CHAPTER 6

MOMENT DISTRIBUTION

In the analysis of the stresses in loaded beams and structures it is necessary to know the manner in which moments and forces vary throughout their length. A single bay beam resting upon two simple supports is *statically determinate*. It will be shown that the application of equilibrium principles alone is sufficient to determine their force and moment distributions. Beams with encastre fixings, two or more bays and many types of structure are *statically indeterminate*. Their analysis becomes more complex requiring both equilibrium and compatibility conditions to be satisfied. Two methods are outlined: (i) the theorem of three moments, which employs known relationships between moments, slopes and deflections and (ii) moment distribution in which moments at supports are balanced by trial. Method (ii) is more versatile and will be applied to both continuous beams and structures.

6.1 Single Span Beams

6.1.1 Relationships Between F and M

In any beam the shear force F and bending moment M obey a differential relationship. Consider an elemental length δz of beam, over which the force and moments vary by δF and δM in the manner shown in Fig. 6.1. Taking moments about O and applying vertical equilibrium,

$$(M + \delta M) + \delta z\,(w\delta z)/2 = M + (F + \delta F)\delta z$$

$$F + w\,\delta z = F + \delta F$$

$$F = \delta M/\delta z \quad \text{and} \quad w = \delta F/\delta z \tag{6.1a,b}$$

Equations (6.1a,b) show that w is the derivative of F and F is the derivative of M, or that M is the area beneath the F- diagram. Note also from eq(6.1b) that the maximum M will occur where F is zero.

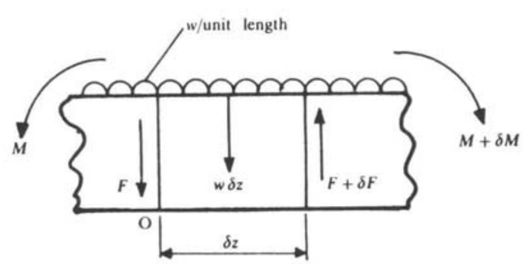


Figure 6.1 Beam element

The following examples show how the distributions in shear force F and bending moment M may be presented graphically.

Example 6.1 Draw the F and M diagrams for the simply supported beam in Fig. 6.2.

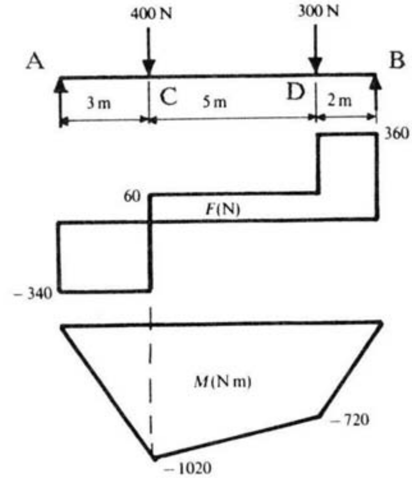


Figure 6.2 F and M distributions

The support reactions R_A and R_B are found from force and moment equilibrium. Take moments about the left support to find the right support reaction,

$$(3 \times 400) + (8 \times 300) = 10\,R_B \;\Rightarrow\; R_B = 360 \text{ N}$$

Apply vertical force equilibrium to find the left support reaction,

$$400 + 300 = R_A + R_B \;\Rightarrow\; R_A = 340 \text{ N}$$

F - diagram: Plot F (↑ + ve) from right to left along the length. The F - diagram must close on the datum line at the l.h. end. The ordinate in the diagram is the shear force at a given section. It may also be calculated independently as the net force arising from all forces to the right (or left) of that section.

M - diagram: An ordinate in the M - diagram is the sum of the moments exerted by all forces lying to one side of a beam at a given position. This net moment may be calculated from working to the left or right depending upon which is easier. A convention applies that a hogging moment is positive and a sagging moment is negative, as follows:

@ r.h. & l.h. ends $M = 0$
@ C to left, $M_C = -\,(3 \times 340) = -\,1020$ Nm
@ D to right, $M_D = -\,(2 \times 360) = -\,720$ Nm

Both M_C and M_D will sag the beam. M_{max} = 1020 Nm where the F - diagram crosses the horizontal datum. In this example, the bending moment diagram is all sagging under concentrated forces and simple end-supports.

6.1.2 Point of Contraflexure

With other forms of loading, the bending moment diagrams may show both hogging and sagging of the beam. A point of contraflexure (or inflection) lies at positions of zero bending moment. This is the point in the length of a beam where its curvature changes from hogging to sagging.

Example 6.2 Sketch the F and M diagrams for the beam in Fig. 6.3 and determine the position and magnitude of the maximum bending moment and the position of the point of contraflexure.

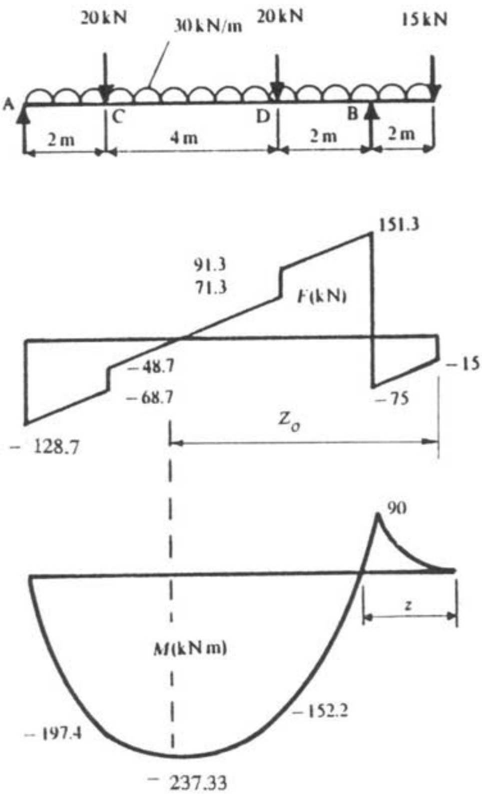


Figure 6.3 Inflection in M - diagram

Reactions R_A and R_B

$(2 \times 20) + (6 \times 20) + (10 \times 15) + (30 \times 10 \times 5) = 8R_B \Rightarrow R_B = 226.3$ kN
$R_A = [20 + 20 + 15 + (30 \times 10)] - 226.3 = 128.7$ kN

F - diagram: Use the construction method where the distributed loading defines the gradient of the F - diagram (from eq 6.1a). Discontinuities arise at points of concentrated forces. The diagram reveals the position of zero F as

$F = 0 = -15 + 226.3 - 30\,z_o - 20 = 0\,,\ \Rightarrow\ z_o = 6.376$ m

M - diagram: Working to left or right, the ordinates are

$M_B = + (2 \times 15) + (2 \times 30 \times 1) = 90$ kNm,
$M_C = - (128.7 \times 2) + (30 \times 2 \times 1) = - 197.4$ kNm,
$M_D = - (128.7 \times 6) + (20 \times 4) + (6 \times 30 \times 3) = - 152.2$ kNm
$M_{max} = (15 \times 6.376) + (2.376 \times 20) - (4.376 \times 226.3) + 20(6.376)^2/2$
$= - 237.33$ kNm, where $F = 0$

Point of Contraflexure: let this lie distance z from RH end

$M = 15z - 226.3(z - 2) + 30z^2/2 = 0$
$15z^2 - 211.3z + 452.6 = 0$
$\therefore z = 2.64$ m.

6.2 Clapeyron's Theorem of Three Moments

Clapeyron's theorem applies to any continuous beam but is derived by taking spans in pairs. The term *three moments* refers to the unknown moments at the central support and at the two ends of any pair of adjacent spans. To derive the theorem it is first necessary to revise Mohr's two theorems for finding the slope and deflection of beams.

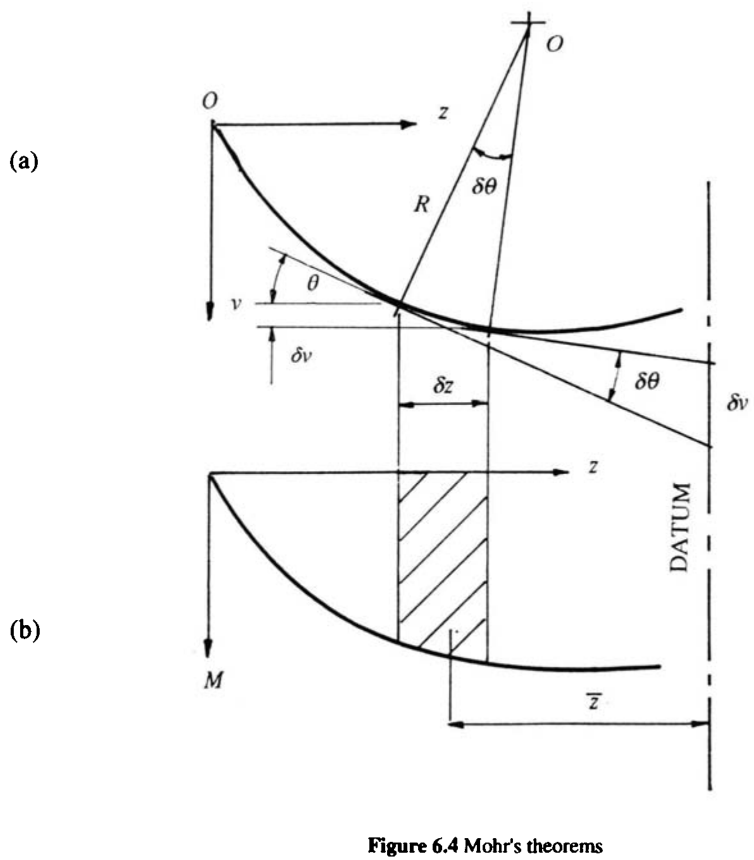


Figure 6.4 Mohr's theorems

Referring to Fig. 6.4a, the theorems provide the change in slope θ and deflection v over length δz of the beam as

$$\delta\theta = \delta z/R = (M\ \delta z)/(EI) \tag{6.2a}$$

$$\delta\theta = (1/EI)(\text{Area of the } M \text{ diagram over } \delta z) \tag{6.2b}$$

$$\delta v = \delta\theta\,\bar{z} = (\delta z/R)\,\bar{z} = (1/EI)\,(M\delta z)\bar{z} \tag{6.3a}$$

$$\delta v = (1/EI)(\text{Moment of area of the } M \text{ diagram over } \delta z) \tag{6.3b}$$

The geometric interpretations (6.2b) and (6.3b) require that I is constant, so that $M \times \delta z$ becomes the shaded area of the M diagram in the region δz (see Fig. 6.4b). In taking the moment of that area, its centroidal distance $\bar{z}$ is measured from a datum where v is required.

