Chapter 1

Toroidal Configuration

1.1 INTRODUCTION

Because of the divergence free nature of the magnetic field, the simplest topological configuration it can assume with no field lines exiting from a fixed volume is toroidal. Since particle trajectories, to lowest order in gyro radius, follow field lines, this is also the simplest configuration which can give complete confinement within this approximation.

In addition, because of the very high particle velocity along the field lines, the field must, at least approximately, possess magnetic surfaces, *i.e.* the field lines trace out a nested set of toroidal surfaces, to prevent rapid particle transport from the inside of the torus to the exterior, and thus allow the existence of pressure and temperature gradients. The temperature must be sufficiently high so that ion collisions are strong enough to overcome the repulsive Coulomb potential and permit nuclear fusion by means of the strong short range nuclear forces. The fuel which is most easily brought into such conditions is a mixture of Deuterium and Tritium, which fuses through the reaction

$$D + T \rightarrow H_e^4(3.5Mev) + n(14.1Mev). \tag{1.1}$$

The cross section for this reaction has a peak near 50 keV, and the plasma must be sustained at a temperature on the order of 15 keV, or around 100 million degrees centigrade, allowing the higher energy particles in the distribution to collide and fuse. For a plasma containing an optimum mixture of half Deuterium and half Tritium the ignition condition (Wesson 1997) is given by

$$nT\tau_E > 3 \times 10^{21} keV s/m^3 \tag{1.2}$$

where n is the ion density, T the ion temperature, and τ_E the energy confinement time, defined by

$$\tau_E = \frac{3}{2}\frac{\int n(T_i + T_e)}{P} \tag{1.3}$$

with P the total power input and subscripts refer to ions and electrons.

Heating of a confined plasma to the necessary temperature is routinely obtained in fusion laboratories using neutral beam injection and radio-frequency heating, and densities $n > 10^{20}/m^3$ as well as plasmas with sufficient confinement times are also produced, but obtaining and sustaining a plasma with all three parameters sufficiently large requires, at least with present designs, a device approximately three times larger than has been constructed. Present research consists of extending the limits of maximum plasma temperature, density, length of discharge, and fusion power produced, and well as improving understanding of confined plasmas, to facilitate the search for design modifications permiting a reduction in the size and complexity of a device confining a burning plasma.

The first conceptual design of toroidal devices with controlled thermonuclear fusion in mind appeared in the early 1950's. In the United States L. Spitzer initiated a research effort to build a Stellarator, a device with helically wound external coils. In the Soviet Union I. Tamm and A. Sakharov sketched a design for a Tokamak, a device with a toroidal plasma current. The difficulty and complexity of the task of constructing a practical fusion device was vastly underestimated, and the ensuing fifty years of theoretical and experimental investigation by thousands of scientists of many nations revealed layer upon layer of sophisticated means by which a high temperature plasma will attempt to avoid confinement.

The story is by no means finished, and it still cannot be said with complete confidence that the goal of economic fusion power is within reach. Nevertheless, a large body of knowledge concerning the physics of plasma confinement now exists, and enormous progress has been made. In Fig. 1.1 (Jassby and Meade, 2000) is shown the achieved fusion power in Kilowatts as a function of year. The points are a composite of data from tokamak devices in the United States, Europe, and Japan. In recent history tokamak reactors have the only significant record for fusion power production, but

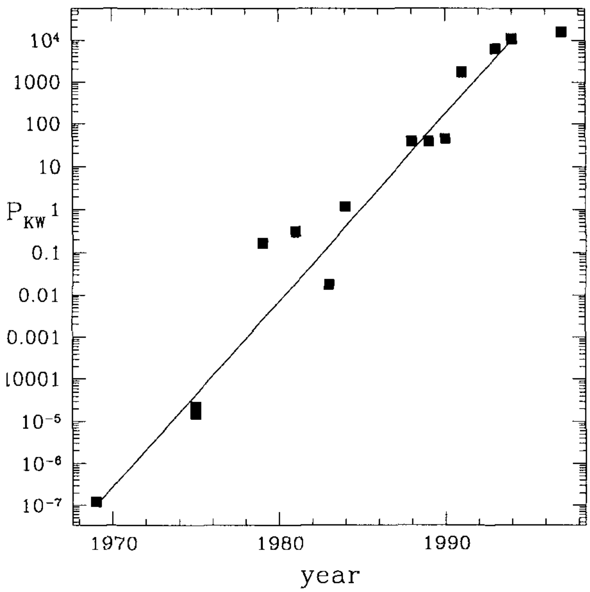

Fig. 1.1 Fusion power output history, Kilowatts.

this may be only a historical accident, it is possible that stellarators or other devices may be just as successful.

This book is restricted to the physics of toroidal confinement. The presentation reflects the order of concerns in designing a toroidal plasma device. Chapter 1 concerns the nature of a toroidal magnetic field configuration and its destruction by perturbations. Chapter 2 discusses the form of, and the requirements for, plasma equilibrium, followed by a brief introduction to single particle orbits in such equilibria in Chap. 3. The questions of linear stability of these equilibria to ideal modes and to resistive modes are discussed in Chaps. 4 and 5. The nonlinear evolution of these modes and the consequences for maintaining plasma discharges is treated in Chap. 6. The effect of a high energy particle population on magnetohydrodynamic stability is discussed in Chap. 7. Chapter 8 gives an introduction to the subject of transport, and Chap. 9 an introduction to the method of phase integrals, a very useful tool for the investigation of plasma instabilities.

The equations defining resistive magnetohydrodynamics (MHD) are

Ampere's law,

$$\nabla \times \vec{B} = \vec{j} \tag{1.4}$$

Faraday's law

$$\nabla \times \vec{E} = -\partial_t \vec{B} \tag{1.5}$$

and Ohm's law

$$\vec{E} + \vec{v} \times \vec{B} = \eta \vec{j}. \tag{1.6}$$

In addition, for the description of the plasma, which is approximated as a fluid, we need the equation of continuity for the density ρ,

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0 \tag{1.7}$$

an equation of state to describe the evolution of the scalar plasma pressure

$$(\partial_t + \vec{v} \cdot \nabla) \left(\frac{p}{\rho^\gamma} \right) = 0, \tag{1.8}$$

and the equation of motion for the fluid

$$\rho(\partial_t + \vec{v} \cdot \nabla)\vec{v} = -\nabla p + \vec{j} \times \vec{B}. \tag{1.9}$$

Some problems are treated in the approximation of neglecting the plasma resistivity (a high temperature plasma has roughly the same conductivity as metallic copper) in whch case the equations describe ideal MHD.

