Chapter 1

Qubits

A single *qubit* is a two-state system, such as a two-level atom. The states (kets) $|h\rangle$ and $|v\rangle$ of the horizontal and vertical polarization of a photon can also be considered as a two-state system. Another example is the relative phase and intensity of a single photon in two arms of an interferometer. The underlying Hilbert space for the qubit is $\mathbb{C}^2$. An arbitrary orthonormal basis for $\mathbb{C}^2$ is denoted by $\{\, |0\rangle, |1\rangle \,\}$, where (scalar product)

$$\langle 0|0\rangle = \langle 1|1\rangle = 1, \qquad \langle 0|1\rangle = \langle 1|0\rangle = 0\,.$$

Any pure quantum state $|\psi\rangle$ (qubit) of this system can be written, up to a phase, as a *superposition* (linear combination of the states)

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle\,, \quad |\alpha|^2 + |\beta|^2 = 1\,, \quad \alpha, \beta \in \mathbb{C}\,.$$

The classical boolean states, 0 and 1, can be represented by a fixed pair of orthonormal states of the qubit.

If the qubit represents a *mixed state* one uses a two-dimensional *density matrix* for its representation. We therefore express one qubit as

$$\rho = \frac{1}{2}(I_2 + \mathbf{n}\cdot\boldsymbol{\sigma}) \equiv \frac{1}{2}(I_2 + n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3)$$

where $\mathbf{n} \in \mathbb{R}^3$, $\mathbf{n}\cdot\mathbf{n} \equiv n_1^2 + n_2^2 + n_3^2 \leq 1$, and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ denote the Pauli spin matrices. For pure states we have $\mathbf{n}\cdot\mathbf{n} = 1$ and $\rho = |\psi\rangle\langle\psi|$.

Problem 1. Any state (qubit) in $\mathbb{C}^2$ can be written as

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \qquad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1\,.$$

Find a parameter representation (i) if the underlying field is the set of real numbers (ii) if the underlying field is the set of complex numbers.

Solution 1. (i) Using $\alpha = \cos\theta$, $\beta = \sin\theta$ and the identity $\cos^2\theta + \sin^2\theta \equiv 1$ for all $\theta \in \mathbb{R}$ we have

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

(ii) We have as a representation

$$\begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix}$$

where $\theta, \phi \in \mathbb{R}$ and $e^{i\phi}e^{-i\phi} = 1$.

Problem 2. Consider the normalized states ($\theta_1, \theta_2 \in [0, 2\pi)$)

$$\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \qquad \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}.$$

Find the condition on θ_1 and θ_2 such that

$$\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} + \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}$$

is normalized.

Solution 2. From the condition that the vector

$$\begin{pmatrix} \cos\theta_1 + \cos\theta_2 \\ \sin\theta_1 + \sin\theta_2 \end{pmatrix}$$

is normalized it follows that

$$(\sin\theta_1 + \sin\theta_2)^2 + (\cos\theta_1 + \cos\theta_2)^2 = 1\,.$$

Thus we have

$$\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 = -\frac{1}{2}\,.$$

It follows that

$$\cos(\theta_1 - \theta_2) = -\frac{1}{2}\,.$$

Therefore, $\theta_1 - \theta_2 = 2\pi/3$ or $\theta_1 - \theta_2 = 4\pi/3$.

Problem 3. Let $\{\, |0\rangle, |1\rangle \,\}$ be an orthonormal basis in the Hilbert space $\mathbb{R}^2$. Let

$$A := |0\rangle\langle 0| + |1\rangle\langle 1|.$$

Consider the three cases

$$\begin{aligned}
&\text{(i)}\ |0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, && |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&\text{(ii)}\ |0\rangle := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, && |1\rangle := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&\text{(iii)}\ |0\rangle := \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, && |1\rangle := \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}.
\end{aligned}$$

Find the matrix representation of A in these bases.

Solution 3. We find

$$\begin{aligned}
\text{(i)} \quad A &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\text{(ii)} \quad A &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\text{(iii)} \quad A &= \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} + \begin{pmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

For all three cases $A = I_2$, where I_2 is the 2×2 unit matrix. Obviously, the third case contains the first two as special cases. This is the *completeness relation.*

Problem 4. Let $\{\, |0\rangle, |1\rangle \,\}$ be an orthonormal basis in the Hilbert space $\mathbb{C}^2$. The *NOT operation* (unitary operator) is defined as

$$|0\rangle \rightarrow |1\rangle, \qquad |1\rangle \rightarrow |0\rangle\,.$$

(i) Find the unitary operator U_{NOT} which implements the NOT operation with respect to the basis $\{\, |0\rangle, |1\rangle \,\}$.
(ii) Let

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Find the matrix representation of U_{NOT} for this basis.
(iii) Let

$$|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

be the *Hadamard basis*. Find the matrix representation of U_{NOT} for this basis.