Clapeyron applied Mohr's theorems to a beam resting on more than two supports. Consider any two successive spans L_1 and L_2 in Fig. 6.5a under arbitrary loading where the moments existing at their ends A, B and C are M_A, M_B and M_C respectively.

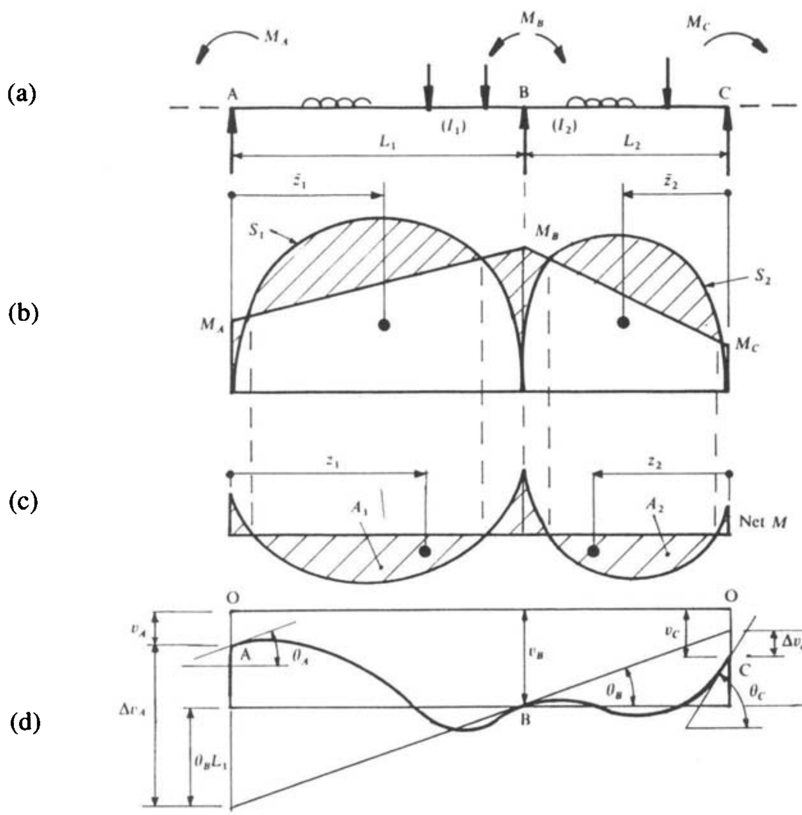


Figure 6.5 Continuous beam

The corresponding free and fixing moment diagrams are superimposed in Fig. 6.5b and take opposing signs. Thus, shaded areas A_1 and A_2 in each bay, that are not common to both diagrams, form the net moment diagram, given in Fig. 6.5c. In the general case let the beam rest on non-level supports at positive vertical distances v_A, v_B and v_C from any horizontal datum OO, as shown in Fig. 6.5d. Furthermore, let the second moments of area I_1 and I_2 for each bay differ.

The centroidal positions $\bar{z}_{1,2}$ and $z_{1,2}$ refer to the free and net moment areas S and A in each bay of Figs 6.5b,c respectively. The slope of the tangent at B is θ_B and the intercepts made by this tangent relative to the ends A and C are Δv_A and Δv_C. Taking the origin at B and with the datum in turn at A and C, eq(6.3a) yields

$$\Delta v_A = A_1 z_1/(EI_1) = \theta_B L_1 + (v_B - v_A) \quad (6.4a)$$

$$\Delta v_C = A_2 z_2/(EI_2) = -[\theta_B L_2 - (v_B - v_C)] \quad (6.4b)$$

in which Δv_C is negative upwards. Eliminating θ from eqs(6.4a,b) leads to

$$A_1 z_1/(EI_1 L_1) + A_2 z_2/(EI_2 L_2) = (v_B - v_A)/L_1 + (v_B - v_C)/L_2 \quad (6.5)$$

where, from Figs 6.5b,c, with hogging positive,

$$A_1 z_1 = \{[M_A L_1^2/2 + (M_B - M_A)L_1^2/3] - S_1 \bar{z}_1\} \quad (6.6a)$$

$$A_2 z_2 = \{[M_C L_2^2/2 + (M_B - M_C)L_2^2/3] - S_2 \bar{z}_2\} \quad (6.6b)$$

Substituting eqs(6.6a,b) into eq(6.5) leads to the theorem of three moments,

$$M_A L_1/I_1 + 2M_B(L_1/I_1 + L_2/I_2) + M_C L_2/I_2 = 6[S_1 \bar{z}_1/(I_1 L_1) + S_2 \bar{z}_2/(I_2 L_2)] + 6E[(v_B - v_A)/L_1 + (v_B - v_C)/L_2] \quad (6.7a)$$

By taking the spans in pairs, sufficient equations are obtained to solve them simultaneously for all the fixing moments. The method is simplified when a continuous beam of uniform section ($I_1 = I_2$) rests on level supports ($v_A = v_B = v_C$), when eq(6.7a) becomes

$$M_A L_1 + 2M_B(L_1 + L_2) + M_C L_2 = 6(S_1 \bar{z}_1/L_1 + S_2 \bar{z}_2/L_2) \quad (6.7b)$$

Consider now the application of Mohr's first theorem to Fig. 6.5. From eq(6.2b), with the origin at B for each bay,

$$\theta_A - \theta_B = A_1/(EI_1) = -[1/(EI_1)][(M_A + M_B)L_1/2 - S_1] \quad (6.8a)$$

$$\theta_C - \theta_B = A_2/(EI_2) = -[1/(EI_2)][(M_B + M_C)L_2/2 - S_2] \quad (6.8b)$$

These changes in slope are both shown to be negative for v positive downwards with positive z originating from B within each bay. Within the first bay, θ_B in eq(6.8a) is

$$\theta_B = \Delta v_A/L_1 - (v_B - v_A)/L_1 = A_1 z_1/(EI_1 L_1) - (v_B - v_A)/L_1 \quad (6.8c)$$

and within the second bay, θ_B in eq(6.8b) is

$$\theta_B = \Delta v_C/L_2 + (v_B - v_C)/L_2 = A_2 z_2/(EI_2 L_2) + (v_B - v_C)/L_2 \quad (6.8d)$$

Substituting eqs(6.6a,b) and (6.8c,d) into eqs (6.8a,b), the slopes become

$$\theta_A = [S_1/(EI_1)](1 - \bar{z}_1/L_1) - [L_1/(6EI_1)](M_B + 2M_A) - (v_B - v_A)/L_1 \quad (6.9a)$$

$$\theta_B = -S_1 \bar{z}_1/(EI_1 L_1) + [L_1/(6EI_1)](M_A + 2M_B) - (v_B - v_A)/L_1 \quad (6.9b)$$

$$= -S_2 \bar{z}_2/(EI_2 L_2) + [L_2/(6EI_2)](M_C + 2M_B) + (v_B - v_C)/L_2 \quad (6.9c)$$

$$\theta_C = [S_2/(EI_2)](1 - \bar{z}_2/L_2) - [L_2/(6EI_2)](M_B + 2M_C) + (v_B - v_C)/L_2 \quad (6.9d)$$

Equations (6.9a-d) are again simplified for a uniform beam resting on level supports, when they become

$$\theta_A = [S_1/(EI)](1 - \bar{z}_1/L_1) - [L_1/(6EI)](M_B + 2M_A) \quad (6.10a)$$

$$\theta_B = -S_1\bar{z}_1/(EIL_1) + [L_1/(6EI)](M_A + 2M_B)$$

$$= -S_2\bar{z}_2/(EIL_2) + [L_2/(6EI)](M_C + 2M_B) \quad (6.10b,c)$$

$$\theta_C = [S_2/(EI)](1 - \bar{z}_2/L_2) - [L_2/(6EI)](M_B + 2M_C) \quad (6.10d)$$

When the moments M_A, M_B and M_C have been found from Clapeyron's theorems (6.7a,b), the slopes θ_A, θ_B and θ_C at the support points follow from eqs(6.9) or (6.10). The deflection and slope at any other points in a continuous beam may be found by integrating Mohr's theorems with the origin O located at a support point of known slope. Let the datum lie at a point within the first bay, distance z from O co-incident with A. Equations (6.2b) and (6.3b) become

$$\theta = \theta_A + [1/(EI)][\text{Area of net } M \text{ diagram from O to } z] \quad (6.11a)$$

$$v = \theta_A z + [1/(EI)][\text{Moment of net } M \text{ diagram from O to } z \text{ about the datum}] \quad (6.11b)$$

The areas and moment of areas of net M diagrams refer to the difference between the areas and moments of areas for the free and fixing moment diagrams. Normally these are sagging (negative) and hogging (positive) diagrams, so that, according to this convention, the shaded areas in Figs 6.5b,c compose the net bending moment diagram.

Example 6.3 Determine the bending moment diagram for the two-bay beam in Fig. 6.6a using the three-moment theorem. Find the support reactions, the slopes and deflections beneath the 10 kN force in the first bay and at the centre of the second bay. Take $EI = 100$ MNm^2.