For the purposes of thermonuclear fusion, the plasma must remain approximately in equilibrium for times long compared to either ideal or resistive motion. It is Eq. 1.9 which limits the form such equilibria can take, and for a time independent flow-free state it can be conveniently written in tensor form

$$\partial_{x_i} T_{ik} = 0 \tag{1.10}$$

with

$$T_{ik} = p_\perp \left(\delta_{ik} - \frac{B_i B_k}{B^2} \right) + p_\parallel \frac{B_i B_k}{B^2} \tag{1.11}$$

where $p_\perp = p + B^2/2$ and $p_\parallel = p - B^2/2$, as can be verified by substitution. This formulation of the equilibrium condition makes it apparent that the magnetic field pressure, $B^2/2$, acts as though it exerts an expansion force in the transverse direction and a tension in the parallel direction. The form such equilibria can take will be examined in Chap. 2.

It is beyond the scope of this book to discuss the domain of validity of resistive magnetohydrodynamics as defined by the above six equations. They will simply be accepted as a starting point. Roughly speaking, the results of the theory are valid provided the system size and the wavelengths of the modes discovered are large compared to particle gyroradii, the modes have growth rates which are larger than diamagnetic drift frequencies, and the particle collisionality is large enough so that the pressure is isotropic, but sufficiently collisionless so that the resistivity is small. Some sections of the book, regarding for example the modification of modes to include a real frequency due to Larmor radius effects or to coupling of the plasma with energetic particles, are in fact outside the domain of resistive magnetohydrodynamics, but only modestly so.

Derivations of ideal MHD from the Boltzman equation and from single particle orbit theory along with further references, can be found in the review by Freidberg (1982). It is known that the predictions of MHD are valid in a domain of collisionality much larger than one has a right to expect. Some of the generalizations of ideal MHD are discussed briefly in Chap. 4, the main result being that stability conditions derived from MHD are not significantly changed by the extensions.

Rationalized units with $c = 1$ are used throughout. In addition, by expressing distances in terms of the characteristic scale length of the problem considered and magnetic field in terms of the on-axis toroidal field, equations quickly reduce to expressions involving the fundamental Alfvèn time for the system under consideration.

1.2 GENERAL COORDINATES

To describe a toroidal magnetic configuration it is convenient to use coordinates defined by the field itself. In addition to simplifying the description of the field, and providing a general theory for the description of all toroidal fusion devices (tokamak, spheromak, toroidal pinch, stellarator, heliac, *etc.*) the magnetic coordinates are closely related to canonical variables for the Hamiltonian description of the first order guiding center particle motion in the field.

The advantages gained by using a coordinate system which is defined

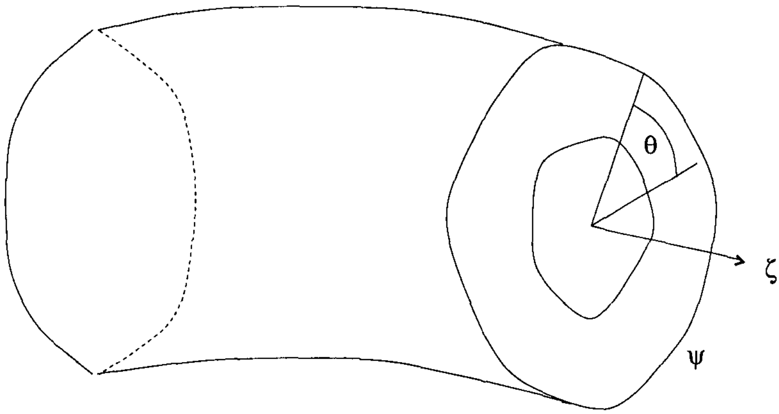

Fig. 1.2 General toroidal coordinates.

by, and is natural for, the description of the magnetic field outweigh those gained by using a standard orthonormal system. The fields and equilibria must ultimately be described in a convenient laboratory coordinate system, so the general coordinates are taken to be functions of the euclidean coordinates. Because of the necessarily toroidal topology of the magnetic field surfaces, the coordinates are taken to be toroidal in form.

Introduce general coordinates ψ, θ, ζ, as shown in Fig. 1.2. Surfaces of constant ψ are taken to consist topologically of nested tori, which necessarily possess an axis which will usually be designated by $\psi = 0$. Surfaces of constant θ define a general poloidal angle and ζ defines the "toroidal" direction. In some cases ζ will be simply equal to the geometrical toroidal angle ϕ, but it is convenient to keep it as a general toroidal coordinate. The surface label ψ is taken to be increasing outward, and thus the system ψ, θ, ζ defines a right handed coordinate system.

1.3 BASIS VECTORS, METRIC TENSOR

Begin with functions defining the transformation from general coordinates to orthonormal euclidean coordinates, $\vec{r}(\psi, \theta, \zeta)$ with $\vec{r} = (x, y, z)$. Define a covariant basis

$$\vec{e}_\psi = \partial_\psi \vec{r} \quad \vec{e}_\theta = \partial_\theta \vec{r} \quad \vec{e}_\zeta = \partial_\zeta \vec{r} \tag{1.12}$$

with Jacobian

$$\mathcal{J} = \vec{e}_\psi \cdot (\vec{e}_\theta \times \vec{e}_\zeta). \tag{1.13}$$

Define also the contravariant basis

$$\vec{e}^\psi = \nabla\psi, \qquad \vec{e}^\theta = \nabla\theta, \qquad \vec{e}^\zeta = \nabla\zeta. \tag{1.14}$$

The matrices of the differential coordinate transformation

$$\begin{pmatrix} \frac{\partial x}{\partial \psi} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \psi} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \psi} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \zeta} \end{pmatrix} \qquad \begin{pmatrix} \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} & \frac{\partial \psi}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{pmatrix} \tag{1.15}$$

are inverses of one another, $\vec{e}^\alpha \cdot \vec{e}_\beta = \delta^\alpha_\beta$ and $\vec{e}^\alpha \vec{e}_\alpha = I$ where I is the unit matrix, $I_{jk} = \delta_{jk}$ and α, β refer to ψ, θ, ζ, the indices j, k refer to x, y, z, and the summation convention is used for repeated indices. In addition we find

$$\vec{e}_\alpha \times \vec{e}_\beta = \mathcal{J}\epsilon_{\alpha\beta\gamma}\vec{e}^\gamma, \qquad \vec{e}^\alpha \times \vec{e}^\beta = \frac{1}{\mathcal{J}}\epsilon^{\alpha\beta\gamma}\vec{e}_\gamma \tag{1.16}$$

with $\epsilon_{\alpha\beta\gamma}$ antisymmetric under exchange of indices and $\epsilon_{\psi\theta\zeta} = 1$. These tensors satisfy the relation $\epsilon_{\alpha\beta\gamma}\epsilon_{\alpha\rho\sigma} = \delta_{\beta\rho}\delta_{\gamma\sigma} - \delta_{\beta\sigma}\delta_{\gamma\rho}$.

