2. Single Particle Motion

Plasmas are collections of very large numbers of electrically charged particles. It is the charged state of the particles that distinguishes plasmas from other particle collections such as normal gases or fluids. The electric charge couples the particles to the electromagnetic field, which affects their motions.

In a situation where the charged particles do not directly interact with each other and where they do not affect the external magnetic field significantly, the motion of each individual particle can be treated independently. This *single particle approach* is only valid in very rarified plasmas where collective effects are negligible. Furthermore, the external magnetic field must be rather strong, much greater than the magnetic field produced by the electric current due to the charged particle motion.

We will see later that the single particle approach can be used only in some plasmas of geophysical interest. However, in order to understand the collective behavior of the plasma, i.e., the motion of the charge carriers under the influence of electric and magnetic fields generated by the motion itself, it is very instructive to study first the motion of charged particles in prescribed electric and magnetic fields.

2.1. Field Equations

Before describing the particle motion in external electric and magnetic fields, we introduce the electromagnetic field equations. There is a twofold coupling between electric charges and electromagnetic fields. Charged particles at rest are the sources of the electrostatic field, **E**, which is the origin of the *Coulomb force*

$$\mathbf{F}_C = q\,\mathbf{E} \tag{2.1}$$

they feel in the combined electrostatic field of all the other particles. On the other hand, charged particles moving with a velocity, **v**, are current elements generating a magnetic field, **B**, which is the origin of the *Lorentz force*

$$\mathbf{F}_L = q\,(\mathbf{v} \times \mathbf{B}) \tag{2.2}$$

The motion of charged particles is strongly influenced by the presence of the electromagnetic field, while at the same time it is also the source of the fields. The relation between fields and particles is described by *Maxwell's equations* (see App. A.5)

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \tag{2.3}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.4}$$

where $\mathbf{j}$ is the electric current density in the plasma, and ϵ_0 and μ_0 are the vacuum permittivity and susceptibility, respectively.

These equations show that the electric and magnetic fields are not independent, but are coupled by their spatial and temporal variations. Moreover, the electric current density turns out to be the source of the magnetic field and of fast fluctuations of the electric field. Since $\epsilon_0 \mu_0 = c^{-2}$ is equal to the inverse square of the light velocity, the latter will be negligible in a plasma as long as we do not consider propagation of electromagnetic waves. Hence, the second term on the right-hand side of Eq. (2.3) is small as long as no fast oscillations appear in the electric field.

In order to close the system, the first two equations have to be supplemented by two more equations, namely the conditions

$$\nabla \cdot \mathbf{B} = 0 \tag{2.5}$$

$$\nabla \cdot \mathbf{E} = \rho / \epsilon_0 \tag{2.6}$$

The first of these expressions indicates that there are no sources of the magnetic field and thus the magnetic field lines are always closed. The second condition shows that the source of the electric field is the electric space charge density, $\rho = e\,(n_i - n_e)$, which is the difference between the charge densities of the ion and the electrons. Similarly, the electric current is defined as the difference between the electron and ion fluxes as $\mathbf{j} = e\,(n_i \mathbf{v}_i - n_e \mathbf{v}_e)$ where for simplicity we have assumed that the ions are singly charged.

2.2. Gyration

The equation of motion for a particle of charge q under the action of the Coulomb and Lorentz forces given in Eqs. (2.1) and (2.2) can be written as

$$m \frac{d\mathbf{v}}{dt} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{2.7}$$

where m represents the particle mass and $\mathbf{v}$ the particle velocity. Under the absence of an electric field this equation reduces to

$$m \frac{d\mathbf{v}}{dt} = q\,(\mathbf{v} \times \mathbf{B}) \tag{2.8}$$

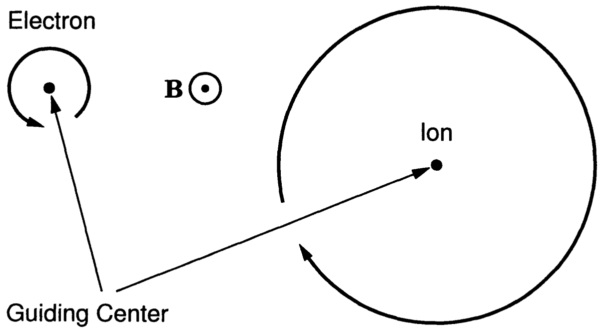

Fig. 2.1. Gyration of charged particles around a guiding center.

Taking the dot product of Eq. (2.8) with $\mathbf{v}$ and noting that $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) = 0$ (useful vector relations are found in App. A.4), we obtain

$$m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = \frac{d}{dt}\left(\frac{mv^2}{2}\right) = 0 \tag{2.9}$$

which shows that the particle kinetic energy as well as the magnitude of its velocity are constants. A static magnetic field, whatever its spatial variance is, does not change the particle kinetic energy.

