Chapter 4

Central Conservative Forces

The most important forces of this type are the gravitational and electrostatic forces, which obey the 'inverse square law'. Much of this chapter will therefore be devoted to this special case. We begin, however, by discussing a problem for which it is particularly easy to solve the equations of motion — the three-dimensional analogue of the harmonic oscillator discussed in §2.2.

4.1 The Isotropic Harmonic Oscillator

We consider a particle moving under the action of a central restoring force proportional to its distance from the origin,

$$\boldsymbol{F} = -k\boldsymbol{r}, \tag{4.1}$$

with k constant. The corresponding equation of motion is

$$m\ddot{\boldsymbol{r}} + k\boldsymbol{r} = \mathbf{0},$$

or, in terms of components,

$$m\ddot{x} + kx = 0, \qquad m\ddot{y} + ky = 0, \qquad m\ddot{z} + kz = 0.$$

The equation of motion for each co-ordinate is thus identical with the equation (2.13) for the simple harmonic oscillator. This oscillator is called *isotropic* because all directions (not only those along the axes!) are equivalent. The *anisotropic* oscillator (which we shall not discuss) is described by similar equations, but with *different* constants in the three equations.

The general solution is again given by (2.19):

$$x = c_x \cos\omega t + d_x \sin\omega t,$$
$$y = c_y \cos\omega t + d_y \sin\omega t,$$
$$z = c_z \cos\omega t + d_z \sin\omega t,$$

where $\omega^2 = k/m$. In vector notation, the solution is

$$\boldsymbol{r} = \boldsymbol{c}\cos\omega t + \boldsymbol{d}\sin\omega t. \tag{4.2}$$

Clearly, the motion is periodic, with period $\tau = 2\pi/\omega$. (Other forms of the solution can be given, analogous to (2.20) or (2.23).)

As in the one-dimensional case, the arbitrary constant vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ can be fixed by the initial conditions. If at $t = 0$ the particle is at $\boldsymbol{r}_0$ and moving with velocity $\boldsymbol{v}_0$, then

$$\boldsymbol{c} = \boldsymbol{r}_0, \qquad \boldsymbol{d} = \boldsymbol{v}_0/\omega. \tag{4.3}$$

We can easily verify the conservation laws for angular momentum and energy. From (4.2) it is clear that $\boldsymbol{r}$ always lies in the plane of $\boldsymbol{c}$ and $\boldsymbol{d}$, so that the direction of $\boldsymbol{J}$ is fixed. Using

$$\dot{\boldsymbol{r}} = -\omega\boldsymbol{c}\sin\omega t + \omega\boldsymbol{d}\cos\omega t,$$

we find explicitly

$$\boldsymbol{J} = m\boldsymbol{r}\wedge\dot{\boldsymbol{r}} = m\omega\boldsymbol{c}\wedge\boldsymbol{d} = m\boldsymbol{r}_0\wedge\boldsymbol{v}_0, \tag{4.4}$$

which is obviously a constant.

The potential energy function corresponding to the force (4.1) — or (3.10) — is just (3.9), namely

$$V = \tfrac{1}{2}kr^2 = \tfrac{1}{2}k(x^2 + y^2 + z^2). \tag{4.5}$$

Thus, evaluating the energy, we find

$$E = \tfrac{1}{2}m\dot{\boldsymbol{r}}^2 + \tfrac{1}{2}k\boldsymbol{r}^2 = \tfrac{1}{2}k(\boldsymbol{c}^2 + \boldsymbol{d}^2) = \tfrac{1}{2}m\boldsymbol{v}_0^2 + \tfrac{1}{2}k\boldsymbol{r}_0^2, \tag{4.6}$$

which is again a constant.

To find the shape of the particle orbit, it is convenient to rewrite (4.2) in a slightly different, but equivalent, form. If θ is any fixed angle, we can write

$$\boldsymbol{r} = \boldsymbol{a}\cos(\omega t - \theta) + \boldsymbol{b}\sin(\omega t - \theta), \tag{4.7}$$

where

$$\begin{aligned} \boldsymbol{c} &= \boldsymbol{a}\cos\theta - \boldsymbol{b}\sin\theta, \\ \boldsymbol{d} &= \boldsymbol{a}\sin\theta + \boldsymbol{b}\cos\theta. \end{aligned} \tag{4.8}$$

or, equivalently,

$$\begin{aligned} \boldsymbol{a} &= \boldsymbol{c}\cos\theta + \boldsymbol{d}\sin\theta, \\ \boldsymbol{b} &= -\boldsymbol{c}\sin\theta + \boldsymbol{d}\cos\theta. \end{aligned}$$

We can now choose θ so that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular. This requires

$$0 = \boldsymbol{a}\cdot\boldsymbol{b} = -(\boldsymbol{c}^2 - \boldsymbol{d}^2)\sin\theta\cos\theta + \boldsymbol{c}\cdot\boldsymbol{d}(\cos^2\theta - \sin^2\theta),$$

or

$$\tan 2\theta = \frac{2\boldsymbol{c}\cdot\boldsymbol{d}}{\boldsymbol{c}^2 - \boldsymbol{d}^2}.$$

We are now free to choose our axes so that the x-axis is in the direction of $\boldsymbol{a}$ and the y-axis in the direction of $\boldsymbol{b}$. Equation (4.7) then becomes

$$x = a\cos(\omega t - \theta), \qquad y = b\sin(\omega t - \theta), \qquad z = 0.$$

The equation of the orbit is obtained by eliminating the time from these equations. It is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad z = 0. \tag{4.9}$$

This is the well-known equation of an ellipse with centre at the origin, and semi-axes a and b. (See Appendix B, Eq. (B.2), and Fig. 4.1.) The vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ are what is known in geometry as a pair of conjugate semi-diameters of the ellipse.

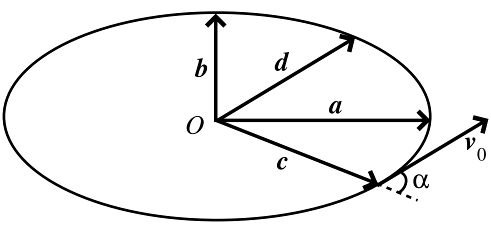

Fig. 4.1

The simplest way to determine the magnitudes of the semi-axes a and b from the initial conditions (that is, from $\boldsymbol{c}$ and $\boldsymbol{d}$) is to use the constancy of E and $\boldsymbol{J}$. Equating the expressions for these quantities, (4.6) and (4.4), to the corresponding expressions in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, we obtain

$$a^2 + b^2 = c^2 + d^2,$$
$$\boldsymbol{a} \wedge \boldsymbol{b} = \boldsymbol{c} \wedge \boldsymbol{d}.$$

(These equations may also be obtained directly from (4.8).) If α is the angle between $\boldsymbol{c}$ and $\boldsymbol{d}$, the second equation yields $ab = cd \sin\alpha$. Hence eliminating b (or a) between these two equations, we obtain for a^2 (or b^2) the quadratic equation

$$a^4 - (c^2 + d^2)a^2 + c^2d^2 \sin^2\alpha = 0. \tag{4.10}$$

Conventionally, a is taken to be the larger root, and called the *semi-major axis*.

