alpha f_1 + \beta f_2$ by

$$(\alpha f_1 + \beta f_2)(g) = \alpha f_1(g) + \beta f_2(g).$$

This makes V into a complex vector space.

For each $g \in G$, define the linear operator $T(g)$ on V as follows: For any f, $T(g)f$ is the function defined by

$$[T(g)f](g') = f(g^{-1}g').$$

For $g, \bar{g} \in G$,

$$\begin{aligned}
[T(g)T(\bar{g})f](g') &= [T(\bar{g})f](g^{-1}g') \\
&= f(\bar{g}_{-1}g^{-1}g') \\
&= f[(g\bar{g})^{-1}g'] \\
&= [T(g\bar{g})f](g').
\end{aligned}$$

So

$$T(g)T(\bar{g}) = T(g\bar{g}),$$

and $\{T(g)\}$ is a representation of G, called the *left regular representation*.

Now define $S(g)$ on V by

$$[S(g)f](g') = f(g'g).$$

Then

$$\begin{aligned}
[S(g)S(\bar{g})f](g') &= [S(\bar{g})f](g'g) \\
&= f(g'g\bar{g}) \\
&= [S(g\bar{g})f](g')
\end{aligned}$$

or

$$S(g)S(\bar{g}) = S(g\bar{g}).$$

Thus $\{S(g)\}$ is a representation of G, called the *right regular representation*.

For finite groups, define the scalar product

$$(\alpha,\ \beta) = \sum_g \alpha^*(g)\beta(g)$$

for α, $\beta \in V$.

Then

$$\begin{aligned}(T(\bar{g})\alpha, T(\bar{g})\beta) &= \Sigma_g \alpha^*(\bar{g}^{-1}g)\beta(\bar{g}^{-1}g)\\ &= (\alpha, \beta).\end{aligned}$$

Thus $T(g)$'s (and similarly $S(g)$'s) are unitary in this scalar product.

Definition. Given two representations

$$\Gamma_1 = \{D^{(1)}(g)\} \text{ and } \Gamma_2 = \{D^{(2)}(g)\}$$

on vector spaces V_1, V_2, they are *equivalent* if there exists a non-singular $(1-1$, onto) linear operator S from V_1 onto V_2 such that

$$SD^{(1)}(g)S^{-1} = D^{(2)}(g).$$

LECTURE 6

i) Reducibility of representations

Suppose one is given a representation Γ on a vector space V. A subspace V_0 of V is invariant under a linear transformation T if $TV_0 \subset V_0$, that is, if $x_0 \in V_0$ implies $Tx_0 \in V_0$. A subspace V_0 is invariant under $\Gamma = \{D(g)\}$ if it is invariant under every $D(g)$. A representation Γ is reducible if there is a subspace $V_0(\neq \{0\}$ or $V)$ of V which is invariant under Γ.

Suppose Γ is reducible. Choose a basis $e_1, \ldots, e_n$ for V where $e_1, \ldots, e_m$ span V_0. The matrix $D(g)$ in this basis is of the form

$$D(g) = \begin{bmatrix} D_1(g) & \alpha(g) \\ 0 & D_2(g) \end{bmatrix}$$

where $D_1(g)$ is $m \times m$, $\alpha(g)$ is $m \times (n-m)$ and $D_2(g)$ is $(n-m) \times (n-m)$. Conversely if $D(g)$ does take this form in some basis, then vectors of the form $(\xi_1, \ldots, \xi_m, 0, \ldots, 0)$ are clearly invariant under Γ, that is, they span an invariant subspace. Thus Γ is reducible iff all $D(g)$'s are of the above form in some basis.

Note that

$$\begin{aligned} D(g_1)D(g_2) &= \begin{bmatrix} D_1(g_1) & \alpha(g_1) \\ 0 & D_2(g_1) \end{bmatrix} \begin{bmatrix} D_1(g_2) & \alpha(g_2) \\ 0 & D_2(g_2) \end{bmatrix} \\ &= \begin{bmatrix} D_1(g_1)D_1(g_2) & D_1(g_1)\alpha(g_2) + \alpha(g_1)D_2(g_2) \\ 0 & D_2(g_1)D_2(g_2) \end{bmatrix} \\ &= D(g_1g_2) \end{aligned}$$

by the group property of $D(g)$'s. Therefore

$$\begin{aligned} D_1(g_1)D_1(g_2) &= D_1(g_1g_2) \\ D_2(g_1)D_2(g_2) &= D_2(g_1g_2). \end{aligned}$$

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Thus $\Gamma_1 = D_1(g)$ and $\Gamma_2 = D_2(g)$ are themselves representations.

