Preface

The purpose of this book

The purpose of this foreword is to explain not only what this book is but also what it is not. The last few years have seen an exponential development of research on photonic crystals, *i.e.* periodic structures, used to control the propagation of light. From the solid state physicist's point of view, such dielectric structures mimic for light what happens in a silicon crystal for electrons, while for the classical optician, this is extending to 2D or 3D structures what is done in Bragg mirrors and in gratings! Anyway, in such structures, propagation may be forbidden in any direction for intervals of frequencies called *photonic band gaps*. The category of microstructured optical fibres became quite recently a substantial subtopic with promising, amongst others, technological applications of the concept of the photonic band gap.

With the invention of lasers and the development of optical fibres able to carry light over hundreds of kilometres came the advent of a new era of worldwide high speed telecommunications, with consequences reaching from benefits in everyday life to the reshaping of the world economy. The first visible effects were cheaper transcontinental phone connections, with the latest being the "Internet revolution". Although one can bemoan the fact that the technologically unprecedented possibilities for people and populations to communicate don't yet seem to have brought better understanding and tolerance between cultures, the optimist will say that this is only a matter of time. Indeed it is easy to forget (especially for younger generations) that optical fibres "transparent" enough to make long haul data transfer possible were invented only about 30 years ago or that the first transatlantic optical fibre was laid less than 20 years ago.

For a decade, the speed (or bandwidth) at which a single optical fibre could carry data seemed pretty much unlimited. Every now and then the bandwidth of telecommunication networks became insufficient, but it appeared that the limitation was not so much due to the intrinsic limits of the fibre, but much more to the limited speed of the signal sources and receivers, so that increasing the bandwidth was for a long time a matter of improving sources and receivers. Intrinsic limits to the bandwidth of optical fibres had been predicted as early as the 1970s, but seemed as theoretical and out of reach as the speed of light for aeroplanes. Research on increasing the bandwidth of optical networks in those times was more a question of how to get the best out of fibres through injecting more information, *e.g.* through multiplexing, than a question of improving optical fibres. Of course the new data injection principles that were developed required new types of fibres, but these were merely more or less adapted versions of the original step index fibre.[1] Given the incredible possibilities offered by existing fibres, there was little eagerness to try to find something radically new, as there was little reason to believe that anything better could ever exist.

It is only in recent years, with the exponentially increasing demand for bandwidth (essentially due to the popularization of the Internet and multimedia contents) along with the progress made in high speed electronics and optoelectronics, that the intrinsic limits of optical fibres have been reached. New, higher density data injection methods are now avaible, but because of non-linear effects, polarization mode dispersion and other effects that we will describe in more detail in the following, good old optical fibres can't keep pace anymore.

Fortunately, at about the same time, the principles of photonic crystals were discovered, leading to the suggestion of radically new mechanisms of light guidance. In the early 1990s the idea of optical fibres using photonic crystal claddings emerged, and after a few years the first photonic crystal fibre was demonstrated. The very first experimental work on these fibres already showed that they could have unprecedented properties and overcome many limitations intrinsic to step index fibres. With photonic crystal fibres, almost everything seems feasible, from *guiding light in vacuum*, hence overcoming all limitations inherent to interactions between light and matter, to achieving dispersion properties unthinkable with step index

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[1] What we claim here is that the fibres were based on the same principle of guidance as step index fibres, which doesn't in any way detract from the work and ingenuity needed for the task of designing such fibres.

fibres, from enhancing non-linear effects through to extreme confinement of light and minimizing the same non-linear effects through very large core single mode fibres. The discovery of these possibilities brought prospects of totally new fields of application for fibre optics, such as optical fibres for high power applications, optical fibres for non-conventional wavelength ranges (*e.g.* far infrared, ultra violet), revolutionary optical fibre sensors, particle guidance through hollow core optical fibres, extremely versatile dispersion management, compact high precision metrology, and low-threshold non-linear optics.

