Chapter 1

Basic Fluid Equations

Fluid dynamics is the continuum description of the flow of a large number of particles. Such a description is widely applicable in astrophysical problems, and fluid dynamical processes play a key role in many areas of astrophysics. In this book the fluid under consideration will generally be a gas, though the equations of fluid dynamics can also be applied to describing the motion of a collection of stars, or even galaxies, provided that one is interested in the collective behaviour on sufficiently large scales.

The term *fluid* in general refers to gases and liquids. Fluids are distinguished from solids in that solids have rigidity. Both solids and fluids deform when a stress is applied to them; but, unlike a solid, a simple fluid has no tendency to return to its original state when the applied stress is removed.

The continuum description is fundamental to the fluid approach to describing the dynamics of a collection of particles. The domain of validity of the continuum description is determined by comparing the collisional mean free path l of the particles with the macroscopic length scale L of interest in the problem. If $l \ll L$ then it is reasonable to introduce the concept of a fluid volume element, whose linear size is much larger than l but much smaller than L. The number of particles inside a fluid element is large, and we can associate with the fluid element a bulk velocity $\boldsymbol{u}$. Individual particle velocities have a random component in addition to $\boldsymbol{u}$ but, because the mean free path is small, the random motion does not immediately take the particle far from its neighbouring particles because the particle travels only a distance of order l before undergoing a collision and changing its direction. We can also associate with the fluid element other macroscopic properties such as a density ρ (total mass of the particles inside the element, divided by its volume). Over a short time interval from time t to time $t + \delta t$

we may define the fluid element to transform by translating each point of the element by an amount $\boldsymbol{u}(\boldsymbol{r},t)\delta t$, where $\boldsymbol{u}(\boldsymbol{r},t)$ is the local mean velocity at the position $\boldsymbol{r}$ of the element. By virtue of the above considerations, the fluid element will still contain essentially the same number of particles at $t+\delta t$ as it did at time t, and moreover they will be almost all the same particles as before. Hence the macroscopic properties of the fluid element will evolve only slowly, and by a diffusive process.

For further discussion of the continuum description and the fluid approach, see e.g. Batchelor (1967) and Shu (1992).

If the mean free path of the particles is not much smaller than the macroscopic scale of interest, then the appropriate description of the collective properties of the particles is kinetic theory. The equations of fluid dynamics can indeed be derived from the microscopic basis of kinetic theory. For a presentation of this approach, see Shu (1992). Here we shall instead assume the continuum description from the outset and see how simple considerations of the motion of the fluid, and the forces acting on it, lead to the fluid dynamical equations.

1.1 The Material Derivative

The fluid properties, such as its density ρ and velocity $\boldsymbol{u}$, will in general be functions of position $\boldsymbol{r}$ and of time t. We shall always use $\partial/\partial t$ to denote the rate of change of some quantity with respect to time *at a fixed position in space*. In describing fluids it is also very useful to define the *material derivative*, which will be denoted D/Dt: this is the rate of change of some quantity with respect to time but travelling along with the fluid.

Let $f(\boldsymbol{r},t)$ be any quantity, for example, temperature of the fluid. It may happen that the temperature of all individual parcels of fluid is not changing with time, so the material derivative Df/Dt is zero; but if some fluid is hotter than other fluid then the temperature at a fixed point in space may still change with time as fluid of different temperature passes the point at which the temperature is measured. In fact, in that case, $\partial f/\partial t = -\boldsymbol{u}\cdot\nabla f$ where $\boldsymbol{u}(\boldsymbol{r},t)$ is the velocity of the fluid. More generally, the material derivative is related to the rate of change at a fixed point in space as

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \boldsymbol{u}\cdot\nabla f \, . \tag{1.1}$$

Equation (1.1) can be derived by considering the change in f when following the fluid, over a short period of time δt, in the limit as δt tends to zero. Since (correct to first order in δt) the fluid element will have moved from $\boldsymbol{r}$ at time t to $\boldsymbol{r} + \boldsymbol{u}\delta t$ at time $t + \delta t$, the material derivative is

$$\frac{Df}{Dt} = \lim_{\delta t \to 0} \left(\frac{f(\boldsymbol{r} + \boldsymbol{u}\,\delta t, t + \delta t) - f(\boldsymbol{r}, t)}{\delta t} \right)$$

and Eq. (1.1) follows.

1.2 The Continuity Equation

Consider a volume V, which is fixed in space, enclosed by a surface S on which $\boldsymbol{n}$ is the outward-pointing normal vector (Fig. 1.1). The total mass of fluid in V is $\int_V \rho \mathrm{d}V$, where $\rho(\boldsymbol{r}, t)$ is the density of the fluid. The time derivative of the mass in V is the mass flux into V across its surface S, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \,\mathrm{d}V = - \int_S (\rho \boldsymbol{u}) \cdot \boldsymbol{n} \,\mathrm{d}S \,. \tag{1.2}$$

Since V is a volume fixed in space, the time derivative on the left of Eq. (1.2) can be taken inside the integral and becomes a derivative at fixed position in space. The surface term on the right-hand side of the equation can be re-expressed as a volume integral using the divergence theorem. Hence we obtain

$$\int_V \frac{\partial \rho}{\partial t} \,\mathrm{d}V = - \int_V \nabla \cdot (\rho \boldsymbol{u}) \,\mathrm{d}V \,.$$