Examples:

1) Let $\mathcal{T}_1 = \{T(a)\}_{a\in\mathbb{R}^1}$ be translations on the vector space $V_1 = \{x\}_{x\in\mathbb{R}^1}$ such that

$$T(a)x = x + a.$$

The $T(a)$'s are not linear operators on V_1. Consider now

$$\Gamma = \left\{ D(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \right\}.$$

Then

$$\begin{aligned} D(a)D(b) &= \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} \\ &= D(a+b). \end{aligned}$$

Thus Γ is a reducible representation of $\mathcal{T}_1$.

The two-dimensional translation group is $\mathcal{T}_2 = \{T(a_1, a_2)\}$ where

$$T(a_1,\ a_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + a_1 \\ x_2 + a_2 \end{pmatrix}.$$

A two-dimensional representation for $\mathcal{T}_2$ is

$$\left\{ D(a_1, a_2) = \begin{bmatrix} 1 & a_1 + ia_2 \\ 0 & 1 \end{bmatrix} \right\}.$$

A four-dimensional representation is

$$\left\{ \begin{bmatrix} 1 & a_1 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & a_2 \\ 0 & 1 \end{bmatrix} \right\} = \begin{bmatrix} 1 & a_2 & a_1 & a_1a_2 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 1 & a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2) Suppose one is given two representations $\Gamma_i = \{D_i(g)\}$ on vector spaces V_i where D_i are matrices. The direct sum $\Gamma_1 \oplus \Gamma_2$ is the reducible representation given by

$$\left\{ \begin{bmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{bmatrix} \right\}.$$

The preceding construction can be stated in a coordinate-free way as follows: If one is given two vector spaces V_1 and V_2, their direct sum $V =$

$V_1 \oplus V_2$, is the linear combination of vectors in V_1 and V_2 in a well-known sense.

The dimension of V, dim V, is dim V_1+dim V_2. If T_1 is a linear operator on V_1, and T_2 is a linear operator on V_2, then the direct sum $T = T_1 \oplus T_2$ is a linear operator on V defined as follows: If $z \in V$, then $z = x + y$, where $x \in V_1$ and $y \in V_2$. We set

$$Tz = T_1 x + T_2 y.$$

Now given representations $\Gamma_i = \{T_i(g)\}$ on V_i, their direct sum Γ is $\Gamma_1 \oplus \Gamma_2 = \{T(g) = T_1(g) \oplus T_2(g)\}$. Note that

$$\begin{aligned} T(g_1)T(g_2)z &= T(g_1)(T_1(g_2)x + T_2(g_2)y) \\ &= T_1(g_1)T_1(g_2)x + T_2(g_1)T_2(g_2)y \\ &= T_1(g_1g_2)x \qquad + T_2(g_1g_2)y \\ &= T(g_1g_2)z. \end{aligned}$$

Here $z \in V_1 \oplus V_2,\ x \in V_1,\ y \in V_2$. Thus Γ is also a representation.

Now choose a basis $e_1,\ \ldots,\ e_m,\ f_1,\ \ldots,\ f_n$ where e_i span V_1, f_i span V_2. Then in this basis,

$$T(g) = \begin{bmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{bmatrix},$$

or Γ is reducible.

ii) Full or complete reducibility

Let $\Gamma = D(g)$ be a reducible representation on a vector space V with invariant subspace V_1. Now if there exists another invariant subspace V_2 such that $V = V_1 \oplus V_2$, then Γ is *fully reducible* into a direct sum $\Gamma_1 \oplus \Gamma_2$ where Γ_i is the *restriction* of Γ to V_i. [In the preceding discussion, $\Gamma_i = \{D_i(g)\}$.]

Now Γ_1 may or may not be reducible on V_1. If it is reducible, we can try writing V_1 itself as the direct sum of two invariant subspaces.

iii) Irreducible representations

Definition. A representation Γ on a vector space V is irreducible (IRR) if V has no invariant subspace besides $\{0\}$ and V itself.