Solution 4. (i) Obviously,

$$U_{NOT} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

since $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$.
(ii) For the standard basis we find

$$U_{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(iii) For the Hadamard basis we find

$$U_{NOT} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus we see that the respective matrix representations for the two bases are different.

Problem 5. The *Walsh-Hadamard transform* is a 1-qubit operation, denoted by H, and performs the following linear transform

$$|0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |1\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

(i) Find the unitary operator U_H which implements H with respect to the basis $\{\, |0\rangle, |1\rangle \,\}$.
(ii) Find the inverse of this operator.
(iii) Let

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

be the standard basis in $\mathbb{C}^2$. Find the matrix representation of U_H for this basis.
(iv) Let

$$|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

be the Hadamard basis in $\mathbb{C}^2$. Find the matrix representation of U_H for this basis.

Solution 5. (i) Obviously,

$$\begin{aligned} U_H &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\langle 0| + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\langle 1| \\ &= \frac{1}{\sqrt{2}}|0\rangle(\langle 0| + \langle 1|) + \frac{1}{\sqrt{2}}|1\rangle(\langle 0| - \langle 1|). \end{aligned}$$

(ii) The operator U_H is unitary and the inverse is given by $U_H^{-1} = U_H^* = U_H$, where * denotes the adjoint.
(iii) For the standard basis we find

$$U_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

(iv) For the Hadamard basis we find

$$U_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

We see that the matrix representations for each of the two bases are the same.

Problem 6. The *Hadamard operator* on one qubit can be written as

$$U_H = \frac{1}{\sqrt{2}}((|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|) .$$

(i) Calculate the states $U_H|0\rangle$ and $U_H|1\rangle$.
(ii) Calculate $U_H U_H$.

Solution 6. (i) We obtain the normalized states

$$U_H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad U_H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

(ii) Since $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$ we obtain

$$U_H U_H = |0\rangle\langle 0| + |1\rangle\langle 1| = I$$

where I is the identity operator (2×2 unit matrix).

Problem 7. Consider the Hilbert space $\mathbb{C}^2$ and the linear operator (2×2 matrix)

$$\Pi(\mathbf{n}) := \frac{1}{2}\left(I_2 + \sum_{j=1}^{3} n_j \sigma_j \right)$$

where $\mathbf{n} := (n_1, n_2, n_3)$ ($n_j \in \mathbb{R}$) is a unit vector, i.e., $n_1^2 + n_2^2 + n_3^2 = 1$. Here σ_1, σ_2, σ_3 are the *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and I_2 is the 2×2 unit matrix.
(i) Describe the properties of $\Pi(\mathbf{n})$, i.e., find $\Pi^\dagger(\mathbf{n})$, $\mathrm{tr}(\Pi(\mathbf{n}))$ and $\Pi^2(\mathbf{n})$.
(ii) Find the vector ($\phi, \theta \in \mathbb{R}$)

$$\Pi(\mathbf{n}) \begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix} .$$

Discuss.

Solution 7. (i) For the Pauli matrices we have $\sigma_1^\dagger = \sigma_1$, $\sigma_2^\dagger = \sigma_2$, $\sigma_3^\dagger = \sigma_3$. Thus $\Pi(\mathbf{n}) = \Pi^\dagger(\mathbf{n})$. Since $\mathrm{tr}\sigma_1 = \mathrm{tr}\sigma_2 = \mathrm{tr}\sigma_3 = 0$, $\mathrm{tr}I_2 = 2$, and the trace operation is linear, we obtain $\mathrm{tr}(\Pi(\mathbf{n})) = 1$. Since

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I_2$$

and

$$[\sigma_1, \sigma_2]_+ = 0, \quad [\sigma_2, \sigma_3]_+ = 0, \quad [\sigma_3, \sigma_1]_+ = 0$$

where $[A, B]_+ := AB + BA$ denotes the *anticommutator*, the expression

$$\Pi^2(\mathbf{n}) = \frac{1}{4}\left(I_2 + \sum_{j=1}^{3} n_j\sigma_j\right)^2 = \frac{1}{4}I_2 + \frac{1}{2}\sum_{j=1}^{3} n_j\sigma_j + \frac{1}{4}\sum_{j=1}^{3}\sum_{k=1}^{3} n_j n_k \sigma_j \sigma_k$$

simplifies to

$$\Pi^2(\mathbf{n}) = \frac{1}{4}I_2 + \frac{1}{2}\sum_{j=1}^{3} n_j\sigma_j + \frac{1}{4}\sum_{j=1}^{3} n_j^2 I_2 .$$