Since this holds for any arbitrary volume V in the fluid, it follows that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \,. \tag{1.3}$$

This is the *continuity equation* (or mass conservation equation). Combining Eqs. (1.1) and (1.3) the continuity equation can also be expressed as

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{u} = 0 \,. \tag{1.4}$$

1.3 The Momentum Equation

One can similarly derive a momentum equation, or equation of motion, for the fluid by considering the rate of change of the total momentum of the fluid inside a volume V. It turns out to be easiest to consider a volume

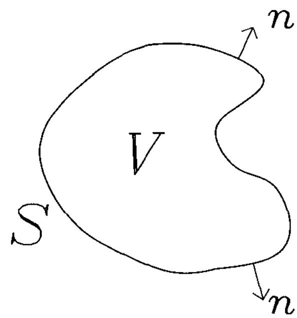

Fig. 1.1 An arbitrary volume of fluid V, with surface S and outward-pointing normal $\boldsymbol{n}$.

moving with the fluid, so that no fluid is flowing across its surface into or out of V. The momentum of the fluid in V is $\int_V \rho \boldsymbol{u}\,\mathrm{d}V$, and the rate of change of this momentum is equal to the net force acting on the fluid in volume V. These are of two kinds. First there are body forces, such as gravity, which act on the particles inside V: their net effect is a force

$$\int_V \rho \boldsymbol{f}\,\mathrm{d}V$$

where $\boldsymbol{f}$ is the body force per unit mass. (Note that force per unit mass has dimensions of acceleration.) For example, $\boldsymbol{f}$ could be the gravitational acceleration $\boldsymbol{g}$. The second kind of forces acting are surface forces – forces exerted on the surface S of V by the surrounding fluid. In an *inviscid* fluid, such as we shall mostly be considering, the surface force acts normally to the surface and its net effect is

$$\int_S -p\boldsymbol{n}\,\mathrm{d}S\ ,$$

p being the pressure. There is no flux of momentum across the surface carried by fluid parcels moving, since by definition none crosses the surface of a material volume. Equating force to change of momentum we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \boldsymbol{u}\,\mathrm{d}V = \int_S -p\boldsymbol{n}\,\mathrm{d}S + \int_V \rho \boldsymbol{f}\,\mathrm{d}V\ . \tag{1.5}$$

Since V is a material volume, when the time derivative is taken inside the integral it becomes a material derivative; but the product $\rho\mathrm{d}V$ is the mass of a fluid element and is invariant following the motion so its material derivative is zero. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\boldsymbol{u}\,\mathrm{d}V \;=\; \int_V \rho\frac{D\boldsymbol{u}}{Dt}\,\mathrm{d}V \tag{1.6}$$

and hence, applying the divergence theorem to the surface integral in Eq. (1.5), we obtain

$$\int_V \rho\frac{D\boldsymbol{u}}{Dt}\,\mathrm{d}V \;=\; \int_V (-\nabla p \,+\, \rho\boldsymbol{f})\,\mathrm{d}V\,.$$

Since this holds for any arbitary material volume V, it follows that

$$\rho\frac{D\boldsymbol{u}}{Dt} \;\equiv\; \rho\left(\frac{\partial\boldsymbol{u}}{\partial t} \,+\, (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right) \;=\; (-\nabla p \,+\, \rho\boldsymbol{f})\;. \tag{1.7}$$

This is the momentum equation for an inviscid fluid.

In a general viscous fluid (it doesn't need to be as extreme an example as treacle!) the ith component of the force exerted on surface S by the surrounding fluid is not just $\int_S -p\,n_i\,\mathrm{d}S$ but is $\int_S \sigma_{ij}n_j\,\mathrm{d}S$, where σ_{ij} is the *stress tensor.* (Note here that the summation convention is used, so that if an index is repeated it should be summed over. A non-repeated index denotes a component of a vector or tensor. See Appendix A. Also note that throughout this book we shall use $\boldsymbol{r}$ or $\boldsymbol{x}$ to denote vector position; but for its ith component form we always write x_i.) For gases and simple liquids it is found that

$$\sigma_{ij} \;=\; -p\delta_{ij} \,+\, \mu\left(\frac{\partial u_i}{\partial x_j} \,+\, \frac{\partial u_j}{\partial x_i} \,-\, \frac{2}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right) \tag{1.8}$$

where μ is the so-called dynamical viscosity: see e.g. Batchelor (1967), Landau & Lifshitz (1959). Also δ_{ij} is the Kronecker delta: see Appendix A. Now

$$\int_S \sigma_{ij}\,n_j\,\mathrm{d}S \;=\; \int_V \frac{\partial}{\partial x_j}\sigma_{ij}\,\mathrm{d}V$$

(divergence theorem), and so if μ is a constant it follows that the equation of motion for a viscous fluid is

$$\rho\frac{D\boldsymbol{u}}{Dt} \;=\; -\nabla p \,+\, \rho\boldsymbol{f} \,+\, \mu\left(\nabla^2\boldsymbol{u} \,+\, \frac{1}{3}\nabla(\nabla\cdot\boldsymbol{u})\right)\;. \tag{1.9}$$

With viscosity included, Eq. (1.9) is called the Navier-Stokes equation. Its inviscid form, Eq. (1.7), is called the Euler equation. Throughout most of this book we shall neglect viscosity: the justification of this approximation in astrophysical contexts will be seen in Section 2.8. However, viscosity plays a key role in some applications, notably in astrophysical accretion disks which we discuss in Chapter 9.

