Chapter 1

Solid Continuum Mechanics

This chapter summarizes the essence of solid continuum mechanics, putting an emphasis on the separation of physical principles and mathematical treatment. The physical principles are deformation, equilibrium and material properties. These principles are simple. However, the mathematics which describes the principles can become complicated[1] when the object of study is a continuum which serves as a model of geological structures or a building structure. By separating the physical principles and the mathematical treatment, it is shown that there is a common framework for all mechanical problems, ranging from a simple spring problem to a general solid continuum problem. Only the mathematical treatment becomes more sophisticated as the problem becomes more complicated. The separation of the physical principles and the mathematical treatment is particularly helpful in studying earthquake wave propagation phenomena; see Appendix A for a brief summary on the earthquake mechanisms.

Tree example problems are provided to summarize the essence of solid continuum mechanics in the next three sections. These problems are a spring problem, a pole problem and a continuum problem, and they are described as an algebraic equation, a boundary value problem with an ordinary differential equation, and a boundary value problem with a set of partial differential equations. While the form of the mathematical problems is different, these three problems are posed by suitably writing the three physical principles that govern the mechanics of solid continuum. This point should be emphasized. Also, it is shown that the problems of a different form can be recast into an *optimization problem* in which a solution is found by minimizing a suitable function or a suitable functional.

There have been numerous textbooks on continuum mechanics which are aimed at readers with various backgrounds. Only a few classical textbooks are mentioned in this book; the readers are recommended to find a text which matches their level of interest in continuum mechanics. As a basic textbook on continuum mechanics, [Chadwick (1976)] and [Spencer (1980)] are recommended; see also [Fung (1965)].

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[1] Solving a boundary value problem and using tensor for field variables are the two major difficulties among the mathematical treatments. Computational mechanics transforms a boundary value problem as a matrix equation and tensor quantities to vector quantities.

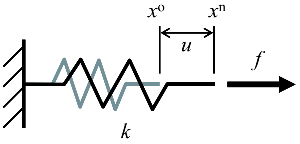

Fig. 1.1 A spring problem.

[Marsden *et al.* (1983)] is strongly recommended for readers who are interested in a more mathematical description of continuum mechanics. Although written in Japanese, [Yamamoto (1997)] is the best introduction for those who are willing to study a more physics-oriented description of classical mechanics that covers continuum mechanics.

1.1 Spring Problem

The spring problem is the simplest problem for solid mechanics. The problem setting is shown in Fig. 1.1. A spring with spring constant k is considered. The left end of the spring is fixed and the right end is subjected to external force f. The problem is to find the location of the right end. The solution is easily found, and f/k is the amount of movement of the right end. The framework of continuum mechanics, however, can be extracted even from this simple problem by carefully deriving the solution; the framework captures the three physical principles that govern the deformation of the spring. These principles provide necessary and sufficient conditions for the spring problem to have a unique solution.

First, how to express the deformation or elongation of the spring is considered. Since the left end is fixed, the amount of the elongation is represented by the movement of the right end. The elongation is written as $u=x^{\rm n}-x^{\rm o}$ with $x^{\rm n}$ and $x^{\rm o}$ being the location of the right end after and before the application of loading, respectively; $x^{\rm n}$ is unknown but $x^{\rm o}$ is known. Next, equilibrium is considered. Internal force acts in the spring; if the spring is cut in the middle, some force needs to be applied so that it is in equilibrium with the internal force which is acting. The amount of the applied force equals f. The internal force is denoted by s, and the equilibrium of the spring is thus expressed as $s=f$. Finally, the material property of the spring is considered. The property is simple. The internal force linearly increases with the elongation, and the spring constant is the ratio of the internal force and the elongation. Thus, the material property is expressed as $s=ku$.

Unknown quantities are the new right end location $x^{\rm n}$, the elongation u, and the internal force s. There are a set of three equations for these three unknowns, i.e.,

$$\begin{cases} u = x^{\mathrm{n}} - x^{\mathrm{o}}, \\ s = f, \\ s = ku. \end{cases} \tag{1.1}$$

It thus follows that a relation between f and x^{n} is $f{=}k(x^{\mathrm{n}}{-}x^{\mathrm{o}})$, and hence the solution is given as $x^{\mathrm{n}}{=}f/k{+}x^{\mathrm{o}}$ by using[2] $u{=}f/k$. The key point is that

$$f = ku \tag{1.2}$$

is mathematically derived from the set of the three physical equations, Eq. (1.1). No physics is involved in deriving Eq. (1.2) from Eq. (1.1). The physical principles that govern the spring deformation are fully described in terms of the three equations of Eq. (1.1).