Examples:

1) The representation

$$\left\{\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}\right\}$$

of $\mathcal{T}_1$ is reducible, but not fully reducible. Here

$$V_1 = \left\{\begin{pmatrix}x\\ 0\end{pmatrix}\right\},\ x \in \mathbb{R}^1$$

is invariant. The general form of V_2 (if it exists) will be

$$V_2 = \left\{\begin{pmatrix}\lambda y\\ y\end{pmatrix}\ \mid \lambda \text{ fixed}, y \in \mathbb{R}^1\right\}.$$

But

$$\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}\begin{pmatrix}\lambda y\\ y\end{pmatrix} = \begin{pmatrix}y(\lambda + a)\\ y\end{pmatrix} \notin V_2 \text{ if } a \neq 0.$$

So V_2 is not invariant under $\mathcal{T}_1$.

Theorem

2) Let H be a Hilbert space with scalar product: $(\cdot,\ \cdot)$. Let $\Gamma = \{U(g)\}$ be a unitary representation of G on H:

$$(x,\ U(g)y) = (U^{\dagger}(g)x,\ y) = (U^{-1}(g)x,\ y).$$

Then Γ is the direct sum of irreducible representations: $\Gamma = \oplus_i \Gamma_i$. Also, if $H = \oplus_i H_i$ is the corresponding decomposition of H, then this direct sum of H can be chosen to be an orthogonal sum, i.e., if $x_\alpha \in H_\alpha,\ x_\beta \in H_\beta$, then $(x_\alpha,\ x_\beta) = 0$ if $\alpha \neq \beta$.

Proof :

Note that:

a) If H_0 is a subspace of H, then we have the decomposition $H = H_0 \oplus H_1$ where the sum is an orthogonal direct sum. H_1 is the orthogonal complement of H_0 in H.

b) If U is unitary, defined on H, and H_0 is invariant under U, then so is H_1.

For since H_0 is invariant under U and U is unitary, $\exists\ \mathrm{e}_1,\ \ldots,\ \mathrm{e}_\mathrm{m}$ spanning H_0 such that $Ue_i = \lambda_i e_i,\ |\lambda_i| = 1$. This implies that $U^{-1}e_i = \lambda_i^{-1} e_i$

so that H_0 is invariant under U^{-1}. Now let $x_0 \in H_0$, $x_1 \in H_1$. By definition, $(x_1,\ x_0) = 0$. To prove $(Ux_1,\ x_0) = 0$, we note that $(Ux_1,\ x_0) = (x_1,\ U^\dagger x_0) = (x_1,\ U^{-1}x_0) = (x_1,\ x_0') = 0$ since $x_0' \in H_0$.

c) Thus if H_0 is invariant under Γ, we can write $H = H_0 \oplus H_1$ where H_1 is orthogonal to H_0 and invariant under Γ. If $H_i (i = 0, 1)$ has an invariant subspace under Γ, one repeats the process.

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LECTURE 7

i) Schur's lemma

Given two vector spaces V and W, we can think of linear operators L from V to W:

$$V \ni x \xrightarrow{L} Lx \in W$$

such that

$$L(x_1 + x_2) = Lx_1 + Lx_2$$

and

$$L(\lambda x) = \lambda Lx, \quad \lambda = \text{any complex number}.$$

For example $V = \{(\xi_1, \ \dots, \ \xi_m)\}$, $W = \{(\eta_1, \ \dots, \ \eta_m)\}$ and L is an $n \times m$ matrix.

Let $L : V \to W$ be a linear operator. The null space $N(L)$ of L is $\{x|Lx = 0 \in W\}$ and the range $R(L)$ of L is $\{y|y = Lx, \ x \in V\}$.

1. The map $L : V \to W$ is $1-1$ iff $N(L)$ consists of the zero vector. If: $Lx_1 = Lx_2 \Rightarrow L(x_1 - x_2) = 0 \Rightarrow x_1 - x_2 \in N(L) \Rightarrow x_1 = x_2$ since $N(L)$ consists of the zero vector. *Only if*: $\xi \in N(L) \Rightarrow Lx = L(x + \xi)$.
2. The map $L : V \to W$ is onto iff $R(L) = W$.
3. L is invertible iff $N(L) = 0$ and $R(L) = W$.
4. If L is invertible, $\dim W = \dim V$.