1.4 Newtonian Gravity

A mass m' at position $\boldsymbol{r}'$ exerts on any other mass m at position $\boldsymbol{r}$ an attractive force proportional to the product of the two masses and inversely proportional to the square of the distance between them, directed towards mass m':

$$\boldsymbol{F} = m\boldsymbol{g}(\boldsymbol{r}) \equiv -\frac{Gmm'}{|\boldsymbol{r}-\boldsymbol{r}'|^2}\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \equiv -\frac{Gmm'(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} . \tag{1.10}$$

Note that $(\boldsymbol{r}-\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|$ is the unit vector along the line of action of the force. Now

$$\nabla\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) = \frac{-(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \tag{1.11}$$

(the derivatives are with respect to $\boldsymbol{r}$: they treat $\boldsymbol{r}'$ as a constant vector), so the gravitational acceleration $\boldsymbol{g}(\boldsymbol{r})$ can be written as the gradient of a potential function $\psi(\boldsymbol{r})$:

$$\boldsymbol{g} = -\nabla\psi , \quad \text{where} \quad \psi = \frac{-Gm'}{|\boldsymbol{r}-\boldsymbol{r}'|} . \tag{1.12}$$

Similarly, the gravitational field due to a fluid can be written as a potential, namely the sum of the potentials due to all the fluid elements. The mass of a fluid element of volume $\mathrm{d}V'$ at position $\boldsymbol{r}'$ is $\rho(\boldsymbol{r}')\mathrm{d}V'$, so the total gravitational potential is

$$\psi(\boldsymbol{r}) = \int_{V'} \frac{-G\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V' , \tag{1.13}$$

where the integration is over the whole volume of the fluid. The gravitational acceleration is $-\nabla\psi$.

Using the result

$$\nabla^2\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) = -4\pi\,\delta(\boldsymbol{r}-\boldsymbol{r}') \tag{1.14}$$

(δ being the Dirac delta function in 3-D space), Eq. (1.13) can be rewritten as a partial differential equation, *Poisson's equation*:

$$\nabla^2\psi = 4\pi G\rho \, . \tag{1.15}$$

1.5 The Mechanical and Thermal Energy Equations

If one takes Newton's third law, $F = ma = m(dv/dt)$ and multiplies by velocity v, one obtains that rate of work of the forces, Fv, is equal to the rate of change of kinetic energy, $d(\frac{1}{2}mv^2)/dt$. Similarly, taking the dot product of the equation of motion for a fluid, (1.7), with the fluid velocity $\boldsymbol{u}$ yields

$$\frac{D}{Dt}\left(\frac{1}{2}\boldsymbol{u}^2\right) = -\frac{1}{\rho}\boldsymbol{u}\cdot\nabla p + \boldsymbol{u}\cdot\boldsymbol{f} \, . \tag{1.16}$$

Equation (1.16) says that the rate of change of the kinetic energy of a unit mass of fluid is equal to the rate at which work is done on the fluid by pressure and body forces. This is sometimes called the mechanical energy equation.

An equation for the total energy — kinetic and internal thermal energy — can be derived in the same manner as was the momentum equation in Section 1.3. Let the internal energy per unit mass of fluid be U. Then the rate of change of kinetic plus internal energy of a material volume (i.e. one moving with the fluid) must be equal to the rate of work done on the fluid by surface and body forces, plus the rate at which heat is added to the fluid. Heat can be added in two ways: one is by its being generated at a rate ϵ per unit mass within the fluid volume (e.g. by nuclear reactions), while the second is by the heat flux $\boldsymbol{F}$ across the surface S (e.g. radiative heat flux). Thus

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_V \left(\frac{1}{2}\boldsymbol{u}^2 + U\right)\rho\,\mathrm{d}V \\ &= \int_S \boldsymbol{u}\cdot(-p\boldsymbol{n})\,\mathrm{d}S + \textstyle\int_V \boldsymbol{u}\cdot\boldsymbol{f}\rho\,\mathrm{d}V + \int_V \epsilon\rho\mathrm{d}V - \int_S \boldsymbol{F}\cdot\boldsymbol{n}\mathrm{d}S \, . \end{aligned} \tag{1.17}$$

In the same way as for the momentum equation, one rewrites all the surface integrals in this equation as volume integrals, using the divergence theorem. The resulting equation holds for an arbitrary volume V and so one deduces that

$$\rho\left(\frac{D}{Dt}\left(\frac{1}{2}\boldsymbol{u}^2\right) + \frac{DU}{Dt}\right) = -\nabla\cdot(p\boldsymbol{u}) + \rho\boldsymbol{u}\cdot\boldsymbol{f} + \rho\epsilon - \nabla\cdot\boldsymbol{F}\,. \tag{1.18}$$

One can derive an equation for the thermal energy alone by dividing Eq. (1.18) by the density and then subtracting the kinetic energy equation (1.16) to obtain

$$\frac{DU}{Dt} = \frac{p}{\rho^2}\frac{D\rho}{Dt} + \epsilon - \frac{1}{\rho}\nabla\cdot\boldsymbol{F}\,. \tag{1.19}$$

The divergence of $\boldsymbol{u}$ has been replaced by $-\rho^{-1}D\rho/Dt$ using the continuity equation (1.4).