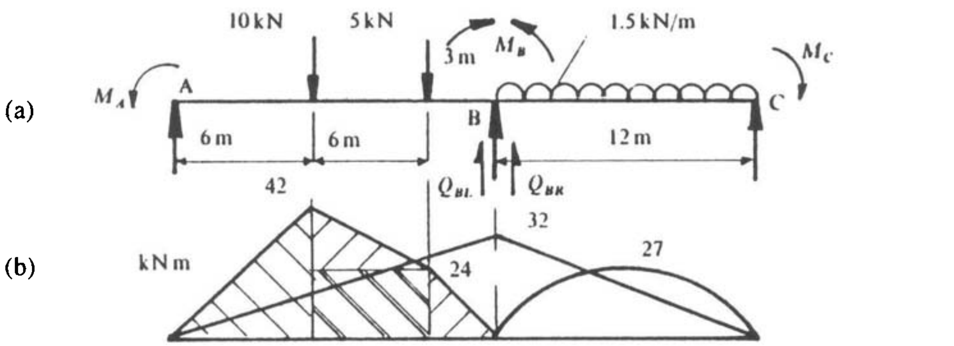


Figure 6.6 Two-bay, simply supported beam

Since $M_A = M_C = 0$ for a two-span beam, eq(6.7b) becomes

$$2M_B(L_1 + L_2) = 6(S_1\bar{z}_1/L_1 + S_2\bar{z}_2/L_2) \quad \text{(i)}$$

Taking the corresponding free moment diagrams from Fig. 6.6b, eq(i) becomes

$2M_B(15 + 12) = 6[(1/15)$(Moment of area of free LH M diagram about A)
$+ (1/12)$(Moment of area of free RH M - diagram about C)] (ii)

Taking LH triangular and rectangular constituent areas in kNm^2 and noting that the r.h. area of a parabola is $2Lh/3$ (centroid $5L/8$ from tip), eq(ii) yields

$$(54/6)\,M_B = (1/15)[(42 \times 6 \times 2 \times 6) / (2 \times 3) + (18 \times 6 \times 8/2) + (24 \times 6 \times 9) + (24 \times 3 \times 13/2)\} + (1/12)\{2 \times 12 \times 27 \times 6/3]$$

$$9M_B = 2700/15 + 1296/12, \;\Rightarrow\; M_B = 32 \text{ kNm} \qquad \text{(iii)}$$

This value is confirmed later from the moment distribution solution (see Table 6.2 in Example 6.5). Let the support reactions and fixing moment M_B act as shown in Fig. 6.6a. The central reaction R_B is composed of its left- and right-hand bay components Q_{BL} and Q_{BR}. Although M_A and M_C are both zero here, they are included to extend the generality to any continuous beam.

Moments about B in bay AB:

$(5 \times 3) + (10 \times 9) - 15R_A + (M_A - M_B) = 0, \quad \Rightarrow R_A = 4.867$ kN

Moments about A in bay AB:

$(6 \times 10) + (12 \times 5) + 32 - 15Q_{BL} = 0, \quad \Rightarrow Q_{BL} = 10.13$ kN

Moments about B in bay BC:

$(1.5 \times 12 \times 6) + (M_C - M_B) - 12R_C = 0, \quad \Rightarrow R_C = 6.333$ kN

Moments about C in bay BC:

$(1.5 \times 12 \times 6) + (M_B - M_C) - 12Q_{BR} = 0, \quad \Rightarrow Q_{BR} = 11.67$ kN

$\therefore R_B = Q_{BL} + Q_{BR} = 10.13 + 11.67 = 21.80$ kN.

Check: $\sum F = [10 + 5 + (12 \times 1.5)] - [4.867 + 21.80 + 6.333] = 0$

The slopes at A and C follow from eqs(6.10a) and (6.10d) with $M_A = M_C = 0$

$$\theta_A EI = S_1(1 - \bar{z}_1/L_1) - L_1 M_B/6 \qquad \text{(iv)}$$
$$\theta_C EI = S_2(1 - \bar{z}_2/L_2) - L_2 M_B/6 \qquad \text{(v)}$$

where $S_1\,\bar{z}_1/L_1 = 180$ kNm^2 and $S_2\,\bar{z}_2/L_2 = 108$ kNm^2 were previously calculated in the three moment expression (iii). Referring again to Fig. 6.6b, the free moment diagram areas are

$$S_1 = (42 \times 6/2) + (18 \times 6/2) + (24 \times 6) + (24 \times 3/2) = 360 \text{ kNm}^2$$
$$S_2 = 2Lh/3 = 2 \times 12 \times 27/3 = 216 \text{ kNm}^2$$

Note that fixing moments of 12.8 and 16 kNm exist at each point within bay 1 and 2 respectively in proportion to $M_B = 32$ kNm. Then, from eqs(iv) and (v), the slopes are

$$\theta_A EI = (360 - 180) - (15 \times 32/6) = 100 \text{ kNm}^2$$
$$\theta_C EI = (216 - 108) - (12 \times 32/6) = 44 \text{ kNm}^2$$

Now from eqs(6.11a,b) at the 10 kN force point, where $z = 6$ m from A,

$\theta EI = 100 + [(12.8 \times 6/2) - (42 \times 6/2)] = 12.4 \text{ kNm}^2$

$\theta = 12.4 \times 10^{-5}$ rad

$vEI = (100 \times 6) + [(12.8 \times 6 \times 6) / (2 \times 3) - (42 \times 6 \times 6) / (2 \times 3)] = 424.8 \text{ kNm}^3$

$v = 4.25$ mm

and at the centre of the second bay, where $z = 6$ m from C,

$\theta EI = 44 + [(16 \times 6/2) - (2 \times 6 \times 27/3)] = -16 \text{ kNm}^2$

$\theta = -16 \times 10^{-5}$ rad

$vEI = (44 \times 6) + [(16 \times 6 \times 6) / (2 \times 3) - (2 \times 6 \times 27 \times 3 \times 6) / (3 \times 8)] = 117 \text{ kNm}^3$

$v = 1.17$ mm

Example 6.4 Determine the fixing moments for the beam in Fig. 6.7a using Clapeyron's theorem.

(a)

(b)

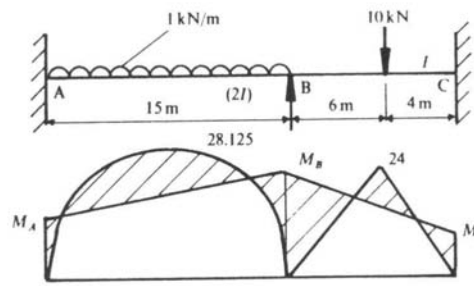


Figure 6.7 Two-bay encastre beam

The free moment diagrams are given in Fig. 6.7b. The fixing moments M_A and M_C, together with the central support moment M_B, are unknowns for this continuous beam. M_C is assumed to be the greatest of these to correspond with Fig. 6.5b so allowing the application of the three-moment theorem. Since $S_1 \bar{z}_1$ and $S_2 \bar{z}_2$ are the moments of area of the free M diagrams about the left- and right-hand ends respectively, eq(6.7a) becomes

$$15M_A/(2I) + 2M_B[15/(2I) + 10/I] + 10M_C/I = 6\{(2 \times 15 \times 28.125 \times 7.5) / (3 \times 2I \times 15) + [(24 \times 6 \times 6/2) + (24 \times 4 \times 2 \times 4)/(2 \times 3)] / (I \times 10)\}$$

$$M_A + 4.67M_B + 1.33M_C = 101.05 \qquad \text{(i)}$$

Two further relationships are found from the application of eq(6.3) to each bay. The intercept between the tangents at A and B is zero. Taking the origin at A and the datum at B,

$$\Delta v_B = [1/(2EI)](\text{Moment of } M \text{ diagram from A to B about B}) = 0$$

$$(15M_A \times 7.5) + (M_B - M_A)(15 \times 15)/(2 \times 3) - (2 \times 15 \times 28.125 \times 7.5/3) = 0$$

$$2M_A + M_B = 56.25 \qquad \text{(ii)}$$

Furthermore, the intercept between the tangents at C and B is zero with the origin at C and the datum at B. Thus,

$\Delta v_B = [1/(EI)]$(Moment of M diagram from C to B about B) = 0

$(10M_C \times 5) + (M_B - M_C)(10\times10)/(2\times3) - [(24\times4\times7.333/2) + (24\times6\times2\times6)/(2\times3)] = 0$

$M_B + 2M_C = 38.4$ (iii)

Solving eqs(i) - (iii) gives $M_A = 21.365$, $M_B = 13.52$ and $M_C = 12.44$ kNm (least). The values are confirmed in Example 6.6 (Table 6.3) using the moment distribution method.

6.3 The Moment Distribution Method

Where a beam has two or more spans, it is necessary to ensure that the moments to the left and right sides of all its inner supports are equal. In a structure the sum of the moments at all joints should be zero. Because an initial estimate of these moments does not balance, it is necessary to distribute the imbalance until the equality is achieved. The method is as follows:

(i) Fix the beam at all supports and calculate the LH and RH fixing moments for each bay from Table 6.1
(ii) Where a beam has simply supported ends, release the fixing moment and carry over (eqs 6.14a,b)
(iii) Use the distribution factors (eqs 6.18a,b) to distribute unbalanced moments at inner supports and carry over. Moments are only carried over to ends that are encastré
(iv) Construct the fixing moment diagram
(v) Isolate each bay and construct the free M-diagram (see Section 6.1)
(vi) The net M-diagram is then found from the difference between ordinates in the fixing and free moment diagrams
(vii) Calculate the supporting reactions from the balanced fixing moments and construct the F-diagram

6.3.1 Fixing Moments (F.M.)

Standard expressions given in Table 6.1 provide fixing moments for the single-span encastre beams shown. They act in the directions shown to hog the beam. The convention adopted is that clockwise F.M's are positive. Therefore the signs of the F.M. alternate from negative to positive within a bay and on either side of each support.