Proofs to 2-4:

Here 2 and 3 are obvious while 4 can be proved as follows.

Choose a basis $x_1, \ \dots, \ x_m \in V$. If $y \in W$, $\exists \ x \in V$ such that $y = Lx$. Then $x = \sum_i \xi_i x_i \Rightarrow y = \sum_i \xi_i (Lx_i) \Rightarrow Lx_i$ is complete in W. Also

$$\sum_i \eta_i Lx_i = 0 \Rightarrow \eta_i = 0$$

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because

$$L\left(\sum \eta_i x_i\right) = 0 \Rightarrow \sum_i \eta_i x_i \in N(L) \Rightarrow \sum_i \eta_i x_i = 0 \Rightarrow \eta_i = 0$$

by linear independence of $\{x_i\}$. Thus the $\{Lx_i\}$ are linearly independent. As $\{Lx_i\}$ is complete as well, $\{Lx_i\}$ is a basis for W. Hence dim $V =$ dim W.

Statement of Schur's Lemma:

1) Let $\Gamma_A = \{A(g)\}$ and $\Gamma_B = \{B(g)\}$ be two IRR's of a group $G = \{g\}$ on complex vector spaces V_A, V_B. Let $L : V_A \to V_B$ be a linear operator such that

$$LA(g) = B(g)L, \quad g \in G.$$

(One says that L intertwines the two representations.) Then either $L = 0$ or L is invertible so that

$$B(g) = LA(g)L^{-1}$$

and the two representations are equivalent. (The possibility that L is singular, but not zero is excluded.)

2) Let $\Gamma = \{D(g)\}$ be an IRR on a vector space V, and let $L : V \to V$ be a linear operator such that $[L, D(g)] = 0$. Then L is a multiple of the unit operator.

See H. Weyl: *Classical Groups*, Chap. 5. The following proof is from this chapter.

1) We can assume that $L \neq 0$. We then show that $N(L)$ and $R(L)$ are invariant subspaces of Γ_A and Γ_B. By irreducibility it will follow that $N(L) = \{0\}$ or V_A and $R(L) = \{0\}$ or V_B. Then $L \neq 0 \Rightarrow N(L) = \{0\}$ and $R(L) = V_B \Rightarrow L$ is invertible.

Let $x \in N(L)$, then

$$LA(g)x = B(g)Lx = 0$$

or

$$A(g)x \in N(L).$$

But this implies that $N(L)$ is invariant under Γ_A.

Let $y \in R(L)$, then $\exists\ x \in V_A$ such that $y = Lx$. Therefore

$$\begin{aligned} B(g)y &= B(g)Lx \\ &= L(A(g)x). \end{aligned}$$

Consequently, $B(g)y \in R(L)$, and this implies that $R(L)$ is invariant under Γ_B.

2) L has at least one eigenvector x with $Lx = \lambda x$. Therefore $(L - \lambda 1)x = 0 \Rightarrow (L - \lambda 1)$ is singular. But $[L - \lambda 1,\ D(g)] = 0$, so by 1), $L - \lambda 1 = 0$ (since $L - \lambda 1$ is singular) or $L = \lambda 1$.

Example :

All IRR's of an Abelian group are one-dimensional. For let $\Gamma = D(a)$ be such an IRR. Then $D(a)D(b) = D(b)D(a) \Rightarrow$ for any $a_0, D(a_0)$ commutes with every $D(a)$. So 2) $\Rightarrow D(a_0) = \lambda(a_0)1$. This is true for all a_0. Since Γ is IRR, here 1 must be a 1×1 matrix.