Noting that the volume per unit mass is just the reciprocal of the density, i.e. $V = \rho^{-1}$, we recognise the thermal energy equation (1.19) as a statement of the first law of thermodynamics:

$$\mathrm{d}U = (-p)\mathrm{d}V + \delta Q, \tag{1.20}$$

that is, the change in the internal energy is equal to the work $(-p)\mathrm{d}V$ done (on the fluid) plus the heat added. Note that V, U, p are properties of the fluid (in fact they are thermodynamic state variables) and we denote changes in them with the symbol "d". In contrast, there is no such property as the heat content and so we cannot speak of the change of heat content. Instead, we can only speak of the heat added, and we therefore use a different notation, i.e. δQ.

Equations (1.16)-(1.19) can be generalized to include a viscous stress term. In that case, $-\rho^{-1}\boldsymbol{u}\cdot\nabla p$ in Eq. (1.16) and $\boldsymbol{u}\cdot(-p\boldsymbol{n})$ in equation (1.18) are replaced by $\rho^{-1}u_i\partial\sigma_{ij}/\partial x_j$ and $u_i\sigma_{ij}n_j$ respectively, where σ_{ij} is the stress tensor as in Eq. (1.8). The consequence for the thermal energy equation (1.19) is that kinetic energy is converted to heat by viscosity, so that one obtains an additional heating term similar to ϵ. This is discussed in more detail in Chapter 9.

With some effort, one can use the above equations to derive an integral equation (sometimes also referred to as the total energy equation) for the rate of change of the total energy (kinetic plus internal plus gravitational potential energy) for the whole fluid volume:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V\left(\frac{1}{2}\boldsymbol{u}^2 + U + \frac{1}{2}\psi\right)\rho\mathrm{d}V + \int_V\nabla\cdot\left[\left(\frac{1}{2}\boldsymbol{u}^2 + U + \frac{p}{\rho} + \psi\right)\rho\boldsymbol{u}\right]\mathrm{d}V$$
$$= \int_V(\rho\epsilon - \nabla\cdot\boldsymbol{F})\mathrm{d}V\,, \tag{1.21}$$

where V is a fixed volume enclosing the whole fluid: e.g. Cox (1980). In deriving the above equation it is helpful first to establish from eq. (1.13) that $\int_V (\partial\rho/\partial t)\psi \mathrm{d}V = \int_V (\rho\partial\psi/\partial t)\mathrm{d}V$ where V is the whole region occupied by the fluid.

The volume integrals of divergence terms in Eq. (1.21) can of course be re-expressed as surface integrals. If the flux in square brackets in the second term on the left of Eq. (1.21) vanishes at the surface of V, which might for example represent the interior of a star, then the total energy in V can only change through internal heat sources/sinks (ϵ) or heat flux ($\boldsymbol{F}$) across the surface.

1.6 A Little More Thermodynamics

The second law of thermodynamics states that

$$\delta Q = T\,\mathrm{d}S\,, \tag{1.22}$$

where S is a thermodynamic state variable, the *specific entropy* (i.e. the entropy per unit mass). Combining this with the first law, Eq. (1.20), yields

$$\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V\,. \tag{1.23}$$

From this various relations between thermodynamic derivatives can be deduced. For example, it follows immediately from Eq. (1.23) that

$$T = \left(\frac{\partial U}{\partial S}\right)_V \quad \text{and} \quad -p = \left(\frac{\partial U}{\partial V}\right)_S\,; \tag{1.24}$$

but a property of partial differentiation is that $\partial^2 f/\partial x\partial y = \partial^2 f/\partial y\partial x$, so we find that

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V\,. \tag{1.25}$$

Another useful manipulation that is a general property of partial derivatives is that

$$\frac{\partial f/\partial y)_x}{(\partial f/\partial x)_y} \equiv -\left(\frac{\partial x}{\partial y}\right)_f\,. \tag{1.26}$$

This follows by rearranging $\mathrm{d}f = (\partial f/\partial x)_y\,\mathrm{d}x + (\partial f/\partial y)_x\,\mathrm{d}y$ to make $\mathrm{d}x$ the subject of the formula, and identifying the resulting coefficient of $\mathrm{d}y$ as the derivative $(\partial x/\partial y)_f$. Various thermodynamic relations that are useful in stellar physics and astrophysical fluid dynamics can be found in e.g. Kippenhahn & Weigert (1990).

We define the *adiabatic exponents* γ_1, γ_2, γ_3 by

$$\gamma_1 = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_S, \quad \frac{\gamma_2 - 1}{\gamma_2} = \left(\frac{\partial \ln T}{\partial \ln p}\right)_S, \quad \gamma_3 - 1 = \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_S. \tag{1.27}$$

Note that all these partial derivatives are at constant specific entropy: 'adiabatic' here means without exchange of heat, so $\delta Q = 0 = \mathrm{d}S$, i.e. S is constant. The quantity $(\gamma_2 - 1)/\gamma_2 \equiv (\partial \ln T/\partial \ln p)_S$ is often referred to as ∇_{ad}.