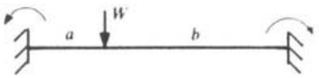
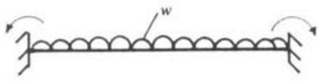
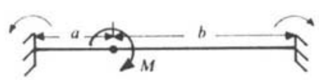
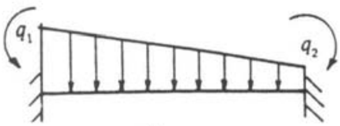
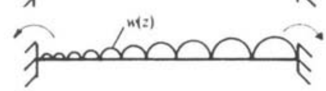
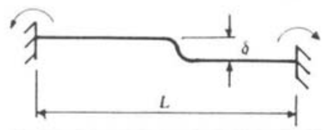
6.3.2 Carry-Over Moments (C.O.M.)

In Fig. 6.8, it is assumed that the adjacent F.M's, as calculated from fixing support B, do not balance. The out-of-balance moment M, at B, is distributed into bays BA and BC as

$$M = M_{BA} + M_{BC} \tag{6.12}$$

where the directions of M_{BA} and M_{BC} must oppose the directions of M to maintain moment equilibrium at B. A further compatibility requirement is that one half of each distributed moment will be carried over in the same direction to the opposite end. That is, M_{AB} is carried over from M_{BA}, and M_{CB} is carried over from M_{BC}.

Table 6.1 Fixed-End Moment Expressions (F.M.)

BEAM	R.H.F.M.	L.H.F.M.
	Wa^2b/L^2	Wab^2/L^2
	$wL^2/12$	$wL^2/12$
	$wa^2(6L^2 - 8aL + 3a^2)/(12L^2)$	$wa^2(4aL - 3a^2)/(12L^2)$
	$(Mb/L)(1 - 3a/L)$	$(Ma/L)(1 - 3b/L)$
	$(2q_1 + 3q_2)L^2/60$	$(3q_1 + 2q_2)L^2/60$
	$(1/L^2)\int wx(L-x)^2\,dx$	$(1/L^2)\int wx^2(L-x)\,dx$
	0	$3EI\delta/L^2$
	$6EI\delta/L^2$	$6EI\delta/L^2$

To show this, both M_{AB} and M_{CB} must ensure that the deflection v, at B, is zero. Applying Mohr's deflection theorem (6.3b) to bay AB where the M-diagram is composed of two parts: (i) the hogging moment M_{BA}, carried over, and (ii) a sagging moment, due to the reaction R_B at B,

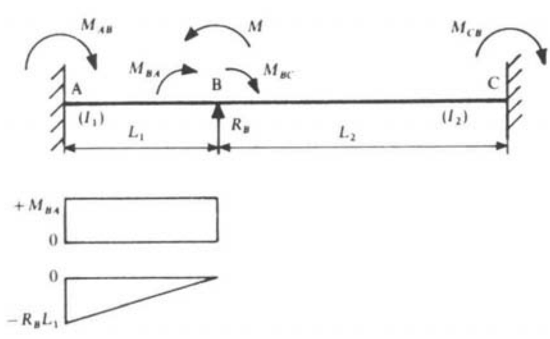


Figure 6.8 Distribution and carry-over of unbalanced moment M

$v_B = [1/(EI)]$(Moment of area of M diagram from A to B about B) = 0

$M_{BA} L_1 (L_1/2) - (R_B L_1^2/2)(2L_1/3) = 0$

$$R_B = 3M_{BA}/(2L_1) \tag{6.13}$$

The net moment applied at A is then

$$M_{BA} - [3M_{BA}/(2L_1)]L_1 = - M_{BA}/2$$

The reacting C.O.M. at A and similarly the C.O.M. at C take the opposite sense to the moments applied at A and C. It follows that each C.O.M. shown in Fig. 6.8 has the same sense with a magnitude,

$$M_{AB} = M_{BA}/2$$
$$M_{CB} = M_{BC}/2 \tag{6.14,b}$$

6.3.3 Distribution Factors (D.F.)

A further compatibility condition is that the slope θ, at B, must be the same for bays AB and BC. If L_1 and I_1 are the length and second moment of area for bay AB, the application of Mohr's slope theorem (6.3a) to Fig. 6.8 gives

$\theta_B = [1/(EI)]$(Area of M diagram from A to B]

$$= [1/(EI_1)](M_{BA} L_1 - R_B L_1^2/2) = - M_{BA} L_1/(2EI_1) \tag{6.15}$$

in which R_B is given by eq(6.13). Similarly, if L_2 and I_2 are the respective length and second moment of area for bay BC, then

$$\theta_B = - M_{BC} L_2/(2EI_2) \tag{6.16}$$

Equating (6.15 and 6.16) leads to

$$M_{BA}/M_{BC} = (I_1 L_2) / (I_2 L_1) \tag{6.17}$$

Combining eqs(6.12) and (6.17) provides the *distribution factors*:

$$\frac{M_{BA}}{M} = \frac{(I_1/L_1)}{(I_1/L_1) + (I_2/L_2)}\ , \quad \frac{M_{BC}}{M} = \frac{(I_2/L_2)}{(I_1/L_1) + (I_2/L_2)} \tag{6.18a,b}$$

That is, M is divided between the two bays in the ratio of their stiffnesses, I/L. It should be noted that eqs(6.18a,b) are modified in the case of a simply supported end bay whose stiffness becomes ¾(I/L).

6.4 Continuous Beams

The following examples illustrate the application of Table 6.1 and eqs(6.14a,b) and (6.18a,b) to continuous beams.

Example 6.5 Construct the net bending moment diagram for the two-bay beam in Fig. 6.9a showing the maximum values. Find the support reactions.

Fixed end-moments from Table 6.1:

Bay AB, L.H.F.M.$=(10 \times 6^2 \times 9/15^2) + (5 \times 12^2 \times 3/15^2) = 24$ kNm
Bay AB, L.H.F.M. $= (10 \times 9^2 \times 6/15^2) + (5 \times 3^2 \times 12/15^2) = 24$ kNm
Bay BC, R.H.F.M. = L.H.F.M. $= (1.5 \times 12^2/12) = 18$ kNm

D.F's at joint B, from eqs(6.18a,b). Note that the ¾ factor cancels for a beam with simple supports throughout.

$M_{BA}/M = (3/4)(I/15)/[(3/4)(I/12 + I/15)] = 4/9$
$M_{BC}/M = (3/4)(I/12)/[(3/4)(I/12 + I/15)] = 5/9$

Fixing Moments: The distribution and carry-over of moments given in Table 6.2 ensure that the moments at B become equal and that there can be no moments at the ends A and C.

Table 6.2 Distributions for Example 6.5

Joint	A	B		C
Member	AB	BA	BC	CA
D.F.		4/9	5/9	
Initial F.E.M.	-24	+24	-18	+18
Release @ A & C & C.O.M.	+24	+12	-9	-18
Modified F.E.M.	0	+36	-27	0
Unbalanced moment		+9		
1st Distribution		-4	-5	
Final fixing moments	0	+32	-32	0

The fixing moment diagram is superimposed upon the inverted free moment diagrams in Fig. 6.9b. The shaded areas in Fig. 6.6b represent the net moment diagram which has been re-based in Fig. 6.9c.

(a)

(b)

(c)

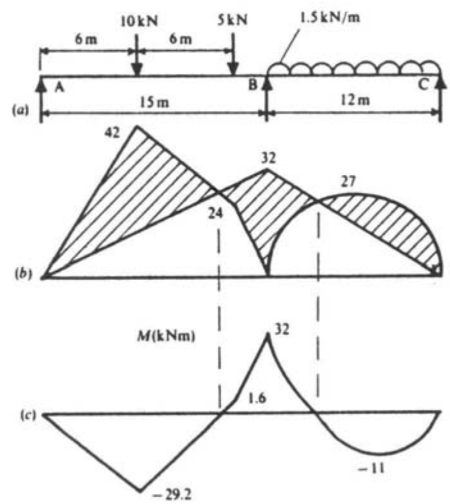


Figure 6.9 Free, fixing and net moment diagrams

The support reactions follow from applying moment equilibrium to the left and right of B. This must include the fixing moments:

$\sum M_B = 0 = +32 + 15R_A - (9 \times 10) - (3 \times 5)$, $\Rightarrow R_A = 4.87$ kN
$\sum M_B = 0 = -32 - 12R_C + (1.5 \times 12 \times 6)$, $\Rightarrow R_C = 6.33$ kN

Apply vertical force equilibrium,

$\sum F = 0 = 4.87 + R_B + 6.33 - 10 - 5 - (12 \times 1.5)$, $\Rightarrow R_B = 21.8$ kN

Example 6.6 Find the fixing moments and support reactions for the beam in Fig. 6.10.

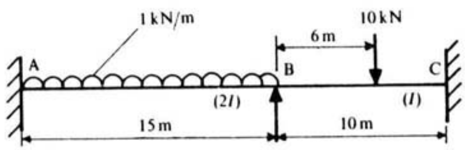


Figure 6.10 Two-bay encastre beam

Fixing moments from Table 6.1:
Bay AB: L.H.F.M.= R.H.F.M. = $1 \times 15^2/12 = 18.75$ kNm
Bay BC: R.H.F.M. = $10 \times 6^2 \times 4/10^2 = 14.4$ kNm
L.H.F.M. = $10 \times 4^2 \times 6/10^2 = 9.6$ kNm

D.F.'s at B from eqs(6.18a,b):
$M_{BA}/M = (2I/15)/[(2I/15) + (I/10)] = 4/7$
$M_{BC}/M = (I/10)/[(2I/15) + (I/10)] = 3/7$

Moment Distribution: Table 6.3 shows how to balance the moments at B and carry over to establish the moments at A and C.