In a uniform magnetostatic field along the z axis, $\mathbf{B} = B\,\hat{\mathbf{e}}_z$, we get the components

$$\begin{aligned} m\dot{v}_x &= qBv_y \\ m\dot{v}_y &= -qBv_x \\ m\dot{v}_z &= 0 \end{aligned} \tag{2.10}$$

The velocity component parallel to the magnetic field, $v_\parallel = v_z$, is constant. Taking the second derivative, we get

$$\begin{aligned} \ddot{v}_x &= -\omega_g^2 v_x \\ \ddot{v}_y &= -\omega_g^2 v_y \end{aligned} \tag{2.11}$$

where ω_g is the *gyrofrequency* or *cyclotron frequency*, which has opposite signs for positive and negative charges (note that ω_g is often defined as a positive number, independent of the sign of the charge; we will also use it this way in later chapters, starting on p. 227)

$$\boxed{\omega_g = \frac{qB}{m}} \tag{2.12}$$

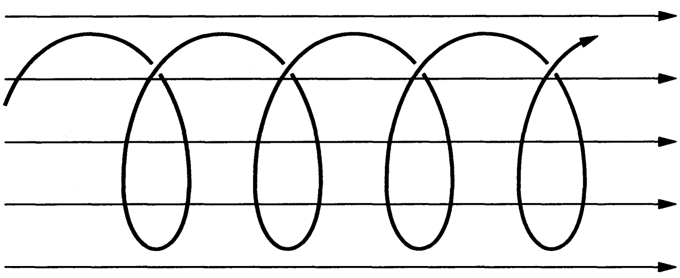

Fig. 2.2. Helicoidal ion orbit in a uniform magnetic field.

Equation (2.11) is a harmonic oscillator equation with solutions of the form

$$\begin{aligned} x - x_0 &= r_g \sin \omega_g t \\ y - y_0 &= r_g \cos \omega_g t \end{aligned} \tag{2.13}$$

Since ω_g carries the sign of the charge, the x component has opposite signs for electrons and ions. r_g is the *gyroradius* defined as

$$\boxed{r_g = \frac{v_\perp}{|\omega_g|} = \frac{m v_\perp}{|q| B}} \tag{2.14}$$

where $v_\perp = (v_x^2 + v_y^2)^{1/2}$ is the constant speed in the plane perpendicular to $\mathbf{B}$.

The components of Eq. (2.13) describe a circular orbit of the particle around the magnetic field, with the sense of rotation depending of the sign of the charge (see Fig. 2.1). The center of the orbit (x_0, y_0) is called the *guiding center*. The circular orbit of the charged particle represents a circular current and the direction of the gyration is such that the magnetic field generated by the circular current is opposite to the externally imposed field. This behavior is called *diamagnetic effect*.

In Fig. 2.1 we have neglected a possible constant velocity of the particle parallel to the magnetic field, $v_\parallel$. Whenever $v_\parallel \neq 0$, the actual trajectory of the particle is three-dimensional and looks like a helix. Such a helicoidal trajectory is shown in Fig. 2.2. The *pitch angle*, α, of the helix is defined as

$$\boxed{\alpha = \tan^{-1}\left(\frac{v_\perp}{v_\parallel}\right)} \tag{2.15}$$

and depends on the ratio between the perpendicular and parallel velocity components.

2.3. Electric Drifts

Taking the electric field into consideration will result in a drift of the particle superimposed onto its gyratory motion. The exact nature of this *electric drift* depends on whether the field is electrostatic or time-varying and whether it is spatially uniform or not.

E × B Drift

Let us now assume that an electrostatic field, **E**, is present. Looking for solutions of Eq. (2.7), we can again treat the perpendicular components and the component parallel to **B** separately. The parallel component

$$m\dot{v}_{\parallel} = qE_{\parallel} \tag{2.16}$$

describes a straightforward acceleration along the magnetic field. However, in geophysical plasmas most parallel electric fields cannot be sustained, since they are immediately canceled out by electrons, which are under most circumstances extremely mobile along the magnetic field lines.

Assuming that the perpendicular electric field component is parallel to the x axis, $\mathbf{E}_{\perp} = E_x\,\hat{\mathbf{e}}_x$, the perpendicular components of Eq. (2.7) are

$$\begin{aligned} \dot{v}_x &= \omega_g v_y + \frac{q}{m}E_x \\ \dot{v}_y &= -\omega_g v_x \end{aligned} \tag{2.17}$$

Taking the second derivative, we obtain

$$\begin{aligned} \ddot{v}_x &= -\omega_g^2 v_x \\ \ddot{v}_y &= -\omega_g^2\left(v_y + \frac{E_x}{B}\right) \end{aligned} \tag{2.18}$$

If we substitute $v_y' = v_y + E_x/B$, we get back to Eq. (2.11), where the particle is gyrating about the guiding center. Thus Eq. (2.18) describes a gyration, but with a superimposed drift of the guiding center in the $-y$ direction. This drift of the guiding center is usually called *E×B drift* and has the general form

$$\boxed{\mathbf{v}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^2}} \tag{2.19}$$

The E×B drift is independent of the sign of the charge and thus electrons and ions move into the same direction.

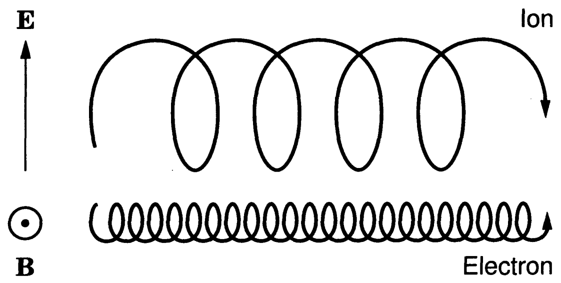

Fig. 2.3. Particle drifts in crossed electric and magnetic fields.

Figure 2.3 shows the acceleration and deceleration effect of a perpendicular electric field. An ion is accelerated into the direction of the electric field, thereby increasing its gyroradius. But it is decelerated during the second half of its gyratory orbit, now with decreasing gyroradius. The different gyroradii shift the position of the guiding center in the $\mathbf{E} \times \mathbf{B}$ direction. The electrons are accelerated when moving antiparallel to the electric field and decelerated when moving parallel. But since their sense of gyration is opposite, too, their guiding centers drift into the same direction.