We define c_p, the specific heat at constant pressure, to be the amount of heat required to make a unit increase in temperature, without the pressure changing: thus, from Eq. (1.22), $c_p = T(\partial S/\partial T)_p$. Similarly we define c_V, the specific heat at constant volume, to be the amount of heat required to make a unit increase in temperature at constant V. The following three useful results relate the amount of heat added to the changes in pairs of thermodynamic variables:

$$\begin{aligned}
\delta Q &= \frac{1}{\rho(\gamma_3 - 1)}\left(\mathrm{d}p - \frac{\gamma_1 p}{\rho}\mathrm{d}\rho\right) \\
\delta Q &= c_p\left(\mathrm{d}T - \frac{\gamma_2 - 1}{\gamma_2}\frac{T}{p}\mathrm{d}p\right) \\
\delta Q &= c_V\left(\mathrm{d}T - (\gamma_3 - 1)\frac{T}{\rho}\mathrm{d}\rho\right) .
\end{aligned} \tag{1.28}$$

For example, the first equation can be derived by noting

$$\begin{aligned}
\delta Q \equiv T\mathrm{d}S &= T\left(\frac{\partial S}{\partial p}\right)_V \mathrm{d}p + T\left(\frac{\partial S}{\partial V}\right)_p \mathrm{d}V \\
&= T\left(\frac{\partial p}{\partial S}\right)_V^{-1}\left[\mathrm{d}p + T\frac{(\partial S/\partial V)_p}{(\partial S/\partial p)_V}\mathrm{d}V\right]
\end{aligned}$$

and using Eq. (1.25) on the factor outside the square brackets and Eq. (1.26) to manipulate the last term, together with the definitions of γ_1 and γ_3. Note that $\ln V = -\ln \rho$. The other two above expressions for δQ are derived similarly.

1.7 Perfect Gases

A perfect gas is one for which

$$pV = RT \tag{1.29}$$

(R being some constant), and

$$U = U(T)\,. \tag{1.30}$$

It follows from Eq. (1.29) that

$$\frac{\mathrm{d}p}{p} + \frac{\mathrm{d}V}{V} = \frac{\mathrm{d}T}{T}\,. \tag{1.31}$$

Now for an adiabatic change of such a gas,

$$0 = \mathrm{d}S = \frac{1}{T}(\mathrm{d}U + p\mathrm{d}V) = \frac{\mathrm{d}U}{\mathrm{d}T}\frac{\mathrm{d}T}{T} + R\frac{\mathrm{d}V}{V}\,. \tag{1.32}$$

From Eq. (1.32) and the definition of γ_3 it follows that

$$\gamma_3 = 1 + \frac{R}{\mathrm{d}U/\mathrm{d}T}\,. \tag{1.33}$$

Eliminating $\mathrm{d}T$ between Eqs. (1.31) and (1.32) gives that γ_1 is given by the same expression (1.33), and likewise for γ_2 (eliminating $\mathrm{d}V$). Thus for a perfect gas, the three adiabatic exponents are equal.

Henceforward in this book, since for a perfect gas all three adiabatic exponents are equal, we shall use γ to denote all of them when no confusion can arise.

In fact, for a monatomic gas (in which the molecules are simply point masses) one can show that γ is equal to 5/3, as follows. For a monatomic gas, the internal energy is just the translational kinetic energy of all the molecules. Assuming the gas to be isotropic (all directions equivalent) and all the molecules identical, the total internal energy of the gas in volume V is

$$U = \frac{1}{2}Nm\left(\overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2}\right) = \frac{3}{2}Nm\overline{v_x^2}\,, \tag{1.34}$$

where m is the mass of a molecule, N is the number of molecules in V, and (for example) $\overline{v_x^2}$ is the mean squared velocity in the x direction. Suppose the volume V is enclosed by a rigid rectangular box of length l in the x-direction (and of cross-sectional area $A = V/l$). The force on the end of the box is pA. Consider a single molecule. It has some x-velocity v_x. In time $\Delta t \equiv 2l/v_x$ it bounces off that end of the box once. In bouncing, its x-momentum changes by an amount $2mv_x$ (assuming an elastic collision). Thus, since force (= pressure × area) is equal to the rate of change of momentum, summing over all molecules gives

$$pA = \sum \frac{2mv_x}{\Delta t} = \sum \frac{m}{l} v_x^2 = \frac{Nm}{l}\overline{v_x^2} . \qquad (1.35)$$

Hence $pV = Nm\overline{v_x^2}$ and so, from Eqs. (1.29), (1.33) and (1.34),

$$U = \frac{3}{2}RT \quad \text{and} \quad \gamma = \frac{5}{3} . \qquad (1.36)$$

If a gas is undergoing ionization, $\mathrm{d}U/\mathrm{d}T$ is greater than it would otherwise be, because energy goes into ionizing the gas; so from e.g. Eq. (1.33) the adiabatic exponents are reduced in value.

1.8 The Virial Theorem

The velocity $\boldsymbol{u}$ is the rate of change of position following the fluid:

$$\boldsymbol{u} = \frac{D\boldsymbol{r}}{Dt} . \qquad (1.37)$$

Hence Eq. (1.7), with $\boldsymbol{f}$ replaced by gravitational acceleration and using Eq. (1.12), can be rewritten

$$\rho \frac{D^2\boldsymbol{r}}{Dt^2} = -\nabla p - \rho\nabla\psi . \qquad (1.38)$$

Taking the dot product with $\boldsymbol{r}$ and integrating over the whole volume of the fluid gives

$$\int_V \boldsymbol{r}\cdot\frac{D^2\boldsymbol{r}}{Dt^2}\rho \mathrm{d}V = -\int_V \boldsymbol{r}\cdot\nabla p \,\mathrm{d}V - \int_v \boldsymbol{r}\cdot\nabla\psi\,\rho\mathrm{d}V . \qquad (1.39)$$