If $T_n = \{a\}_{a \in \mathbb{R}^n}$ is the n-dimensional translation group, its IRR's are given by

$$a = (a_1,\ \ldots,\ a_n) \to \exp \sum \lambda_i a_i,\ (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$$

with λ_i fixed in a given IRR. [Prove this as an exercise. Here $\mathbb{C}^n = n$ - dimensional complex vector space.]

ii) Operations with representations and with groups

1) Let $\Gamma = D(g)$ be a representation of a group G in terms of matrices.

$$\begin{aligned}(a)\ D(g_1)^* D(g_2)^* &= [D(g_1)D(g_2)]^* \\ &= D(g_1 g_2)^* \\ \Rightarrow g &\to D(g)^*\end{aligned}$$

gives a representation $\Gamma^* = \{D(a)^*\}$, the complex conjugate of Γ.

$$\begin{aligned}(b)\ D(g_1)^T D(g_2)^T &= [D(g_2)D(g_1)]^T \\ &= D(g_2 g_1)^T.\end{aligned}$$

So

$$\begin{aligned}D(g_1^{-1})^T D(g_2^{-1})^T &= D((g_1 g_2)^{-1})^T \\ \Rightarrow g &\to D(g^{-1})^T\end{aligned}$$

gives a representation called the representation contragradient to Γ.

$$(c)\ g \to D(g^{-1})^\dagger$$

also gives a representation, which can be called $\Gamma^\dagger$. If the representation is unitary,

$$D(g^{-1})^\dagger = D(g^{-1})^{-1} = D(g).$$

So (c) is the same as Γ, and (b) is the same as (a).

Γ and Γ^* may or may not be equivalent.

Example:

For $SU(2)$ or $SO(3)$, Γ and Γ^* are equivalent for any Γ. [See Problem 19.]

For $SU(3)$, $G = \{g\}$ where g is a 3×3 matrix with $\det g = 1$ and $g^\dagger = g^{-1}$. The representation $\underline{3}$ for $SU(3)$ associates the matrix g itself with g, i.e.,

$$SU(3) \ni g \to D(g) \equiv g \in \underline{3}.$$

So

$$SU(3) \ni g \to D(g)^* \equiv g^* \in \underline{3}^*.$$

Now

$$z = \begin{bmatrix} e^{2\pi i/3} & & \\ & e^{2\pi i/3} & \\ & & e^{2\pi i/3} \end{bmatrix} \in SU(3).$$

We have $D(z) = z$, but

$$D(z)^* = z^* = \begin{bmatrix} e^{4\pi i/3} & & \\ & e^{4\pi i/3} & \\ & & e^{4\pi i/3} \end{bmatrix}.$$

Since $D(z)$, $D(z)^*$ have different spectra (their eigenvalues differ), they cannot be related by a similarity transformation. This implies that $\underline{3}$ and $\underline{3}^*$ are not equivalent. Therefore, * is an outer automorphism. (An inner automorphism would be a map via conjugation by an element of G itself.)

2) Similarity transformations on representations give equivalent representations.

3) Direct sums of representations give representations.

iii) Direct product of representations

Consider two vector spaces V_i $(i = 1, 2)$, V_1 with basis $e_1, \ldots, e_m$ and V_2 with basis $f_1, \ldots, f_n$. Then $V_1 \otimes V_2$ has a basis

$$e_i \otimes f_j,\ i = 1,\ 2,\ \ldots,\ m;\ j = 1,\ 2,\ \ldots,\ n.$$

A general element in $V_1 \otimes V_2$ is

$$x = \sum a_{ij}\ e_i \otimes f_j.$$

The dimension dim $[V_1 \otimes V_2]$ is mn.

If L_i are linear operators on V_i, then we define $L = L_1 \otimes L_2$ by

$$Lx = \sum_{i,j} a_{ij}(L_1 e_i) \otimes (L_2 f_j).$$

If $M = M_1 \otimes M_2$ is another linear operator, then clearly

$$ML = (M_1 L_1) \otimes (M_2 L_2).$$

Suppose $D^{(i)}$ is the matrix of L_i in the above basis:

$$L_1 e_i = D^{(1)}_{ji} e_j,$$
$$L_2 f_i = D^{(2)}_{ji} f_j.$$

Then the matrix of L in basis $\{e_1 \otimes f_j\}$ is defined by

$$\begin{aligned} L(e_i \otimes f_j) &= D_{i'j',ij} e_{i'} \otimes f_{j'} \\ &= L_1 e_i \otimes L_2 f_j \\ &= D^{(1)}_{i'i} D^{(2)}_{j'j} e_{i'} \otimes f_{j'}. \end{aligned}$$