It is instructive to note that the E×B drift has a fundamental physical root in the *Lorentz transformation* of the electric field into the moving system of the particle. In this system the electric field is given by

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} \tag{2.20}$$

For a free particle this field must vanish, $\mathbf{E}' = 0$, which yields for the electric field

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} \tag{2.21}$$

Solving for the velocity immediately yields the expression for the electric drift in Eq. (2.19). Because the Lorentz transformation does not depend on the charge of the particles, the E×B drift is also independent of the sign of the charge.

Polarization Drift

We could have derived Eq. (2.19) directly from Eq. (2.7). Taking the cross-product of both sides of Eq. (2.7) with $\mathbf{B}/B^2$, we obtain

$$\mathbf{v} - \frac{\mathbf{B}\,(\mathbf{v} \cdot \mathbf{B})}{B^2} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} - \frac{m}{q}\frac{d\mathbf{v}}{dt} \times \frac{\mathbf{B}}{B^2} \tag{2.22}$$

We can recognize the left-hand side as a perpendicular velocity vector and the first term on the right-hand side as the E×B drift. Averaging over the gyroperiod and thus neglecting temporal changes of the order of the gyroperiod or faster allows us to take the perpendicular velocity as the perpendicular drift velocity, $\mathbf{v}_d$. Remembering that the magnetic field is assumed time independent, we rewrite

$$\mathbf{v}_d = \mathbf{v}_E - \frac{m}{qB^2}\frac{d}{dt}(\mathbf{v} \times \mathbf{B}) \tag{2.23}$$

which yields with Eqs. (2.12) and (2.21)

$$\mathbf{v}_d = \mathbf{v}_E + \frac{1}{\omega_g B}\frac{d\mathbf{E}_\perp}{dt} \tag{2.24}$$

Equation (2.24) describes the drift of a charged particle in crossed homogeneous magnetic and electric fields, where the electric field is allowed to vary slowly. The last term in this equation is called *polarization drift*.

$$\boxed{\mathbf{v}_P = \frac{1}{\omega_g B}\frac{d\mathbf{E}_\perp}{dt}} \tag{2.25}$$

There is an important qualitative difference between the polarization drift and the E×B drift. The E×B drift does neither depend on the charge nor on the mass of the particle, since it can be viewed as a result of the Lorentz transformation. Thus electrons, protons, and heavier ions all move into the same direction perpendicular to **B** and **E** with the same velocity. The polarization drift, on the other hand, increases proportional to the mass of the particle. It is directed along the electric field, but oppositely for electrons and ions. Accordingly, it creates a current

$$\mathbf{j}_P = n_e e\,(\mathbf{v}_{Pi} - \mathbf{v}_{Pe}) = \frac{n_e(m_i + m_e)}{B^2}\frac{d\mathbf{E}_\perp}{dt} \tag{2.26}$$

which carries electrons and ions into opposite directions and polarizes the plasma. Since $m_i \gg m_e$, the *polarization current* is mainly carried by the ions.

Electric Drift Corrections

Equation (2.24) can also describe the drift due to inhomogeneities of the electric field if the total time derivative is taken as the sum of the temporal and the convective derivative $d/dt = \partial/\partial t + \mathbf{v}\cdot\nabla$, where the velocity vector can, to a good approximation, be replaced by the E×B velocity. The convective term becomes proportional to E^2. It is a nonlinear contribution and is usually much smaller than the time derivative and therefore neglected in most cases.

The convective derivative takes into account spatial variations of the electric field in the direction of the E×B drift under the assumption that the electric field changes only gradually. When this is not the case and the electric field changes considerably over one gyroradius, there is a further correction on the electric field drift, known as *finite Larmor radius effect*. This correction is a second order effect in r_g and leads to the following more complete expression for the electric field drift

$$\mathbf{v}_E = \left(1 + \tfrac{1}{4} r_g^2 \nabla^2\right) \frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{2.27}$$

The second spatial derivative takes account of the spatial variation of the electric field, averaged over the gyration orbit. Finite Larmor radius effects are normally neglected in macroscopic applications of particle motion but may become important in the vicinity of plasma boundaries, plasma transitions and small scale structures in a plasma.

2.4. Magnetic Drifts

When analyzing Eq. (2.8), we have assumed that the magnetic field is homogeneous. This is often not the case. A typical magnetic field has gradients and often field lines are curved. This inhomogeneity of the magnetic field leads to a *magnetic drift* of charged particles. Time variations of the magnetic field itself cannot impart energy to a particle, since the Lorentz force is always perpendicular to the velocity of the particle. However, since $\partial \mathbf{B}/\partial t = -\nabla \times \mathbf{E}$, the associated inhomogeneous electric field may accelerate the particles in the way described in the previous section.

Gradient Drift

Let us now assume that the magnetic field is weakly inhomogeneous, for example increasing in the upward direction. As visualized in Fig. 2.4, the gyroradius of a particle decreases in the upward direction and thus the gyroradius of a particle will be larger at the bottom of the orbit than at the top half. As a result, ions and electrons drift into opposite directions, perpendicular to both $\mathbf{B}$ and ∇B.