4.2 The Conservation Laws

We now consider the general case of a central, conservative force. It corresponds, as we saw in §3.7, to a potential energy function $V(r)$ depending only on the radial co-ordinate r. There are two conservation laws, one for energy,

$$\tfrac{1}{2}m\dot{\boldsymbol{r}}^2 + V(r) = E = \text{constant},$$

and one for angular momentum

$$m\boldsymbol{r} \wedge \dot{\boldsymbol{r}} = \boldsymbol{J} = \text{constant}.$$

According to the discussion of §3.4, the second of these laws implies that the motion is confined to a plane, so that the problem is effectively two-dimensional. Introducing polar co-ordinates r, θ in this plane, we may write the two conservation laws in the form

$$\begin{aligned} \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) &= E, \\ mr^2\dot{\theta} &= J. \end{aligned} \tag{4.11}$$

A great deal of information about the motion can be obtained directly from these equations, without actually having to solve them to find r and

θ as functions of the time. We note that $\dot{\theta}$ may be eliminated to yield an equation involving only r and $\dot{r}$:

$$\tfrac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E. \tag{4.12}$$

We shall call this the *radial energy equation*. For a given value of J, it has precisely the same form as the one-dimensional energy equation with an *effective potential energy* function

$$U(r) = \frac{J^2}{2mr^2} + V(r). \tag{4.13}$$

It is easy to understand the physical significance of the extra term $J^2/2mr^2$ in this 'potential energy' function. It corresponds to a 'force' J^2/mr^3. This is precisely the '*centrifugal force*' $mr\dot{\theta}^2$ (the first term on the right hand side of the radial equation in (3.48)), expressed in terms of the constant J rather than the variable $\dot{\theta}$.

We can use the radial energy equation just as we did in §2.1. Since $\dot{r}^2$ is positive, the motion is limited to the range of values of r for which

$$U(r) = \frac{J^2}{2mr^2} + V(r) \le E. \tag{4.14}$$

The maximum and minimum radial distances are given by the values of r for which the equality holds.

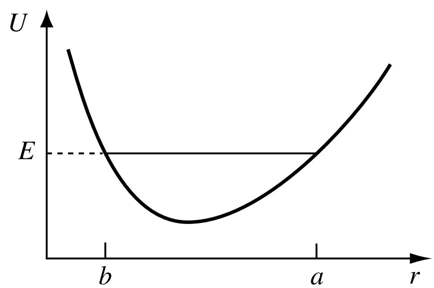

Fig. 4.2

As an example, let us consider again the case of the isotropic harmonic oscillator, for which $V(r) = \frac{1}{2}kr^2$. The corresponding effective potential energy function $U(r)$ is shown in Fig. 4.2. It has a minimum (corresponding to a position of stable equilibrium in the one-dimensional case) at

$$r = \left(\frac{J^2}{mk}\right)^{1/4}. \tag{4.15}$$

When the value of E is equal to the minimum value of U, $\dot{r}$ must always be zero, and r is fixed at the minimum position. In this case, the particle must move in a circle around the origin. It is interesting to note that we could also obtain (4.15) by equating the attractive force kr to the centrifugal force in the circular orbit, J^2/mr^3.

For any larger value of E, the motion is confined to the region $b \le r \le a$ between the two limiting values of r, given by the solutions of the equality in (4.14). If the particle is initially at a distance r_0 from the origin, and moving with velocity v_0 in a direction making an angle α with the radius vector (as in Fig. 4.1), then the values of E and J are

$$E = \tfrac{1}{2}mv_0^2 + \tfrac{1}{2}kr_0^2, \qquad J = mr_0v_0 \sin\alpha.$$

Thus the equation whose roots are a and b becomes (on multiplying by $2r^2/k$)

$$r^4 - \left(r_0^2 + \frac{m}{k}v_0^2\right) r^2 + \frac{m}{k}r_0^2v_0^2 \sin^2\alpha = 0.$$

By (4.3), this is identical with the equation (4.10) found previously.

4.3 The Inverse Square Law

We now consider a force

$$\boldsymbol{F} = \frac{k}{r^2}\hat{\boldsymbol{r}},$$

where k is a constant. The corresponding potential energy function is $V(r) = k/r$. The constant k may be either positive or negative; in the first case, the force is repulsive, and in the second, attractive.

The radial energy equation for this case is

$$\tfrac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + \frac{k}{r} = E. \tag{4.16}$$

It corresponds to the 'effective potential energy' function

$$U(r) = \frac{J^2}{2mr^2} + \frac{k}{r}.$$

Repulsive case

We suppose first that $k > 0$. Then $U(r)$ decreases monotonically from $+\infty$ at $r = 0$ to 0 at $r = \infty$ (see Fig. 4.3). Thus it has no minima, and circular

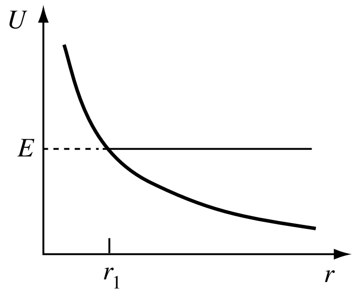

Fig. 4.3

motion is impossible, as is physically obvious. For any positive value of E, there is a minimum value of r, r_1 say, which is the unique positive root of $U(r) = E$, but no maximum value. If the radial velocity is initially inward, the particle must follow an orbit on which r decreases to r_1 (at which point the velocity is purely transverse), and then increases again without limit. As is well known, and as we shall show in the next section, the orbit is actually a hyperbola.

Let us consider an example.

***Example:* Distance of closest approach**

> A charged particle of charge q moves in the field of a fixed point charge q', with $qq' > 0$. Initially it is approaching the centre of force with velocity v (at a large distance) along a path which, if continued in a straight line, would pass the centre at a distance b. Find the distance of closest approach.

This distance b is known as the *impact parameter*, and will appear frequently in our future work. (See Fig. 4.4.) Since the particle starts from a great distance, its initial potential energy is negligible. (Note that we have chosen the arbitrary additive constant in V so that $V(\infty) = 0$.) Thus the total energy is simply

$$E = \tfrac{1}{2}mv^2. \tag{4.17}$$

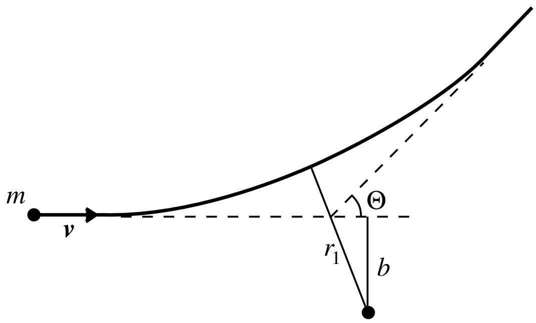

Fig. 4.4

Moreover, since the component of $\boldsymbol{r}$ perpendicular to $\boldsymbol{v}$ is b, the angular momentum is

$$J = mvb. \tag{4.18}$$

The distance of closest approach, r_1, is obtained by substituting these values, and $k = qq'/4\pi\epsilon_0$, into the radial energy equation, and then setting $\dot{r} = 0$. This yields

$$r_1^2 - 2ar_1 - b^2 = 0, \qquad \text{where} \qquad a = \frac{qq'}{4\pi\epsilon_0 mv^2}. \tag{4.19}$$

The required solution is the positive root

$$r_1 = a + \sqrt{a^2 + b^2}. \tag{4.20}$$

Attractive case

We now suppose that $k < 0$, as in the gravitational case. It will be useful to define a quantity l, with the dimensions of length, by

$$l = \frac{J^2}{m|k|}. \tag{4.21}$$

Then the effective potential energy function is

$$U(r) = |k|\left(\frac{l}{2r^2} - \frac{1}{r}\right).$$

It is plotted in Fig. 4.5. Evidently, $U(\frac{1}{2}l) = 0$, and $U(r)$ has a minimum at $r = l$, with minimum value $U(l) = -|k|/2l$.