The left-hand side of Eq. (1.39) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \boldsymbol{r}\cdot\frac{D\boldsymbol{r}}{Dt}\rho\mathrm{d}V - \int_V \left(\frac{D\boldsymbol{r}}{Dt}\right)^2 \rho\mathrm{d}V = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_V |\boldsymbol{r}|^2\rho\mathrm{d}V - 2\mathcal{T} , \qquad (1.40)$$

where $\mathcal{T} \equiv \frac{1}{2}\int_V \rho\boldsymbol{u}^2\,\mathrm{d}V$ is the total kinetic energy of the fluid. (Note that here and in similar expressions we write $\boldsymbol{u}^2$ when what is meant is $|\boldsymbol{u}|^2$, i.e. $\boldsymbol{u}\cdot\boldsymbol{u}$ — the quantity is a scalar.)

Using the divergence theorem and the identity $\nabla \cdot \boldsymbol{r} \equiv \partial x_i/\partial x_i = 3$, the pressure term in Eq. (1.39) can be rewritten as

$$-\int_V \boldsymbol{r} \cdot \nabla p \, \mathrm{d}V = -\int_S p\boldsymbol{r} \cdot \boldsymbol{n} \mathrm{d}S + 3\int_V p \, \mathrm{d}V . \qquad (1.41)$$

We suppose that the pressure vanishes at the boundary of the fluid volume (this can be a good approximation for a star, for example) so that the surface term is zero.

Finally,

$$\begin{aligned} -\int_V \boldsymbol{r} \cdot \nabla \psi \, \rho \mathrm{d}V &= G \int_V \int_{V'} \boldsymbol{r} \cdot \nabla \Big(\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \Big) \rho(\boldsymbol{r}) \, \mathrm{d}V \, \mathrm{d}V' \\ &= -G \int_V \int_{V'} \frac{\boldsymbol{r} \cdot (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \rho(\boldsymbol{r}) \mathrm{d}V \, \rho(\boldsymbol{r}') \mathrm{d}V' \\ &= -\frac{1}{2} G \int_V \int_{V'} \Big(\frac{\boldsymbol{r} \cdot (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} + \frac{\boldsymbol{r}' \cdot (\boldsymbol{r}' - \boldsymbol{r})}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \Big) \rho(\boldsymbol{r}) \mathrm{d}V \, \rho(\boldsymbol{r}') \mathrm{d}V' \\ &= -\frac{1}{2} G \int_V \int_{V'} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \rho(\boldsymbol{r}) \mathrm{d}V \, \rho(\boldsymbol{r}') \mathrm{d}V' \\ &= \Psi \end{aligned} \qquad (1.42)$$

(the later steps exploit the symmetry between $\boldsymbol{r}$ and $\boldsymbol{r}'$) where

$$\Psi \equiv \frac{1}{2} \int_V \psi \, \rho(\boldsymbol{r}) \mathrm{d}V \qquad (1.43)$$

is the total gravitational energy. Putting all this together yields

$$\frac{1}{2} \frac{d^2 \mathcal{I}}{dt^2} = 2\mathcal{T} + 3 \int_V p \, \mathrm{d}V + \Psi , \qquad (1.44)$$

where $\mathcal{I} \equiv \int_V \rho r^2 \, \mathrm{d}V$. Equation (1.44) is the scalar form of the *virial theorem.*

One can also derive a tensor virial theorem, by taking the ith component of Eq. (1.38) and multiplying by the jth component of $\boldsymbol{r}$:

$$\rho x_j \frac{D^2 x_i}{Dt^2} = -x_j \frac{\partial p}{\partial x_i} - \rho x_j \frac{\partial \psi}{\partial x_i} . \qquad (1.45)$$

It can then be shown that

$$\frac{1}{2} \frac{\mathrm{d}^2 I_{ij}}{\mathrm{d}t^2} = 2T_{ij} + \delta_{ij} \int_V p \, \mathrm{d}V + \Psi_{ij} , \qquad (1.46)$$

where

$$\begin{aligned} I_{ij} &= \int_V \rho x_i x_j \mathrm{d}V \, , \\ T_{ij} &= \frac{1}{2}\int_V \rho u_i u_j \mathrm{d}V \, , \\ \Psi_{ij} &= -\frac{1}{2}G\int_V\int_{V'} \frac{(x_i - x_i')(x_j - x_j')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\rho(\boldsymbol{r})\mathrm{d}V\,\rho(\boldsymbol{r}')\mathrm{d}V' \, . \end{aligned} \tag{1.47}$$

Note that if Eq. (1.46) is contracted over i and j (i.e. multiplied by δ_{ij} and summed over i and j) then the scalar virial theorem (1.44) is recovered.

Derivations of the virial theorem in different forms can be found in Chandrasekhar (1969) and Tassoul (1978).