Since we assume that the typical scale length of a magnetic field gradient is much larger than the particle gyroradius, we can Taylor expand the magnetic field vector about the guiding center of the particle

$$\mathbf{B} = \mathbf{B}_0 + (\mathbf{r} \cdot \nabla)\mathbf{B}_0 \tag{2.28}$$

where $\mathbf{B}_0$ is measured at the guiding center and $\mathbf{r}$ is the distance from the guiding center. Inserting this relation into Eq. (2.8) we obtain

$$m\frac{d\mathbf{v}}{dt} = q\,(\mathbf{v} \times \mathbf{B}_0) + q\,[\mathbf{v} \times (\mathbf{r} \cdot \nabla)\mathbf{B}_0] \tag{2.29}$$

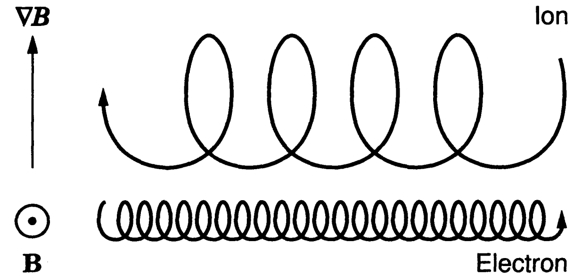

Fig. 2.4. Particle drifts due to a magnetic field gradient.

Expanding the velocity term into a gyration and a drift motion, $\mathbf{v} = \mathbf{v}_g + \mathbf{v}_\nabla$, and noting $v_\nabla \ll v_g$ yields

$$m\frac{d\mathbf{v}_\nabla}{dt} = q\,(\mathbf{v}_\nabla \times \mathbf{B}_0) + q\,(\mathbf{v}_g \times [\mathbf{r}\cdot\nabla)\mathbf{B}_0] \tag{2.30}$$

where we have omitted the terms describing gyration in a homogeneous field and neglected $\mathbf{v}_\nabla \times (\mathbf{r}\cdot\nabla)\mathbf{B}_0$ as a small quantity.

Since we are interested in time scales much larger than the gyroperiod, we can average over one gyration. Upon this the left side vanishes since any acceleration a particle experiences when moving into the weak field region is balanced by a deceleration when moving into the strong field region in the other half of its gyratory orbit. Since we know that $\mathbf{v}_\nabla$ lies in the plane perpendicular to the magnetic field, we can follow the same line as on p. 16 and take the cross-product with $\mathbf{B}_0/B_0^2$. Then we find

$$\mathbf{v}_\nabla = \frac{1}{B_0^2}\left\langle (\mathbf{v}_g \times (\mathbf{r}\cdot\nabla)\mathbf{B}_0) \times \mathbf{B}_0 \right\rangle \tag{2.31}$$

where the angle brackets denote averaging over one gyroperiod. Assuming $\mathbf{B}$ to vary only in the x direction, $\mathbf{B}_0 = B_0(x)\,\hat{\mathbf{e}}_z$, we obtain

$$\mathbf{v}_\nabla = -\frac{1}{B_0}\left\langle \mathbf{v}_g x \frac{dB_0}{dx} \right\rangle \tag{2.32}$$

Replacing $\mathbf{v}_g$ and x by the expressions given in Eq. (2.13), we get

$$\begin{aligned} v_{\nabla x} &= -\frac{v_\perp r_g}{B_0}\left\langle \sin\omega_g t \cos\omega_g t \frac{dB_0}{dx} \right\rangle \\ v_{\nabla y} &= \frac{v_\perp r_g}{B_0}\left\langle \sin^2\omega_g t \frac{dB_0}{dx} \right\rangle \end{aligned} \tag{2.33}$$

Taking the gyroperiod average, $v_{\nabla x}$ will vanish since it contains the product of sine and cosine terms. Averaging over the $\sin^2$ term yields a factor 1/2. Thus the drift will have only a y component

$$\mathbf{v}_\nabla = \pm \frac{v_\perp r_g}{2B_0} \frac{\partial B_0}{\partial x} \hat{\mathbf{e}}_y \tag{2.34}$$

where the direction of the motion depends on the sign of the charge. Since the direction of the magnetic field gradient was chosen arbitrarily, this can be generalized

$$\boxed{\mathbf{v}_\nabla = \frac{m v_\perp^2}{2qB^3} (\mathbf{B} \times \nabla B)} \tag{2.35}$$

showing that a magnetic field gradient leads to a *gradient drift* perpendicular to both the magnetic field and its gradient, as sketched in Fig. 2.4.

Equation (2.35) shows that ions and electrons drift into opposite directions and that, furthermore, the gradient drift velocity is proportional to the perpendicular energy of the particle, $W_\perp = \frac{1}{2} m v_\perp^2$. More energetic particles drift faster, since they have a larger gyroradius and experience more of the inhomogeneity of the field.

As in the case of the polarization drift, the opposite drift directions of electrons and ions lead to a transverse current. This *gradient drift current* has the form

$$\mathbf{j}_\nabla = n_e e \left(\mathbf{v}_{\nabla i} - \mathbf{v}_{\nabla e} \right) = \frac{n_e (\mu_i + \mu_e)}{B^2} \mathbf{B} \times \nabla B \tag{2.36}$$

when using the *magnetic moment*

$$\boxed{\mu = \frac{m v_\perp^2}{2B} = \frac{W_\perp}{B}} \tag{2.37}$$

to describe the ratio between perpendicular particle energy and magnetic field.