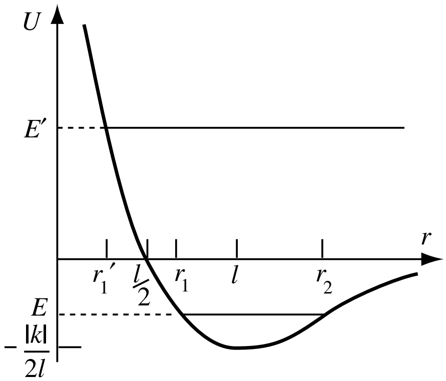

Fig. 4.5

Here, different types of motion are possible according to the value of E. We may distinguish four cases:

(a) $E = -\dfrac{|k|}{2l}$.

This is the minimum value of U. Hence, $\dot{r}$ must always be zero, and the particle must move in a circle of radius l. It is easy to find the orbital velocity v from the kinetic energy $T = E - V$. Since the potential energy is $V = -|k|/l$, we must have

$$T = \frac{|k|}{2l}, \qquad \text{whence} \qquad v = \sqrt{\frac{|k|}{ml}}. \tag{4.22}$$

This can also be found by equating the attractive force $|k|/l^2$ to the 'centrifugal force' mv^2/l.

Note the interesting result that for a circular orbit the potential energy is always twice as large in magnitude as the kinetic energy (see Problem 21).

(b) $-\dfrac{|k|}{2l} < E < 0$.

This case is illustrated by the lower line (E) in Fig. 4.5. The radial distance is limited between a minimum distance r_1 and a maximum distance r_2. As we shall see, the orbit is in fact an ellipse.

(c) $E = 0$.

In this case, there is a minimum distance, $r_1 = \frac{1}{2}l$, but the maximum distance r_2 is infinite. Thus the particle has just enough energy to escape to infinity, with kinetic energy tending to zero at large distances. The orbit will be shown to be a parabola.

(d) $E > 0$.

This corresponds to the upper line (E') in Fig. 4.5. Again, there is a minimum distance but no maximum distance. Now, however, the particle can escape to infinity with non-zero limiting velocity. The orbit is in fact a hyperbola.

As a simple example of the use of these equations, we compute the escape velocity from the Earth.

***Example:* Escape velocity**

What is the minimum velocity with which a projectile launched from the surface of the Earth (taken to be a sphere of mass M and radius R) can escape, and how does this depend on the angle of launch?

Suppose the projectile is launched with velocity v at an angle α to the vertical. The energy and angular momentum are

$$E = \tfrac{1}{2}mv^2 - \frac{GMm}{R}, \qquad J = mRv\sin\alpha, \tag{4.23}$$

where G is Newton's gravitational constant. To express the energy in terms of more familiar quantities, we note that the gravitational force on a particle at the Earth's surface is

$$mg = \frac{GMm}{R^2},$$

from which we obtain the useful result

$$GM = R^2 g. \tag{4.24}$$

Thus $E = \frac{1}{2}mv^2 - Rgm$. The projectile will escape to infinity provided that $E \geq 0$; or equivalently that its velocity exceeds the *escape velocity*,

$$v_{\mathrm{e}} = \sqrt{2Rg}.$$

Note that this condition is *independent* of the angle of launch α (so long as we neglect atmospheric drag). Using the values $R = 6370\,\mathrm{km}$, $g = 9.81\,\mathrm{m\,s^{-2}}$, we find for the escape velocity from the Earth

$$v_\mathrm{e} = 11.2\,\mathrm{km\,s^{-1}}.$$

If the projectile is launched with a speed less than escape velocity, it will reach a maximum height and then fall back. (This is a generalization of the problem discussed in §3.2.) The problem of finding this distance provides another useful example.

***Example:* Maximum height of projectile**

If the launch speed v is equal to the circular orbit speed v_c in an orbit just above the Earth's surface, how high will the projectile reach for a given angle of launch α?

The maximum distance r_2 from the centre is the larger root of the equation $U(r) = E$, or, with the values (4.23) for E and J,

$$(2Rg - v^2)r^2 - 2R^2gr + R^2v^2\sin^2\alpha = 0.$$

The circular orbit velocity, by (4.22) and (4.24), is

$$v_\mathrm{c} = \sqrt{Rg} = 7.9\,\mathrm{km\,s^{-1}}.$$

Thus the equation reduces to $r^2 - 2Rr + R^2\sin^2\alpha = 0$, so that

$$r_2 = R(1 + \cos\alpha).$$

For a vertical launch, the maximum distance is $2R$. For any other angle, it is less, and in the limit of an almost horizontal launch, the orbit is nearly a circle of radius R.

This example illustrates the kind of problem which may readily be solved by using the radial energy equation. It is particularly useful when we are interested only in r, and not in the polar angle θ or the time t.

Energy levels of the hydrogen atom

This is not a problem that can be solved by the methods of classical mechanics alone. However, the energy levels derived from quantum mechanics can also be obtained by imposing on the classical orbits an *ad hoc* 'quantization rule'. Indeed, historically, this is how they were first obtained. According to Bohr's 'old quantum theory', the electron in an atom cannot occupy any orbit, but only a certain discrete set of orbits. In the case of circular orbits, the quantization rule he imposed was that the angular momentum J be an integer multiple of $\hbar$ (Planck's constant divided by 2π). The constant k in this case is $k = -e^2/4\pi\epsilon_0$, where e is the electric charge on the proton, or minus the charge on the electron. Thus for $J = n\hbar$ the radius a_n of the orbit is, by (4.21),

$$a_n = \frac{4\pi\epsilon_0 J^2}{me^2} = n^2 a_1,$$

where a_1, the radius of the first Bohr orbit (usually denoted by a_0), is

$$a_1 = \frac{4\pi\epsilon_0 \hbar^2}{me^2} = 5.29 \times 10^{-11}\,\mathrm{m}.$$

The corresponding energy levels are

$$E_n = -\frac{e^2}{8\pi\epsilon_0 a_n} = -\frac{1}{n^2}\frac{e^2}{8\pi\epsilon_0 a_1}.$$

These values agree well with the energies of atomic transitions, as determined from the spectrum of hydrogen.

4.4 Orbits

We now turn to the problem of determining the *orbit* of a particle moving under a central conservative force. This can be done by eliminating the time from the two conservation equations, (4.11), to obtain an equation relating r and θ. The simplest way of doing this is to work not with r itself, but with the variable $u = 1/r$, and look for an equation determining u as a function of θ. Now

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta},$$

whence

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\dot{\theta} = -r^2\dot{\theta}\frac{\mathrm{d}u}{\mathrm{d}\theta} = -\frac{J}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}.$$

Thus substituting for $\dot{r}$ in the radial energy equation (4.12), we obtain

$$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + V = E, \tag{4.25}$$

in which of course V is to be regarded as a function of $1/u$. This equation can be integrated (numerically, if not analytically) to give the equation of the orbit.

We shall consider explicitly only the case of the inverse square law, for which $V = ku$. We shall treat both cases $k > 0$ and $k < 0$ together, by writing $V = \pm|k|u$. It will be useful, as in the preceding section, to define the length parameter

$$l = \frac{J^2}{m|k|} \tag{4.26}$$

Then, multiplying (4.25) by $2/|k|$, we obtain

$$l\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + lu^2 \pm 2u = \frac{2E}{|k|},$$

where the upper sign refers to the repulsive case, $k > 0$, and the lower to the attractive case, $k < 0$.

To solve this equation, we multiply both sides by l and add 1 to 'complete the square'. We now introduce the new variable

$$z = lu \pm 1, \qquad \text{so that} \qquad \frac{\mathrm{d}z}{\mathrm{d}\theta} = l\frac{\mathrm{d}u}{\mathrm{d}\theta}.$$

Then we may write the equation as

$$\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + z^2 = \frac{2El}{|k|} + 1 = e^2, \tag{4.27}$$

say, defining a new dimensionless constant e. Note that since the left hand side is a sum of squares, it can have a solution only when the the right hand side is positive, in agreement with our earlier result that the minimum value of E is $-|k|/2l$.