Using $n_1^2 + n_2^2 + n_3^2 = 1$ we obtain $\Pi^2(\mathbf{n}) = \Pi(\mathbf{n})$. Thus $\Pi(\mathbf{n})$ is a projection matrix.
(ii) We find

$$\Pi(\mathbf{n}) \begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (1+n_3)e^{i\phi}\cos\theta + (n_1 - in_2)\sin\theta \\ (n_1 + in_2)e^{i\phi}\cos\theta + (1 - n_3)\sin\theta \end{pmatrix} .$$

Problem 8. The *qubit trine* is defined by the following states

$$|\psi_0\rangle = |0\rangle, \quad |\psi_1\rangle = -\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle, \quad |\psi_2\rangle = -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

where $\{\,|0\rangle, |1\rangle\,\}$ is an orthonormal basis. Find the probabilities

$$|\langle\psi_0|\psi_1\rangle|^2, \quad |\langle\psi_1|\psi_2\rangle|^2, \quad |\langle\psi_2|\psi_0\rangle|^2 .$$

Solution 8. Using $\langle 0|0\rangle = 1$, $\langle 1|1\rangle = 1$ and $\langle 0|1\rangle = 0$ we find

$$|\langle\psi_0|\psi_1\rangle|^2 = \frac{1}{4}, \quad |\langle\psi_1|\psi_2\rangle|^2 = \frac{1}{4}, \quad |\langle\psi_2|\psi_0\rangle|^2 = \frac{1}{4} .$$

Problem 9. The kets $|h\rangle$ and $|v\rangle$ are states of horizontal and vertical polarization, respectively. Consider the normalized states

$$\begin{aligned}
|\psi_1\rangle &= -\frac{1}{2}(|h\rangle + \sqrt{3}|v\rangle)\\
|\psi_2\rangle &= -\frac{1}{2}(|h\rangle - \sqrt{3}|v\rangle)\\
|\psi_3\rangle &= |h\rangle\\
|\phi_1\rangle &= \frac{1}{\sqrt{3}}(-|h\rangle + \sqrt{2}e^{-2\pi i/3}|v\rangle)\\
|\phi_2\rangle &= \frac{1}{\sqrt{3}}(-|h\rangle + \sqrt{2}e^{+2\pi i/3}|v\rangle)\\
|\phi_3\rangle &= \frac{1}{\sqrt{3}}(-|h\rangle + \sqrt{2}|v\rangle)\,.
\end{aligned}$$

Give an interpretation of these states.

Solution 9. Since $\langle h|h\rangle = \langle v|v\rangle = 1$ and $\langle v|h\rangle = \langle h|v\rangle = 0$ we find

$$\langle\psi_1|\psi_2\rangle = -\frac{1}{2}, \qquad \langle\psi_1|\psi_3\rangle = -\frac{1}{2}, \qquad \langle\psi_2|\psi_3\rangle = -\frac{1}{2}\,.$$

Since the solution to $\cos(\alpha) = -1/2$ is given by $\alpha = 120^o$ or $\alpha = 240^o$ we find that that the first three states $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$ correspond to states of linear polarization separated by 120^o. We find

$$\langle\phi_1|\phi_2\rangle = -\frac{i}{\sqrt{3}}\,.$$

The states $|\phi_1\rangle$ and $|\phi_2\rangle$ correspond to elliptic polarization and the third state $|\phi_3\rangle$ corresponds to linear polarization.

Problem 10. Let

$$|\psi\rangle = \begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix}$$

where $\phi, \theta \in \mathbb{R}$.
(i) Find the density matrix $\rho := |\psi\rangle\langle\psi|$.
(ii) Find $\mathrm{tr}\rho$.
(iii) Find ρ^2.

Solution 10. (i) Since

$$\langle\psi| = (e^{-i\phi}\cos\theta, \sin\theta)$$

we obtain the 2 × 2 density matrix

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \cos^2\theta & e^{i\phi}\sin\theta\cos\theta \\ e^{-i\phi}\sin\theta\cos\theta & \sin^2\theta \end{pmatrix} .$$

(ii) Since $\cos^2\theta + \sin^2\theta = 1$ we obtain from (i) that $\mathrm{tr}\rho = 1$.
(iii) Since $\langle\psi|\psi\rangle = 1$ we have

$$\rho^2 = (|\psi\rangle\langle\psi|)^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = \rho .$$