Any vector can be written $\vec{v} = v^\alpha \vec{e}_\alpha = v_\alpha \vec{e}^\alpha$ and v_α, v^α are called the covariant and contravariant components, respectively, given by $v_\alpha = \vec{e}_\alpha \cdot \vec{v}$ and $v^\alpha = \vec{e}^\alpha \cdot \vec{v}$. The notation is consistent with the transformation properties of the vectors under coordinate transformation. Quantities with raised indices are called contravariant. The differential distance produced by displacements in ψ, θ, ζ is given by the chain rule, $d\vec{r} = \vec{e}_\alpha d\alpha$ so the differential distance is given by $ds^2 = d\vec{r} \cdot d\vec{r} = \vec{e}_\alpha \cdot \vec{e}_\beta d\alpha d\beta$ and thus the metric tensor is given by

$$g_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta, \qquad \textit{with} \qquad g^{\alpha\beta} = \vec{e}^\alpha \cdot \vec{e}^\beta. \tag{1.17}$$

It follows immediately that $g_{\alpha\beta}g^{\beta\kappa} = \delta^\kappa_\alpha$ and that $\vec{e}_\alpha = g_{\alpha\beta}\vec{e}^\beta$. Also for any vector $\vec{v} = v^\alpha \vec{e}_\alpha = v_\alpha \vec{e}^\alpha$ we have $v_\alpha = g_{\alpha\beta}v^\beta$ *i.e.* the metric tensor can be used to raise and lower indices of basis vectors and components. Note that $\vec{v} = v^\alpha \vec{e}_\alpha = v_\alpha \vec{e}^\alpha$ is coordinate system invariant, with the basis vectors transforming covariantly (contravariantly) while the components transform contravariantly (covariantly). This can easily lead to confusion in nomenclature. Normally a representation of a vector is named a covariant or contravariant representation according to how the components transform.

Note that

$$\frac{1}{\mathcal{J}} = \nabla\psi \cdot (\nabla\theta \times \nabla\zeta). \tag{1.18}$$

The relation of $\mathcal{J}$ to the volume element follows easily by calculating the volume of a small parallelopiped constructed with $d\vec{r}_1 = \vec{e}_\psi d\psi$, $d\vec{r}_2 = \vec{e}_\theta d\theta$, $d\vec{r}_3 = \vec{e}_\zeta d\zeta$. Then $dV = d\vec{r}_1 \cdot (d\vec{r}_2 \times d\vec{r}_3) = \mathcal{J} d\psi d\theta d\zeta$.

The Jacobian is related to the metric tensor through $\mathcal{J}^{-2} = det(g^{\alpha\beta})$. To see this consider the matrix

$$M_l^\alpha = e_l^\alpha \tag{1.19}$$

with $\alpha = \psi, \theta, \zeta$ and $l = x, y, z$. Then we have

$$detM = \frac{1}{\mathcal{J}}. \tag{1.20}$$

Further note that

$$g^{\alpha\beta} = (MM^T)^{\alpha\beta} \tag{1.21}$$

from which it follows that

$$det(g^{\alpha\beta}) = (detM)^2 = \frac{1}{\mathcal{J}^2} \tag{1.22}$$

and

$$det(g_{\alpha\beta}) = \mathcal{J}^2. \tag{1.23}$$

For most purposes in plasma physics second rank tensors can be written as dyads, *i.e.* the product of two vectors. For this reason the technique of using either a covariant or a contravariant representation for the vector depending on the nature of the calculation to be done is sufficient to simplify the result. Here we introduce the Christoffel symbols, which describe the differential spatial dependence of the basis vectors, and provide an alternate means of calculation. The basis vectors are functions of space, so we define $\partial_\alpha \vec{e}_\beta = \{_{\alpha\beta}\ {}^\tau\}\vec{e}_\tau$, where $\{_{\alpha\beta}\ {}^\tau\}$ is the Christoffel symbol of the second kind. Take the dot product with $\vec{e}^\kappa$, giving $\{_{\alpha\beta}\ {}^\kappa\} = \vec{e}^\kappa \cdot \partial_\alpha \vec{e}_\beta$, but $\vec{e}_\beta = \partial_\beta \vec{r}$ and so $\{_{\alpha\beta}\ {}^\kappa\} = (1/2)\vec{e}^\kappa \cdot [\partial_\alpha \vec{e}_\beta + \partial_\beta \vec{e}_\alpha]$.

Use $g_{\kappa s}$ to lower the index, giving $g_{\kappa s}\{_{\alpha\beta}\ {}^s\} = [\alpha\beta, \kappa]$ which is the Christoffel symbol of the first kind. Thus

$$[\alpha\beta, \kappa] = \frac{1}{2}\vec{e}_\kappa \cdot [\partial_\alpha \vec{e}_\beta + \partial_\beta \vec{e}_\alpha] \tag{1.24}$$

or

$$[\alpha\beta,\kappa] = \frac{1}{2}[\partial_\alpha g_{\kappa\beta} - \vec{e}_\beta \cdot \partial_\alpha \vec{e}_\kappa + \partial_\beta g_{\alpha\kappa} - \vec{e}_\alpha \cdot \partial_\beta \vec{e}_\kappa] =$$
$$\frac{1}{2}[\partial_\alpha g_{\kappa\beta} + \partial_\beta g_{\alpha\kappa} - \partial_\kappa(\partial_\beta \vec{r} \cdot \partial_\alpha \vec{r})] \qquad (1.25)$$

giving an expression for the Christoffel symbol in terms of the metric tensor,

$$[\alpha\beta,\kappa] = \frac{1}{2}[\partial_\alpha g_{\kappa\beta} + \partial_\beta g_{\alpha\kappa} - \partial_\kappa g_{\alpha\beta}]. \qquad (1.26)$$

1.4 VECTOR OPERATORS

Vector calculus in general coordinates is facilitated by using a representation for the vectors involved which is appropriate for the calculation to be done. Simplification arises because of the properties of the gradient operator, namely $\nabla \times \nabla f = 0$ and $\nabla \cdot (\nabla A \times \nabla B) = 0$. Thus to calculate the divergence of a vector write it in terms of a covariant basis.

The gradient of a scalar is given by $\nabla f = \vec{e}^\alpha \partial_\alpha f$. The divergence of a vector is simply expressed in terms of its components. Write

$$\vec{v} = \mathcal{J}(\vec{v} \cdot \nabla\psi)\nabla\theta \times \nabla\zeta + \mathcal{J}(\vec{v} \cdot \nabla\theta)\nabla\zeta \times \nabla\psi + \mathcal{J}(\vec{v} \cdot \nabla\zeta)\nabla\psi \times \nabla\theta. \qquad (1.27)$$

Taking the divergence of this expression we find

$$\nabla \cdot \vec{v} = \frac{1}{\mathcal{J}}\partial_\alpha(\mathcal{J}\vec{v} \cdot \nabla\alpha). \qquad (1.28)$$

The curl of a vector is easily calculated if written in terms of a contravariant basis. Write

$$\vec{v} = v_\psi \nabla\psi + v_\theta \nabla\theta + v_\zeta \nabla\zeta. \qquad (1.29)$$

Taking the curl of this expression and then taking the dot product with the three contravariant basis vectors we find

$$(\nabla \times \vec{v}) \cdot \nabla\zeta = \frac{1}{\mathcal{J}}(\partial_\psi v_\theta - \partial_\theta v_\psi) \qquad (1.30)$$

and cyclic permutations of ψ, θ, ζ.