Table 6.3 Distributions for Example 6.6

Joint	A	B		C
Member	AB	BA	BC	CB
D.F.		4/7	3/7	
Initial F.E.M.	- 18.75	+18.75	- 9.6	+14.40
Unbalanced moment		+9.15		
Distribution & C.O.M.	- 2.62	-5.23	- 3.92	- 1.96
Final fixing moments	- 21.37	+13.52	- 13.52	+12.44

These F.M.'s are employed in calculations for the reactions. Take moments about B to right (positive clockwise),

$\sum M_B = 0 = (10 \times 6) + 12.44 - 10R_C - 13.5$, $\Rightarrow R_C = 5.894$ kN
Moments about B to left,
$\sum M_B = 0 = -(15 \times 1 \times 7.5) - 21.37 + 15R_A + 15.32$, $\Rightarrow R_A = 8.03$ kN

Vertical force equilibrium,

$\sum F = 0 = 8.023 + R_B + 5.894 - 25, R_B = 11.083$ kN

Example 6.7 Construct the moment diagram for the beam in Fig. 6.11a. Find the four support reactions.

F.M. from Table 6.1:
Bay AB, L.H.F.M. = $5 \times 3^2 \times 2/5^2 = 3.6$ kNm, R.H.F.M. = $5 \times 2^2 \times 3/5^2 = 2.4$ kNm
Bay BC, L.H.F.M. = R.H.F.M. = $1 \times 5^2/12 = 2.08$ kNm
Bay CD, L.H.F.M. = $(3 \times 3^2 \times 2/5^2) + (3 \times 1^2 \times 4/5^2) = 2.64$ kNm
R.H.F.M. = $(3 \times 2^2 \times 3/5^2) + (3 \times 4^2 \times 1/5^2) = 3.36$ kNm

D.F. at B and C. Introduce ¾ factor for end bays within eqs(6.18a,b),

$M_{BA}/M = (3/4)(I/5)/[(3/4)(I/5)+I/5] = 3/7$
$M_{BC}/M = (I/5)/[(3/4)(I/5) + I/5] = 4/7$
$M_{CB}/M = (I/5)/[(3/4)(I/5) + I/5] = 4/7$
$M_{CD}/M = (3/4)(I/5)/[(3/4)(I/5) + I/5] = 3/7$

Moment distribution: In Table 6.4 the moments are distributed until they balance at B and C and equal zero at A and D.

Table 6.4 Distributions for Example 6.7

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
D.F.		3/7	4/7	4/7	3/7	
Initial F.E.M.	- 3.6	+2.4	- 2.08	+2.08	- 2.64	+3.36
Release A & D & C.O.	+3.6	+1.8			- 1.68	- 3.36
Net fixing moments	0	+4.2	- 2.08	+2.08	- 4.32	0
Unbalanced moment		+2.12		-2.24		
1st Distribution		- 0.91	- 1.21	+1.28	+0.96	
C.O.M.		0	+0.64	-0.61	0	
Unbalanced moment		+0.64		- 0.61		
2nd Distribution		- 0.27	- 0.37	+0.34	+0.26	
C.O.M.		0	+0.17	-0.19	0	
Unbalanced moment		+0.17		-0.19		
3rd Distribution		- 0.07	- 0.10	+0.11	+0.08	
C.O.M.		0	+0.06	-0.05	0	
Unbalanced moment		+0.06		-0.05		
4th Distribution		- 0.025	- 0.035	+0.03	+0.02	
C.O.M.		0	+0.015	-0.017		
Final fixing moments	0	+2.925	-2.91	+2.98	- 3.0	0

The free and fixing moments are superimposed in Fig. 6.11b using average F.M. values. The net moment diagram is given by the shaded regions.

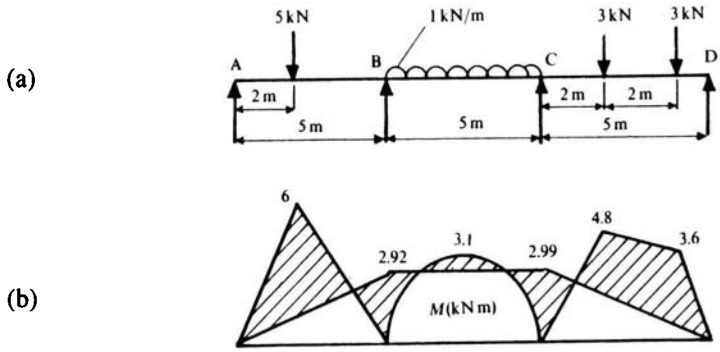


Figure 6.11 Free, fixing and net moment diagrams for a three-bay s.s. beam

Reactions: Take moments to left and right of B and C including the final F.M.'s

$M_C = 0 = (3 \times 2) + (3 \times 4) - 2.99 - 5R_D,$ $\Rightarrow$ $R_D = 3$ kN
$M_B = 0 = 5R_A + 2.92 - (3 \times 5),$ $\Rightarrow$ $R_A = 2.42$ kN
$M_C = 0 = 2.99 + 5R_B + (10 \times 2.42) - (8 \times 5) - (5 \times 1 \times 2.5),$ $\Rightarrow$ $R_B = 5.06$ kN
$M_B = 0 = -\ 2.92 + (5\times1\times2.5) - 5R_C + (7\times3) + (9\times3) - (10\times3)$ $\Rightarrow$ $R_C = 5.52$ kN

Example 6.8 Draw the bending moment diagram for the continuous beam in Fig. 6.12a. Calculate the support reactions and draw the F - diagram.

Fixed-end moments from Table 6.1:
Bay AB: F.M's = 0
Bay BC: L.H.F.M. = R.H.F.M. = $6 \times 1^3/2^2 = 1.5$ kNm
Bay CD: " " $= 3 \times 2^2/12 = 1.0$ kNm
Bay DE: $M_D = 3 \times 1^2/2 = 1.5$ kNm (hogging), $M_E = 0$

D.F.'s at B and C, from eqs(6.18a,b):
$M_{BA}/M = (2I/2)/[(2I/2) + (2I/2)] = 0.5 = M_{AB}/M$
$M_{CB}/M = (2I/2)/[(2I/2) + (3/4)(I/2)] = 0.73$
$M_{CD}/M = (3/4)(I/2)[(2I/2) + (3/4)(I/2)] = 0.27$

Free Sagging Moments:
Bay BC, $M_{max} = Wl/4 = 6 \times 2/4 = 3$ kNm,
Bay CD, $M_{max} = wl^2/8 = 3 \times 2^2/8 = 1.5$ kNm

In Table 6.5 the moments are balanced at B and C and D. Further, it must be ensured that M_D = 1.5 kNm remains the net moment at D. These calculations reveal a net bending moment diagram within the shaded regions of Figs 6.12b. The re-based net moments in Fig. 6.12c are consistent with the convention that hogging moments are positive. The shear force diagram (see Fig. 6.12d) construction starts from the RH end, taking downward forces to be negative.

Table 6.5 Distributions fro Example 6.8

Joint	A	B		C		D	E	
Member	AB	BA	BC	CB	CD	DC	DE	ED
D.F.	0	0.5	0.5	0.73	0.27			
Initial F.E.M.	0	0	-1.5	+1.50	-1.00	+1.0	-1.5	0
Balance @ D						+0.5		
C.O.M.					+0.25			
Net fixing moments	0	0	-1.5	+1.5	-0.75	+1.5	-1.5	0
Unbalanced moment		-1.5		+0.75				
1st Distribution		+0.75	+0.75	-0.55	-0.20			
C.O.M.	+0.37		-0.28	+0.37				
2nd Distribution		+0.14	+0.14	-0.27	-0.10			
C.O.M.	+0.07		-0.14	+0.07				
3rd Distribution		+0.07	+0.07	-0.05	-0.02			
C.O.M.	+0.035		-0.025	+0.035				
4th Distribution		+0.012	+0.012	-0.026	-0.01			
C.O.M.	+0.006		-0.013	+0.006				
Final F.M's	+0.48	+0.972	-0.986	+1.085	-1.080	+1.5	-1.5	0

$\sum M_C = 0 = -1.082 + (3 \times 3 \times 1.5) - 2R_D$, from which reaction is $\Rightarrow R_D = 6.2$ kN
$\sum M_B = 0 = -0.98 + (6 \times 1) + (3 \times 3 \times 3.5) - 2R_C - (4 \times 6.2)$, $\Rightarrow R_C = 5.86$ kN
$\sum M_A = 0 = +0.48 + (6 \times 3) + (3 \times 3 \times 5.5) - 2R_B - (4 \times 5.86) - (6 \times 6.2)$, $\Rightarrow R_B = 3.67$ kN
$\sum F = 0 = 6 + (3 \times 3) - 3.67 - 5.86 - 6.2 - R_A$, $\Rightarrow R_A = -0.73$ kN

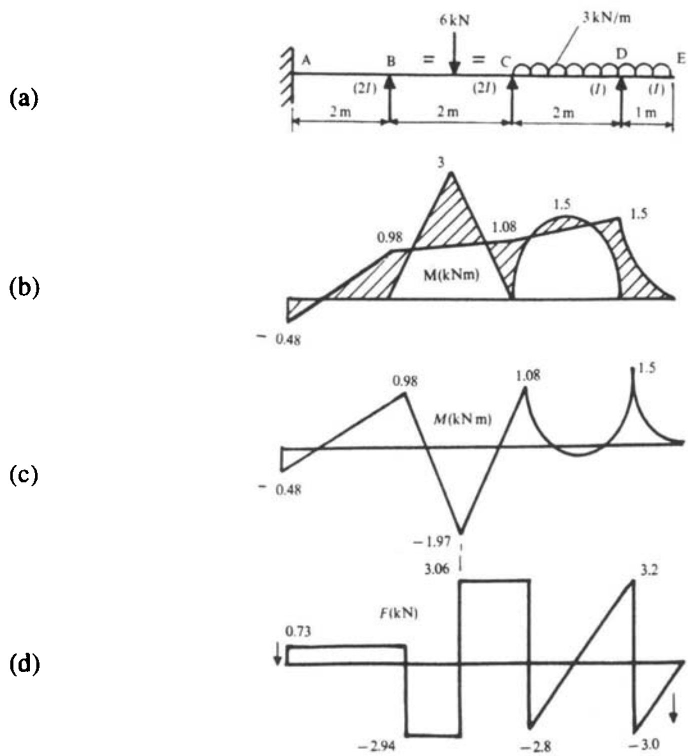


Figure 6.12 Continuous beam with overhang

6.5 Beams With Misaligned Supports

If the supports for a continuous beam are not level, moments are induced at these points. This applies to all misaligned interior supports and to the ends if these are encastré.