Unsurprisingly, the field of photonic crystal fibres became extremely popular, and soon numerous research groups around the world started drawing all kinds of photonic crystal fibres, with hollow or solid cores, with regular or irregular structures, using silica or polymers. Inevitably the pioneers of the field all gave those fibres different names – photonic crystal fibres (PCF), microstructured optical fibres (MOF), crystal fibres (CF), holey fibres (HF) – each having different connotations; we will discuss the meaning of each of these in the next Chapter.

We felt that it is important to produce a document that can be used as a short introduction to this research topic for new or not so new researchers in the field. This book is not a complete catalog of all the existing computation methods. The main reason is that we want to keep it short and not loose the reader in an endless enumeration of all the variants. Once you have understood a few methods, the new ones can be more easily learned. The methods presented here are clearly a deliberate choice of the authors and may be therefore questionable but this choice is committed since they are precisely the methods that the authors use in their own research. But before presenting the solution methods, a preliminary task is to clearly explain the problem. Roughly speaking this can be stated as solving the Maxwell equations in dielectric structures with an axis of invariance. In fact, we seek propagating solutions *i.e.* such that t (time) and z (coordinate along the invariant axis) dependencies are of the form $e^{i(\omega t-\beta z)}$ where ω and β are constants (respectively the pulsation and the propagating constant). At this moment the fundamental problem appears: not all the (ω, β) pairs are feasible! The set of pairs for which a solution of the propagation problem exists are called *dispersion relations* and can be drawn as dispersion curves in the (ω, β) plane. The electromagnetic field associated with a feasible pair is called a *propagating mode*.

The usual way to find a pair is to choose one of the numbers, either ω or β and search for the other one. This kind of problem where the existence of a solution is conditional on the value of a parameter often leads as in the case of this book, to what is called a *spectral problem.* A subtle question appears here: what do we call *a solution* to a problem *e.g.* a partial differential equation with specific conditions (boundary conditions, imposed behaviour at infinity, etc.)? For physicists, any function satisfying the equation and the boundary condition is usually sufficient and it may be given in various forms: explicit analytic closed form, infinite series and in fact any numerical or algorithmic form that gives you the values. Further mathematical sophistication is very often considered as pedantry. In our problem and especially when a dielectric waveguide in open space is considered, a propagating mode must be carefully defined as it must be associated with an eigenvalue *i.e.* a value in the discrete spectrum. Here the use of the framework of functional analysis where functions are elements of infinite dimensional vector spaces is compulsory and the question of whether a number is in the discrete, continuous or residual spectrum is physically meaningful. In the case of holey fibres, the treatment of the propagation of a signal in a low index part of the fiber raises the problem of the *leaky modes.* These are associated with complex propagation constant and here again the very definition of a propagating mode must be generalized and made mathematically accurate. And yet another problem appears: mode propagation is associated with translational invariance of the fiber, so what is a mode propagating in a curved fiber or in a twisted fiber where the translational invariance is lost? Once again, the definition of a propagation mode must be reexamined rather than just given arbitrarily, but this in turn is related to the possibility of performing a Fourier transform of a function that is constant along a co-ordinate... Therefore, the first part of the book is devoted to defining the problem as simply, precisely and as clearly as possible. In order to avoid an intricate mathematical formalism, as many features as possible will be discussed as one-dimensional cases. To complete the discussion, it must be said that if photonic crystals are periodic structures, interesting things happen in defects *i.e.* where the periodicity of the structure is broken, just as doping makes semiconductors more interesting. Moreover, real structures are of finite extension. Nevertheless, the study of ideal crystals is of the highest interest as it allows the determination of forbidden gaps and may be performed by making use of the Floquet-Bloch theory that will be added to our theoretical tools: solutions to wave propagation problems in periodic media are in the form of quasi-periodic functions obtained by multiplying a function having the same periodicity as the media by the wave-function of a plane wave. So much theory may seem superfluous but correctly defining a problem is half of the solution and it is worth the effort. Throughout the book, we try to be as accurate as possible but avoid drowning the reader in mathematical pedantry.