As will be shown in Sec. 1.3, all problems of solid continuum mechanics have the identical structure as the above spring problem. That is, there are a set of three equations for the physical principles, and an equation is derived from the set so that the solution of the problem is obtained. This structure is called a *framework* in this book. Also, the mathematically derived equation is called a *governing equation* and the three equations for the physical principles are called *field equations.* The three physical principles are mutually independent, in the sense that the three equations hold by themselves.

While Eq. (1.2) is readily solved, it is interesting to transform this problem to an equivalent optimization problem. First, Eq. (1.2) is rewritten as

$$\delta u(ku - f) = 0,$$

for arbitrary δu. Replacing δu by du and integrating the right side with respect to u, the following function for u is derived:

$$J(u) = \frac{1}{2}ku^2 - fu. \tag{1.3}$$

As is seen, u that minimizes this J is the solution of Eq. (1.2). It appears tricky, but what has been done is to define a function J so that the problem of minimizing J has the same solution as Eq. (1.2). That is, an optimization problem of minimizing J is equivalent with an algebraic equation of the spring problem.

1.2 Pole Problem

The next example is a pole problem, the simplest problem of structure mechanics. The problem setting is shown in Fig. 1.2. A pole with height H and cross section area A is considered. It consists of an elastic material of Young's modulus E. The bottom end is fixed and the top end is traction-free. The x-coordinate is taken

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[2] As is seen, x^{o} plays a role similar to a boundary condition.

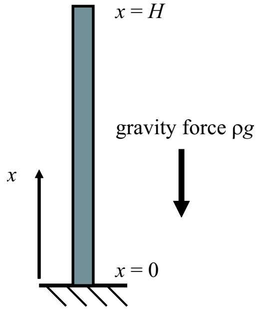

Fig. 1.2 A pole problem.

vertically, with x=0 and H corresponding to the bottom and the top of the pole, respectively. The pole problem is to find the deformation of the pole when it is subjected to gravity.

The three physical principles that govern the deformation of the pole are considered, and the corresponding field equations are described. First, denoting a displacement function by $u(x)$, strain $\epsilon(x)$ is derived as the derivative of $u(x)$. This strain is a measure[3] of local deformation of a point at x. Next, denoting stress by $\sigma(x)$, the equilibrium for a thin portion at x is expressed as

$$A\sigma(x + dx) - A\sigma(x) = -\rho g(A\,dx)$$

with ρ and g being the density and the gravity constant and dx being the height of the portion; $\sigma(x)$ is introduced to define internal force which acts at a point of x. Finally, the material property is described as a linear relation between strain $\epsilon(x)$ and stress $\sigma(x)$. Since the pole consists of a uniform material, E gives the coefficient of this relation at any x.

Unknown field variables of the pole problem are displacement $u(x)$, strain $\epsilon(x)$ and stress $\sigma(x)$. The three physical principles are described as the following set of field equations that these field variables must satisfy:

$$\begin{cases} \epsilon(x) = u'(x), \\ A\sigma'(x) = -\rho g A, \\ \sigma(x) = E\epsilon(x). \end{cases} \tag{1.4}$$

The governing equation is readily derived from the set, and a differential equation for $u(x)$ is obtained. That is,

$$EA\,u''(x) = -\rho g A. \tag{1.5}$$

There are two boundary conditions, $u(0)$=0 and $\sigma(H)$=0. Thus, a well-posed

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[3] In general, displacement is not a good measure of deformation since it can include rigid-body motion (translation or rotation).

boundary value problem for the displacement function $u(x)$ is described, i.e.,

$$\begin{cases} E\,u''(x) = -\rho g & 0 < x < H, \\ u(x) = 0 & x = 0, \\ u'(x) = 0 & x = H. \end{cases} \tag{1.6}$$

The solution is $u(x){=}\rho g/2E\,((x{-}H)^2{-}H^2)$, from which stress of the pole is found as $\sigma(x){=}\rho g(x{-}H)$. Thus, it is seen that the top goes down by $u(H){=}{-}\rho g H^2/2E$; the displacement increases linearly to the square of the height H. Also, the solution shows the distribution of internal forces acting in the pole. For instance, the pole bottom carries the cross section force of $A\sigma(0){=}{-}\rho g A H$, which coincides with the total weight of the pole, as it should be.