The general solution of this equation is

$$z = lu \pm 1 = e\cos(\theta - \theta_0),$$

where θ_0 is an arbitrary constant of integration. Thus finally, replacing u by $1/r$ and multiplying by r, we find that the orbit equation is, in the repulsive case

$$r[e\cos(\theta - \theta_0) - 1] = l, \qquad (k > 0) \tag{4.28}$$

and in the attractive case

$$r[e\cos(\theta - \theta_0) + 1] = l. \qquad (k < 0) \tag{4.29}$$

These are the polar equations of *conic sections*, referred to a *focus* as origin (see Appendix B). The constant e, the *eccentricity*, determines the shape of the orbit; l, called the *semi-latus rectum*, determines its scale; and θ_0 its orientation relative to the coordinate axes.

In the repulsive case, e must be greater than unity, since otherwise the square bracket in (4.28) is always negative. Therefore, by (4.27), E must be positive.

In the attractive case, $e = 0$ when E has its minimum value, $-|k|/2l$; the orbit is then the circle $r = l$. So long as $e < 1$, or $E < 0$, the square bracket in (4.29) is always positive, and there is a value of r for every θ. Thus the orbit is closed. When $e \geq 1$, or $E \geq 0$, r can become infinite when the square bracket vanishes. This is in agreement with our previous conclusion that the particle will escape to infinity if and only if its total energy is positive.

Note that r takes its minimum value when $\theta = \theta_0$. Thus θ_0 specifies the direction of the point of closest approach. In the attractive case, the constant l also has a simple geometrical meaning. It is the radial distance at right angles to this direction; that is, $r = l$ when $\theta = \theta_0 \pm \frac{1}{2}\pi$.

Elliptic orbits ($E < 0$, $0 \leq e < 1$)

For most applications, it is better to use the orbit equation directly in polar form. It is, however, straightforward to rewrite it in the possibly more familiar Cartesian form (see Appendix B). If we choose the axes so that $\theta_0 = 0$, we obtain after some algebra

$$\frac{(x + ae)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where

$$a = \frac{l}{1 - e^2} = \frac{|k|}{2|E|} \qquad \text{and} \qquad b^2 = al = \frac{J^2}{2m|E|} \tag{4.30}$$

This is the equation of an *ellipse* with centre at $(-ae, 0)$, and semi-axes a and b. (See Fig. 4.6) It is useful to note that the *semi-major axis* a is

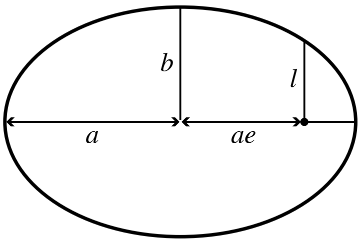

Fig. 4.6

determined by the value of the energy E, while the semi-latus rectum l is fixed by the angular momentum J.

The time taken by the particle to traverse any part of its orbit may be found from the relation (3.26) between angular momentum and rate of sweeping out area,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{J}{2m}.$$

All we have to do is to evaluate the area swept out by the radius vector, and multiply by $2m/J$. In particular, since the area of the ellipse is $A = \pi ab$, the orbital period is $\tau = 2m\pi ab/J$. Thus by (4.26) and (4.30),

$$\left(\frac{\tau}{2\pi}\right)^2 = \frac{m^2a^2b^2}{J^2} = \frac{m^2a^3l}{m|k|l},$$

and thus

$$\left(\frac{\tau}{2\pi}\right)^2 = \frac{m}{|k|}a^3. \tag{4.31}$$

(It is easy to verify directly from (4.22) that this gives the correct period for a circular orbit of radius a.) For a planet or satellite orbiting a central body of mass M,

$$\left(\frac{\tau}{2\pi}\right)^2 = \frac{a^3}{GM}. \tag{4.32}$$

This yields *Kepler's third law* of planetary motion: the square of the orbital period is proportional to the cube of the semi-major axis.

Example: **Time spent in the two halves of Earth's orbit**

If the Earth's orbit is divided in two by its minor axis, how much longer does it spend in one half than the other?

The eccentricity of Earth's orbit is $e = 0.0167$. It is clear from Fig. 4.6 that the areas swept out by the position vector of the Earth in the two halves are equal to half the area of the ellipse $\pm$ the area of a triangle with base $2b$ and height ae, namely $\frac{1}{2}\pi ab \pm aeb$. Thus the times taken are $(\frac{1}{2} \pm e/\pi)$ years. The difference between the two is $2e/\pi$ years, or 3.88 days.

Hyperbolic orbits ($E > 0$, $e > 1$)

For both the repulsive and attractive cases, the Cartesian equation of the orbit is

$$\frac{(x-ae)^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where

$$a = \frac{l}{e^2-1} = \frac{|k|}{2E} \qquad \text{and} \qquad b^2 = al = \frac{J^2}{2mE}. \tag{4.33}$$

This is the equation of a *hyperbola* with centre at $(+ae, 0)$ and semi-axes a and b. (See Fig. 4.7.) One branch of the hyperbola (on the left in the figure) corresponds to the orbit in the attractive case and the other to the orbit in the repulsive case. As before, a is determined by the energy E, and l by

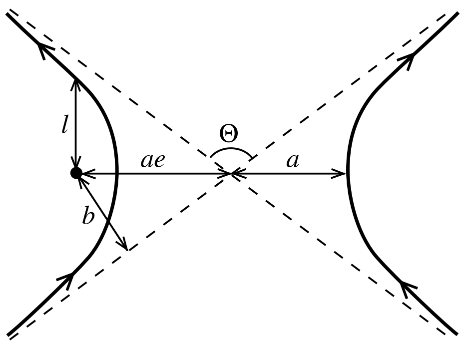

Fig. 4.7

the angular momentum J. Note that by (4.17) and (4.18), the *semi-minor axis* b is identical with the impact parameter introduced earlier.

The directions in which r becomes infinite are, in the repulsive case, $\theta = \pm\arccos(1/e)$, and, in the attractive case, $\theta = \pm[\pi - \arccos(1/e)]$. In both cases, therefore, the angle through which the particle is deflected from its original direction of motion is

$$\Theta = \pi - 2\arccos(1/e). \tag{4.34}$$

This angle Θ is called the *scattering angle*. For later use, it will be helpful to find the relation between the impact parameter b, the scattering angle Θ, and the limiting velocity v at infinity. From (4.34), we have $e = \sec\frac{1}{2}(\pi-\Theta)$. Substituting in (4.33), this gives

$$b^2 = a^2(e^2-1) = a^2[\sec^2\tfrac{1}{2}(\pi-\Theta) - 1] = a^2\cot^2\tfrac{1}{2}\Theta.$$

But also from (4.33), using $E = \frac{1}{2}mv^2$, we have $a = |k|/mv^2$. Thus we obtain

$$b = \frac{|k|}{mv^2}\cot\tfrac{1}{2}\Theta. \tag{4.35}$$

Let us complete this section by considering a rather special orbit.

Example: **Orbit of a comet**

The minimum distance of a comet from the Sun is half the radius of the Earth's orbit (assumed circular), and its velocity at that point is twice the orbital velocity of the Earth. Find its velocity when it crosses the Earth's orbit, and the angle at which the orbits cross. Will the comet subsequently escape from the solar system? What kind of orbit does it follow?