Once the problem is correctly set up, the next step is to solve it *i.e.* finding the (ω, β) pairs together with the corresponding eigenmodes and drawing the dispersion curves, which contain much information about the physics of the system. As stated above, only two methods are presented in this book: the *Multipole Method (MM)* where the fields are expanded in Fourier-Bessel series and a generalized scattering problem is solved, and the *Finite Element Method (FEM)* based on a weak formulation of the problem together with an expansion on a basis of elementary functions (typically low order polynomials) with bounded supports (and the sizes of these supports are small with respect to the size of the problem). How can we justify the choice of these two methods? The advantages of having two completely different methods to solve the same problem is that comparing the results gives some confidence in their accuracy if they match and also that the methods may have complementary advantages so that they can rescue each other if one of them encounters a difficulty in the solution of a particular problem. Therefore, it seems natural to choose two methods that are as different as possible. This is quite the case between the multipole and the finite element methods. As a common feature, the two methods share the fact that the unknown fields or functions are approximated using a linear combination of given basis functions. Nevertheless, one can say that the approach of the multipole method is global while the finite element method is local. On the one hand the basis functions in the Multipole Method are solutions of the Helmholtz equations to be solved (except at the source point, but that is not included in the domain where the solution is considered) as they are the field produced by a multipolar radiating source. Their linear combinations are themselves solutions and the problem is to pick the good one by choosing the right coefficients (*i.e.* the values of the equivalent multipolar charges). This is performed via the numerical expression of local conditions: continuity conditions of the tangential fields at the interfaces. On the other hand the basis functions in the finite element methods are simple (*e.g.* piecewise linear) functions with bounded support (of small size). In the case of the Whitney family of finite elements, if basis functions do not match the Helmholtz equation, they locally respect the continuity conditions at the interfaces *e.g.* tangential continuity for electric and magnetic fields represented by edge elements (that is the geometrical nature of the fields that determines the kind of element to be used) and so do their linear combinations. In this case, the Helmholtz equation itself (in weak variational form) is used to compute the numerical coefficients of the approximation.

Let us compare the respective advantages of the methods that show their complementarity. On the one hand, the Multipole Method is specific to structures including matrix with homogeneous linear characteristics and inclusions with linear characterictics, furthermore the method is extremely efficient in case of circular boundaries for which analytical formula are available. It allows fast and accurate computations of the dispersion curves necessary for parametric study and optimization of the fibres. On the other hand, the finite element method is extremely flexible from both the geometric point of view, since it allows an easy treatment of inclusions of any shape (and permits also a 2D treatment of twisted fibres in helicoidal coordinates), and from the material point of view as it allows anisotropic, inhomogeneous (and possibly non-linear) characteristics to be incorporated.

There are of course many other methods that we discuss briefly here to explain why they are less adapted to our problem. First of all, we must agree on what we call a "method": for us, it is a complete algorithmic procedure that allows you to compute the values of the electric and/or magnetic fields at any point (at least in the vicinity of the structure).

- With respect to this definition, the *Effective Index Method* is not a method but rather an approximation. It tells you to replace an inhomogeneous structure (*e.g.* a holey part of the fiber) by an equivalent homogeneous index. Although, this kind of model may be extremely useful in some cases, it is not adapted to the study of microstructured fibre. Moreover, it is not a method but an ancillary trick as it just gives a new simplified structure to be computed by a numerical method of your choice.
- To some extent, the problem is the same with the very popular *Supercell Method.* Imagine that you have to find the modes propagating in a defect of a periodic structure. For the periodic structure itself, the Floquet-Bloch theory gives you the general form of the solution as a quasi-periodic Bloch function. The solution may be computed on the cell using special Bloch boundary conditions. As a consequence of the simple geometry of the cell, the solutions themselves may me computed as Fourier series. In the case of a defect (*e.g.* a missing element in the infinite repetition of a pattern), the