General Force Drift

By using Eq. (2.1) to replace $\mathbf{E}$ in Eq. (2.19) by $\mathbf{F}/q$, we get a more general form of guiding center drift, which is valid not only for the Coulomb force, but for any force acting on a charged particle in a magnetic field

$$\boxed{\mathbf{v}_F = \frac{1}{\omega_g} \left(\frac{\mathbf{F}}{m} \times \frac{\mathbf{B}}{B} \right)} \tag{2.38}$$

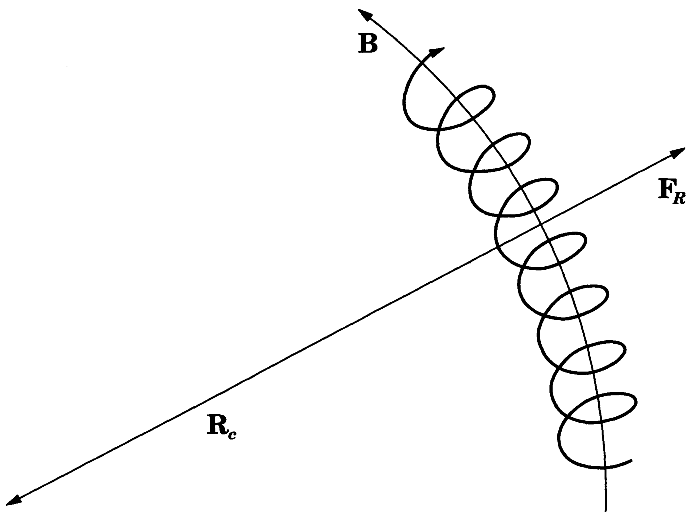

Fig. 2.5. Centrifugal force felt by a particle moving along a curved field line.

All particle drifts can be described this way by using the appropriate force terms, whenever the drift velocity of the particle is much smaller than its gyrovelocity. For the gradient, the polarization, and the gravitational drift, these forces can be written as

$$\mathbf{F}_\nabla = -\mu \nabla B \tag{2.39}$$

$$\mathbf{F}_P = -m \frac{d\mathbf{E}}{dt} \tag{2.40}$$

$$\mathbf{F}_G = -m\,\mathbf{g} \tag{2.41}$$

where $\mathbf{F}_\nabla$ denotes the gradient and $\mathbf{F}_P$ the polarization force. $\mathbf{F}_G$ is the gravitational force, which is typically much weaker than the other forces. Except for processes near the solar surface, it can usually be neglected.

Equation (2.38) shows that all drifts generated by a force other than the Coulomb force depend on the sign of the charge, since ω_g carries this sign. Hence, for these drifts ions and electrons run into opposite directions, creating a transverse current. Moreover, these drifts depend on the mass of the charge ca.riers and thus the drift velocities are typically quite different for ions and electrons.

Curvature Drift

The gradient drift is only one component of the particle drift in an inhomogeneous magnetic field. When the field lines are curved, a *curvature drift* appears. Due to their parallel velocity, $v_\parallel$, the particles experience a centrifugal force

$$\mathbf{F}_R = m v_\parallel^2 \frac{\mathbf{R}_c}{R_c^2} \tag{2.42}$$

where $\mathbf{R}_c$ is the local radius of curvature. This situation is depicted in Fig. 2.5. Inserting Eq. (2.42) into (2.38) yields directly the expression for the curvature drift

$$\boxed{\mathbf{v}_R = \frac{m v_\parallel^2}{q} \frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2 B^2}} \tag{2.43}$$

The curvature drift is proportional to the parallel particle energy, $W_\parallel = \frac{1}{2} m v_\parallel^2$, and perpendicular to the magnetic field and its curvature. It creates a transverse current since ion and electron drifts have opposite signs. The *curvature drift current* has the form

$$\mathbf{j}_R = n_e e \left(\mathbf{v}_{Ri} - \mathbf{v}_{Re}\right) = \frac{2 n_e (W_{i\parallel} + W_{e\parallel})}{R_c^2 B^2} \left(\mathbf{R}_c \times \mathbf{B}\right) \tag{2.44}$$

As the associated drift, the curvature current flows perpendicular to both the curvature of the magnetic field and the magnetic field itself.

In a cylindrically symmetric field, it turns out that $-\nabla B = (B/R_c^2)\,\mathbf{R}_c$. Thus we may add the gradient to the curvature drift to obtain the total magnetic drift

$$\boxed{\mathbf{v}_B = \mathbf{v}_R + \mathbf{v}_\nabla = (v_\parallel^2 + \tfrac{1}{2} v_\perp^2) \frac{\mathbf{B} \times \nabla B}{\omega_g B^2}} \tag{2.45}$$

It is the transverse current associated with this full magnetic drift which creates the magnetospheric ring current mentioned in Sec. 1.3 and further detailed in Secs. 3.2–3.5.

2.5. Adiabatic Invariants

In the preceding section we encountered the magnetic moment $\mu = W_\perp / B$ of a particle. This quantity has been treated as a characteristic constant of the particle. Such quantities are called *adiabatic invariants*. Adiabatic invariants are not absolute constants like total energy or total momentum, but may change both in space and time. There essence is, however, that they change very slowly compared with some typical periodicities of the particle motion.

For particles in electromagnetic fields, adiabatic invariants are associated with each type of motion the particle can perform. The magnetic moment, μ, is associated with the gyration about the magnetic field, the longitudinal invariant, J, with the longitudinal motion along the magnetic field, and the third invariant, Φ, with the perpendicular drift.

Whenever these motions are periodic and changes in the system have angular frequencies much smaller than the oscillation frequency of the particle corresponding to one of the above motions, the action integral

$$J_i = \oint p_i dq_i \tag{2.46}$$

is a constant of the motion and describes an adiabatic invariant. The pair of variables (p_i, q_i) are the generalized momentum and coordinate of Hamiltonian mechanics, and the integration has do be done over one full cycle of q_i.

