COMPLEX FUNCTIONS, RIEMANN SURFACES AND TWO-DIMENSIONAL CONFORMAL FIELD THEORY (AN INTRODUCTION)

CHAPTER 1

The theory of complex variables/complex functions has always been utilized in a better understanding of physics. More recently it has been shown that the complex formulations of physical theories (e.g., string theories, 2-dimensional conformal field theories) lead not only to a better understanding of these theories but are often instrumental in bringing out the elegance of the theory.

We shall see in later chapters (4, 5 and 11) how some of the classical concepts–infinite series, transforms of functions, Möbius transformations, Riemann surfaces, etc.,–are used in affine and current algebras, and string and superstring theories. In this chapter we give a brief account of these basic concepts. In Sec. 1 we give a few definitions, and in Sec. 2 we describe the notion of complex structure on a manifold. In Sec. 3 and Sec. 4 we describe Riemann surfaces and the 2-dimensional conformal field theory, respectively.

1 COMPLEX FUNCTIONS

1.1 Complex Plane

Geometrically, the *XY*-plane is the *complex plane* or the Z-plane, when we choose to treat the *Y* axis as the imaginary axis by replacing the y-coordinate of a point by iy. The coordinates (x, y) of a point in the plane are replaced by a single coordinate $z = x + iy$ in the Z-plane. Some of the geometric configurations are represented more simplistically using the complex coordinates. For instance, the equation of a circle with center z_0 and radius r is $|z - z_0| = r$ where $|z|$ is the absolute value: $\sqrt{z\bar{z}} = \sqrt{(x + iy)(x - iy)}$.

1.2 Analytic Function

Definition 1.1.1: A complex-valued function f of z is called (complex)-analytic or holomorphic at a point z_0 if its derivative exists at z_0 as well as at each point in some neighbourhood of z_0. It is analytic in a region R if it is analytic at every point of R. Note that the function $f(z) = z^n$ is analytic for all positive integral values of n, but $f(z) = |z|^n$ is not. An analytic function is called an *entire* function if it is analytic in the whole of complex plane. $f(z) = z^n$ given above is an entire function. If a function $f(z)$ fails to be analytic at a point z_0 but is analytic at some point in the neighbourhood of z_0, then z_0 is called a *point of singularity*. For instance, for $f(z) = \frac{1}{z}$, *zero* is a point of singularity.

1.3 Harmonic Functions

A function $g : \mathbb{R}^2 \to \mathbb{R}$ is said to be *harmonic* in a given domain D of $\mathbb{R}^2$ if it has continuous first and second partial derivatives there, and satisfies:

$$g_{xx}(x, y) + g_{yy}(x, y) = 0. \tag{1.1.1}$$

Treating D as a domain in the Z-plane, we note that if an analytic function $f(z)$ is written as $u(x, y) + iv(x, y)$, then its real and imaginary components u and v satisfy the *Cauchy-Riemann* equations:

$$\text{(a) } u_x = v_y \qquad \text{(b) } u_y = -v_x \tag{1.1.2}$$

and are harmonic functions, i.e.,

$$\text{(a) } u_{xx} + u_{yy} = 0 \qquad \text{(b) } v_{xx} + v_{yy} = 0. \tag{1.1.3}$$

Note that (1.1.3) (a) and (b) are *Laplace equations* for u and v. If two given functions $u(x, y)$ and $v(x, y)$ in D are harmonic and in addition satisfy the Cauchy-Riemann equations (1.1.2) (a) and (b) throughout D, then v is said to be *harmonic conjugate* of u. Hence a necessary and sufficient condition for a function $f(z) = u + iv$ to be analytic in D is that v be harmonic conjugate of u in D.

If one relates u and v with fluid flow, the curves $v(x, y) = c$ are the *stream lines* of the flow with $v(x, y)$ a *stream function*, and u as the velocity potential (velocity being $u_x + iu_y$), then the analytic function $f(z) = u(x, y) + iv(x, y)$ is called the *complex potential* of the flow.