Problem 11. Given the Hamilton operator

$$\hat{H} = \hbar\omega\sigma_x .$$

(i) Find the solution

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(t=0)\rangle$$

of the *Schrödinger equation*

$$i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle$$

with the initial conditions

$$|\psi(t=0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

(ii) Find the probability

$$|\langle\psi(t=0)|\psi(t)\rangle|^2 .$$

(iii) The solution of the *Heisenberg equation of motion*

$$i\hbar\frac{d\sigma_z}{dt} = [\sigma_z, \hat{H}](t)$$

is given by

$$\sigma_z(t) = e^{i\hat{H}t/\hbar}\sigma_z e^{-i\hat{H}t/\hbar}$$

where $\sigma_z(t=0) = \sigma_z$. Calculate $\sigma_z(t)$.
(iv) Show that

$$\langle\psi(t=0)|\sigma_z(t)|\psi(t=0)\rangle = \langle\psi(t)|\sigma_z|\psi(t)\rangle .$$

Solution 11. (i) The solution of the Schrödinger equation is given by

$$|\psi(t)\rangle = \exp(-i\hat{H}t/\hbar)|\psi(t=0)\rangle .$$

Since $\sigma_x^2 = I_2$ we find the unitary matrix

$$\exp(-i\hat{H}t/\hbar) \equiv U(t) = \begin{pmatrix} \cos(\omega t) & -i\sin(\omega t) \\ -i\sin(\omega t) & \cos(\omega t) \end{pmatrix} .$$

Thus the normalized state at time t is

$$|\psi(t)\rangle = U(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\omega t) \\ -i\sin(\omega t) \end{pmatrix} .$$

(ii) We find the probability

$$|\langle\psi(t=0)|\psi(t)\rangle|^2 = \cos^2(\omega t) .$$

(iii) Since the commutators are given by

$$[\sigma_z, \hat{H}] = \hbar\omega[\sigma_z, \sigma_x] = 2i\hbar\omega\sigma_y, \qquad [\sigma_y, \hat{H}] = \hbar\omega[\sigma_y, \sigma_x] = -2i\hbar\omega\sigma_z$$

we obtain the linear system of matrix-valued differential equations

$$\frac{d\sigma_z}{dt} = 2\omega\sigma_y(t), \qquad \frac{d\sigma_y}{dt} = -2\omega\sigma_z(t)$$

with the initial conditions $\sigma_z(t=0) = \sigma_z$ and $\sigma_y(t=0) = \sigma_y$. Here we used the Heisenberg equation of motion for σ_y to obtain the second differential equation. The solution of this system of matrix-valued linear differential equations is given by

$$\sigma_z(t) = \sigma_z\cos(2\omega t) + \sigma_y\sin(2\omega t)$$

$$\sigma_y(t) = \sigma_y\cos(2\omega t) - \sigma_z\sin(2\omega t) .$$

(iv) We find

$$\langle\psi(t=0)|\sigma_z(t)|\psi(t=0)\rangle = \cos(2\omega t)$$

and

$$\langle\psi(t)|\sigma_z|\psi(t)\rangle = \cos^2(\omega t) - \sin^2(\omega t) = \cos(2\omega t) .$$

Problem 12. Consider a *Mach-Zehnder interferometer* in which the beam pair spans a two-dimensional Hilbert space with orthonormal basis $\{\, |0\rangle, |1\rangle \,\}$. The state vectors $|0\rangle$ and $|1\rangle$ can be considered as orthonormal wave packets that move in two given directions defined by the geometry of the interferometer. We may represent mirrors, beam splitters and relative U_P phase shifts by the unitary matrices

$$U_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U_B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad U_P = \begin{pmatrix} e^{i\chi} & 0 \\ 0 & 1 \end{pmatrix}$$

respectively. Consider the density matrix

$$\rho_{in} = |0\rangle\langle 0|$$

where $\{\,|0\rangle,\ |1\rangle\,\}$ denotes the standard basis. Using this basis find

$$\rho_{out} = U_B U_M U_P U_B \rho_{in} U_B^\dagger U_P^\dagger U_M^\dagger U_B^\dagger\,.$$

Give an interpretation of the result.

Solution 12. Since

$$\rho_{in} = |0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1\ 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$U_B U_M U_P U_B = \frac{1}{2} \begin{pmatrix} e^{i\chi}+1 & e^{i\chi}-1 \\ -e^{i\chi}+1 & -e^{i\chi}-1 \end{pmatrix}$$

we obtain

$$\rho_{out} = \frac{1}{2} \begin{pmatrix} 1+\cos(\chi) & i\sin(\chi) \\ -i\sin(\chi) & 1-\cos(\chi) \end{pmatrix}\,.$$

This yields the intensity along $|0\rangle$ as

$$I \propto 1+\cos(\chi)\,.$$

Thus the relative U_P phase χ could be observed in the output signal of the interferometer.