Example 6.9 Find the moments at B and C for the beam in Fig. 6.13 when the support at B lies 40 mm below the level of A, C and D. Take $E = 200$ GPa, $I = 4 \times 10^6$ mm^4.

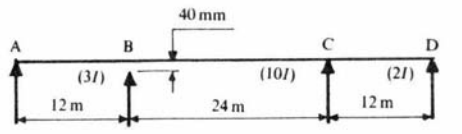


Figure 6.13 Continuous beam with misaligned supports

F.M.'s are found from $M = 6EI\delta/L^2$ (see Table 6.1)

Bay AB, L.H.F.M. = $(6 \times 200 \times 10^3 \times 3 \times 4 \times 10^6 \times 40)/(12^2 \times 10^6)$
$= 4 \times 10^6$ Nmm = 4 kNm = R.H.F.E.M.

Bay BC, L.H.F.M. = $(6 \times 200 \times 10^3 \times 10 \times 4 \times 10^6 \times 40)/(24^2 \times 10^6)$
$= (3.333 \times 10^6)$ Nmm = 3.333 kNm = R.H.F.M.

Bay CD, F.M's = 0

D.F.'s at B and C, from eqs(6.9):

$M_{BA}/M = (3/4)(3I/12)/[(3/4)(3I/12) + 10I/24\ ] = 0.31$

$M_{BC}/M = (10I/24)/[[(3/4)(3I/12) + 10I/24\ ] = 0.69$

$M_{CB}/M = (10I/24)/[(3/4)(2I/12) + 10I/24\ ] = 0.77$

$M_{CD}/M = (3/4)(2I/12)/[(3/4)(2I/12) + 10I/24] = 0.23$

Table 6.6 Distributions for Example 6.9

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
D.F		0.31	0.69	0.77	0.23	
Initial F.E.M.	-4	-4	+3.333	+3.333	0	0
Rel. at A & D & C.O.	+4	+2				
Net F.E.M.	0	-2	+3.333	+3.333	0	0
Unbalanced Moment		+1.333		+3.333		
1st Distribution	0	-0.413	-0.920	-2.566	-0.767	
C.O.M.			-1.283	-0.460		
2nd Distribution	0	+0.398	+0.885	+0.354	+0.106	
C.O.M.			+0.177	+0.443		
3rd Distribution	0	-0.055	-0.122	-0.341	-0.102	
C.O.M.			-0.171	-0.061		
4th Distribution	0	+0.053	+0.118	+0.047	+0.014	
C.O.M.			+0.024	+0.059		
5th Distribution	0	-0.007	-0.017	-0.045	-0.014	
C.O.M.			-0.023	-0.009		
6th Distribution	0	+0.007	+0.016	+0.007	+0.002	
C.O.M.			+0.004	+0.008		
Final F.E.M.	0	-2.017	+2.021	+0.769	-0.761	0

Because the end A is simply supported, it is first necessary to release the moment at this point and carry over. Table 6.6 shows that, without external loading, the initial F.M's will have the same sense within each bay but alternating from bay to bay.

6.6 Moment Distribution for Structures

The moment distribution method can be extended to structures. The F.M.'s are found for all bays from Table 6.1. The joints in a structure may connect two or more bars with various inclinations. The D.F.'s are found from extending eqs(6.18a,b) to all bars at each joint. For $n = 1, 2 \ldots N$ bars the fraction of the joint imbalance moment M, carried by each bar, defines the D.F. as

$$\text{D.F.} = M_n / M = (I/L)_n \Big/ \sum_{n=1}^{N} (I/L)_n \tag{6.19}$$

The following examples illustrate the principles involved in attaining the required moment balance $\sum_{n=1}^{N} M_n = 0$ at joints for structures that are rigid and for those that sway.

Example 6.10 Find the vertical reactions at A and B for the structure in Fig. 6.14a. The ends A and B are encastre, C is pinned and O is a rigid joint.

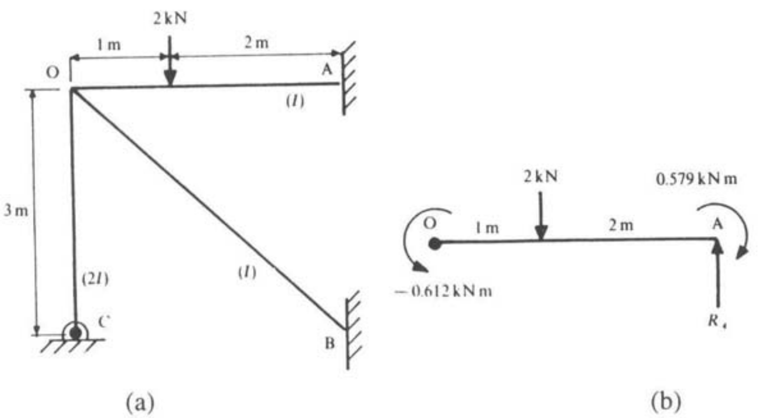


Figure 6.14 Three-bar plane structure

F.M.'s from Table 6.1:

Bay OA: L.H.F.M. = $2 \times 2^2 \times 1/3^2 = 0.89$ kN, R.H.F.M. = $2 \times 1^2 \times 2/3^2 = 0.44$ kN.
Struts OC and OB: F.M's are zero.

D.F.'s at O, from eqs(6.19):

$M_{OA} / M = (I/3) / [(I/3) + I/(3\sqrt{2}) + (3/4)(2I/3)] = 0.313$

$M_{OB} / M = [I/(3\sqrt{2})] / [(I/3) + I/(3\sqrt{2}) + (3/4)(2I/3)] = 0.220$

$M_{OC} / M = (3/4)(2I/3) / [(I/3) + I/(3\sqrt{2}) + (3/4)(2I/3)] = 0.467$

The L.H.F.M. for bay OA is the unbalanced moment for the joint O, which is distributed within Table 6.7:

Table 6.7 Distributions fro Example 6.10

Joint	C		O		A	B
Member	CO	OC	OB	OA	AO	BO
D.F		0.467	0.220	0.313		
Initial F.E.M.	+0	-0	+0	-0.89	+0.44	-0
Unbalanced moment			-0.89			
1st Distribution		+0.416	+0.196	+0.278		
C.O.M.	+0				+0.139	+0.098
Final F.M's	+0	+0.416	+0.196	-0.612	+0.579	+0.098

At the joint O, Table 6.7 shows that the final F.M's in the three bars OC, OB and OA sum to zero. The reaction at A is found by taking moments about O, with OA as a free body, see Fig. 6.14b,

$$\sum M_O = 0 = -\ 0.612 + (2 \times 1) + 0.579 - 3R_A, \qquad \Rightarrow\ R_A = 0.657\ \text{kN}$$

Similarly for OB as a free body,

$$\sum M_O = 0 = +\ 0.196 + 0.098 - 3R_B, \qquad \Rightarrow\ R_B = 0.098\ \text{kN}$$

Example 6.11 The bridge structure in Fig. 6.15 is built in at A, D and F and is hinged at E. Find the bending moments at A, F and D when I is uniform throughout.

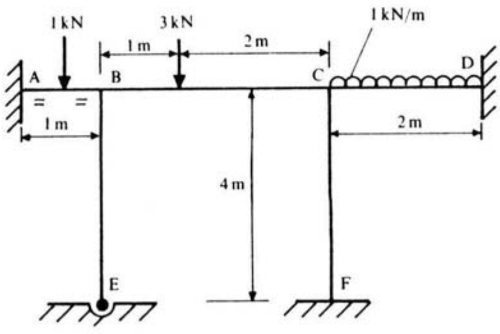


Figure 6.15 Bridge structure

F.M.'s from Table 6.1

Bay AB: L.H.F.M. = R.H.F.M. = $1 \times (1/2) \times (1/2)^2/12 = 0.125$ kNm

Bay BC: L.H.F.M. = $3 \times 2^2 \times 1/3^2 = 1.33$ kNm, R.H.F.M = $3 \times 1^2 \times 2/3^2 = 0.67$ kNm

Bay CD: L.H.F.M. = R.H.F.M. = $1 \times 2^2/12 = 0.33$ kNm

Bays BE and CF: F.M's are zero.

D.F.'s at joint B, from eqs(6.19):

$M_{BA}/M = (I/1)/[(I/1) + (3/4)(I/4) + (I/3)] = 0.658$

$M_{BE}/M = (3/4)(I/4)/[(I/1) + (3/4)(I/4) + (I/3)] = 0.123$

$M_{BC}/M = (I/3)/[(I/1) + (3/4)(I/4) + (I/3)] = 0.219$

D.F.'s at joints C:

$M_{CB}/M = (I/3)/[(I/3) + (I/4) + (I/2)] = 0.308$
$M_{CF}/M = (I/4)/[(I/3) + (I/4) + (I/2)] = 0.231$
$M_{CD}/M = (I/2)/[(I/3) + (I/4) + (I/2)] = 0.461$

The pinned end E cannot fix a moment and therefore need not enter into Table 6.8. If, however, a horizontal force were applied along BE then the F.M. calculated for E would need to be released and carried over to B.