If a function f is analytic at a given point of a domain, then all its derivatives are also analytic at that point. Two important formulae used for analytic functions are:

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz \tag{1.1.4a}$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi !} \int_C \frac{f(z)\, dz}{(z - z_0)^{n+1}} \quad n = 0, 1, 2, \cdots \tag{1.1.4b}$$

where C denotes a contour taken in a counterclockwise way that encloses the region R in which f is analytic and z_0 is an interior point of R. Evidently $n = 0$ gives (1.1.4a). If C is replaced by a circle $C_0 : |z - z_0| = r_0$ enclosing an open disc $|z - z_0| < r$, and function $f(z)$, which is analytic within and on the circle, assumes the maximum value $M \equiv |f(z)|$ on the circle, then using (1.1.4) we have the *Cauchy inequality*:

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r_0^n} \quad (n = 1, 2, \cdots) \tag{1.1.5}$$

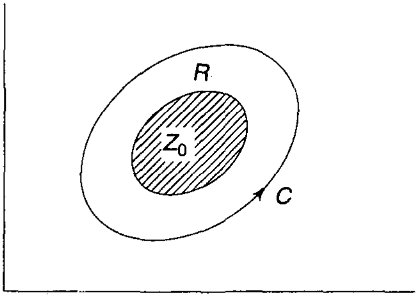

Fig. 1.1 Domain of an analytic function

If f is analytic at z_0 in $|z - z_0|$, there is a circle $|z - z_0| = r_0$ around z_0, such that f is represented as:

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n (z - z_0)^n \quad \text{(Taylor series)} \tag{1.1.6}$$

where $a_0 = f(z_0)$ and $a_n = f^{(n)}(z_0)/n!$ $(n = 1, 2, \cdots)$. If z_0 is a zero of $f(z)$, then $a_0 = 0$; if, in addition

$$f'(z_0) = f''(z_0) = \cdots = f^{(k-1)}(z_0) = 0$$

but $f^{(k)}(z_0) \neq 0$, then z_0 is called a *zero* of order k and (1.1.6) can be written as

$$f(z) = (z - z_0)^k \sum_{n=0}^{\infty} a_{n+k} (z - z_0)^n \quad (a_k \neq 0, |z - z_0| < r_0). \tag{1.1.7}$$

The above discussion implies that if z_0 is a zero of f, then there is a neighbourhood of z_0 in which f has no other zero unless f is identically zero; thus zeros of an analytic function are *isolated* points of a given region.

We now state another series expansion for an analytic function. This series, known as Laurent series, is used in string theories—in particular in the definition of affine Lie algebras.

1.4 Laurent Series

Let C_1 and C_2 be two positively oriented circles $|z - z_0| = r_1$, $|z - z_0| = r_2$ $(r_2 < r_1)$ which enclose an annular domain D, and let f be an analytic function in D and on C_1, C_2, then at each point z in D, $f(z)$ can be expanded as:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \tag{1.1.8}$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(s)\,ds}{(s - z_0)^{n+1}} \qquad (n = 0, 1, 2, \cdots) \tag{1.1.9}$$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(s)\,ds}{(s - z_0)^{-n+1}} \qquad (n = 1, 2, \cdots) \tag{1.1.10}$$

The series given by (1.1.8) is called the Laurent series of $f(z)$.

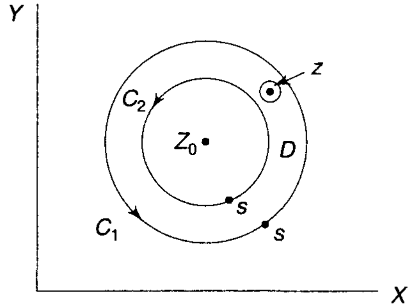

Fig. 1.2 Domain of expansion of a Laurent series

1.5 Simply Connected and Multiply Connected Domain

A domain D in which every simple closed contour encloses only points of D is called a simply connected domain. The *set* of points interior to a simple closed contour is an easy example of such a domain. (Naturally) a domain that is not simply connected is a multiply connected domain. The annular domain in Fig. 1.2 is a multiply connected domain, since the region enclosed by the circle C_2 (a simple closed contour) does not contain the points of domain.