Problem 13. Let $\{\,|0\rangle, |1\rangle\,\}$ be an orthonormal basis in $\mathbb{C}^2$.
(i) Find the commutator

$$\Big[\,|0\rangle\langle 1|\,,\ |1\rangle\langle 0|\,\Big]\,.$$

(ii) Find the operator $\exp(t|0\rangle\langle 1|)$.
(iii) Find the operator $\exp(t|1\rangle\langle 0|)$.
(iv) Find the operator $\exp(t|0\rangle\langle 1|)\exp(t|1\rangle\langle 0|)$.
(v) Find the operator $\exp(t(|0\rangle\langle 1| + |1\rangle\langle 0|))$.
(vi) Is

$$\exp(t(|0\rangle\langle 1| + |1\rangle\langle 0|)) = \exp(t|0\rangle\langle 1|)\exp(t|1\rangle\langle 0|)\ ?$$

Solution 13. (i) We have

$$\Big[\,|0\rangle\langle 1|\,,\ |1\rangle\langle 0|\,\Big] = |0\rangle\langle 0| - |1\rangle\langle 1|$$

since $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$. We see that the commutator is nonzero.
(ii) Since $\langle 0|1\rangle = \langle 1|0\rangle = 0$ we find

$$\exp(t|0\rangle\langle 1|) = \sum_{j=0}^{\infty} \frac{t^j}{j!}(|0\rangle\langle 1|)^j = I_2 + t|0\rangle\langle 1|\,.$$

(iii) Analogously

$$\exp(t|1\rangle\langle 0|) = \sum_{j=0}^{\infty} \frac{t^j}{j!}(|1\rangle\langle 0|)^j = I_2 + t|1\rangle\langle 0|\,.$$

(iv) Multiplying the results found above we obtain

$$\exp(t|0\rangle\langle 1|)\exp(t|1\rangle\langle 0|) = I_2 + t(|0\rangle\langle 1| + |1\rangle\langle 0|) + t^2|0\rangle\langle 0|\,.$$

(v) Since

$$(|0\rangle\langle 1| + |1\rangle\langle 0|)^2 = I_2$$

we obtain

$$\begin{aligned}\exp(t|0\rangle\langle 1| + t|1\rangle\langle 0|) &= \sum_{j=0}^{\infty} \frac{t^{2j}}{(2j)!} I_2 + \sum_{j=0}^{\infty} \frac{t^{2j+1}}{(2j+1)!}(|0\rangle\langle 1| + |1\rangle\langle 0|)\\ &= \cosh(t) I_2 + \sinh(t)(|0\rangle\langle 1| + |1\rangle\langle 0|)\,.\end{aligned}$$

(vi) Clearly when $t \neq 0$ we have

$$\exp(t(|0\rangle\langle 1| + |1\rangle\langle 0|)) \neq \exp(t|0\rangle\langle 1|)\exp(t|1\rangle\langle 0|)\,.$$

Problem 14. Consider the unitary matrix for the NOT gate

$$U_{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Show that we can find a unitary matrix V such that $V^2 = U_{NOT}$. Thus V would be the square root NOT gate. What are the eigenvalues of V?

Solution 14. We find the unitary matrix

$$V = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}.$$

Obviously $-V$ is also a square root. The eigenvalues of V are 1 and i. The eigenvalues of $-V$ are -1 and $-i$. Note that the eigenvalues of U_{NOT} are 1 and -1.

Problem 15. Let $\sigma_1, \sigma_2, \sigma_3$ be the Pauli spin matrices. Let $\mathbf{n}$ be a unit vector in $\mathbb{R}^3$. We define the operator

$$\Sigma := \mathbf{n} \cdot \boldsymbol{\sigma} \equiv n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3 \,.$$

(i) Calculate Σ^2. From this result and the fact that Σ is hermitian show that Σ is unitary.
(ii) Find the eigenvalues of Σ.
(iii) Let

$$|\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

Calculate the state $\Sigma|\psi\rangle$ and the probability $|\langle\psi|\Sigma|\psi\rangle|^2$.