Magnetic Moment

To demonstrate that the magnetic moment of a particle does not change when the particle moves into stronger or weaker magnetic fields, we consider the energy conservation equation

$$W = W_\parallel + W_\perp \tag{2.47}$$

Since W is a constant in the absence of electric fields, its time derivative vanishes

$$\frac{dW_\parallel}{dt} + \frac{dW_\perp}{dt} = 0 \tag{2.48}$$

For the transverse energy, we can use Eq. (2.37) and obtain

$$\frac{dW_\perp}{dt} = \mu\frac{dB}{dt} + B\frac{d\mu}{dt} \tag{2.49}$$

Here $dB/dt = v_\parallel dB/ds$ is the variation of the magnetic field as seen by the particle along its guiding center trajectory. The magnetic field itself is assumed to be constant. The parallel particle energy can be derived from the parallel component of the gradient force in Eq. (2.39) which gives the parallel equation of motion

$$m\frac{dv_\parallel}{dt} = -\mu\nabla_\parallel B = -\mu\frac{dB}{ds} \tag{2.50}$$

Multiplying the left-hand side of Eq. (2.50) with $v_\parallel$ and the right-hand side with its equivalent, ds/dt, one finds for the time derivative of the parallel energy

$$\frac{dW_\parallel}{dt} = -\mu\frac{dB}{dt} \tag{2.51}$$

Adding Eqs. (2.49) and (2.51) and observing Eq. (2.48), we obtain

$$\frac{dW_\parallel}{dt} + \frac{dW_\perp}{dt} = B\frac{d\mu}{dt} = 0 \tag{2.52}$$

which yields immediately that the magnetic moment is an invariant of the particle motion. The magnetic moment is not affected by small changes in the cyclotron frequency or the gyroradius which occur when the magnetic field changes along the particle path.

Up to now we have neglected electric fields, which can accelerate particles. Thus we have neglected temporal changes of the magnetic field, since $\partial \mathbf{B}/\partial t = -\nabla \times \mathbf{E}$. However, when the magnetic field fluctuations are slow enough, the magnetic moment is still conserved. This can be demonstrated by considering the change in perpendicular particle energy caused by an electric field. This change is calculated by taking the dot product of Eq. (2.7) with $\mathbf{v}_\perp$

$$\frac{dW_\perp}{dt} = q\,(\mathbf{E}\cdot\mathbf{v}_\perp) \tag{2.53}$$

The gain in energy over one gyration is obtained by integrating over the gyroperiod

$$\Delta W_\perp = q \int_0^{2\pi/\omega_g} (\mathbf{E}\cdot\mathbf{v}_\perp)\,dt \tag{2.54}$$

If the field changes slowly, the particle orbit is closed and we can replace the time integral by a line integral over the unperturbed orbit. Using Stokes' theorem (see App. A.4) and Maxwell's equations, we obtain

$$\Delta W_\perp = q\oint_C \mathbf{E}\cdot d\mathbf{s} = q\int_A (\nabla\times\mathbf{E})\cdot d\mathbf{A} = -q\int_A \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{A} \tag{2.55}$$

where $d\mathbf{s} = \mathbf{t}\,ds$ is the product of a line element, ds, of the closed gyratory orbit, C, with the line element's tangent vector, $\mathbf{t}$, and $d\mathbf{A} = \mathbf{n}\,dA$ is the product of a surface element, dA of the plane, $\mathbf{A}$, enclosed by the orbit and the surface element's normal vector, $\mathbf{n}$. For changes in the field much slower than the gyroperiod, $\partial\mathbf{B}/\partial t$ can be replaced by $\omega_g \Delta B/2\pi$, with ΔB the average change during one gyroperiod

$$\Delta W_\perp = \tfrac{1}{2} q\omega_g r_g^2 \Delta B = \mu\Delta B \tag{2.56}$$

Here we have inserted the expression for ω_g and r_g from Eqs. (2.12) and (2.14) and used the definition of μ in Eq. (2.37). On the other hand, in Eq. (2.49) the change in perpendicular energy is given by

$$\Delta W_\perp = \mu\Delta B + B\Delta\mu \tag{2.57}$$

Comparing the last two equations, we find again that

$$\Delta\mu = 0 \tag{2.58}$$

demonstrating that in the approximation of slowly variable fields the magnetic moment is invariant even when the particles are accelerated in induction electric fields.

Similarly, slow temporal variations of the electric field do not violate the invariance of the magnetic moment, since they will produce only second order time variations of the magnetic field, $\partial^2 B/\partial t^2$, on the right-hand side of Eq. (2.55), which can be neglected. For all slow variations the magnetic moment is a constant of motion. Changes in the fields merely lead to the different types of particle drifts but conserve μ.

From the adiabatic invariance of the magnetic moment, it follows that also the magnetic flux Φ_μ through the surface encircled by the gyrating particle does not change. This flux is given by $\Phi_\mu = B\pi r_g^2$ (see App. A.5), or inserting the expression for r_g from Eq. (2.14) and using the definition of μ in Eq. (2.37)

$$\boxed{\Phi_\mu = \frac{2\pi m}{q^2}\mu = \text{const}} \tag{2.59}$$

Hence, as a particle moves into a region of stronger magnetic field, the gyroradius of the particle will get increasingly smaller, so that the magnetic flux encircled by the orbit remains constant.

Magnetic Mirror

Let us follow the guiding center of a particle moving along an inhomogeneous magnetic field by considering its magnetic moment

$$\mu = \frac{mv^2 \sin^2\alpha}{2B} \tag{2.60}$$

where we have replaced $v_\perp$ by $v \sin\alpha$, using the pitch angle defined in Eq. (2.15). Since the magnetic moment is invariant and the total energy is a constant of motion, only the pitch angle can change when the magnetic field increases or decreases along the guiding center trajectory. The above equation also shows that the pitch angles of a particle at different locations are directly related to the magnetic field strengths at those locations according to

$$\frac{\sin^2\alpha_2}{\sin^2\alpha_1} = \frac{B_2}{B_1} \tag{2.61}$$

Thus knowing the pitch angle of a particle at one location, we can calculate this quantity at all other locations.