For the Laplacian of f simply take the divergence of the vector $\vec{v} = \nabla f$. We have from above $\nabla \cdot \vec{v} = \frac{1}{\mathcal{J}}\partial_\alpha(\mathcal{J}v^\alpha)$. Since $v_\beta = \partial_\beta f$ we have $v^\alpha =$

$g^{\alpha\beta}\partial_\beta f$ giving

$$\nabla^2 f = \frac{1}{\mathcal{J}}\partial_\alpha(\mathcal{J} g^{\alpha\beta}\partial_\beta f). \tag{1.31}$$

1.5 MAGNETIC FIELD REPRESENTATION

There are many different ways to represent a magnetic field. We derive here a representation of a general magnetic field which is useful for two reasons. First it displays manifestly the Hamiltonian character of the magnetic field line trajectories. Secondly it is the form which we will find most useful for discussing MHD equilibrium and particle trajectories.

Write $\vec{B} = \nabla \times \vec{A}$, since $\nabla \cdot \vec{B} = 0$ (Poincaré, 1892). Write the vector potential in terms of the coordinates ρ, θ, ζ with $\vec{r}(\rho, \theta, \zeta)$

$$\vec{A} = A_\rho \nabla\rho + A_\theta \nabla\theta + A_\zeta \nabla\zeta. \tag{1.32}$$

Then, define $\partial_\rho G = A_\rho$. Since $\nabla G = \partial_\rho G \nabla\rho + \partial_\theta G \nabla\theta + \partial_\zeta G \nabla\zeta$ we have

$$\vec{A} = \nabla G + (A_\theta - \partial_\theta G)\nabla\theta + (A_\zeta - \partial_\zeta G)\nabla\zeta \tag{1.33}$$

or

$$\vec{A} \equiv \nabla G + \psi\nabla\theta - \psi_p \nabla\zeta \tag{1.34}$$

and thus

$$\vec{B} = \nabla\psi \times \nabla\theta - \nabla\psi_p \times \nabla\zeta. \tag{1.35}$$

Using this form we can easily show that the field line trajectories are Hamiltonian in nature. The magnetic field lines are defined by $d\psi/d\zeta = \vec{B}\cdot\nabla\psi/\vec{B}\cdot\nabla\zeta$, $\quad d\theta/d\zeta = \vec{B}\cdot\nabla\theta/\vec{B}\cdot\nabla\zeta$ or

$$\frac{d\psi}{d\zeta} = -\frac{\nabla\psi\cdot(\nabla\psi_p \times \nabla\zeta)}{\nabla\psi \times \nabla\theta)\cdot\nabla\zeta} \qquad \frac{d\theta}{d\zeta} = -\frac{\nabla\theta\cdot(\nabla\psi_p \times \nabla\zeta)}{\nabla\psi \times \nabla\theta)\cdot\nabla\zeta}. \tag{1.36}$$

But the gradient of ψ_p is given by

$$\nabla\psi_p = \partial_\psi\psi_p\nabla\psi + \partial_\theta\psi_p\nabla\theta + \partial_\zeta\psi_p\nabla\zeta \tag{1.37}$$

Thus

$$\frac{d\psi}{d\zeta} = -\partial_\theta\psi_p \qquad \frac{d\theta}{d\zeta} = \partial_\psi\psi_p \tag{1.38}$$

This is of Hamiltonian form with $\psi_p(\psi, \theta, \zeta)$ the Hamiltonian, ψ the momentum, θ the coordinate, and ζ the time.

1.6 MAGNETIC SURFACES

A two dimensional surface defined by a function $f(\vec{r}) = constant$ is said to be a magnetic surface if at any point the magnetic field lies within the surface, ie. $\vec{B} \cdot \nabla f = 0$. The existence of such surfaces in magnetic confinement devices, or at least the existence of approximate magnetic surfaces over a large fraction of the plasma volume, is an essential requirement for long-term confinement. The existence of such surfaces has been shown only under fairly restrictive conditions (Morozov and Solov'ev 1966). They are known to exist everywhere or in all but a small part of the plasma volume only when there exists a symmetry or approximate symmetry. This is easily demonstrated in cylindrical geometry for translational, axial, and helical symmetry. Writing the field in terms of the vector potential, $\vec{B} = \nabla \times \vec{A}$, the surfaces are defined by $A_z(r, \theta) = constant$ in case $\vec{A}$ is translationally invariant in z, $rA_\theta(r, z) = constant$ in case $\vec{A}$ is independent of θ, and $A_z(r, \theta - \alpha z) + \alpha r A_\theta(r, \theta - \alpha z) = constant$ in case $\vec{A}$ is helically invariant; *i.e.*, depends only on the variables $r, \theta - \alpha z$. It is readily verified that these equations define surfaces to which $\vec{B}$ is tangent.

According to a theorem by Kolmogorov (1957), small perturbation of a symmetric case leaves well defined magnetic surfaces existing everywhere except in a small volume proportional to the square root of the perturbation, where the field assumes a stochastic character. The onset of stochasticity will be further discussed in Sec. 1.7.

Consider the representation of $\vec{B}$ given by Eq. 1.35. Forming $\vec{B} \cdot \nabla\psi$ and $\vec{B} \cdot \nabla\psi_p$ it is easily seen that ψ_p is a magnetic surface only if ψ_p is independent of ζ, *i.e.* $\psi_p = \psi_p(\psi, \theta)$, and ψ is a magnetic surface only if ψ_p is independent of θ, *i.e.* $\psi_p = \psi_p(\psi, \zeta)$. Thus both ψ and ψ_p are magnetic surfaces if $\psi_p = \psi_p(\psi)$.