structure is no longer periodic and the Floquet-Bloch theory can not be applied. Nevertheless, one arbitrarily decides that this defect may be correctly represented by an empty cell surrounded by a finite set of layers of regular cells. Moreover, in order to get a convenient problem, one decides again arbitrarily that this defect and its neighbourhood may be repeated infinitely in order to recover a periodic structure where the Floquet-Bloch theory applies but now with a much larger cell that contains the defect and its neighborhood: a supercell! This procedure is quite easy to implement and provides interesting numerical results but we do not use it as it is based on several unjustified hypotheses and it is therefore impossible to prove convergence to the requested solution (since, if it is stable and converges to the solution of the artificial periodic problem, there is no guarantee that this result is a solution of the original problem!). Besides, we cannot hope to estimate the loss of the leaky modes: an electromagnetic field in a supercell cannot run away! Here again, the supercell method is merely a trick, replacing a non-periodic structure by an artificial periodic one, that is usually associated with a Fourier method but could be used with other numerical procedures able to find the Bloch solutions (*e.g.* the FEM).

- The basic principle of the *Plane Wave Method* is to express the unknown fields as a sum of plane waves with various propagation vectors. The only difference between this method and the multipole method is the choice of the basis functions. Plane waves are also solutions of the Helmholtz equations. As they are complex exponential of linear forms of the coordinates, this method is equivalent to a bare Fourier method. Therefore, why choose the MM which involves more tedious computations with Bessel functions? Simply because the plane waves are completely non localized and they slowly converge to localized wave packets (such as the propagating modes). On the contrary, multipole series require a small number of terms to get a reasonable accuracy. Note that a similar situation occurs in the computation of electronic band structures in solid state physics. For conduction bands with nearly free electrons, a plane wave method is used since plane waves correspond to free electrons while a Linear Combination of Atomic Orbitals (LCAO) method is used for valence bands.
- The *Boundary Element Method* (BEM) may be considered as an

intermediate between the finite element method and the multipole method. On the one hand, the geometry is divided into small elements of given shapes where some unknown functions are supposed to have a given behavior (*e.g.* low order polynomial) just like in the finite element method. On the other hand, the problem is set in the form of an integral equation involving the fundamental solution (Green function) of the differential problem. The main difference with the finite element method is that instead of the geometrical domains of the problem, only the boundaries have to be discretized. This leads to a dramatic decrease of the number of unknowns in the problem. As the physical interpretation of the Green function is the field produced by a point charge (Dirac distribution), the BEM can be considered as a member of the equivalent charge method family just as the multipole method (usually one speaks of equivalent or fictitious charge methods when the charges, monopoles or multipoles, are not on the surfaces of discontinuity in order to avoid singular integrals). The BEM is as flexible as the finite element method as far as the geometry is concerned and it allows complex geometries to be dealt with more easily since only the boundaries have to be meshed. Unfortunately the method is well adapted only to homogeneous (and therefore linear) media. From a practical point of view, one of the most difficult steps in the method is the numerical computation of a singular (multiple) integral involving the Green function and/or its derivatives, and some of those integrals are even defined as Cauchy principal values or Hadamard finite parts. The real bottleneck of the boundary element method is the fact that it produces large full matrices. Even if the number of unknowns is much lower than in FEM, the sparsity of matrices in FEM makes it often much more tractable. Nevertheless computational linear algebra is an extremely active field of research and the ever increasing power of computers prevents us from drawing any definitive conclusion on this subject. Anyway, in the present state of the art, the FEM is more flexible than the BEM (since FEM allows almost any kind of material properties and coordinate systems to be dealt with) and the drawback of having to mesh domains is not a heavy one since pretty good free tools are now available that do this work so that it appears as an automatic step after the boundary meshing.

There are two main settings of BEM: collocation methods where the

algebraic equations are obtained by writing the integral equation for particular points of the surface and the Galerkin method where the integral equation is used in a weighted residual process similar to that directly applied to the differential equation to obtain the FEM.
In the context of high frequency electromagnetism, the BEM and other related equivalent charge methods are often called *Method of Moments.* An extremely active field of research is the *Fast Multipole Method (FMM)* where the algorithmic efficiency is optimized using multipole decomposition of the remote action of sets of charges and the idea that the higher the moment the faster it decreases with the distance. The adaptation of such methods to "mode chasing" may be a promising path for future research.