Comparing the two expressions gives

$$D_{i'j',ij} = D^{(1)}_{i'i} D^{(2)}_{j'j}.$$

If we enumerate the basis in the order $e_1 \otimes f_1$, $e_1 \otimes f_2$, …, $e_1 \otimes f_n$; $e_2 \otimes f_1$, …, $e_2 \otimes f_n$; …, $e_m \otimes f_n$, then

$$D = \begin{bmatrix} D^{(1)}_{11}\ D^{(2)} & D^{(1)}_{12}\ D^{(2)} & . & . & D^{(1)}_{1m}\ D^{(2)} \\ D^{(1)}_{21}\ D^{(2)} & . & . & . & D^{(1)}_{2m}\ D^{(2)} \\ & & & & \\ . & . & . & . & \\ . & . & . & . & \\ D^{(1)}_{m1}\ D^{(2)} & . & . & . & D^{(1)}_{mm}\ D^{(2)} \end{bmatrix}.$$

Thus D is the Kronecker product $D^{(1)} \otimes D^{(2)}$ of $D^{(1)}$ and $D^{(2)}$.

If S_i, D_i are matrices, then recall results

$$(S_1 \otimes S_2)(D_1 \otimes D_2) = S_1 D_1 \otimes S_2 D_2$$

and

$$Tr\ D \equiv \sum_{i,j} D_{ij,\ ij} = Tr\ D^{(1)}\ Tr\ D^{(2)}.$$

Given two representations $\Gamma_i = \{D^{(i)}(g)\}$, the direct product of the two representations $\Gamma = \Gamma_1 \otimes \Gamma_2$ is $\Gamma = \{D^{(1)} \otimes D^{(2)}\}$. Γ is a representation of the group (see product rule above). One can define the direct product for any number of representations in this way. Γ may not be irreducible even if the Γ_i's are.

The Clebsch-Gordan Problem

Consider representations Γ_i of G on vector spaces V_i. Then $\Gamma = \Gamma_1 \otimes \Gamma_2$ is defined on $V_1 \otimes V_2 = V$. Assume Γ can be written as the direct sum of IRR's. Then we can pose the following problems:

a) Write Γ as the direct sum $\Gamma = \oplus \Gamma_i$ where Γ_i are IRR.

b) If Γ_α are acting on $V^\alpha \subset V$, find a basis for V^α in terms of those in V_i and V_2.

Example:

$SU(2)$:

Consider <u>*IRR*</u>

$$V_{j_1} \text{ with basis } |j_1\ m_1\rangle,\ m_1 = -j_1,\ \cdots,\ +j_1 \qquad \Gamma_{j_1}$$

and

$$V_{j_2} \text{ with basis } |j_2\ m_2\rangle,\ m_2 = -j_2,\ \cdots,\ +j_2 \qquad \Gamma_{j_2}.$$

Then

$$\Gamma_{j_1} \otimes \Gamma_{j_2} = \oplus_{j=|j_1-j_2|}^{j_1+j_2} \Gamma_j.$$

A basis for Γ_j is $|j_1\ j_2\ j\ m\rangle,\ m = -j,\ \cdots,\ +j$, with

$$|j_1\ j_2\ j\ m\rangle = \sum_{m=m_1+m_2} C(j_1\ j_2\ j, m_1\ m_2\ m)|j_1\ m_1\rangle \otimes |j_2\ m_2\rangle.$$

The $C(j_1\ j_2\ j,\ m_1\ m_2\ m)$ are called Clebsch-Gordan coefficients.

iv) Direct products of groups

Let $G = \{g\}$ and $H = \{h\}$ be groups. Then the direct product group is

$$G \times H = \{(g,\ h)|\ g \in G,\ h \in H\},$$

with multiplication rule

$$(g_1,\ h_1)(g_2,\ h_2) = (g_1 g_2,\ h_1 h_2).$$

If $\{D(g)\},\ \{S(h)\}$ are representations of G and H, then

$$\Gamma = \{D(g) \otimes S(h)\}$$

($\otimes$ is the Kronecker product) is a representation of $G \times H$. However, Γ may not be faithful.

Example:

$SU(2) \times SU(2)$: Consider the mapping of group products into Kronecker products:

$$(g, h) \to g \otimes h.$$

Then

$$(1, 1) \to 1 \otimes 1 \ \textit{and}$$
$$(-1, -1) \to (-1) \otimes (-1) \equiv 1 \otimes 1.$$

Consequently certain different elements in the group product can be the same in the Kronecker product.