The simplest way to guarantee the existence of approximate surfaces is to take the confinement device to be a small perturbation of a system in which $\psi_p = \psi_p(\psi)$. Consider such a system, with ζ the toroidal angle, and ψ_p a function of ψ alone, so that $\vec{B} \cdot \nabla\psi = \vec{B} \cdot \nabla\psi_p = 0$. Let $d\psi_p/d\psi \equiv 1/q(\psi)$. From Eqs. 1.18, 1.35 the toroidal flux is then $(1/2\pi)\int(\vec{B} \cdot \nabla\zeta)\mathcal{J} d\zeta d\theta d\psi = 2\pi\psi$, and the poloidal flux is $(1/2\pi)\int(\vec{B} \cdot$

$\nabla\theta)\mathcal{J}d\zeta d\theta d\psi = 2\pi\psi_p$, where the integration extends from the magnetic axis to the flux surface ψ. Shown in Fig. 1.3 is a poloidal surface, (fixed θ), and a toroidal surface (fixed ζ), each bounded by the magnetic axis and a flux surface defined by ψ. The fluxes ψ and ψ_p are defined by the total magnetic flux passing through these surfaces. This differs from the definition used by some authors in that we have taken both ψ and ψ_p to be zero at the magnetic axis, so the surfaces are limited by the magnetic axis and ψ. Note that the assumption of nested magnetic surfaces does not imply axisymmetry, and also note that generally θ and ζ are not the usual geometrically defined angular variables, it is only required that the position in space be periodic with period 2π in each of them and that the volume defined by them be topologically toroidal.

1.7 MAGNETIC SURFACE DESTRUCTION

To illustrate the way in which nested toroidal magnetic surfaces are destroyed, introduce a small perturbation of the poloidal flux function, Fourier

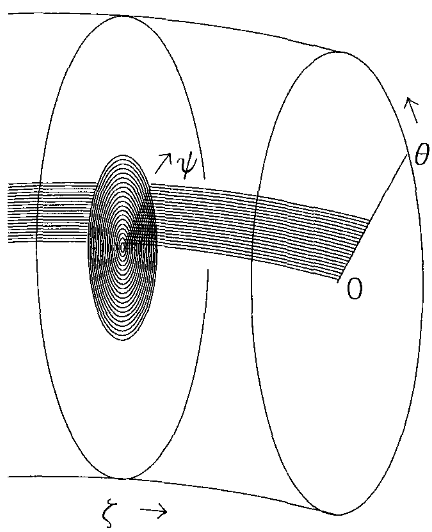

Fig. 1.3 Toroidal and poloidal surfaces defining flux ψ and ψ_p.

decompose it in the variables θ, ζ, and consider a single harmonic

$$\psi_p = \int \frac{d\psi}{q} + V cos(n\zeta - m\theta), \tag{1.39}$$

and the equations for the field lines become, from Eq. 1.38

$$\frac{d\psi}{d\zeta} = -V m sin(n\zeta - m\theta), \qquad \frac{d\theta}{d\zeta} = \frac{1}{q(\psi)}. \tag{1.40}$$

The function $q(\psi)$, referred to as the safety factor in tokamak theory, thus defines the field helicity in the θ, ζ variables on the surface ψ . If $q' = dq/d\psi$ is not zero the field possesses shear. Note from Eq. 1.35 that $\vec{B}\cdot\nabla\psi = mV sin(n\zeta - m\theta)/\mathcal{J}$, *i.e.* the perturbation V introduces a component of $\vec{B}$ directed across the original flux surfaces. If $q(\psi)$ is not close to the rational m/n with $V \neq 0$, ψ is oscillatory and $q(\psi)$ can be approximated as constant, giving $\theta = \zeta/q + \theta_0$ and $\psi - \psi_0 = V cos(n\zeta - m\theta)/(n - m/q)$. The flux surfaces are distorted but they remain topologically nested surfaces. However if $mq(\psi) - n\zeta \simeq 0$ the resonant denominator creates large excursions in ψ and this solution is not valid. In this case expand $q(\psi)$ about ψ_0, with $q(\psi_0) = m/n$, and introduce the variable $Q = n\zeta - m\theta$. We then find

$$\begin{aligned} dQ &= \left(\frac{mq'}{q^2}\right)(\psi - \psi_0)d\zeta \\ d\psi &= -mV sin(Q) d\zeta \end{aligned} \tag{1.41}$$

where $q' = dq/d\psi$, evaluated at $\psi = \psi_0$. Integrating these equations we find

$$(\psi - \psi_0)^2 = \frac{q^2 V}{q'}[cos(Q) + k] \tag{1.42}$$

where the constant k is determined by the initial position. (Note that the flux surfaces cannot be found by setting the Hamiltonian ψ_p constant, since this is a "time" dependent Hamiltonian.)

The points in the ψ, θ plane at $\zeta = 0$ formed by successive transits of a field line are called a Poincaré plot. If q is rational the Poincaré plot will consist only of a finite set of the points on the surface given by Eq. 1.42, but if q is irrational the whole surface will be covered. The points given by $cos(Q) = 1$, $k = -1$, and those with $cos(Q) = -1$, $k = 1$ are periodic points of period m. Expanding the cosine function near $cos(Q) = +1$ we

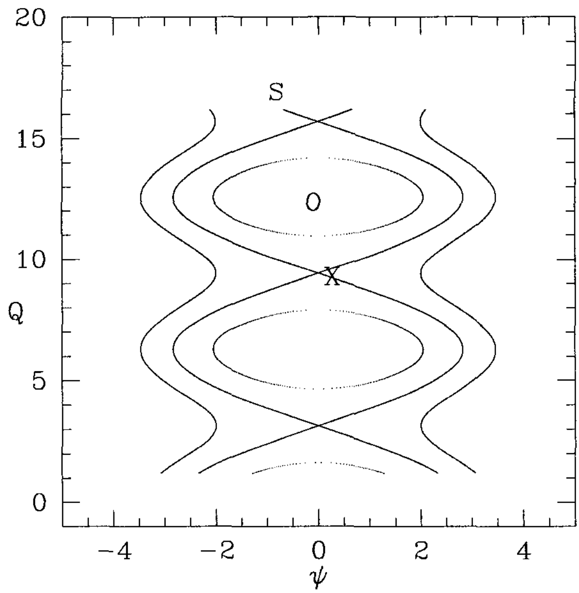

Fig. 1.4 Broken flux surfaces due to a single island chain showing separatrix S, X-point and O-point.

find

$$(\psi - \psi_0)^2 = \frac{q^2 V}{q'}(1 - Q^2/2 + k) \tag{1.43}$$

i.e. for $Vq' > 0$ the points lie on elliptic manifolds about the periodic points. They form either a continuous curve or discrete points depending on the value of q. Similarly, near the points $cos(Q) = -1$ the points lie on hyperbolic manifolds. Thus there is a chain of elliptic and hyperbolic points at $\psi = \psi_0$. This island chain is separated from the topologically toroidal surfaces by a separatrix which passes through the X-points. The separatrix is described by choosing the integration constant $k = 1$, giving islands with width

$$\delta\psi = 4\left(\frac{Vq^2}{q'}\right)^{1/2}. \tag{1.44}$$

In Fig. 1.4 are shown the resulting flux surfaces for a single island chain.