In a converging magnetic field geometry, a particle moving into regions of stronger fields will have its pitch angle increase and, therefore, have its transverse energy $W_\perp$ increase at the expense of its parallel energy $W_\parallel$. If B_m is a point along the field line where the pitch angle reaches $\alpha = 90°$, the particle is reflected from this *mirror point*. Here, all of the particle energy is in $W_\perp$ and the particle cannot penetrate any further,

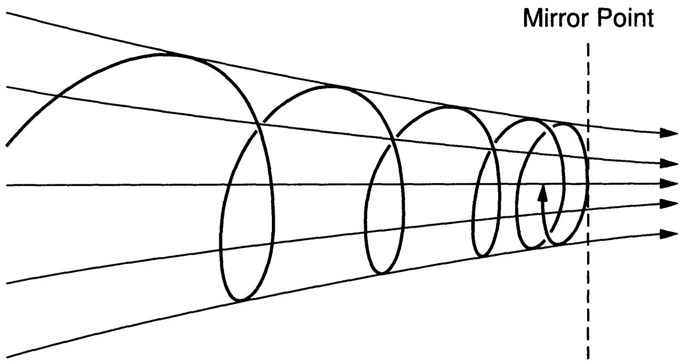

Fig. 2.6. Ion orbit and reflection in a converging magnetic field.

but is pushed back by the parallel component of the gradient force given in Eq. (2.39), the so-called *mirror force*, $-\mu\nabla_{\parallel}B$. This mirroring of a particle is visualized in Fig. 2.6.

In a symmetric magnetic field geometry with a minimum field in the middle and converging magnetic field lines on both sides, like in a dipole field, a particle may bounce back and forth between its two mirror points and become *trapped*. In this case we can describe the particle's pitch angle at a specific location by

$$\boxed{\sin\alpha = \left(\frac{B}{B_m}\right)^{1/2}} \tag{2.62}$$

i.e., the ratio of the field strengths at that location and at its mirror point.

Adiabatic Heating

The mirror effect described above is a consequence of the invariance of the magnetic moment when particles move along magnetic field lines. Conservation of the magnetic moment has also an important effect when particles drift across field lines. Consider a particle moving along its drift path from a region with a magnetic field strength B_1 into a region of increasing field strength B_2. Since the magnetic moment is conserved, we obtain

$$\boxed{\frac{W_{\perp 2}}{W_{\perp 1}} = \frac{B_2}{B_1}} \tag{2.63}$$

with $W_{\perp 2} > W_{\perp 1}$. However, in contrast to the mirror case, the particle will not bounce back and the perpendicular energy increase will remain.

The convective transport of particles into stronger magnetic fields therefore ends up with a gain of energy in the transverse direction. The work required for this process is of course taken from the drift motion which transports the particle into the stronger field. This type of particle energization is called *adiabatic heating* and is some kind of *betatron acceleration*. It increases the perpendicular energy of the particle without affecting its parallel energy. This results in an anisotropy of the particle energy. However, since magnetic fields may vary also in the parallel direction, we will discuss the generation of such an anisotropy in a more general way after introducing the second adiabatic invariant.

Longitudinal Invariant

If the field has a mirror symmetry where the field lines converge on both sides as in a dipole field, there is the possibility for a second adiabatic invariant, J. A particle moving in such a converging field will be reflected from the region of strong magnetic field and can oscillate in the field at a certain *bounce frequency*, ω_b. The *longitudinal invariant* is defined by

$$J = \oint m v_\parallel ds \tag{2.64}$$

where $v_\parallel$ is the parallel particle velocity, ds is an element of the guiding center path and the integral is taken over a full oscillation between the mirror points.

For electromagnetic variations with frequencies $\omega \ll \omega_b$, the longitudinal invariant is a constant, irrespective of weak changes in the path of the particle and its mirror points due to slow changes in the fields.

Energy Anisotropy

Invariance of the magnetic moment in magnetic fields which change in the perpendicular direction may lead to an increase in the perpendicular energy of the particle. When the magnetic field also varies in the direction parallel to the field lines, conservation of the longitudinal invariant implies that the parallel energy of the particle will also change during the combined drift and bounce motion of the particle along and across the field.

Let us define the total length of the field line between the two mirror points of the particle as ℓ, and the average parallel velocity along the field line as $\langle v_\parallel \rangle$. In terms of these quantities the longitudinal invariant can be expressed as

$$J = \oint m v_\parallel ds = 2m\ell \langle v_\parallel \rangle \tag{2.65}$$

During its drift motion from weaker into stronger fields, the particle necessarily moves from one field line of length ℓ_1 onto another field line of length ℓ_2. At the same time its average parallel velocity changes from $\langle v_\parallel \rangle_1$ to $\langle v_\parallel \rangle_2$. Conservation of the longitudinal invariant then implies that the averaged parallel energy, $\langle W_\parallel \rangle$, changes according to

$$\frac{\langle W_\parallel \rangle_2}{\langle W_\parallel \rangle_1} = \frac{\ell_1^2}{\ell_2^2} \tag{2.66}$$

If the length of the bounce path decreases, the parallel energy of the particle increases. This is the basic element of *Fermi acceleration.*

This result can be combined with the simultaneous increase in the perpendicular energy from Eq. (2.63) to determine the anisotropy in the energy attained by the particle. Let us define this anisotropy as

$$A_W = \frac{\langle W_\perp \rangle}{\langle W_\parallel \rangle} \tag{2.67}$$

Then the *energy anisotropy* of the particle changes according to

$$\boxed{\frac{A_{W2}}{A_{W1}} = \frac{B_2}{B_1}\frac{\ell_2^2}{\ell_1^2}} \tag{2.68}$$

which shows that the anisotropy increases if only the square of the field line length decreases less than the magnetic field increase.