Table 6.8 Distributions for Example 6.11

Joint	A	B			C			F	D
Member	AB	BA	BE	BC	CB	CF	CD	FC	DC
D.F's		.658	.123	.219	.308	.231	.461		
Init.F.E.M.	-.125	+.125	0	-1.33	+.67	0	-.33	0	+.33
Imbalance			-1.205			+.340			
1st Dibn		+.793	+.148	+.264	-.105	-.079	-.157		
C.O.M.	+.397			-.053	+.132			-.040	-.0785
2nd Dibn		+.033	+.006	+.011	-.040	-.030	-.060		
C.O.M.	+.017			-.020	+.006			-.015	-.030
3rd Dibn		+.013	+.003	+.004	-.0019	-.0014	-.0028		
C.O.M.	+.007			-.001	+.002			-.0007	-.0014
4th Dibn		+.0007	+.00012	+.00022	-.00062	-.00046	-.00092		
C.O.M.	+.0003			-.0003	+.00011			-.0002	-.0005
Final F.M.	+.296	+.965	+.157	-1.127	+.663	-.111	-.551	-.056	+.220

The distribution is continued until the unbalanced carry-over moments at B and C become negligibly small. Final F.M's at A, F and D lie in the first and final two columns respectively.

Example 6.12 Construct the bending moment diagram for the portal frame in Fig. 6.16a. Joints A and B are pinned and joints B and C are rigid. Assume EI is constant.

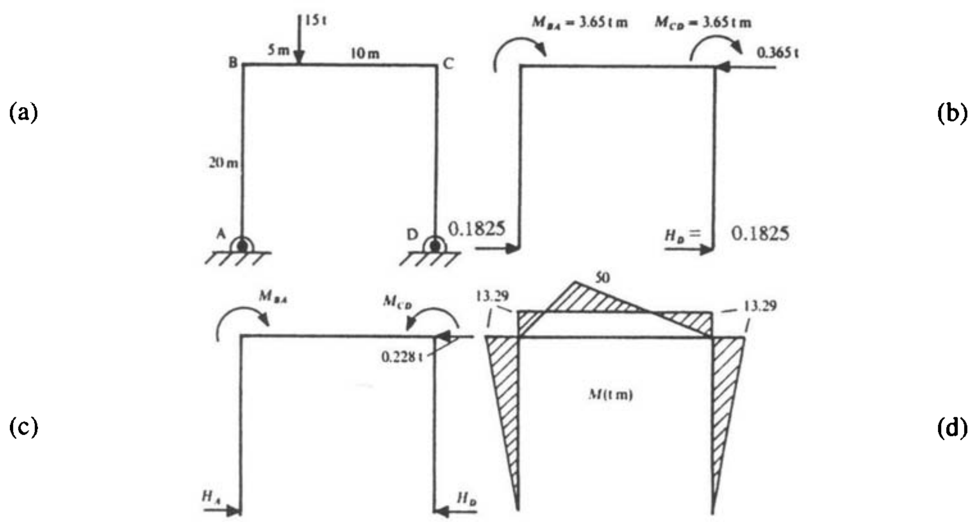


Figure 6.16 Moment analysis of a portal frame

F.M.'s for bay BC

R.H.F.M. = $15 \times 5^2 \times 10/15^2 = 16.67$ tm

L.H.F.M. = $5 \times 10^2 \times 15/15^2 = 33.33$ tm

F.M's = 0 for struts AB and CD.

D.F.'s for joints B and C

$M_{BA}/M = (3/4)(I/20)/[(3/4)(I/20) + (I/15)] = 0.36 = M_{CD}/M$

$M_{BC}/M = (I/15)/[(3/4)(I/20) + (I/15)] = 0.64 = M_{CB}/M$

Table 6.9 shows the F.M.'s are distributed in the usual way, but they must then be corrected to account for the effect of horizontal side sway.

Table 6.9 Distribution for Example 6.12

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
D.F.		0.36	0.64	0.64	0.36	
Initial F.M.	0	0	-33.33	+16.67	0	0
1st Dibn		+12	+21.3	-10.7	-6	
C.O.M.			-5.4	+10.7		
2nd Dibn		+1.95	+3.45	-6.85	-3.85	
C.O.M.			-3.43	+1.73		
3rd Dibn		+1.24	+2.19	-1.10	-0.63	
C.O.M.			-0.55	+1.10		
4th Dibn		+0.20	+0.35	-0.70	-0.40	
C.O.M.			-0.40	+0.20		
5th Dibn		+0.15	+0.25	-0.13	-0.07	
C.O.M.			-0.07	+0.13		
6th Dibn		+0.03	+0.04	-0.08	-0.05	
Non sway F.M.		+15.57	-15.57	+11.0	-11.0	0
Init sway *M*	0	+5		+5		0
1st Dibn		-1.8	-3.20	-3.20	-1.80	
C.O.M.			-1.60	-1.60		
2nd Dibn		+0.60	+1.00	+1.00	+0.60	
C.O.M.			+0.50	+0.50		
3rd Dibn		-0.20	-0.30	-0.30	-0.20	
C.O.M.			-0.15	-0.15		
4th Dibn		+0.05	+0.10	+0.10	+0.05	
Sway *M*	0	+3.65	-3.65	-3.65	+3.65	0
Corr sway *M*	0	-2.28	+2.28	+2.28	-2.28	0
Final F.M.	0	+13.29	-13.29	+13.28	+13.28	0

The non-sway F.M's enable the horizontal forces at A and D to be found by treating AB and DC as free bodies. From Fig. 6.16b,

$\sum M_B = 0 = M_{BA} - 20H_A, \quad \Rightarrow H_A = 15.57/20 = 0.778$ t

$\sum M_C = 0 = M_{CD} - 20H_D, \quad \Rightarrow H_D = 11.0/20 = 0.55$ t

Horizontal force equilibrium of the frame yields

$$\sum H = 0 = H_A - H_D - H_C, \quad \Rightarrow \quad H_C = 0.778 - 0.55 = 0.228 \text{ t}$$

The non-sway F.M's in Table 6.9 require a horizontal force of 0.228 t to be applied at B. Without this force, the frame sways sideways by an amount δ at C. If δ is known, moments M_{BA} and M_{CD} at B and C are given by $M = 3EI\delta/L^2$ (see Table 6.1), each with the same sense. Since δ is unknown, we assume a value for M_{BA} and M_{CD} (say 5 tm) for a second moment distribution within Table 6.9 in order to find the sway moments of 3.65 tm at the rigid joints at B and C. These sway moments produce equal horizontal reactions $H_A = H_D =$ 0.1825 t, balanced by a force of 0.365 t applied to C (see Fig. 6.16c). However, there can be no horizontal force at C since the horizontal reactions at A and D must balance. That is, between Figs 5.17 b and c, $0.228 + 0.365C = 0$, where $C = -0.625$ is the multiplication factor used to correct the sway moments in Table 6.8. The sign of C depends upon the sense assumed for M_{BA} and M_{CD}. The final F.M.'s become the sum of the corrected sway and non-sway moments in Table 6.9. The B.M. diagram is constructed in Fig. 6.16d. For bay BC, the net moments lie within the shaded areas, these being the difference between the free and fixing moment diagrams.

EXERCISES

Clapeyron's Theorem

6.1 Verify the solutions previously found from moment distribution for the continuous beams in Figs 6.11 and 6.12, using the three-moment theorem.

6.2 Draw the shear force and bending moment diagrams, showing maximum values for the continuous beam in Fig. 6.17. Determine also the slope and deflection beneath the 2 kN force. Take $E = 200$ GPa, $I = 80 \times 10^6$ mm^4.

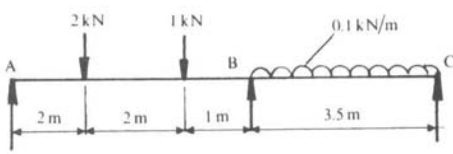


Figure 6.17

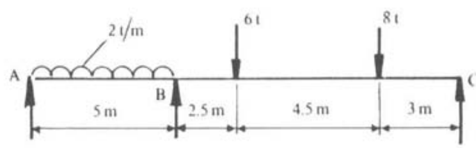


Figure 6.18

6.3 Find the central fixing moment and the position and magnitude of (i) the maximum bending moment and (ii) the maximum deflection for the continuous beam in Fig. 6.18. Take $EI = 15$ MNm^2.

6.4 A beam of flexural stiffness 30 kNm^2 is fixed horizontally at the left end A and is simply supported at the same level at distances of 4 m and 6 m from that end, as shown in Fig. 6.19. A uniformly distributed load of 2500 N/m is carried between supports B and C. Determine the deflection at a distance of 3 m from A.

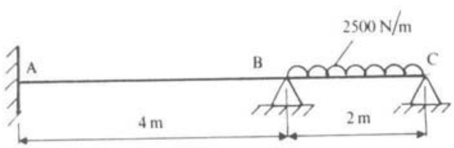


Figure 6.19

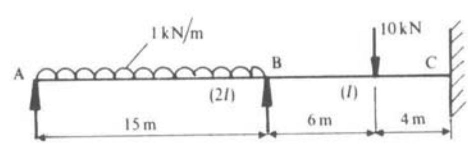


Figure 6.20

6.5 Use the three-moment theorem to construct the bending moment diagram for the beam in Fig. 6.20. Derive from the shear force diagram from the moment diagram.

Moment Distribution – Continuous Beams

6.6 A simply supported beam of constant cross-section is loaded as shown in Fig. 6.21. Draw the *F* and *M* diagrams, showing the maximum values.