An equivalent pendulum for this structure is given by the canonical transformation generated by $F_3(p,Q,t) = -npt/m + pQ/m$ with $p,q,t = \psi - \psi_r, \theta, \zeta$ and $q = -\partial_p F_3$, $P = -\partial_Q F_3$ giving $P = -p/m, Q = n\zeta - m\theta$ (see Goldstein, p 242). The new Hamiltonian is $K = H + \partial_t F_3$. Hamilton's equations are also left invariant with the transformation $t \rightarrow at$, $P \rightarrow aP$. Taking $a = m\sqrt{q'/q^2}$ gives

$$K = -\frac{P^2}{2} + V cosQ \tag{1.45}$$

with $P^2 = mq'(\psi - \psi_r)^2/q^2$. The equations of motion are then $dP/dt = -\partial K/\partial Q$, $dQ/dt = \partial K/\partial P = P$ and the phase plot is given by surfaces of constant $K = E$. The period for an orbit trapped in the island is then

$$T = \frac{2}{\sqrt{2V}} \int_{-Q_b}^{Q_b} \frac{dQ}{\sqrt{cosQ - cosQ_b}} \tag{1.46}$$

with $2V cosQ_b = -E$, and Q_b defining the maximum excursion in Q. Letting $cosQ = 1 - sin^2(Q/2)$ with $k = sin(Q_b/2)$ and $k sin\phi = sin(Q/2)$ find that

$$T = \frac{4}{\sqrt{V}} \int_0^{\pi/2} \frac{d\phi}{1 - k^2 sin^2\phi} = \frac{4}{\sqrt{V}} K(k) \tag{1.47}$$

with $K(k)$ an elliptic integral. In the deeply trapped limit we find the period to be $T \simeq 2\pi/\sqrt{V}$. Using the relation between the time and ζ we find the internal q_I for rotation about the island O-point,

$$q_I = \frac{d\zeta}{d\theta_I} = \frac{q}{m\sqrt{q'V}}. \tag{1.48}$$

with θ_I the angle around the island O-point. For an orbit near the separatrix $k \simeq 1$ and we find

$$t = \frac{1}{\sqrt{2V}} \int_0^Q \frac{dQ}{\sqrt{cosQ + 1}} = \frac{1}{\sqrt{V}} ln[tan(\pi/4 + Q/4)]. \tag{1.49}$$

or equivalently $Q = 4atan(e^{\omega_0 t} - \pi)$ with $\omega_0 = \sqrt{V}$. The period for a complete oscillation near the X-point is approximately $T = 4ln(4/k')/\sqrt{V}$ with $k' = \sqrt{1-k^2} \ll 1$.

The Hamiltonian describing an exact pendulum, Eq. 1.45 gives rise to an exactly defined separatrix between the island interiors and the topologically unperturbed exterior. However if the system is additionally perturbed the separatrix is no longer exact, and becomes a band of chaotic trajectories of nonzero width. The separatrix width can be estimated by the addition of a small perturbation to the pendulum equation, and finding the modification of an orbit near the separatrix. Thus consider the Hamiltonian

$$H = \frac{p^2}{2} + VcosQ - \epsilon cos(aQ - t - t_0). \tag{1.50}$$

Use an unperturbed orbit near the separatrix to calculate changes due to ϵ,

$$\Delta H = \int_{-\infty}^{t} dt \partial_t H. \tag{1.51}$$

The unperturbed orbit is $Q = 4atan[e^{\omega_0 t}] - \pi$ with $\omega_0 = \sqrt{V} \ll 1$. This gives

$$\partial_t H = -\epsilon[sin(aQ - t)cos(t_0) - sin(t_0)cos(aQ - t)] \tag{1.52}$$

and the first term contributes nothing in the limit of large t since $Q(t)$ is odd. Let $s = \omega_0 t$, giving

$$\Delta H = \frac{\epsilon sin(t_0)}{\omega_0} \int_{-\infty}^{s} cos[aQ(s) - s/\omega_0]ds \tag{1.53}$$

and $Q(s) = 4atan(e^s) - \pi$. This integral is called a Melnikov-Arnold integral (Melnikov, 1963, Arnold 1964) , and it has both oscillatory and secular parts. As an example, consider the integral

$$\mathcal{A} = \int_{-\infty}^{s} cos[Q(s)/2 - Q_0 s]ds. \tag{1.54}$$

Let $x = 2atan(e^s) = Q/2 + \pi/2$, giving $e^s = (e^{ix} - 1)/(ie^{ix} + i)$, and $e^{iQ/2} = -ie^{ix} = (1 + ie^s)/(i + e^s)$.

$$\mathcal{A} = Re \int_{-\infty}^{s} e^{-iQ_0 s} \frac{1 + ie^s}{i + e^s} ds. \tag{1.55}$$

This integral has simple poles at $e^s = -i$, $s = -i\pi/2 - 2i\pi n$. Closing the contour in the lower half plane, $\mathcal{A}$ is the integral over the contour minus contributions from vertical pieces. The contributions from the vertical parts of the contour are, letting $s = S + iy$ with $S \gg 1$

$$\int_{0}^{-\infty} idy e^{-iQ_0 S} e^{Q_0 y} i \sim \frac{1}{Q_0} e^{-iQ_0 S} \tag{1.56}$$

giving an oscillatory contribution. The contribution from the dominant pole at $s = -i\pi/2$ is $\mathcal{A} \simeq -4\pi e^{-\pi Q_0/2}$, and $Q_0 = 1/\omega_0$, giving

$$\Delta H \sim \frac{\epsilon}{\sqrt{V}} sin(t_0) e^{-\frac{\pi}{2\sqrt{V}}} + oscillatory\ \ terms. \tag{1.57}$$

Orbits near the separatrix thus suffer a shift in position given by ΔH of effectively random sign because of the phase t_0, and the fact that the period is becoming logarithmically infinite as one approaches the separatrix. This leads to a small stochastic band about the original separatrix, with a width which is nonanalytic in V. As the magnitude of V increases the separatrix itself broadens, eliminating good KAM surfaces in its vicinity.

In the presence of many Fourier harmonics the magnetic flux surfaces break up into island chains at each rational surface where $\delta_{nm} \neq 0$ with the island widths given above. In between the island chains the flux surfaces remain topologically toroidal although distorted. Actually nonlinear interaction of two island chains with $q = m/n$ and m'/n' produces a smaller island structure at the surface $q = (m+m')/(n+n')$ and so on. Thus for arbitrarily small perturbations all rational surfaces in this Fibonacci sequence produced by the mode numbers of the original perturbations break up into island chains. However as long as the perturbations are small, the total volume of the island structure is small, proportional to $\sqrt{\delta}$, and magnetic surfaces which are topologically toroidal exist almost everywhere. This was shown rigorously by Kolmogorov, Arnold and Moser, and the surviving toroidal surfaces are called KAM surfaces. The KAM surfaces are important because no field lines can cross them. Their existence prevents the magnetic field from wandering in ψ. As the perturbation strength increases, island widths grow until neighboring island chains overlap. Chirikov (1979) gave this overlap as a criterion for the loss of the last KAM surface and the stochastic wandering of the magnetic field. In fact the last KAM surface vanishes significantly earlier due to the nonlinearly produced secondary islands, but the overlap criterion provides a rough first estimate for the onset of stochasticity.