- The *Finite Difference Time Domain Method (FDTD)* is one of the most powerful methods designed to tackle electromagnetic wave propagation. The principle of the finite difference method is to replace the derivatives of functions in differential operators by finite differences *i.e.* by differences of the values of the function computed on two ends of a small interval (of time or of space). A fundamental question in using the method is of course "How small?". Doing this, algebraic equations, instead of differential equations, can be obtained directly, and these can be solved via some computational algebra. The FDTD algorithm proposed by Yee in 1966 is in fact much more than that since it relies on the deep structure of Maxwell equations. Instead of using a single grid to produce the finite differences, two dual grids are used. First take a 3D rectangular grid, the primal grid, and then build the dual grid so that its nodes are the centers of the parallelepipeds of the primal grid. Note that there is a cross-correspondence not only between nodes and parallelepipeds of the two grids but also between edges and faces. The unknowns of the problem are the components of the electric fields along the edges of the primal mesh and of the magnetic field along the edges of the dual mesh. A development of the Maxwell equations to the first order provides the algebraic system. As for the time variation, duality is also used. One associates the electric field with equally spaced instants (integer index time steps) and the magnetic field with the middle of the time intervals between those instants (half integer index time steps). Here also a first order development provides the finite difference algebraic equations.

The time integration scheme thus obtained is called "leapfrog" as one "jumps" from the values of electric field for an integer time step to the values of magnetic field for the following half integer time step and so on... The algorithm is extremely powerful as it is purely explicit: no matrix storage and computation is necessary! This method is probably the most used currently to solve very large electromagnetic propagation problems. The main drawback is the use of a structured mesh that makes the accurate representation of complex geometries difficult. The modern analysis of the FDTD method and of edge FEM shows that the two methods are in fact very close in spirit: the unknowns are associated with the edges of a mesh covering the whole domain and they rely on duality. Current research is still looking for FDTD on an unstructured mesh *e.g.* by lumping finite element matrices on the diagonal to avoid heavy matrix computations in time domain FEM. Nevertheless, in our case, the FDTD suffers a major flaw: it is intrinsically a time domain method, but we need a frequency domain method to directly obtain the modes (even though the Fast Fourier Transform is a magic algorithm to convert time domain results to frequency domain results).

- There are many other variants of the numerical methods. For instance, giving up the inter-element continuity in "finite-element-like" methods leads to the *Discontinuous Galerkin Method* that is receiving an increasing amount of interest for its efficacy in the solution of hyperbolic problems. FE-FD methods lead to similar schemes starting from different points of view. Another approach is to emphasize conservation laws for fluxes, leading to the *Finite Volume Method.*

  There are also the meshless methods or fuzzy element methods where the approximation is based on a linear combination of functions with overlapping supports.

  Another category of methods comprises those close to geometrical optics and ray tracing such as the *Geometrical Theory of Diffraction* (GTD). They are applicable when the wavelength is small with respect to the geometry but this is not the case in our problem.

This brief overview of numerical methods in electromagnetism shows that it is hopeless to give any complete account. The two main methods presented in this book are naturally suited to the problem of mode determination in microstructured fibres are, quite complementary, and cover a large part of the basic techniques necessary to understand numerical modelling in this context.

The adaptation of those methods to periodic problems in the context of Floquet-Bloch theory is also discussed in detail.

In the final chapter, some numerical examples demonstrate how the methods can be used to study microstructured fibres and they also emphasize some physical phenomena at the edge of the current understanding of microstructured fibres.

We hope we have convinced the reader that it is not a waste of time for her/him to read the rest of this book and wish her/him to enjoy it.

Vade mecum

From the start, our intention was to write a book for an audience as wide as possible, say from graduate students to researchers. We hope that we have achieved our aims! Be that as it may and despite all the efforts of the authors especially on the homogeneity of the notations some chapters are merely aimed at graduate students and some others are intended for experienced researchers... Additionally, our aim was to gather not only practical features of a certain class of new fibres but also theoretical aspects associated with such fibres. This short section should allow the reader to find efficiently what he is looking for.

