

Contents

Preface	v
1 Preliminaries From Calculus	1
1.1 Functions in Calculus	1
1.2 Variation of a Function	4
1.3 Riemann Integral and Stieltjes Integral	9
1.4 Lebesgue's Method of Integration	14
1.5 Differentials and Integrals	14
1.6 Taylor's Formula and Other Results	15
2 Concepts of Probability Theory	21
2.1 Discrete Probability Model	21
2.2 Continuous Probability Model	28
2.3 Expectation and Lebesgue Integral	33
2.4 Transforms and Convergence	37
2.5 Independence and Covariance	39
2.6 Normal (Gaussian) Distributions	41
2.7 Conditional Expectation	43
2.8 Stochastic Processes in Continuous Time	47
3 Basic Stochastic Processes	55
3.1 Brownian Motion	56
3.2 Properties of Brownian Motion Paths	63
3.3 Three Martingales of Brownian Motion	65
3.4 Markov Property of Brownian Motion	67
3.5 Hitting Times and Exit Times	69
3.6 Maximum and Minimum of Brownian Motion	71
3.7 Distribution of Hitting Times	73
3.8 Reflection Principle and Joint Distributions	74
3.9 Zeros of Brownian Motion. Arcsine Law	75

3.10	Size of Increments of Brownian Motion	78
3.11	Brownian Motion in Higher Dimensions	81
3.12	Random Walk	81
3.13	Stochastic Integral in Discrete Time	83
3.14	Poisson Process	86
3.15	Exercises	88
4	Brownian Motion Calculus	91
4.1	Definition of Itô Integral	91
4.2	Itô Integral Process	100
4.3	Itô Integral and Gaussian Processes	103
4.4	Itô's Formula for Brownian Motion	105
4.5	Itô Processes and Stochastic Differentials	108
4.6	Itô's Formula for Itô Processes	111
4.7	Itô Processes in Higher Dimensions	117
4.8	Exercises	120
5	Stochastic Differential Equations	123
5.1	Definition of Stochastic Differential Equations	123
5.2	Stochastic Exponential and Logarithm	128
5.3	Solutions to Linear SDEs	130
5.4	Existence and Uniqueness of Strong Solutions	133
5.5	Markov Property of Solutions	135
5.6	Weak Solutions to SDEs	136
5.7	Construction of Weak Solutions	138
5.8	Backward and Forward Equations	143
5.9	Stratanovich Stochastic Calculus	145
5.10	Exercises	147
6	Diffusion Processes	149
6.1	Martingales and Dynkin's Formula	149
6.2	Calculation of Expectations and PDEs	153
6.3	Time Homogeneous Diffusions	156
6.4	Exit Times from an Interval	160
6.5	Representation of Solutions of ODEs	165
6.6	Explosion	166
6.7	Recurrence and Transience	167
6.8	Diffusion on an Interval	169
6.9	Stationary Distributions	170
6.10	Multi-Dimensional SDEs	173
6.11	Exercises	180

7	Martingales	183
7.1	Definitions	183
7.2	Uniform Integrability	185
7.3	Martingale Convergence	187
7.4	Optional Stopping	189
7.5	Localization and Local Martingales	195
7.6	Quadratic Variation of Martingales	198
7.7	Martingale Inequalities	200
7.8	Continuous Martingales. Change of Time	202
7.9	Exercises	209
8	Calculus For Semimartingales	211
8.1	Semimartingales	211
8.2	Predictable Processes	212
8.3	Doob-Meyer Decomposition	214
8.4	Integrals with respect to Semimartingales	215
8.5	Quadratic Variation and Covariation	218
8.6	Itô's Formula for Continuous Semimartingales	220
8.7	Local Times	222
8.8	Stochastic Exponential	224
8.9	Compensators and Sharp Bracket Process	228
8.10	Itô's Formula for Semimartingales	234
8.11	Stochastic Exponential and Logarithm	236
8.12	Martingale (Predictable) Representations	237
8.13	Elements of the General Theory	240
8.14	Random Measures and Canonical Decomposition	244
8.15	Exercises	247
9	Pure Jump Processes	249
9.1	Definitions	249
9.2	Pure Jump Process Filtration	250
9.3	Itô's Formula for Processes of Finite Variation	251
9.4	Counting Processes	252
9.5	Markov Jump Processes	259
9.6	Stochastic Equation for Jump Processes	261
9.7	Explosions in Markov Jump Processes	263
9.8	Exercises	265

10 Change of Probability Measure	267
10.1 Change of Measure for Random Variables	267
10.2 Change of Measure on a General Space	271
10.3 Change of Measure for Processes	274
10.4 Change of Wiener Measure	279
10.5 Change of Measure for Point Processes	280
10.6 Likelihood Functions	282
10.7 Exercises	285
11 Applications in Finance: Stock and FX Options	287
11.1 Financial Derivatives and Arbitrage	287
11.2 A Finite Market Model	293
11.3 Semimartingale Market Model	297
11.4 Diffusion and the Black-Scholes Model	302
11.5 Change of Numeraire	310
11.6 Currency (FX) Options	312
11.7 Asian, Lookback and Barrier Options	315
11.8 Exercises	319
12 Applications in Finance: Bonds, Rates and Options	323
12.1 Bonds and the Yield Curve	323
12.2 Models Adapted to Brownian Motion	325
12.3 Models Based on the Spot Rate	326
12.4 Merton's Model and Vasicek's Model	327
12.5 Heath-Jarrow-Morton (HJM) Model	331
12.6 Forward Measures. Bond as a Numeraire	336
12.7 Options, Caps and Floors	339
12.8 Brace-Gatarek-Musiela (BGM) Model	341
12.9 Swaps and Swaptions	345
12.10 Exercises	347
13 Applications in Biology	351
13.1 Feller's Branching Diffusion	351
13.2 Wright-Fisher Diffusion	354
13.3 Birth-Death Processes	356
13.4 Branching Processes	360
13.5 Stochastic Lotka-Volterra Model	366
13.6 Exercises	373

14 Applications in Engineering and Physics	375
14.1 Filtering	375
14.2 Random Oscillators	382
14.3 Exercises	388
Solutions to Selected Exercises	391
References	407
Index	413