1.6 Residues and Poles

From the definition of an isolated singularity z_0 of an analytic function, we already know that there is a neighbourhood of z_0, where f is analytic at all points except at z_0, thus there is a positive number r such that f is analytic at each point for which $0 < |z - z_0| < r$. The function in this domain can be expressed in terms of Laurent series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \cdots \tag{1.1.11}$$

where $b_1, b_2, b_3, \cdots$ are given by the integral (1.1.10).

The complex number

$$b_1 = \frac{1}{2\pi i} \int_C f(z)\, dz$$

where C is a positively oriented simple contour such that f is analytic at C and at all points interior to C except at z_0, is called the *residue* of f at z_0. Suppose that f is analytic on a positively oriented simple closed contour as well as on points interior to it except at a finite number of singular points $z_1, \cdots, z_k$, then

$$\int_C f(z)\, dz = 2\pi i\, (B_1 + B_2 + \cdots + B_k) \tag{1.1.12}$$

where B_i denotes the residue of f at z_i, $(i = 1, 2, \cdots, k)$.

If the analytic function f at an isolated singular point z_0 has Laurent expansion (1.1.11), then the sum with negative powers of $(z - z_0)$ is called the *principal part* of f at z_0. Moreover, if this principal part contains a finite number of non-zero terms the last being $b_k \neq 0$ and $b_{k+1}, b_{k+2}, \cdots$ are all zero, then the isolated singular point z_0 is called a *pole* of order k of the function f. When $k = 1$, it is called a simple pole. For instance, the function:

$$f(z) = \frac{z^2 - 3z + 5}{z - 3} = 3 + (z - 3) + \frac{5}{z - 3} \quad (|z - 3| > 0) \tag{1.1.13}$$

has an isolated singularity at $z = 3$, which is a simple pole. The residue there is 5, whereas the function

$$\frac{\sinh z}{z^6} = \frac{1}{z^5} + \frac{1}{3!z^3} + \frac{1}{5!z} + \frac{z}{7!} + \frac{z^3}{9!} + \cdots \quad (|z| > 0) \tag{1.1.14}$$

has an isolated singularity at $z = 0$ which is a pole of order 5, and the residue there is $\frac{1}{5!}$.

When the principal part of f at z_0 has an infinite number of terms, z_0 is called an *essential singularity*. An example of essential singularity is given by the function:

$$\exp\left(\frac{1}{z}\right) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\frac{1}{z^n} \quad (|z|) > 0 \tag{1.1.15}$$

where $z = 0$ is the singular point. Details on results listed above can be found in any elementary text on complex variables (see for instance [1] and [7]).

Finally we give two examples to illustrate some of the ideas introduced in this section.

Example 1.1.2: Let λ be a real constant, the transformation

$$z = \frac{\lambda i - w}{\lambda i + w} \tag{1.1.16}$$

between the W-plane and the Z-plane satisfies:

$$\text{(a)} \ |z| < 1 \text{ iff im } w > 0, \qquad \text{(b)} \ |z| = 1 \text{ iff im } w = 0 \tag{1.1.17}$$