At perihelion, $r = a_{\mathrm{E}}/2$ and $v = 2v_{\mathrm{c,E}}$. Hence the energy is

$$E = 2mv_{\mathrm{c,E}}^2 - \frac{2GM_{\mathrm{S}}m}{a_{\mathrm{E}}} = 0,$$

using (4.22). It follows that the comet's velocity v when it crosses the Earth's orbit is given by

$$\tfrac{1}{2}mv^2 = \frac{GM_{\mathrm{S}}m}{a_{\mathrm{E}}} = mv_{\mathrm{c,E}}^2,$$

whence $v = \sqrt{2}v_{\mathrm{c,E}}$. The angular momentum is $J = ma_{\mathrm{E}}v_{\mathrm{c,E}}$, whence by (4.26), $l = a_{\mathrm{E}}$. Thus the angle α at which the orbits cross is given by

$$ma_{\mathrm{E}}v\cos\alpha = J = ma_{\mathrm{E}}v_{\mathrm{c,E}},$$

which yields $\alpha = 45°$.

Since $E = 0$, the comet has just enough energy to escape from the solar system. Its orbit is a parabola.

4.5 Scattering Cross-sections

One of the most important ways of obtaining information about the structure of small bodies (for example atomic nuclei) is to bombard them with particles and measure the number of particles scattered in various directions. The angular distribution of scattered particles will depend on the shape of the target and on the nature of the forces between the particles and the target. To be able to interpret the results of such an experiment, we must know how to calculate the expected angular distribution when the forces are known.

We shall consider first a particularly simple case. We suppose that the target is a fixed, hard (that is, perfectly elastic) sphere of radius R, and that a uniform, parallel beam of particles impinges on it. Let the particle flux in the beam, that is, the number of particles crossing unit area normal to the beam direction per unit time, be f. Then the number of particles which strike the target in unit time is

$$w = f\sigma, \tag{4.36}$$

where σ is the cross-sectional area presented by the target, namely

$$\sigma = \pi R^2. \tag{4.37}$$

Now let us consider one of these particles. We suppose that it impinges on the target with velocity v and impact parameter b. Then, as is clear from Fig. 4.8, it will hit the target at an angle α to the normal given by

$$b = R\sin\alpha.$$

The force on the particle is an impulsive central conservative force, corresponding to a potential $V(r)$ which is zero for $r > R$, and rises very sharply in the neighbourhood of $r = R$. Thus the kinetic energy and angular momentum must be the same before and after the collision. Let us take the

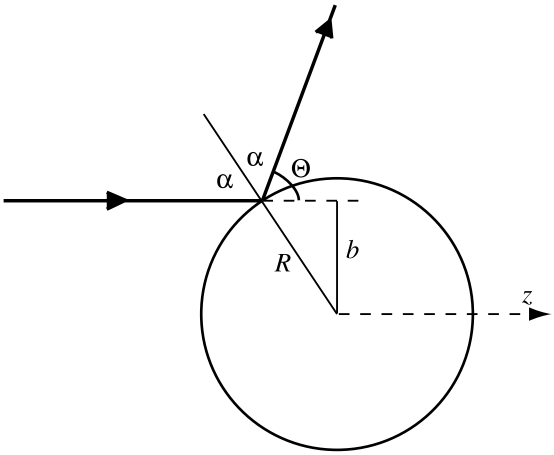

Fig. 4.8

positive z direction ($\theta = 0$) to be the direction of motion of the incoming particles. Then, by the axial symmetry of the problem, the particle must move in a plane $\varphi = $ constant. From energy conservation, its velocity must be the same in magnitude before and after the collision. Then, from angular momentum conservation, it follows that the particle will bounce off the sphere at an angle to the normal equal to the incident angle α. Thus the particle is deflected through an angle $\theta = \pi - 2\alpha$, related to the impact parameter by

$$b = R\cos\tfrac{1}{2}\theta. \tag{4.38}$$

We can now calculate the number of particles scattered in a direction specified by the polar angles θ, φ, within angular ranges $\mathrm{d}\theta, \mathrm{d}\varphi$. The particles scattered through angles between θ and $\theta + \mathrm{d}\theta$ are those that came in with impact parameters between b and $b + \mathrm{d}b$, where

$$\mathrm{d}b = -\tfrac{1}{2}R\sin\tfrac{1}{2}\theta\,\mathrm{d}\theta.$$

(Note that in our case $\mathrm{d}b$ is actually negative.)

Consider now a cross-section of the incoming beam. The particles we are interested in are those which cross a small element of area

$$\mathrm{d}\sigma = b\,|\mathrm{d}b|\,\mathrm{d}\varphi. \tag{4.39}$$

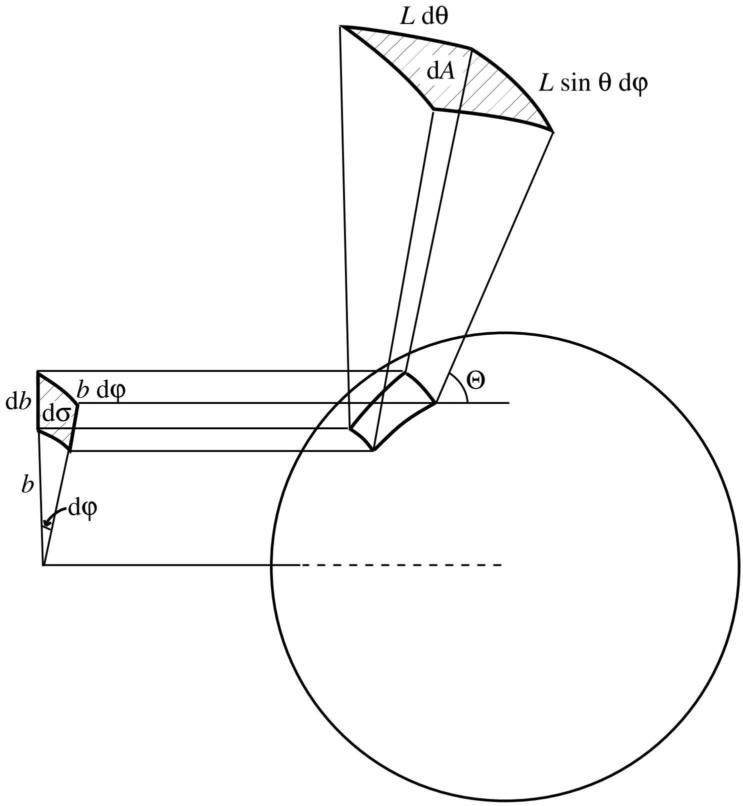

Fig. 4.9

(See Fig. 4.9.) Inserting the values of b and db, we find

$$d\sigma = \tfrac{1}{4}R^2 \sin\theta \, d\theta \, d\varphi. \tag{4.40}$$

The rate at which particles cross this area, and therefore the rate at which they emerge in the given angular range, is

$$dw = f d\sigma. \tag{4.41}$$

In order to measure this rate, we may place a small detector at a large distance from the target in the specified direction. We therefore want to express our result in terms of the cross-sectional area dA of the detector, and its distance L from the target (which we assume to be much greater than R). Now from the discussion of §3.5 (see the paragraph preceding (3.30)), we see that the element of area on a sphere of radius L is

$$\mathrm{d}A = L\,\mathrm{d}\theta \times L\sin\theta\,\mathrm{d}\varphi.$$

We define the *solid angle* subtended at the origin by the area dA to be

$$\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi, \tag{4.42}$$

so that

$$\mathrm{d}A = L^2\mathrm{d}\Omega. \tag{4.43}$$

The solid angle is measured in *steradians* (sr). It plays the same role for a sphere as does the angle in radians for a circle; Eq. (4.43) is the analogue of the equation d$s = L\,\mathrm{d}\theta$ for a circle of radius L. Just as the total angle subtended by an entire circle is 2π, so the total solid angle subtended by an entire sphere is

$$\iint \mathrm{d}\Omega = \frac{1}{L^2}\iint \mathrm{d}A = \int_0^\pi \sin\theta\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi = 4\pi.$$

The important quantity is not the cross-sectional area dσ itself, but rather the ratio dσ/dΩ, which is called the *differential cross-section*. By (4.41) and (4.43), the rate dw at which particles enter the detector is

$$\mathrm{d}w = f\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\frac{\mathrm{d}A}{L^2}. \tag{4.44}$$

It is useful to note an alternative definition of the differential cross-section, which is applicable even if we cannot follow the trajectory of each individual particle, and therefore cannot say just which of the incoming particles are those that emerge in a particular direction. We may *define* dσ/dΩ to be the ratio of scattered particles per unit solid angle to the number of incoming particles per unit area. Then the rate at which particles are detected is obtained by multiplying the differential cross-section by the flux of incoming particles, and by the solid angle subtended at the target by the detector, as in (4.44). Note that dσ/dΩ has the dimensions of area (solid angle, like angle, is dimensionless); it is measured in square metres per steradian ($\mathrm{m}^2\,\mathrm{sr}^{-1}$).