Solution 15. (i) Using $n_1^2 + n_2^2 + n_3^2 = 1$, $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I_2$ and

$$\sigma_1\sigma_2 + \sigma_2\sigma_1 = 0, \quad \sigma_1\sigma_3 + \sigma_3\sigma_1 = 0, \quad \sigma_2\sigma_3 + \sigma_3\sigma_2 = 0$$

we obtain

$$\begin{aligned}\Sigma^2 &= (n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3)^2 \\ &= (n_1^2 + n_2^2 + n_3^2)I_2 \\ &\quad + n_1n_2(\sigma_1\sigma_2 + \sigma_2\sigma_1) + n_1n_3(\sigma_3\sigma_1 + \sigma_1\sigma_3) + n_2n_3(\sigma_2\sigma_3 + \sigma_3\sigma_2) \\ &= I_2 \,.\end{aligned}$$

Since Σ is hermitian, i.e. $\Sigma = \Sigma^*$ and $\Sigma^2 = I_2$ we find that Σ is a unitary matrix with $\Sigma = \Sigma^{-1}$.
(ii) Since Σ is hermitian and unitary the eigenvalues λ_1, λ_2 can only be ± 1. Since $\mathrm{tr}\Sigma = 0 = \lambda_1 + \lambda_2$ we obtain that the eigenvalues are $+1$ and -1.
(iii) We find

$$\Sigma|\psi\rangle = n_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + n_2 \begin{pmatrix} 0 \\ i \end{pmatrix} + n_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

It follows that

$$|\langle\psi|\Sigma|\psi\rangle|^2 = n_3^2 \,.$$

Problem 16. Let $\mathbf{n}$ be a unit vector in $\mathbb{R}^3$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and

$$\mathbf{n} \cdot \boldsymbol{\sigma} := n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3 \,.$$

(i) Find the unitary matrix $\exp(i\theta\mathbf{n} \cdot \boldsymbol{\sigma})$, where $\theta \in \mathbb{R}$.
(ii) Find the state

$$\exp(i\theta\mathbf{n} \cdot \boldsymbol{\sigma}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

Solution 16. (i) Since

$$\sigma_j \sigma_k = \delta_{jk} I_2 + i \sum_{\ell=1}^{3} \epsilon_{jk\ell} \sigma_\ell$$

where $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$ and 0 otherwise, we obtain

$$\begin{aligned} \exp(i\theta \mathbf{n} \cdot \boldsymbol{\sigma}) &= I_2 \cos\theta + i(\mathbf{n} \cdot \boldsymbol{\sigma}) \sin\theta \\ &= \begin{pmatrix} \cos\theta + i n_3 \sin\theta & i(n_1 - i n_2)\sin\theta \\ i(n_1 + i n_2)\sin\theta & \cos\theta - i n_3 \sin\theta \end{pmatrix}. \end{aligned}$$

Note that we could also use $(\mathbf{n} \cdot \boldsymbol{\sigma})^2 = I_2$ to find the result.
(ii) Using (i) we find

$$\exp(i\theta \mathbf{n} \cdot \boldsymbol{\sigma}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta + i n_3 \sin\theta \\ i(n_1 + i n_2)\sin\theta \end{pmatrix}.$$

Problem 17. Consider the Pauli spin matrix σ_z and the state in $\mathbb{C}^2$

$$|\psi\rangle = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

Calculate the *variance*

$$V_{\sigma_z}(\psi) := \langle\psi|\sigma_z^2|\psi\rangle - (\langle\psi|\sigma_z|\psi\rangle)^2$$

and discuss the dependence on θ.

Solution 17. Using that $\sigma_z^2 = I_2$ we have

$$\begin{aligned} V_{\sigma_z}(\psi) &= \langle\psi|I_2|\psi\rangle - (\langle\psi|\sigma_z|\psi\rangle)^2 \\ &= 1 - \begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \\ &= 1 - 2\cos(\theta)\sin(\theta). \end{aligned}$$

For $\theta = 0$ we have $V_{\sigma_z}(\psi) = 1$. The minimum value is 0, for example at $\theta = \pi/4$. The maximum value is 2, for example at $\theta = 3\pi/4$.

Problem 18. Consider the Hamilton operator

$$\hat{H} = \hbar\omega \begin{pmatrix} 0 & \alpha \\ \alpha & 1 \end{pmatrix}$$

where $\alpha \geq 0$. Find α where the energy gap between the two energy levels is the smallest.

Solution 18. From the eigenvalue equation we find

$$E^2 - \hbar\omega E = \hbar^2\omega^2\alpha^2 .$$

Consequently

$$E_0 = \hbar\omega(1 - \sqrt{1+\alpha^2}), \qquad E_1 = \hbar\omega(1 + \sqrt{1+\alpha^2}) .$$

Thus

$$E_1 - E_0 = 2\hbar\omega\sqrt{1+\alpha^2} .$$

Therefore the shortest energy gap is for $\alpha = 0$.

Problem 19. Consider the Hamilton operator

$$\hat{H} = \hbar\omega\sigma_z + \Delta\sigma_x$$

where $\Delta \geq 0$.
(i) Find the eigenvalues and the normalized eigenvectors of $\hat{H}$.
(ii) Use the *Cayley-Hamilton theorem* to calculate $\exp(-i\hat{H}t/\hbar)$.