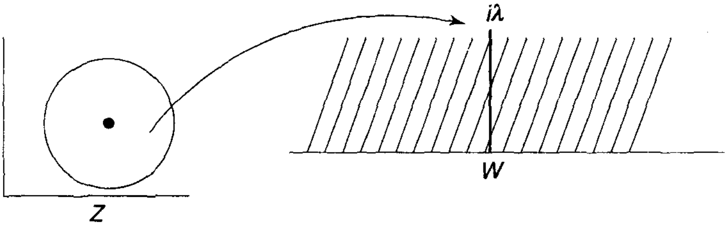

Fig. 1.3 Cayley transform of the circle

and

$$\frac{dz}{iz} = \frac{2\lambda}{\lambda^2 + w^2}\,dw. \tag{1.1.18}$$

The above transformation is called the *Cayley transform* of the circle into the upper half plane. From (1.1.16) we have:

$$z = -1 + \frac{2\lambda i}{\lambda i + w}$$

therefore

$$dz = -\frac{2\lambda i}{(\lambda i + w)^2}\,dw$$

or

$$\frac{dz}{z} = -\frac{2\,\lambda i}{(\lambda i + w)^2} \cdot \frac{(\lambda i + w)}{(\lambda i - w)}\,dw = +\frac{2\,\lambda i}{\lambda^2 + w^2}\,dw.$$

Statements (1.1.17) (a) and (b) can be established by writing $z = x + iy$ and $w = u + iv$ and simplifying thereafter. Note that w as a function of z can be written as

$$w = \lambda i \frac{1-z}{1+z}$$

and that when $\lambda = \frac{i}{2}$ in (1.1.16), it becomes the Möbius transformation (see Sec. 3).

Example 1.1.3: The line integral $(dx^2 + dy^2)$ in complex coordinates can be expressed as a product $dzd\bar{z}$. This is immediate from the definition $z = x + iy$ which implies $dz = dx + idy$ and $d\bar{z} = dx - idy$.

1.7 Elliptic Curves

Let $\Gamma = \Gamma_1 + i\Gamma_2$ be an element of $\mathbb{C}$ with $\Gamma_2 > 0$. The set of points defined by:

$$\mathbb{C}/(2\pi\mathbb{Z} + 2\pi\Gamma\mathbb{Z}) \tag{1.1.19}$$

is called an elliptic curve (denoted E_Γ) associated to the complex number Γ. Given below is a simple diagram pertaining to the area enclosed by E_Γ which can easily be seen to be $4\pi^2\Gamma_2$ (see p. 15 in 5.[14] for its use in string path integrals).

The area of the parallelogram $OACB$ is evidently the same as that of $B'C'CB$ with arm lengths 2π and $2\pi\Gamma_2$, respectively.

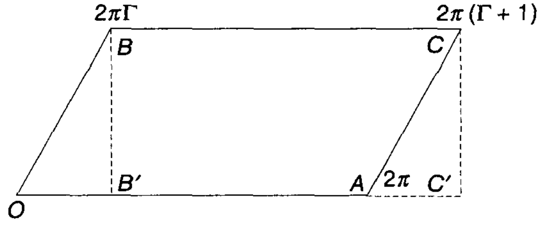

Fig. 1.4

2 COMPLEX STRUCTURES ON A MANIFOLD, KÄHLER METRIC

2.1 Complex Manifold *M*

Definition 1.2.1: In layman's language, an n-dimensional complex manifold is a real manifold of dimension $2n$ if there can (always) be found complex coordinates on it with holomorphic (i.e., analytic on real manifold) transition functions. More precisely, let M be a complex manifold of (complex) dimension n and z^λ, $(\lambda = 1, 2, \cdots, n)$ a system of complex local coordinates on an open subset U of M. Let $z^\lambda = x^\lambda + iy^\lambda$; then $(x^1, y^1, \cdots, x^n, y^n)$ is a system of (real) local coordinates of the differentiable manifold M on U.

2.2 Complex Structure on M

For each $\mathbf{x} \in U$ we define a linear transformation $J_\mathbf{x}$ of the tangent space $T_\mathbf{x}(M)$ that transforms the pair

$$\left(\frac{\partial}{\partial x^\lambda}, \frac{\partial}{\partial y^\lambda}\right)$$