While the boundary value problem of Eq. (1.6) is well-posed, it is often better to transform it to another problem of a different form in order to find a solution by means of numerical analysis. A variational problem is used to this end. Like the boundary value problem, the variational problem is to find a function as a solution. The solution is a function which makes a certain functional[4] stationary. The boundary value problem of the pole is transformed to the variational problem that uses the following functional:

$$J(u) = \int_0^H \frac{1}{2} EA(u'(x))^2 - \rho g A u(x)\,dx, \tag{1.7}$$

for $u(x)$ satisfying $u(0){=}0$; the boundary condition of $u'(H){=}0$ is naturally satisfied in stationarizing[5] J. The variational problem of Eq. (1.7) is equivalent with the boundary value problem of Eq. (1.6) in the sense that the solution of Eq. (1.7) coincides with that of Eq. (1.6). Due to this equivalence, a numerical solution of Eq. (1.7) gives an approximate solution of Eq. (1.6). The variational problem of J is an optimization problem since the solution of the variational problem minimizes J due to $E{>}0$.

1.3 Continuum Problem

As the last example, a solid continuum problem is considered. Before studying this problem, the notation used in this book is explained. The Cartesian coordinate

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[4] A functional is understood as a function for functions, in the sense that when a function is input to a functional, it outputs a value. Integration is often used to describe a functional.

[5] Stationarization means vanishing of the first variation, i.e.,

$$\delta J(u) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(J(u + \varepsilon \delta u) - J(u))$$

for arbitrary δu satisfying $\delta u(0){=}0$. The limit is readily calculated as

$$\delta J(u) = \int_0^H \delta u(x)(-Eu''(x) - \rho g)\,A\,dx + \left[\delta u(x)(EA\,u'(x))\right]_0^H,$$

and hence the function that makes $\delta J{=}0$ coincides with the solution of Eq. (1.6).

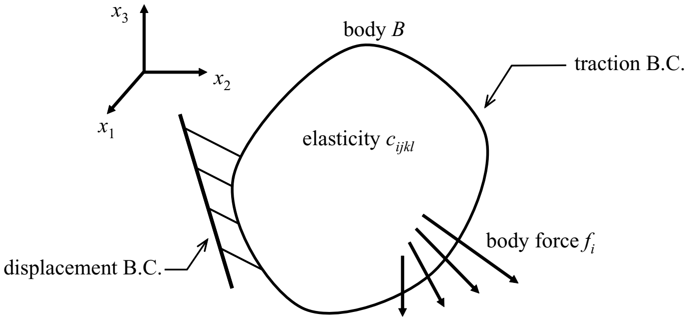

Fig. 1.3 A continuum problem.

system, $\{x_i\}$, is used, and vector and tensor quantities are designated by adding subscripts; for instance, u_i is a displacement vector, ϵ_{ij} and σ_{ij} are strain and stress tensors, and c_{ijkl} is an elasticity tensor. An argument of a function is omitted in the text; for instance, a displacement vector function is denoted by u_i instead of $u_i(\boldsymbol{x})$. Other rules are 1) subscript following a comma stands for the partial differentiation, i.e., $(.)_{,i} = \partial(.)/\partial x_i$; 2) summation convention is employed; and 3) δ_{ij} is Kronecker's delta, i.e., $\delta_{ij}{=}1$ for $i{=}j$ or 0 for $i{\neq}j$.

The problem setting is shown in Fig. 1.3. A deformable body with configuration B is the object of study. The body consists of a homogeneous and linear elastic material with elasticity c_{ijkl}. On some parts of the boundary denoted by ∂B_u, displacement $u_i^{\rm o}$ is prescribed, and on other parts denoted by ∂B_t, traction $t_i^{\rm o}$ is prescribed. External body force f_i is given. The continuum problem is to find the deformation of B. The major difference of the solid continuum problem from the previous two example problems is that the problem is stated in a two- or three-dimensional setting. This leads to the use of the following two mathematical tools:

1) a tensor[6] is used to describe strain, stress and elasticity;
2) a partial differential equation[7] is used to describe field equations.