In the particular case of scattering from a hard sphere, the differential cross-section is, by (4.40) and (4.42),

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \tfrac{1}{4}R^2. \tag{4.45}$$

It has the special feature of being *isotropic*, or independent of the scattering angle. Thus the rate at which particles enter the detector is, in this case, independent of the direction in which it is placed. We note that the total cross-section (4.37) is correctly given by integrating (4.45) over all solid angles; in this case, we have merely to multiply by the total solid angle, 4π.

4.6 Mean Free Path

The total cross-section σ is useful in discussing the attenuation of a beam of particles passing through matter. Let us consider first a particle moving through a material containing n atoms per unit volume, and suppose that the total cross-section for scattering by a single atom is σ. Consider a cylinder whose axis is the line of motion of the particle, and whose cross-sectional area is σ. Then the particle will collide with any atoms whose centres lie within the cylinder. Now, the number of such atoms in a length x of the cylinder is $n\sigma x$. This is therefore the average number of collisions made by the particle when it travels a distance x. Thus the mean distance travelled between collisions — the *mean free path* λ — is $x/n\sigma x$, *i.e.*

$$\lambda = \frac{1}{n\sigma}. \tag{4.46}$$

Now consider a beam of particles, with flux f, impinging normally on a wall. How far will they penetrate? We need to calculate the flux of particles that penetrate to a depth x without suffering a collision. Let this flux be $f(x)$, and consider a thin slice of wall of thickness $\mathrm{d}x$ and area A at the depth x (see Fig. 4.10). The rate at which unscattered particles enter the slice is $Af(x)$, and the rate at which they emerge on the other side is $Af(x + \mathrm{d}x)$. The difference between the two must be the rate at which collisions occur within the slice. Now the total number of atoms in the slice is $nA\mathrm{d}x$, and the total cross-sectional area presented by all of them is $\sigma nA\,\mathrm{d}x$ (assuming $\mathrm{d}x$ is small enough that none of them overlap). Thus the rate at which collisions occur in the slice is $f(x)\sigma nA\,\mathrm{d}x$. Equating the two quantities, we have

$$Af(x) - Af(x + \mathrm{d}x) = f(x)\sigma nA\,\mathrm{d}x = f(x)\frac{A}{\lambda}\mathrm{d}x,$$

by (4.46). Equivalently,

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = -\frac{1}{\lambda}f(x).$$

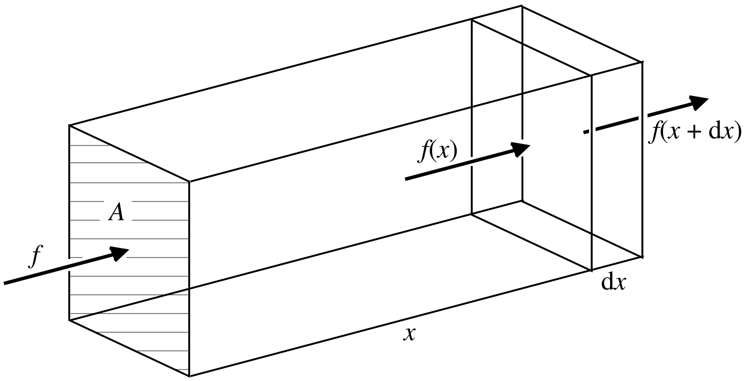

Fig. 4.10

This equation may be integrated to give

$$f(x) = C\mathrm{e}^{-x/\lambda},$$

where C is an arbitrary constant. But $f(0) = f$, so finally

$$f(x) = f\mathrm{e}^{-x/\lambda}. \tag{4.47}$$

Thus the number of particles in the beam decreases by a factor of 1/e in one mean free path.

Very often, when we wish to study the structure of a small object, like an atom, it would be quite impractical to use a target consisting of only a single atom. Instead, we have to use a target containing a large number N of atoms. If the thickness of the target is x, the rate at which collisions occur within the target will be

$$Af(0) - Af(x) = A(1 - \mathrm{e}^{-x/\lambda})f.$$

If x is small compared to the mean free path, then we may retain only the linear term in the expansion of the exponential. In this case, the rate is approximately $Axf/\lambda = Axn\sigma f$. Since $N = nAx$, this shows that for a thin target the number of particles scattered will be just N times the number scattered by a single atom. For a thick target, it would be less, because the atoms would effectively screen each other.

When the target is thin, the probability of multiple collisions, in which a particle strikes several target atoms in succession, will be small. If we assume that it is negligible, then we can conclude that the angular distribution of scattered particles will also be the same as that from a single target atom. This will be the case if all the dimensions of the target are small in comparison with λ. If we use a detector whose distance from the target, L, is large compared to the target size (so that the scattering angle does not depend appreciably on which of the atoms in the target is struck), then the rate at which particles enter the detector will be

$$\mathrm{d}w = Nf\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\frac{\mathrm{d}A}{L^2}, \tag{4.48}$$

that is, just N times the rate for a single target atom.

4.7 Rutherford Scattering

We discuss in this section a problem which was of crucial importance in obtaining an understanding of the structure of the atom. In a classic experiment, performed in 1911, Rutherford bombarded atoms with α-particles (helium nuclei). Because these particles are much heavier than electrons, they are deflected only very slightly by the electrons in the atom (see Chapter 7), and can therefore be used to study the heavy atomic nucleus. From observations of the angular distribution of the scattered α-particles, Rutherford was able to show that the law of force between α-particle and nucleus is the inverse square law down to very small distances. Thus he concluded that the positive charge is concentrated in a very small nuclear volume rather than being spread out over the whole volume of the atom.

Let us now calculate the differential cross-section for the scattering of a particle of charge q and mass m by a fixed point charge q'. The impact parameter b is related to the scattering angle θ by (4.35), *i.e.*

$$b = a\cot\tfrac{1}{2}\theta, \qquad a = \frac{qq'}{4\pi\epsilon_0 mv^2}. \tag{4.49}$$

Thus

$$\mathrm{d}b = -\frac{a\,\mathrm{d}\theta}{2\sin^2\frac{1}{2}\theta},$$

so that, substituting in (4.39), we obtain

$$\mathrm{d}\sigma = \frac{a^2\cos\frac{1}{2}\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi}{2\sin^3\frac{1}{2}\theta}.$$

Dividing by the solid angle (4.42), we find for the differential cross-section

$$\frac{d\sigma}{d\Omega} = \frac{a^2}{4\sin^4 \frac{1}{2}\theta}. \tag{4.50}$$

This is the *Rutherford scattering cross-section.*

We note that, in contrast to the differential cross-section for hard-sphere scattering, this cross-section is strongly dependent both on the velocity of the incoming particle and on the scattering angle. It also increases rapidly with increasing charge. For scattering of an α-particle on a nucleus of atomic number Z, $qq' = 2Ze^2$ (where $-e$ is the electronic charge). Thus we expect the number of particles scattered to increase like Z^2 with increasing atomic number.