The separation between two neighboring rational surfaces is $\delta\psi \simeq (\Delta q)/q'$. Equating this to the island width Eq. 1.44 we find the overlap condition

$$V = \frac{1}{16}\left(\frac{\Delta q}{q}\right)^2 \frac{1}{q'}. \tag{1.58}$$

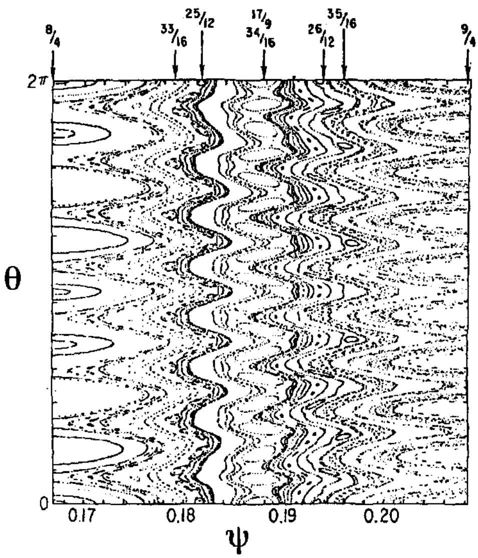

Fig. 1.5 Magnetic field structure due to perturbations.

Thus for fixed mode spectrum Δq, island overlap, and thus the onset of stochasticity, occurs at smaller perturbation amplitude for large shear, q' large.

Above stochastic threshold the field trajectories, although deterministic, are apparently chaotic, and the behavior is described in terms of statistical quantities. Two such quantities are the magnetic diffusion, D, giving the rate at which field trajectories diffuse across the original flux surfaces ψ, and the Kolmogorov entropy h, describing the rate at which nearby trajectories on average exponentially diverge. The fact that the Kolmogorov entropy is positive above stochastic threshold guarantees that trajectories are exponentially sensitive to initial conditions, introducing apparent chaos in a deterministic system.

In Fig. 1.5 is shown an example of magnetic surface destruction due to perturbations δ_{nm} for $m/n = 8/4, 9/4$. The higher order islands are due to nonlinear interaction of these two modes.

An island chain at the rational surface $q = m/n$ has associated with it a periodic orbit of length m, made up of the elliptic points of the islands in the chain, that is, m toroidal circuits are required before returning to the initial elliptic point. Consider the discrete map of points in the ψ, θ plane consisting of the transformation produced by one toroidal circuit. Orbits in the neighborhood of the elliptic points can be computed in the linear differential approximation. The domain of this approximation is called the tangent space. A toroidal transit is then represented by a matrix M acting on the initial vector $(\delta\psi, \delta\theta)$. Since $\nabla \cdot \vec{B} = 0$, this map is area preserving and hence $\det M = 1$. Equivalently, from the Hamiltonian nature of the trajectories, the Liouville theorem guarantees that phase space area is conserved. The eigenvalues of M depend only on its trace. Greene (1979) has investigated the criterion for the vanishing of the last KAM surface for some discrete maps. He has defined the residue

$$R = \frac{2 - traceM}{4}. \tag{1.59}$$

Clearly in the limit of $\delta\vec{B} = 0$ the residue is zero. The eigenvalues of M are given in terms of the residue through

$$\lambda = 1 - 2R \pm 2[R(R-1)]^{1/2}. \tag{1.60}$$

As $\delta\vec{B}$ increases from zero, the residue increases monotonically from zero. When R is between zero and one, λ is complex with magnitude one, and the tangent-space orbits may take many toroidal circuits to rotate about the elliptic periodic points. Writing $\lambda = e^{i2\pi/q_I}$, q_I is the ratio of toroidal transits divided by the number of revolutions about the periodic points. As the residue increases monotonically, q_I decreases monotonically from infinity. As R approaches one, the ellipses forming the nearby orbits flatten until at $R = 1$ the opposite sides touch, and the original elliptic point becomes an X-point, the island thus bifurcating into two elliptic points separated by the new X-point. At this point the two complex conjugate eigenvalues coalesce at $\lambda = -1$ and become real, and $q_I = 2$. This process is referred to as period doubling (Lichtenberg and Lieberman, 1992) since the two new elliptic points have a periodicity twice that of the original elliptic point, and generally after the doubling some orbits exist with the new periodicity. This means that in the power spectrum of the system (the Fourier transform in time of the amplitude squared) there appears a new frequency at half the existing one.

As the strength of the peturbation is increased, elliptic points with larger values of m and n are destabilized, becoming X-points, and at each

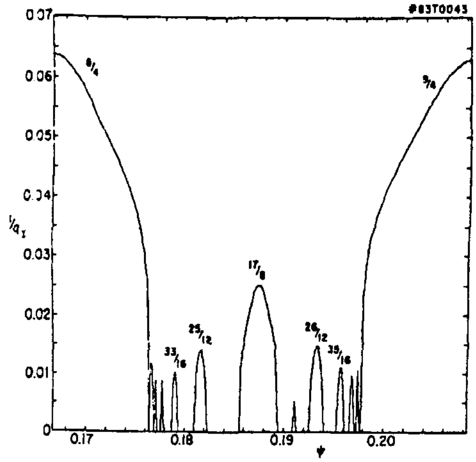

Fig. 1.6 Local q_I due to magnetic islands.

such destabilization stable orbits of double the periodicity of the destroyed elliptic point appear. A KAM surface at an irrational q is destabilized when all island chains m/n are destabilized in the limit $m/n \rightarrow q$. Examination of the destabilization of island chains with very large values of m and n then gives an accurate estimate of stochastic threshold (Greene, 1979). This behavior is to a large degree independent of the details of the map M.

In the case of analytically given discrete maps, the residue associated with an island chain can be calculated analytically, and the bifurcation of various order periodic orbits studied as a function of perturbation amplitude. This permits the calculation of the extent of stochastic domains. In the case of maps defined by toroidal magnetic field configurations the matrix M must be obtained by integration of the differential equations, and even the location of the periodic elliptic points is generally beyond analytic representation. However, the different domains; toroidal, island, and stochastic- can be differentiated numerically. Consider two nearby points in the ψ, θ plane with separation

$$\vec{\delta} = \vec{r}_1 - \vec{r}_2 = \delta_0(cos\theta_I, sin\theta_I). \tag{1.61}$$

Advance the points together toroidally. If they are within an island, they rotate about each other on the average with $\Delta\theta_I = \Delta\zeta/q_I$. If they are within a stochastic domain on the average they separate exponentially with

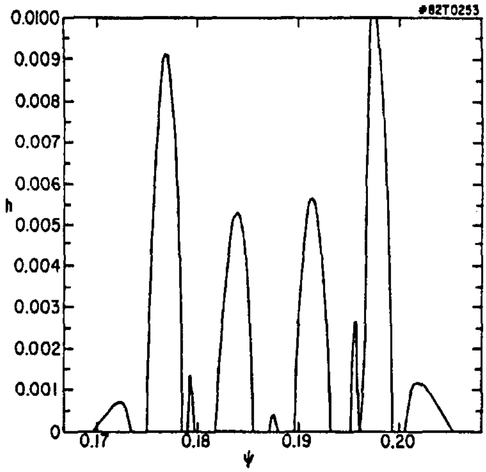

Fig. 1.7 Kolmogorov entropy in stochastic bands near separatrices.