Solution 19. (i) The Hamilton operator is given by

$$\hat{H} = \begin{pmatrix} \hbar\omega & \Delta \\ \Delta & -\hbar\omega \end{pmatrix} .$$

From $\det(\hat{H} - EI_2) = 0$ we find the two eigenvalues

$$E_\pm = \pm\sqrt{\hbar^2\omega^2 + \Delta^2} .$$

We set $E := \sqrt{\hbar^2\omega^2 + \Delta^2}$. Then from the eigenvalue equation

$$\begin{pmatrix} \hbar\omega & \Delta \\ \Delta & -\hbar\omega \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = E_+ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

for the eigenvalue $E_+ = E$ we find $\Delta u_2 = (E - \hbar\omega)u_1$. Thus the eigenvector is given by

$$\begin{pmatrix} \Delta \\ E - \hbar\omega \end{pmatrix} .$$

After normalization we have

$$\frac{1}{\sqrt{\Delta^2 + (E - \hbar\omega)^2}} \begin{pmatrix} \Delta \\ E - \hbar\omega \end{pmatrix} .$$

Analogously we find for the eigenvalue $E_- = -E$ the normalized eigenvector

$$\frac{1}{\sqrt{\Delta^2 + (E + \hbar\omega)^2}} \begin{pmatrix} \Delta \\ -E - \hbar\omega \end{pmatrix} .$$

(ii) Since $E_+ \neq E_-$ and $E_+ = E$, $E_- = -E$ we have to solve the system of equations

$$e^{-iEt/\hbar} = c_0 + c_1 E, \qquad e^{iEt/\hbar} = c_0 - c_1 E$$

for c_0 and c_1. Then

$$e^{-i\hat{H}t/\hbar} = c_0 I_2 + c_1 \hat{H} = \begin{pmatrix} c_0 + c_1\hbar\omega & c_1\Delta \\ c_1\Delta & c_0 - c_1\hbar\omega \end{pmatrix} .$$

The solution of the system of equations is given by

$$\begin{aligned} c_0 &= \cos(Et/\hbar) \\ c_1 &= \frac{e^{-iEt/\hbar} - e^{iEt/\hbar}}{2E} = \frac{-i\sin(Et/\hbar)}{E} \end{aligned}$$

Thus

$$e^{-i\hat{H}t/\hbar} = \begin{pmatrix} \cos(Et/\hbar) - i\sin(Et/\hbar)\hbar\omega/E & -i\sin(Et/\hbar)\Delta/E \\ -i\sin(Et/\hbar)\Delta/E & \cos(Et/\hbar) + i\sin(Et/\hbar)\hbar\omega/E \end{pmatrix} .$$

Obviously, $\exp(-i\hat{H}t/\hbar)$ is a unitary matrix.

Problem 20. Consider the Pauli spin matrices σ_x, σ_y and σ_z. Can one find an $\alpha \in \mathbb{R}$ such that

$$\exp(i\alpha\sigma_z)\sigma_x \exp(-i\alpha\sigma_z) = \sigma_y\, ?$$

Solution 20. We have

$$\exp(i\alpha\sigma_z)\sigma_x \exp(-i\alpha\sigma_z) = \begin{pmatrix} 0 & e^{2i\alpha} \\ e^{-2i\alpha} & 0 \end{pmatrix} .$$

Thus we have to solve

$$\exp(2i\alpha) = -i, \qquad \exp(-2i\alpha) = i\,.$$

For $\alpha \in [0, 2\pi)$ we obtain $\alpha = 3\pi/4$.

Problem 21. Consider the *unary gates* (2×2 unitary matrices)

$$N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$V = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

and the normalized state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Calculate the state $NHVW|\psi\rangle$ and the expectation value $\langle\psi|NHVW|\psi\rangle$.

Solution 21. We find the unitary matrix

$$NHVW = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -e^{i3\pi/4} \\ 1 & e^{i3\pi/4} \end{pmatrix}.$$

Thus we obtain the state

$$NHVW|\psi\rangle = \frac{1}{2} \begin{pmatrix} 1 - e^{i3\pi/4} \\ 1 + e^{i3\pi/4} \end{pmatrix}.$$

It follows that

$$\langle\psi|NHVW|\psi\rangle = \frac{1}{\sqrt{2}}.$$

Problem 22. Let $\mathbf{n}$ and $\mathbf{m}$ be a unit vectors in $\mathbb{R}^3$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and

$$\mathbf{n} \cdot \boldsymbol{\sigma} := n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3 .$$

Calculate the commutator $[\mathbf{n} \cdot \boldsymbol{\sigma}, \mathbf{m} \cdot \boldsymbol{\sigma}]$.