1.9 Vorticity

An important derived quantity for a fluid flow is the vorticity

$$\boldsymbol{\omega} = \nabla\times\boldsymbol{u} \, . \tag{1.48}$$

For a fluid rotating rigidly with angular velocity $\boldsymbol{\Omega}$, for example, $\boldsymbol{u} = \boldsymbol{\Omega}\times\boldsymbol{r}$ and

$$\boldsymbol{\omega} = 2\boldsymbol{\Omega} \, , \tag{1.49}$$

using a standard vector identity (cf. Appendix A). Generally, in a fluid flow $\boldsymbol{u}$ the vorticity at any location is equal to twice the local rotation rate of a fluid line element at that location, as is proved below. This does not mean however that streamlines have to be curved for the fluid to possess non-zero vorticity. For example, consider the shear flow $\boldsymbol{u} = Cy\boldsymbol{e}_x$ where C is a non-zero constant: this is a unidirectional shear flow in the x-direction with magnitude proportional to y. Here as elsewhere we use $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$ to denote unit vectors in the x-, y- and z-directions. It is a straightforward exercise to show that the vorticity of such a flow is $\boldsymbol{\omega} = -C\boldsymbol{e}_z$, which is non-zero although the streamlines (lines everywhere parallel to the flow) are straight lines.

To see the relationship between vorticity and local rotation of the fluid, we shall now analyse the relative motion of a fluid in the vicinity of a point. Let A be a point moving with the fluid, which at an initial time t is at position $\boldsymbol{r}$; and let B be another point which at time t is at nearby position $\boldsymbol{r} + \delta\boldsymbol{r}$ (Fig. 1.2). At time t therefore the position of B relative to A is $\delta\boldsymbol{r}$.

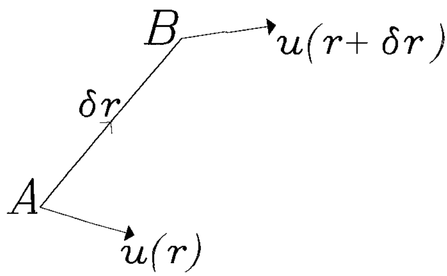

Fig. 1.2 The motion of neighbouring points A (at position $\boldsymbol{r}$) and B (at postion $\boldsymbol{r}+\delta\boldsymbol{r}$), which leads to the evolution of the material line element $\delta\boldsymbol{r}$ with time.

The fluid velocity at A is $\boldsymbol{u}(\boldsymbol{r})$ and at B it is $\boldsymbol{u}(\boldsymbol{r}+\delta\boldsymbol{r})$; therefore after a short time δt the separation of B from A has changed to

$$\{\boldsymbol{r}+\delta\boldsymbol{r}+\delta t\,\boldsymbol{u}(\boldsymbol{r}+\delta\boldsymbol{r})\} \; - \; \{\boldsymbol{r}+\delta t\,\boldsymbol{u}(\boldsymbol{r})\} \;\equiv\; \delta\boldsymbol{r}+\delta t\,\{\boldsymbol{u}(\boldsymbol{r}+\delta\boldsymbol{r})-\boldsymbol{u}(\boldsymbol{r})\}$$

correct to $\mathrm{O}(\delta t)$. The last term can be simplified by expanding $\boldsymbol{u}(\boldsymbol{r}+\delta\boldsymbol{r})$ in a Taylor series about $\boldsymbol{r}$ and keeping only terms up to $\delta\boldsymbol{r}$. Treating $\delta\boldsymbol{r}$ now as a function of time, an equation for the rate of change of $\delta\boldsymbol{r}$ with time can be can be obtained from here by dividing the difference between the new separation and the old one by δt and taking the limit as $\delta t \to 0$:

$$\frac{D\delta\boldsymbol{r}}{Dt} = \delta\boldsymbol{r}\cdot\nabla\boldsymbol{u} \tag{1.50}$$

where, since we are following the separation between material points, we write the derivative as a material derivative. This equation therefore describes the evolution of a material element $\delta\boldsymbol{r}$. The right-hand side can be expressed in index notation as $(\delta\boldsymbol{r})_j\partial u_i/\partial x_j$. The tensor $\nabla\boldsymbol{u}$, like any other second-rank tensor, can be split into a symmetric part and an anti-symmetric part:

$$\frac{\partial u_i}{\partial x_j} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right). \tag{1.51}$$

The second term on the right of Eq. (1.51) is anti-symmetric: using the definition (1.48) of vorticity, it can be written as $-\frac{1}{2}\epsilon_{ijk}\omega_k$ (cf. Appendix A). Hence, substituting into Eq. (1.50) it makes a contribution to the right-hand side which is equal to $-\frac{1}{2}\delta\boldsymbol{r}\times\boldsymbol{\omega} \equiv \frac{1}{2}\boldsymbol{\omega}\times\delta\boldsymbol{r}$. Thus, comparing this with the velocity due to a solid-body rotation, we deduce that the antisymmetric part of $\nabla\boldsymbol{u}$ contibutes to the motion of point B relative to point A a motion which is a rotation with angular velocity $\frac{1}{2}\boldsymbol{\omega}$. Equivalently, the vorticity is given by Eq. (1.49) where $\boldsymbol{\Omega}$ is interpreted as the local rotation rate.

The symmetric part of $\partial u_i/\partial x_j$, i.e. the first term on the right of Eq. (1.51), is called the rate of strain tensor e_{ij} :

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) . \tag{1.52}$$

Its trace e_{kk} is equal to $\nabla\cdot\boldsymbol{u}$ and the isostropic part of e_{ij}, namely $\frac{1}{3}e_{kk}\delta_{ij}$, represents an expansion or compression of the fluid in the region of point A. The remainder of e_{ij}, namely $e_{ij}-\frac{1}{3}e_{kk}\delta_{ij}$, has zero trace and represents a local shear of the fluid.