These two mathematical tools are not an easy subject. As will be shown later, however, these tools are transformed to vector and matrix operation in numerical analysis of computational mechanics. Thus, the readers do not have to be an expert in using these mathematical tools although understanding these tools at a proper level is essential.

The three physical principles that govern the deformation of the body are considered. The first principle involves the *kinematics* that describe the state of deformation. It is assumed that deformation is infinitesimally small; this assumption is often

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[6] Understanding tensor and tensor algebra is important. However, tensors are not essential for computational mechanics when fixed Cartesian coordinates are used.

[7] Analysis of partial differential equations is essential for classical physics on which earthquake engineering is based; see [Farlow (1982)] for the application of partial differential equations to a broad class of engineering problems.

made for solids. Denoting displacement by u_i, strain ϵ_{ij} is defined as the symmetric part of displacement gradient $u_{i,j}$; strain is a measure of deformation. The second principle accounts for *statics*[8] that describes equilibrium. Assuming a quasi-static state, the equilibrium can be expressed as a balance between the gradient of stress and the external body forces, $\sigma_{ij,i}{=}f_j$, and the symmetry of stress, $\sigma_{ij}{=}\sigma_{ji}$; these two conditions are used to satisfy the equilibrium of force and moment, respectively. Stress is a measure of internal force that is generated by the deformation. The final principle designates the material property. Since B is linearly elastic, the property is described as a linear relation between σ_{ij} and ϵ_{ij} through c_{ijkl}.

Unknown field variables of the continuum problem are the displacement vector, u_i, and the strain and stress tensors, ϵ_{ij} and σ_{ij}. Like the previous problems, the physical principles that govern the continuum are expressed as the following set of field equations that the field variables must satisfy:

$$\begin{cases} \epsilon_{ij}(\boldsymbol{x}) = \frac{1}{2}(u_{i,j}(\boldsymbol{x}) + u_{j,i}(\boldsymbol{x})), \\ \sigma_{ij,i}(\boldsymbol{x}) + f_j(\boldsymbol{x}) = 0, \\ \sigma_{ij}(\boldsymbol{x}) = c_{ijkl}\epsilon_{kl}(\boldsymbol{x}). \end{cases} \tag{1.8}$$

Here, the symmetry of stress, $\sigma_{ij}{=}\sigma_{ji}$, is automatically satisfied by the symmetry of elasticity, $c_{ijkl}{=}c_{jikl}$; actually, elasticity is defined to satisfy this symmetry condition together with the other two symmetry conditions, $c_{ijkl}{=}c_{ijlk}$ and $c_{ijkl}{=}c_{klij}$. The governing equation[9] for u_i is thus derived from Eq. (1.8), as

$$(c_{ijkl}u_{k,l}(\boldsymbol{x}))_{,i} + f_j(\boldsymbol{x}) = 0. \tag{1.9}$$

Displacement u_i° is prescribed on some part of the boundary, ∂B_u, and traction t_i° is prescribed on the other part of the boundary, ∂B_t. These two conditions serve as boundary conditions. Hence, the following boundary value problem is posed for the displacement function u_i:

$$\begin{cases} (c_{ijkl}u_{k,l}(\boldsymbol{x}))_{,i} + f_j(\boldsymbol{x}) = 0 & \text{in } B, \\ u_i(\boldsymbol{x}) = u_i^{\circ}(\boldsymbol{x}) & \text{on } \partial B_u, \\ n_i(\boldsymbol{x})c_{ijkl}u_{k,l}(\boldsymbol{x}) = u_j^{\circ}(\boldsymbol{x}) & \text{on } \partial B_t. \end{cases} \tag{1.10}$$

Here, n_i is the unit outer normal of the boundary, and $n_i\sigma_{ij}$ gives traction acting on the boundary. As is seen, the flow from the three field equations to the governing equation is common to the previous examples, and this is the framework of continuum mechanics.

Like the pole problem, it is often the case that the boundary value problem is transformed to an equivalent variational problem so that the solution is found by means of numerical analysis. The variational problem is for a vector function u_i

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[8] Dynamics should be used instead of statics when inertia is included.