Solution 22. We find

$$\begin{aligned} [\mathbf{n} \cdot \boldsymbol{\sigma}, \mathbf{m} \cdot \boldsymbol{\sigma}] &= 2i((n_2m_3 - m_2n_3)\sigma_1 + (n_3m_1 - m_3n_1)\sigma_2 \\ &\quad +(n_1m_2 - m_1n_2)\sigma_3) \\ &= 2i(\mathbf{n} \times \mathbf{m}) \cdot \boldsymbol{\sigma} \end{aligned}$$

where × denotes the vector product. Thus the vector $\mathbf{n}\times\mathbf{m}$ is perpendicular to the plane spanned by the vectors $\mathbf{n}$ and $\mathbf{m}$.

Problem 23. We define a linear bijection, h, between $\mathbb{R}^4$ and $\mathbf{H}(2)$, the set of complex 2×2 hermitian matrices, by

$$(t, x, y, z) \to \begin{pmatrix} t + x & y - iz \\ y + iz & t - x \end{pmatrix}.$$

We denote the matrix on the right hand side by H.

(i) Show that the matrix can be written as a linear combination of the Pauli spin matrices and the identity matrix I_2.
(ii) Find the inverse map.
(iii) Calculate the determinant of the 2×2 hermitian matrix H. Discuss.

Solution 23. (i) We have

$$H = tI_2 + x\sigma_z + y\sigma_x + z\sigma_y \,.$$

(ii) Consider ($a, b \in \mathbb{R}$)

$$\begin{pmatrix} a & c \\ c^* & b \end{pmatrix} = \begin{pmatrix} t+x & y-iz \\ y+iz & t-x \end{pmatrix} .$$

Comparing the entries we obtain

$$t = \frac{a+b}{2}, \quad x = \frac{a-b}{2}, \quad y = \frac{c+c^*}{2}, \quad z = \frac{c^*-c}{2i} .$$

(iii) We obtain

$$\det H = t^2 - x^2 - y^2 - z^2 \,.$$

This is the *Lorentz metric*. Let U be a unitary 2×2 matrix. Then $\det(UHU^*) = \det(H)$.

Problem 24. Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be two normalized states in a Hilbert space $\mathcal{H}$. A distance d with $0 \le d \le \pi/2$ can be defined as

$$\cos^2 d = |\langle\psi_1|\psi_2\rangle|^2 \,.$$

Let $\mathcal{H} = \mathbb{C}^2$ and consider the states

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

Find d.

Solution 24. Since $\langle\psi_1|\psi_2\rangle = 0$ we have $\cos^2 d = 0$ and therefore $d = \pi/2$.

Problem 25. Let ρ_1 and ρ_2 be density matrices in the same Hilbert space. The *Bures distance* between the two density matrices is defined as

$$D_B(\rho_1, \rho_2) := \sqrt{2(1 - \mathrm{tr}((\rho_1^{1/2}\rho_2\rho_1^{1/2})^{1/2}))} \,.$$

Consider the density matrices

$$\rho_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \rho_2 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

acting in the Hilbert space $\mathbb{C}^2$. Find the Bures distance.

Solution 25. Since

$$\rho_1^{1/2} = \rho_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

we obtain

$$\rho_1^{1/2}\rho_2\rho_1^{1/2} = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix} .$$

Thus

$$D_B(\rho_1, \rho_2) = \sqrt{2(1 - 1/\sqrt{2})} .$$

Problem 26. Consider the Hilbert space $\mathbb{C}^2$. Show that

$$\Pi_S = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \qquad \Pi_A = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

are *projection matrices* and decompose the Hilbert space into sub-Hilbert spaces.

Solution 26. We have

$$\Pi_S = \Pi_S^*, \qquad \Pi_S^2 = \Pi_S,$$

$$\Pi_A = \Pi_A^*, \qquad \Pi_A^2 = \Pi_A$$

and

$$\Pi_S + \Pi_A = I_2, \qquad \Pi_S \Pi_A = 0_2 .$$

Consider the state

$$|\psi\rangle = \begin{pmatrix} e^{i\phi}\sin\theta \\ \cos\theta \end{pmatrix} .$$

Then

$$\Pi_S|\psi\rangle = \begin{pmatrix} e^{i\phi}\sin\theta - i\cos\theta \\ ie^{i\phi}\sin\theta + \cos\theta \end{pmatrix}, \quad \Pi_A|\psi\rangle = \begin{pmatrix} e^{i\phi}\sin\theta + i\cos\theta \\ -ie^{i\phi}\sin\theta + \cos\theta \end{pmatrix}$$

with

$$\langle\psi|\Pi_A\Pi_S|\psi\rangle = 0 .$$