An evolution equation for vorticity can be derived by taking the curl of the momentum equation. Initially we shall consider the inviscid case. First we rewrite the $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ term in Eq. (1.7) using a vector identity to obtain

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{u}\times\boldsymbol{\omega} - \nabla\left(\frac{1}{2}\boldsymbol{u}^2\right) - \frac{1}{\rho}\nabla p + \boldsymbol{f} . \tag{1.53}$$

Taking the curl of this equation gives

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{\omega}) + \frac{1}{\rho^2}\nabla\rho\times\nabla p + \nabla\times\boldsymbol{f} \tag{1.54}$$

since the curl of a gradient vector is zero. The $\nabla\times(\boldsymbol{u}\times\boldsymbol{\omega})$ can be expanded using identity (A.4) from Appendix A. Noting that $\nabla\cdot\boldsymbol{\omega}$ vanishes by its definition (1.48), Eq. (1.54) becomes

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} - (\nabla\cdot\boldsymbol{u})\,\boldsymbol{\omega} + \frac{1}{\rho^2}\nabla\rho\times\nabla p + \nabla\times\boldsymbol{f} . \tag{1.55}$$

Finally, using the continuity equation (1.3) to eliminate $\nabla\cdot\boldsymbol{u}$, we obtain

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) \equiv \frac{\partial}{\partial t}\left(\frac{\boldsymbol{\omega}}{\rho}\right) + \boldsymbol{u}\cdot\nabla\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\right)\cdot\nabla\boldsymbol{u} + \frac{1}{\rho^3}\nabla\rho\times\nabla p + \frac{1}{\rho}\nabla\times\boldsymbol{f} . \tag{1.56}$$

Equation (1.56) is called the *vorticity equation.* It describes how vorticity evolves in a fluid.

A fluid for which $\nabla\rho\times\nabla p = 0$ everywhere is called *barotropic*: since the vector gradients of density and pressure are everywhere parallel, the surfaces of constant density and of constant pressure coincide, and it is possible to write either variable as a function solely of the other variable, e.g. $\rho = \rho(p)$. Conversely, if density and pressure are just functions one of the other, then the fluid is barotropic. If the fluid is barotropic and any body force $\boldsymbol{f}$ that is present is conservative (i.e. $\nabla \times \boldsymbol{f} = 0$, or equivalently $\boldsymbol{f}$ can be written as the gradient of a scalar potential), so in particular there are no viscous forces, then the last two terms in Eq. (1.56) vanish and the vorticity equation (1.56) is of the same form as the equation (1.50) for the evolution of a material line element. We deduce that in this case vortex lines move with the fluid. (A vortex line is a line everywhere parallel to the vorticity.) We can define a *vortex tube* to be, loosely speaking, a bundle of vortex lines or more precisely a tube whose surface is nowhere crossed by vortex lines and whose surface is itself composed of vortex lines. In the present case, then, the walls of a vortex tube form a material surface moving with the fluid. We define the strength of a vortex tube to be $\int_S \boldsymbol{\omega} \cdot \boldsymbol{n}\, \mathrm{d}S$, where S is any cross-section area cut across the tube and $\boldsymbol{n}$ is a vector normal to that area. Since no vortex lines cross the walls of a vortex tube, and vorticity is divergence-free, it follows from the divergence theorem that the strength of a vortex tube is a well-defined quantity, i.e. it is independent of which cross-section area we choose on which to evaluate it.

Generally in a fluid we can define the *circulation* about a closed curve C contained within the fluid to be

$$\Gamma = \oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{r} . \tag{1.57}$$

If we choose C to be a material curve, moving with the fluid, then

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \oint_C \frac{D\boldsymbol{u}}{Dt} \cdot \mathrm{d}\boldsymbol{r} + \oint_C \boldsymbol{u} \cdot \frac{D}{Dt}\mathrm{d}\boldsymbol{r} . \tag{1.58}$$

The last term can be rewritten as the integral around C of $\nabla(\frac{1}{2}\boldsymbol{u}^2)$ using Eq. (1.50), and for an inviscid fluid we can replace $D\boldsymbol{u}/Dt$ in the first term on the right of Eq. (1.58) using the momentum equation (1.7). We consider only 3-D fluid domains for which curve C can be spanned by a surface S wholly contained in the fluid domain; so Stokes's theorem can be applied. Hence Eq. (1.58) becomes

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \int_S \left(\frac{1}{\rho^2} \nabla\rho \times \nabla p + \nabla \times \boldsymbol{f} \right) \cdot \boldsymbol{n}\, \mathrm{d}S . \tag{1.59}$$

We can immediately deduce from this that for a barotropic fluid with only conservative body forces $\boldsymbol{f}$, the circulation around a material curve is invariant with time. This is a statement of *Kelvin's circulation theorem.*

Moreover, the strength of a vortex tube can be expressed, using Stokes's theorem, as a flow around a material curve embedded in the walls of the tube and encircling the tube's axis. Hence the strength of a vortex tube in such a flow is invariant also: if the fluid motion is such as to cause the vortex tube to become narrower (known as vortex stretching), the magnitude of $\boldsymbol{\omega}$ must increase so that $\int \boldsymbol{\omega} \cdot \boldsymbol{n}\, \mathrm{d}S$ over the cross-section of the tube is constant.

A fluid for which $\nabla\rho\times\nabla p \neq 0$ is called *baroclinic.* This means that surfaces of constant density are inclined to the surfaces of constant pressure.