Chapter 1 Everybody should start with this chapter. After a short review of conventional optical fibre properties (section 1.1). It briefly defines what is a photonic crystal (section 1.2), it also sets the terminology used (microstructured optical fibres and photonic crystal fibres). The skilled reader may directly start with section 1.3. This section explains how it is possible to guide light in a fibre using photonic crystal. Section 1.4 is devoted to solid core microstructured optical fibres dealing with their guidance mechanism and some of their properties. This chapter finishes with section 1.5, this section details what is a leaky mode through several approaches. Some parts of this section may require careful thought for the beginner but it is worth it.

Chapter 2 This chapter is aimed at all readers. Section 2.1 is a brief survey of Maxwell equations both *in vacuo* and in idealized matter. Section 2.2 is devoted to one-dimensional structures; these simple structures permit the adoption of a pedestrian approach to the concepts encountered in the rest of this book in a more general context. Amongst other things, both structures of finite and infinite extent are considered. The former are studied in two different ways: through the poles of the scattering matrix and trough the spectrum of an operator. The latter are studied by making use of the so-called Bloch wave theory. Section 2.3 is more essential ; relations between transverse and axial components are derived in quite a general context. Section 2.5 probably will interest readers to which the theoretical aspects of Physics appeal. Finally, we found it necessary to address Bloch wave theory; in the section 2.6 attention has been drawn especially to the Bloch wave decomposition and the so-called Wannier transform.

Chapter 3 This chapter is a general presentation of the finite element method in electromagnetism together with a detailed account of its applications to inhomogeneous waveguides. Section 3.1 is a very general presentation of the finite element method. It starts with a one dimensional example for absolute beginners. It also includes more advanced topics such as mixed formulations and eigenvalue problems that are more mathematically demanding, as this section relies strongly on functional analysis. Some abstract material is relegated to appendix A to lighten the presentation. Section 3.2 is a kind of fresh look at electromagnetism and discrete methods that can be read almost independently from the rest of the book. It presents the geometrical point of view on how FDTD and FEM fit together in this framework. Sections 3.1 and 3.2 are quite parallel and lead to the same conclusion: the necessity to use edge elements to interpolate the electric and magnetic fields. Section 3.3 contains a discussion of some technical problems at the heart of finite element modelling. Each subsection can be read more or less independently (even though there is some logical progression in their ordering) when the reader feels the need to have a deeper look at the considered problem. Section 3.4 is in fact the first section specifically devoted to waveguides and optical fibres. The reader acquainted with finite elements should start reading here. It is a detailed description of the full wave model using an electric formulation and edge elements followed by a short presentation of the variants found in the abundant literature on the subject. Section 3.5 explains how the finite element method can be combined with the Floquet-Bloch theory to study wave propagation in periodic structures. Section 3.6 on twisted fibres is an illustration of the versatility of the finite element method together with a discussion on how the concept of a propagation mode can be generalized when the translational invariance is lost.

Chapter 4 This chapter is intended for all the readers and describes in some detail one of the two main numerical methods used in this book to study MOFs: the Multipole Method (MM). We define the foundations of MM in the first parts of section 4.2. Then we present a simplified approach of the method in order to clearly describe its key ideas and to define the way the electromagnetic fields are expressed. Then, we give the full method. The related sections are more difficult but we try to make it accessible to the majority of interested readers, nearly all the delicate or tedious calculus being done in Appendix B. In section 4.3, we sum up the theoretical work of McIsaac relating to waveguide symmetries and the properties of modal fields, these results being also very useful to classify MOF modes. We give several examples which cover most idealized MOF structures. In the next two sections 4.4 and 4.5, we show how the Multipole Method can be implemented and how it can be validated. In section 4.6 of this chapter, we give complete and accurate results obtained with the Multipole Method for the electromagnetic fields and Poynting vector for three few-hole MOF: a hexagonal MOF with six identical holes, a six hole birefringent MOF and a square MOF with eight holes.