We saw in §4.3 that the minimum distance of approach is given by (4.20). Thus to investigate the structure of the atom at small distances, we must use high-velocity particles, for which a is small, and examine large-angle scattering, corresponding to particles with small impact parameter b. The cross-section (4.50) is large for small values of the scattering angle, but physically it is the large-angle scattering which is of most interest. For, the fact that particles *can* be scattered through large angles is an indication that there are very strong forces acting at short distances. If the positive nuclear charge were spread out over a large volume, the force would be inverse-square-law only down to the radius of the charge distribution. Beyond that point, it would decrease as we go to even smaller distances. Consequently, the particles that penetrate to within this distance would experience a weaker force than the inverse square law predicts, and would be scattered through smaller angles.

A peculiar feature of the differential cross-section (4.50) is that the corresponding *total* cross-section is infinite. This is a consequence of the infinite range of the Coulomb force. However far away from the nucleus a particle may be, it still experiences some force, and is scattered through a non-zero (though small) angle. Thus the total number of particles scattered through any angle, however small, is indeed infinite. We can easily calculate the number of particles scattered through any angle greater than some small lower limit θ_0. These are the particles which had impact parameters b less than $b_0 = a \cot \frac{1}{2}\theta_0$. The corresponding cross-section is, therefore,

$$\sigma(\theta > \theta_0) = \pi b_0^2 = \pi a^2 \cot^2 \tfrac{1}{2}\theta_0. \tag{4.51}$$

(This may also be verified by direct integration of (4.50).)

4.8 Summary

For a particle moving under any central, conservative force, information about the radial motion may be obtained from the radial energy equation, which results from eliminating $\dot{\theta}$ between the conservation equations for energy and angular momentum. The values of E and J can be determined from the initial conditions, and this equation then tells us the radial velocity at any value of r.

When information about the angle θ is needed, we must find the equation of the orbit. For the inverse square law, the orbit is an ellipse or a hyperbola, according as $E < 0$ or $E > 0$. The semi-major axis is fixed by E, and the semi-*latus rectum* by J.

If we are concerned with finding the time taken to traverse part of the orbit, we can use the relation between the angular momentum and the rate of sweeping out area.

When a beam of particles strikes a target, the angular distribution of scattered particles may be found from the differential cross-section $\mathrm{d}\sigma/\mathrm{d}\Omega$. This may be calculated from a knowledge of the relation between the scattering angle and the impact parameter. The attenuation of the beam is related to the total cross-section σ, obtained by integrating $\mathrm{d}\sigma/\mathrm{d}\Omega$ over all solid angles.

Problems

1. The orbits of synchronous communications satellites have been chosen so that viewed from the Earth they appear to be stationary. Find the radius of the orbits.
2. Find the radii of synchronous orbits about Jupiter and about the Sun. [Their mean rotation periods are 10 hours and 27 days, respectively. The mass of Jupiter is 318 times that of the Earth. The semi-major axis of the Earth's orbit, or *astronomical unit* (AU) is 1.50×10^8 km.]
3. The semi-major axis of Jupiter's orbit is 5.20 AU. Find its orbital period in years, and its mean (time-averaged) orbital speed. (Mean orbital speed of Earth $= 29.8\,\mathrm{km\,s^{-1}}$.)
4. The orbit of an asteroid extends from the Earth's to Jupiter's, just touching both. Find its orbital period. (Treat the planetary orbits as circular and coplanar.)

5. Find the maximum and minimum orbital speeds of the asteroid in Problem 4.
6. The Moon's mass and radius are $0.0123\,M_{\rm E}$ and $0.273\,R_{\rm E}$ (E = Earth). For Jupiter the corresponding figures are $318\,M_{\rm E}$ and $11.0\,R_{\rm E}$. Find in each case the gravitational acceleration at the surface, and the escape velocity.
7. Calculate the period of a satellite in an orbit just above the Earth's atmosphere (whose thickness may be neglected). Find also the periods for close orbits around the Moon and Jupiter.
8. The Sun has an orbital speed of about $220\,{\rm km\,s^{-1}}$ around the centre of the Galaxy, whose distance is 28 000 light years. Estimate the total mass of the Galaxy in solar masses.
9. A particle of mass m moves under the action of a harmonic oscillator force with potential energy $\frac{1}{2}kr^2$. Initially, it is moving in a circle of radius a. Find the orbital speed v. It is then given a blow of impulse mv in a direction making an angle α with its original velocity. Use the conservation laws to determine the minimum and maximum distances from the origin during the subsequent motion. Explain your results physically for the two limiting cases $\alpha = 0$ and $\alpha = \pi$.
10. Write down the effective potential energy function $U(r)$ for the system described in Chapter 3, Problem 11. Initially, the particle is moving in a circular orbit of radius $2a$. Find the orbital angular velocity ω in terms of the natural angular frequency ω_0 of the oscillator when not rotating. If the motion is lightly disturbed, the particle will execute small oscillations about the circular orbit. By considering the effective potential energy function $U(r)$ near its minimum, find the angular frequency ω' of small oscillations. Hence describe the disturbed orbit qualitatively.
11. Show that the comet discussed at the end of §4.4 crosses the Earth's orbit at opposite ends of a diameter. Find the time it spends inside the Earth's orbit. (To evaluate the area required, write the equation of the orbit in Cartesian co-ordinates. See Appendix B.)
12. A star of mass M and radius R is moving with velocity v through a cloud of particles of density ρ. If all the particles that collide with the star are trapped by it, show that the mass of the star will increase at a rate

$$\frac{{\rm d}M}{{\rm d}t} = \pi\rho v\left(R^2 + \frac{2GMR}{v^2}\right).$$

Given that $M = 10^{31}$ kg and $R = 10^8$ km, find how the effective cross-sectional area compares with the geometric cross-section πR^2 for velocities of $1000\,\mathrm{km\,s^{-1}}$, $100\,\mathrm{km\,s^{-1}}$ and $10\,\mathrm{km\,s^{-1}}$.

13. *Find the polar equation of the orbit of an isotropic harmonic oscillator by solving the differential equation (4.25), and verify that it is an ellipse with centre at the origin. (*Hint*: Change to the variable $v = u^2$.) Check also that the period is given correctly by $\tau = 2mA/J$.
14. Discuss qualitatively the orbits of a particle under a repulsive force with potential energy function $V = \frac{1}{2}kr^2$ where k is *negative*, using the effective potential energy function U. How would the orbit equation, as found in Problem 13, differ in this case? What shape is the orbit?
15. *If the Earth's orbit is divided in two by the *latus rectum*, show that the difference in time spent in the two halves, in years, is

$$\frac{2}{\pi}\left(e\sqrt{1-e^2} + \arcsin e\right),$$

and hence for small e about twice as large as the difference computed in the example in §4.4. (*Hint*: Use Cartesian co-ordinates to evaluate the required area. The identity $\pi/2 - \arcsin\sqrt{1-e^2} = \arcsin e$ may be useful.)

16. A spacecraft is to travel from Earth to Jupiter along an elliptical orbit that just touches each of the planetary orbits (*i.e.*, the orbit of the asteroid in Problem 4). Use the results of Problems 3, 4 and 5 to find the relative velocity of the spacecraft with respect to the Earth just after launching and that with respect to Jupiter when it nears that planet, neglecting in each case the gravitational attraction of the planet. Where in its orbit must Jupiter be at the time of launch, relative to the Earth? Where will the Earth be when it arrives?
[This semi-elliptical trajectory is known as a *Hohmann transfer* and it is energy-efficient for interplanetary travel using high-thrust rockets in that a discrete boost is required at the beginning and at the end of the journey in order respectively to leave Earth and to arrive at the target planet. A similarly careful choice of timing for initiating a return to Earth is necessary, so that there is an inevitable minimum time that must be spent in the region of (here) Jupiter before this can be done. It is an interesting and important exercise to compare timings for round trips of planetary exploration using this form of transfer. For Mars, an obvious target in the short term, the time required is about 32 Earth months with about 15 months spent at Mars. There are of course other

ways of effecting transfer, but at rather greater cost, either in fuel or in time spent at the destination!]