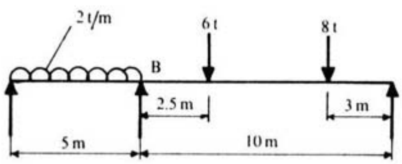


Figure 6.21

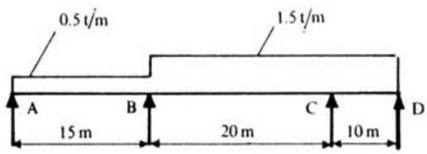


Figure 6.22

6.7 The continuous beam in Fig. 6.22 rests on four level supports. Determine the bending moments and the reaction at each support.

6.8 The beam in Fig. 6.23 is of uniform section. Draw the shear force and bending moment diagrams, indicating principal values.

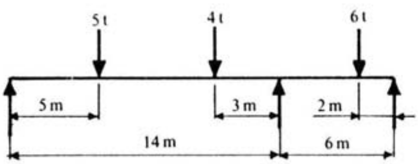


Figure 6.23

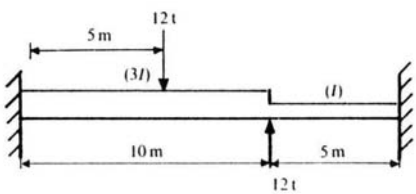


Figure 6.24

6.9 Calculate the fixed-end moments at each end of the continuous beam given in Fig. 6.24.

6.10 The continuous beam in Fig. 6.25 is built in at one end and propped at the three positions shown. If the section varies as shown, draw the *M* and *F* diagrams, indicating salient values.

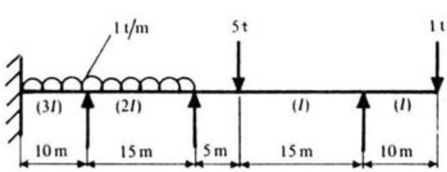


Figure 6.25

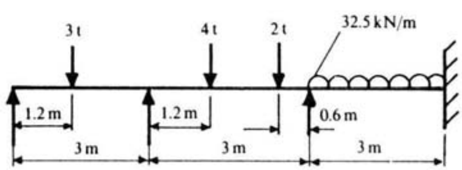


Figure 6.26

6.11 Draw the *F* and *M* diagrams for the propped cantilever in Fig. 6.26. What is the maximum moment?

6.12 The beam in Fig. 6.27 has a uniform section throughout. Draw the F and M diagrams, showing the maximum values.

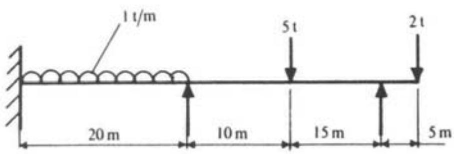


Figure 6.27

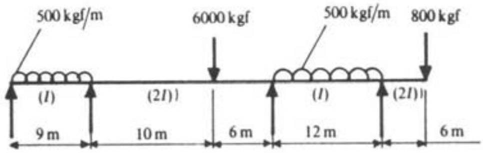


Figure 6.28

6.13 Determine the magnitude and position of the maximum bending moment for the beam in Fig. 6.28.

6.14 Calculate the bending moments at points B and C for the beam in Fig. 6.29 and hence construct the M-diagram, showing maximum values.

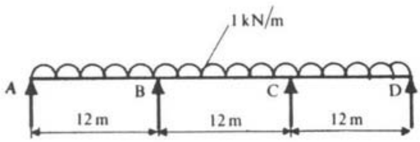


Figure 6.29

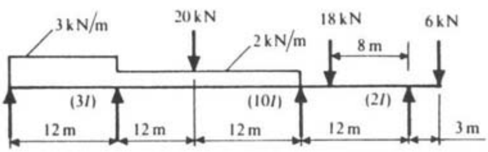


Figure 6.30

6.15 Determine all the fixing moments for the continuous beam in Fig. 6.30.

6.16 A continuous beam rests on three simple supports A, B and C. If the level of B is 20 mm below that of A and C, determine the fixing moment at B, given that lengths AB = 5 m and BC = 10 m. Take $I_{AB} = 12 \times 10^8$ mm^4, $I_{BC} = 40 \times 10^8$ mm^4 and $E = 210$ GPa.

Moment Distribution for Structures

6.17 Construct the bending moment diagrams for the equal-legged portal frame in Fig. 6.31 where EI is constant and the legs are pinned to the foundations.

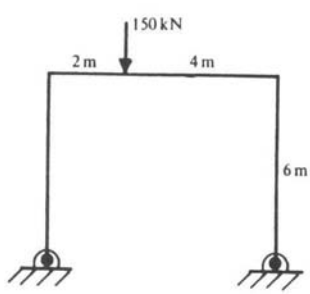


Figure 6.31

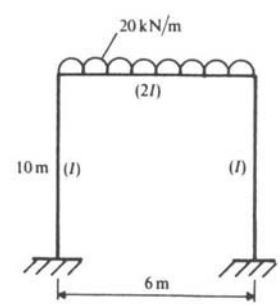


Figure 6.32

6.18 The symmetrical portal frame in Fig. 6.32 is rigidly fixed into its foundations. Establish the moment distribution diagram for the frame.

6.19 The frame ABC in Fig. 6.33 is simply supported at A and pinned at C. Find the horizontal and vertical reactions at A and C when a concentrated moment of 10 kNm is applied at the rigid joint B.

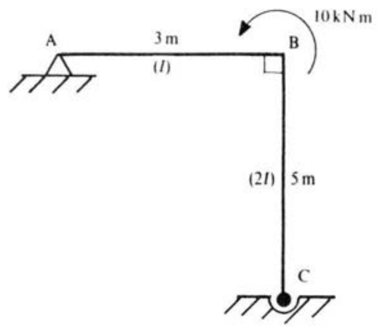


Figure 6.33

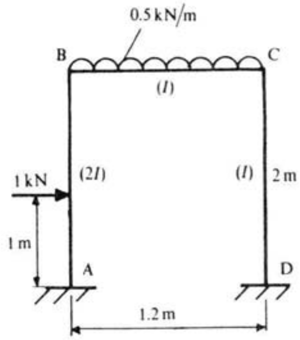


Figure 6.34

6.20 The frame in Fig. 6.34 carries a concentrated horizontal force of 1 kN in addition to uniformly distributed loading of 0.5 kN/m as shown. Determine the reactions at the rigid foundations and the horizontal displacement at C. Take $E = 75$ GPa, $I = 2.5 \times 10^3$ mm^4.

6.21 The two-storey frame in Fig. 6.35 is manufactured with a central horizontal member 4 mm too short. Determine the resulting moment distribution for the frame when this bar is elastically stretched into position. The cross-section of each bar is rectangular 100 mm × 25 mm and $E = 200$ GPa.

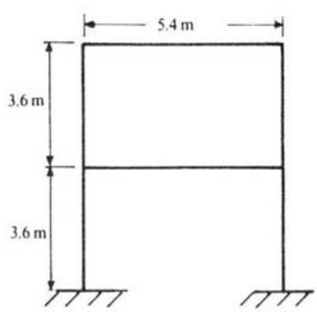


Figure 6.35

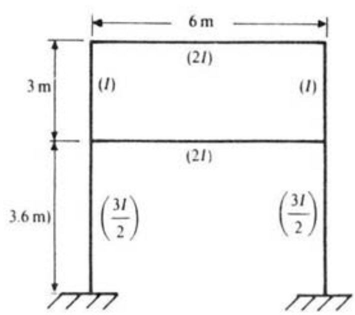


Figure 6.36

6.22 The two-storey frame shown in Fig. 6.36 is subjected to a temperature increase of 25°C. Determine the resulting induced moment distribution given that $\alpha = 15 \times 10^{-6}$/°C, $E = 210$ GPa and $I = 20 \times 10^6$ mm^4.

6.23 Construct the bending moment diagram for the unsymmetrical portal frame in Fig. 6.37, where EI is constant, taking account of side sway. Show the position and magnitude of the maximum bending moment.

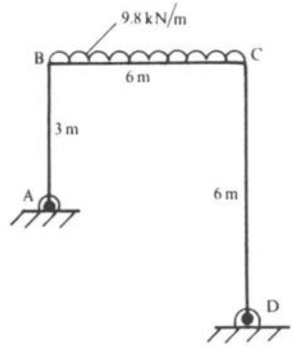


Figure 6.37

6.24 Find the position and magnitude of the maximum bending moment for the structure in Fig. 6.38 when it is fixed at A, rigid at B and hinged at C.

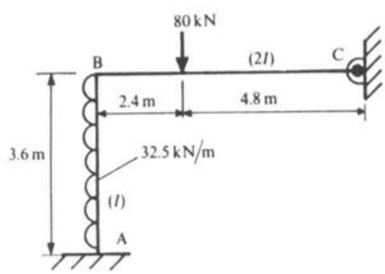


Figure 6.38

6.25 Determine the fixing moments and construct the bending moment diagram for the structure in Fig. 6.39, given that I is constant.

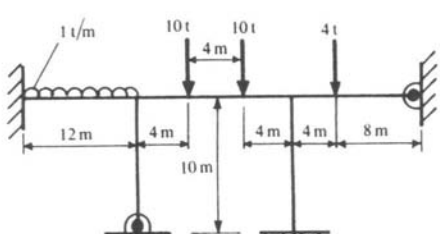


Figure 6.39

6.26 Determine the maximum of the true fixing moments for the structure in Fig. 6.40 when the effect of side sway is accounted for.

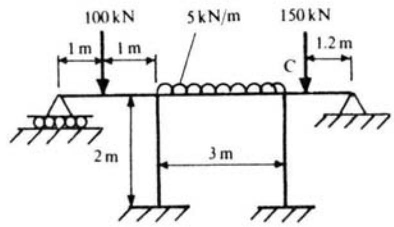


Figure 6.40