[9] There are cases where $(c_{ijkl}u_{k,l})_{,i}{+}f_j{=}0$ is called the equilibrium equation. In this book, however, $\sigma_{ij,i}{+}f_j{=}0$ is called the equilibrium equation as one of the three physical field equations; $(c_{ijkl}u_{k,l})_{,i}{+}f_j{=}0$ is a governing equation which is mathematically derived from the set of the field equations.

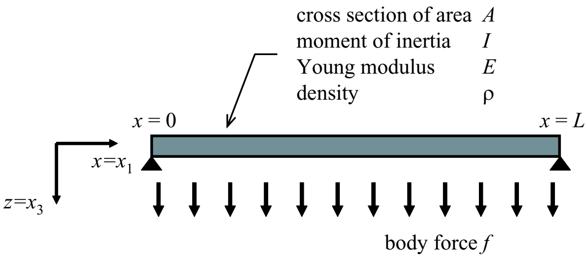

Fig. 1.4 A beam problem.

and uses the following functional:

$$J(\boldsymbol{u}) = \int_B \frac{1}{2} c_{ijkl}(\boldsymbol{x}) u_{i,j}(\boldsymbol{x}) u_{k,l}(\boldsymbol{x}) - f_i(\boldsymbol{x}) u_i(\boldsymbol{x})\, dv_{\boldsymbol{x}} + \int_{\partial B_t} u_i(\boldsymbol{x}) t_i^{\circ}(\boldsymbol{x})\, ds_{\boldsymbol{x}}, \tag{1.11}$$

for u_i satisfying $u_i{=}u_i^{\circ}$ on ∂B_u. The solution that stationarizes J, i.e., $\delta J(u){=}0$, coincides with the solution of the boundary value problem. This variational problem of Eq. (1.11) is the optimization problem that is equivalent with Eq. (1.10). The first term of the integrand, $\frac{1}{2} c_{ijkl} u_{i,j} u_{k,l}$, is called *strain energy density*, and the volume integral of this term gives the *strain energy* which is stored in B. The integral of the second term, $f_i u_i$, is usually called the *external work* done by the body force, and the surface integral is the external work done by the boundary traction.

It is interesting to note that a governing equation for some problems of structure mechanics is derived from that of an elastic continuum, just by assuming a particular form for a displacement vector function. For instance, the governing equation for the pole problem, which is presented in Sec. 1.2, is readily derived just by assuming that a non-zero component of a displacement vector function is u_3 only and that this u_3 does not depend on x_1 nor x_2, i.e.,

$$u_1 = u_2 = 0 \quad \text{and} \quad u_3 = u(x) \quad (x = x_3).$$

Another example is the governing equation for a beam problem. As shown in Fig. 1.4, a horizontal beam with uniform cross section is considered. The x_1-axis is taken along the longitudinal direction and body forces act in the x_3-direction; the body force per unit volume is f. The following form is assumed for a displacement vector function:

$$u_1 = -zw'(x), \quad u_2 = 0 \quad \text{and} \quad u_3 = w(x) \quad (x = x_1, z = x_3)$$

with $z{=}0$ being the neutral axis. There is only one non-zero components of strain, $\epsilon_{11}{=}zw''$; ϵ_{13} vanishes since the assumed displacement vector components satisfy $u_{1,3}{+}u_{3,1}{=}0$. Thus, only c_{1111} needs to be determined to calculate J of Eq. (1.11), and, for an isotropic elastic material, this component is given as $c_{1111}{=}E$. Substitution of these strain and elasticity components into J yields

$$J(w) = \int_0^L \frac{1}{2} EI(w''(x))^2 + Af(x)w(x)\,dx,$$

where I and A are the moment of inertia and the cross section area, i.e., $I{=}\int z^2\,ds$ and $A{=}\int\,ds$. The first variation, $\delta J{=}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(J(w+\varepsilon\delta w){-}J(w))$, is

$$\delta J = \int_0^L \delta w(x)\Big(EIw''''(x) + Af(x)\Big)\,dx \\ + \Big[\delta w(x)EIw'''(x) - \delta w'(x)EIw''(x)\Big]_0^L.$$

A condition of $\delta J{=}0$ leads to a boundary value problem for w; a fourth-order differential equation of w,

$$EIw''''(x) + Af(x) = 0,$$

is obtained, together with boundary conditions at $x{=}0$ and L. As is seen, the governing equation of a beam problem is derived from the functional of an elastic continuum, just by using a displacement vector function of a particular form.