Drift Invariant

The third invariant, Φ, is simply the conserved magnetic flux encircled by the periodic orbit of a particle trapped in an axisymmetric mirror magnetic field configuration when it performs closed *drift shell* orbits around the magnetic field axis. This *drift invariant* can be written as

$$\Phi = \oint v_d r d\psi \tag{2.69}$$

where v_d is the sum of all perpendicular drift velocities, ψ is the azimuthal angle, and the integration must be taken over a full circular drift path of the particle.

Whenever the typical frequency of the electromagnetic fields is much smaller than the drift frequency, $\omega \ll \omega_d$, Φ is invariant and essentially equal to the magnetic flux enclosed by the orbit. This can be written like in Eq. (2.59) as

$$\boxed{\Phi = \frac{2\pi m}{q^2} M = \text{const}} \tag{2.70}$$

where M is the magnetic moment of the axisymmetric field.

Violation of Invariance

So far we have considered variations and motions which conserve the magnetic moment, the longitudinal invariant, and the drift invariant. In nature, however, all the fields may vary in such a way that the adiabatic invariance of one or the other invariant is violated.

Time variations faster than the gyrofrequency of the particle with frequencies $\omega > \omega_g$ violate the first adiabatic invariant μ, the magnetic moment of the particle. These are high-frequency variations in either the magnetic or electric field. In this case, the concept of a gyratory orbit about a guiding center becomes obsolete and the full particle motion must be considered.

On the other hand, for frequencies $\omega_g > \omega > \omega_b$ the magnetic moment μ is conserved and the guiding center approximation is useful for the drift motion. However, under these conditions the longitudinal invariant is not conserved but violated, and the particle motion cannot be described anymore as a simple oscillation along the magnetic field between mirror points.

Finally, for frequencies $(\omega_g, \omega_b) > \omega > \omega_d$ the first two invariants will be conserved while the drift invariant becomes violated. The particles gyrate and bounce but diffuse across drift shells under the influence of the variation in the magnetic field. All these cases are realized in nature. Some of them we will encounter later.

Adiabatic invariants are not only violated when the fields vary in time but also when they abruptly change over a length scale $L < r_a$ shorter than the characteristic radius r_a of the periodic motion related to the adiabatic invariant. This can be proven, for instance, for the gyration of a particle across a magnetic field gradient which is shorter than the particle gyroradius r_g. In such a case, $r_g/L > 1$. Using Eq. (2.14) to write the perpendicular velocity of the particle as

$$v_\perp = \omega_g r_g \tag{2.71}$$

and dividing both sides of this expression by L, the gradient length of the spatial change, one finds

$$\omega = \frac{v_\perp}{L} = \omega_g \frac{r_g}{L} > \omega_g \tag{2.72}$$

Hence, for the gyrating particle the effective frequency of change in the field, $\omega > \omega_g$, is higher than its gyrofrequency, and the magnetic moment of a particle gyrating in a magnetic field which varies strongly over a length of the order $L < r_g$ is not an adiabatic invariant. Similar arguments can be applied also to the two remaining invariants.

Concluding Remarks

In concluding this chapter, a word of caution is mandatory. The guiding center theory assumes that the electromagnetic fields are prescribed. It can thus be used in geophysical plasmas where the external field is strong and will not be changed much by the motion of the particles themselves. It should not be used in weak field regions where the field will be substantially changed by the particle motion and where thus the concept of a guiding center looses its meaning.

Summary of Guiding Center Drifts

For quick reference, we summarize the expressions of the guiding center drifts and the associated transverse currents.

E×B Drift:	$\mathbf{v}_E = \dfrac{\mathbf{E}\times\mathbf{B}}{B^2}$	
Polarization Drift:	$\mathbf{v}_P = \dfrac{1}{\omega_g B}\dfrac{d\mathbf{E}_\perp}{dt}$	$\mathbf{j}_P = \dfrac{n_e(m_i+m_e)}{B^2}\dfrac{d\mathbf{E}_\perp}{dt}$
Gradient Drift:	$\mathbf{v}_\nabla = \dfrac{mv_\perp^2}{2qB^3}(\mathbf{B}\times\nabla B)$	$\mathbf{j}_\nabla = \dfrac{n_e(\mu_i+\mu_e)}{B^2}(\mathbf{B}\times\nabla B)$
Curvature Drift:	$\mathbf{v}_R = \dfrac{mv_\parallel^2}{qR_c^2B^2}(\mathbf{R}_c\times\mathbf{B})$	$\mathbf{j}_R = \dfrac{2n_e(W_{i\parallel}+W_{e\parallel})}{R_c^2B^2}(\mathbf{R}_c\times\mathbf{B})$

Further Reading

For those readers who want to know more about the guiding center approach and to see the actual proof of the second and third invariant, we recommend reading one of the two monographs listed below.

[1] H. Alfvén and C. G. Fälthammar, *Cosmical Electrodynamics, Fundamental Principles* (Clarendon Press, Oxford, 1963).

[2] T. G. Northrop, *The Adiabatic Motion of Charged Particles* (Interscience Publishers, New York, 1963).