17. *Suppose that the asteroid of Problems 4 and 5 approaches the Earth with an impact parameter of $5R_{\rm E}$, where $R_{\rm E}$ = Earth's radius, moving in the same plane and overtaking it. (This is an improbably close encounter for a large asteroid; however the spectacular impact of comet Shoemaker–Levy 9 with Jupiter in July 1994 should prevent us from being too complacent about this threat!) Find the distance of closest approach and the angle through which the asteroid is scattered, in the frame of reference in which the Earth is at rest. (Assume that the asteroid is small enough to have negligible effect on the Earth's orbit.) What is its new velocity v relative to the Sun? Show that the semi-major axis of its new orbit is $a_{\rm E}v_{\rm E}^2/(2v_{\rm E}^2 - v^2)$, where $a_{\rm E}$ and $v_{\rm E}$ are the Earth's orbital radius and orbital velocity. Find the asteroid's new orbital period.
18. *Show that the position of a planet in its elliptical orbit can be expressed, using a frame with x-axis in the direction of *perihelion* (point of closest approach to the Sun), in terms of an angular parameter ψ by $x = a(\cos\psi - e)$, $y = b\sin\psi$. (See Problem B.1. In the literature, ψ is sometimes called the *eccentric anomaly*, while the polar angle θ is the *true anomaly*.) Show that $r = a(1 - e\cos\psi)$, and that the time from perihelion is given by $t = (\tau/2\pi)(\psi - e\sin\psi)$ (Kepler's equation).
19. Use the parametrization of Problem 18 to calculate the time-averaged values of the kinetic and potential energies T and V over a complete period. Hence verify the *virial theorem*, $\langle V\rangle_{\rm av} = -2\langle T\rangle_{\rm av}$.
20. *Find a parametrization similar to that of Problem 18 for a hyperbolic orbit, using hyperbolic functions.
21. On reaching the vicinity of Jupiter, the spacecraft in Problem 16 is swung around the planet by its gravitational attraction — a 'sling-shot' manoeuvre. Consider this encounter in the frame of reference in which Jupiter is at rest. What is the magnitude and direction of the spacecraft's velocity before scattering? What is its magnitude after scattering? If the scattering angle in this frame is 90°, what must be the impact parameter? What is the distance of closest approach to the planet, in terms of Jupiter radii? ($M_{\rm J} = 318M_{\rm E}$, $R_{\rm J} = 11.0R_{\rm E}$.)
22. *If the manoeuvre in Problem 21 is in the orbital plane, so that the final velocity of the spacecraft relative to Jupiter is radially away from the Sun, what is its velocity in magnitude and direction relative to the Sun? Use the radial energy equation to determine the spacecraft's

farthest distance from the Sun (its *aphelion* distance) in astronomical units. Find also its new orbital period. When it returns, what will be its perihelion distance?

23. An alternative to the manoeuvre described in Problem 22 is for the spacecraft to be scattered out of the orbital plane. Assume that relative to Jupiter its velocity after scattering is directed normal to the orbital plane. What is its velocity relative to the Sun? What will be its aphelion distance and orbital period? How far from the orbital plane will it reach? (*Hint*: Immediately after scattering, the radial component of its velocity is zero. This is therefore the perihelion point of the new orbit. The farthest point from the original orbital plane will occur when it is at one end of the semi-minor axis of the orbit.)
24. *A ballistic rocket (one that moves freely under gravity after its initial launch) is fired from the surface of the Earth with velocity $v < \sqrt{Rg}$ at an angle α to the vertical. (Ignore the Earth's rotation.) Find the equation of its orbit. Express the range $2R\theta$ (measured along the Earth's surface) in terms of the parameters l and a, and hence show that to maximize the range, we should choose α so that $l = 2a - R$. (*Hint*: A sketch may help.) Deduce that the maximum range is $2R\theta$ where $\sin\theta = v^2/(2Rg - v^2)$. Given that the maximum range is 3600 nautical miles, find the launch velocity and the angle at which the rocket should be launched. (*Note*: 1 nautical mile = 1 minute of arc over the Earth's surface.)
25. Discuss the possible types of orbit for a particle moving under a central inverse-cube-law force, described by the potential energy function $V = k/2r^2$. For the repulsive case ($k > 0$), show that the orbit equation is $r\cos n(\theta - \theta_0) = b$, where n, b and θ_0 are constants. Show that for the attractive case ($k < 0$), the nature of the orbit depends on the signs of $J^2 + mk$ and of E. Find the equation of the orbit for each possible type. (Include the cases where one of these parameters vanishes.)
26. Show that the scattering angle for particles of mass m and initial velocity v scattered by a repulsive inverse-cube-law force is $\pi - \pi/n$ (see Problem 25). Hence find the differential cross-section.
27. *The potential energy of a particle of mass m is $V(r) = k/r + c/3r^3$, where $k < 0$ and c is a small constant. (The gravitational potential energy in the equatorial plane of the Earth has approximately this form, because of its flattened shape — see Chapter 6.) Find the angular velocity ω in a circular orbit of radius a, and the angular frequency ω' of small radial oscillations about this circular orbit. Hence show that a

nearly circular orbit is approximately an ellipse whose axes precess at an angular velocity $\Omega \approx (c/|k|a^2)\omega$.

28. A beam of particles strikes a wall containing 2×10^{29} atoms per m^3. Each atom behaves like a sphere of radius 3×10^{-15} m. Find the thickness of wall that exactly half the particles will penetrate without scattering. What thickness would be needed to stop all but one particle in 10^6?
29. An α-particle of energy 4 keV ($1\,\mathrm{eV} = 1.6 \times 10^{-19}\,\mathrm{J}$) is scattered by an aluminium atom through an angle of 90°. Calculate the distance of closest approach to the nucleus. (Atomic number of α-particle = 2, atomic number of Al = 13, $e = 1.6 \times 10^{-19}$ C.) A beam of such particles with a flux of $3 \times 10^8\,\mathrm{m}^{-2}\,\mathrm{s}^{-1}$ strikes a target containing 50 mg of aluminium. A detector of cross-sectional area $400\,\mathrm{mm}^2$ is placed 0.6 m from the target in a direction at right angles to the beam direction. Find the rate of detection of α-particles. (Atomic mass of Al = 27 u; $1\,\mathrm{u} = 1.66 \times 10^{-27}$ kg.)
30. *It was shown in §3.4 that Kepler's second law of planetary motion implies that the force is central. Show that his first law — that the orbit is an ellipse with the Sun at a focus — implies the inverse square law. (*Hint*: By differentiating the orbit equation $l/r = 1 + e\cos\theta$, and using (3.26), find $\dot{r}$ and $\ddot{r}$ in terms of r and θ. Hence calculate the radial acceleration.)
31. Show that Kepler's third law, $\tau \propto a^{3/2}$, implies that the force on a planet is proportional to its mass.
[This law was originally expressed by Kepler as $\tau \propto \bar{r}^{3/2}$, where $\bar{r}$ is a 'mean value' of r. For an ellipse, the mean over *angle* θ is in fact b; the mean over *time* is actually $a(1 + \frac{1}{2}e^2)$; it is the mean over *arc length* — or the median — which is given by a! Of course, for most planets in our Solar System these values are almost equal.]