Chapter 5 This chapter presents the analysis of electromagnetic waves propagating through a doubly periodic array of cylindrical channels in oblique incidence. The change of the Multipole Method described in the previous chapter to periodic structure give us the necessary tools. This change amounts to reducing a spectral problem for partial differential equations set on a basic cell with Floquet-Bloch boundary conditions, to a certain algebraic problem of the Rayleigh type. We obtain a formulation in terms of an eigenvalue problem that enables us to construct *dispersion curves* and thereby to look at *photonic band gap* structures in oblique incidence. The mathematics behind the Rayleigh algorithm depend upon the resolution of an integral equation involving a Green's function which has sources at all the lattice points within the array. The idea is then to expand both the electromagnetic field and the Green's function in terms of a basis of functions which are appropriate to the geometry of the problem. Since the cylinders which we consider have circular cross-sections, we expand the field in terms of Bessel functions rather than in a Laurent series contrarily to Rayleigh. We show that these so-called multipole-expansions are absolutely convergent within a prescribed annulus. Projecting these expansions back onto the surface of each cylinder and using the boundary conditions (tangential continuity of the fields) we are led to a *scattering matrix*. It then remains to take into account the contribution of the superposed effects of all singular sources arising from all the other cylinders in the array.

This is done by introducing the so-called *lattice sums* which require special treatment in order to avoid a conditional convergence. Once this is done, we end up with an infinite linear system which, after suitable normalisation, can be truncated to a certain multipole order with a fairly high accuracy.

Chapter 6 This chapter tackles a natural question when dealing with all methods based upon scattering matrices like the Multipole Method. As a result, this part is merely intended for all readers interested in the numerical implementation of the method. The question posed is the following: are we doomed to find the complex poles of the scattering matrix in a point-by-point fashion or else, following the example of the *Brave Little Tailor*, can we obtain seven (or more) poles at one stroke? The answer is affirmative and this fairy tale is told in this Chapter.

Chapter 7 This chapter is intended for all the readers and is the key chapter of the book as far as MOF properties are concerned. We start with a brief review of the basic properties of MOF losses (section 7.1). In section 7.2, we expound the definitive answer concerning the single-modedness of plain core MOF through the study of the description of a cut-off of the second mode. In the following section, 7.3, we show that the fundamental mode exhibits another kind of cut-off. We then explain how using the results of the previous sections we can construct a diagram of the different operational regimes of plain core MOFs. We also give simple physical models of conventional optical fibres which are valid approximations of MOF outside a transition region where MOF-peculiar properties arise. In section 7.4, we describe plain core MOF chromatic dispersion properties and we explain how the chromatic dispersion can be managed. In the last section, we illustrate one of the most striking properties of MOF, the possibility to guide a mode in a hollow core in a way that allows us to systematically reduce the losses. This last section allows us to couple the finite element method for band diagram computations and the multipole one for the study of finite structure.

Appendix A This appendix gives the general abstract framework for mixed finite element methods and discusses the fundamental inf-sup criterion. It can be read as a fruitful complement to Section 3.1.3.

Appendix B This appendix contains some details concerning the formulation of the Multipole Method (MM). Section B.1 deals with the Wijngaard identity. This identity is the crux of the method. In the full derivation of the MM several changes of basis for the Bessel and Hankel functions are required. Section B.2 is devoted to this topic. In the presented implementation of the MM, only circular inclusions are used since in this case analytical results are available for the inclusion scattering matrices even when conical mounting is considered. These results are described in section B.3.

Appendix C This appendix is a short survey of the mathematical tools used in this book. Main notations and definitions of mathematical concepts are given there together with some important theorems and a few fundamental examples. This is not a course but rather an informal lexicon. Vector spaces appear as a powerful unifying concept and it is worth the effort for the physicist to grasp the main ideas of this abstract theory. A first application is the functional analysis that appears as the theory of infinite dimensional vector spaces and is a natural framework to set up distribution theory, spectral analysis, and discrete (*i.e.* finite dimensional) approximations. A second application is differential geometry. Although classical theoretical electromagnetism is most frequently presented using vector analysis, differential geometry gives a sharper view (see Section 3.2) even if vector analysis in Cartesian coordinates remains a valuable tool to perform some explicit computations. Even if this is not really necessary here, we could not resist the pleasure of adding a few words on how the distribution theory merges with differential geometry in the theory of de Rham currents and recalling the basics of complex analysis as an application of differential geometry.

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