$\delta = \delta_0 e^{hR\Delta\zeta}$, with h the Kolmogorov entropy. This exponential separation on average is a result of the fact that phase space has become densely filled with X-points. Finally, if good magnetic surfaces exist on the scale of the separation of the points, they will not rotate, and will separate only linearly due to the difference in shear at the two surfaces.

The Kolmogorov entropy is a statistical quantity and must be averaged over various initial conditions. The island rotational transform q_I, is, on the other hand, well defined locally. In Fig. 1.6 is shown the resulting numerical determination of the island internal q_I for the field of Fig. 1.5.

An accurate determination of q_I is made by advancing in ζ until the trajectory rotates completely around the periodic point. The largest value of q_I that can be observed is restricted by the number of toroidal circuits made. The method suffices to pick out the first four orders in the Fibonacci series. In Fig. 1.7 is shown the determination of the Kolmogorov entropy using the same procedure. In this example it is found that significant stochastic domains exist in the vicinity of the separatrices of the largest islands, that is, the 8/4 and 9/4 chains, as well as the 17/8. Owing to the finite number of toroidal circuits used to determine h, there is a minimum value, given by nonstochastic orbit excursions, below which h is not significant.

Although toroidal flux surfaces can be destroyed by an arbitrarily small perturbation, for small perturbations the island and stochastic domains are confined between surviving toroidal flux surfaces. This means that an appropriately defined concept of approximate closed flux surfaces is stable with respect to small perturbations. It is in this sense that the equilibrium fields of Chap. 2 must be understood.

1.8 THE STANDARD MAP

It has been found that the approach to chaos, or stochasticity, of deterministic Hamiltonian systems exhibits quite universal behavior, independent of the form of the Hamiltonian. Thus both the behavior of the magnetic field and of the particle orbits can be better understood by the study of simple Hamiltonian systems. The simplest such system, the standard map, introduced independently by Chirikov (1979) and Taylor (unpublished), can be obtained by simplifying the field stepping equations of Sec. 1.7 for a single harmonic perturbation by considering steps in ζ of 2π and taking constant shear. Suitably normalized variables then gives

$$\begin{aligned} x' &= x + v \\ v' &= v + \epsilon sin(x') \end{aligned} \tag{1.62}$$

where v is the cross field (ψ) direction and x the poloidal angle. The prime on the x in the second equation is necessary to ensure that the map has determinant one. The safety factor is $q = 2\pi/v mod(2\pi)$ and ranges from ∞ to 1 in the interval $v = 0, 2\pi$.

Islands appear at $v = 2k\pi$, k integer, of width $\Delta v = 4\sqrt{\epsilon}$. The island overlap criterion then gives for the vanishing of the last KAM surface $\epsilon = \pi^2/4 \simeq 2.47$. The nonlinear effects lower this threshold considerably, and toroidal flux surfaces exist only for $\epsilon < 0.9716...$ (Greene 1979). The most stable KAM surface is at $q = (\sqrt{5} - 1)/2$, which is that irrational which is "farthest away" from the rationals.

For large ϵ it is possible to analytically calculate both the diffusion in v and the Kolmogorov entropy. The diffusion rate is simply that obtained by making a random phase approximation for the displacement in v, $D = (\Delta v)^2/2t$ with t the number of steps, and using $\overline{sin^2 x} = 1/2$ we find

$$D_{QL} = \frac{\epsilon^2}{4} \tag{1.63}$$

where QL stands for quasilinear. Similarly the Kolmogorov entropy is

$$h = ln(\epsilon/2). \tag{1.64}$$

The period doubling point can be found analytically. From Eq. 1.62 we find the tangent map to be given by the matrix

$$M = \begin{pmatrix} 1 & 1 \\ k & 1+k \end{pmatrix} \tag{1.65}$$

with $k = \epsilon cos x$. The elliptic fixed point is at $x = \pi$ and the eigenvalues of M are $\lambda = 1 + k/2 + \sqrt{k^2/4 + k}$. Period doubling occurs at $\epsilon = 4$ where $q_I = 2$.

Long time correlations modify D significantly from the quasilinear value, and the result can be obtained analytically (Rechester-White, 1980). Write the Chirikov Taylor map as $x_j = x_{j-1} + v_{j-1}, v_j = v_{j-1} + \epsilon sin(x_j)$. The distribution of particles (field lines) at position x_j, v_j at time t_j is given by

$$\begin{aligned} P(x_j, v_j, t_j) &= \int dv_{j-1} dx_{j-1} \delta(v_j - v_{j-1} - \epsilon sin(x_j)) \\ &\times \delta(x_j - x_{j-1} - v_{j-1}) P(x_{j-1}, v_{j-1}, t_{j-1}). \end{aligned} \tag{1.66}$$

Now iterate this expression, obtaining

$$\begin{aligned} &P(x_T, v_T, t_T) = \\ &\int dv_{T-1} dx_{T-1} \delta(v_T - v_{T-1} - \epsilon sin(x_T)) \delta(x_T - x_{T-1} - v_{T-1}) \\ &\quad \cdots dv_0 dx_0 \delta(v_1 - v_0 - \epsilon sin(x_1)) \delta(x_1 - x_0 - v_0) P(x_0, v_0, t_0). \end{aligned} \tag{1.67}$$

Now take the initial distribution to be uniform in x but located at $v = 0$, $P(x_0, v_0, t_0) = \delta(v_0)$. We wish to calculate the diffusion in v. Use the delta functions to do the integrals in v_k, giving

$$P(x_T, v_T, T) = \int dx_{T-1} \cdots dx_0 \delta(v_T - S_T) \prod_{k=1}^{T} \delta(x_k - x_{k-1} - S_{k-1}) \tag{1.68}$$

with $S_k = \epsilon \sum_1^k sin(x_p)$. The diffusion constant D is given by the limit of $T \to \infty$ of

$$D = \frac{1}{2T} \int dv_T \int \frac{dx_T}{2\pi} v_T^2 P(x_T, v_T, T) \tag{1.69}$$

or

$$D = \frac{1}{2T} \int \frac{dx_0 \cdots dx_T}{(2\pi)} S_T^2 \prod_{k=1}^{T} \delta(x_k - x_{k-1} - S_{k-1}). \tag{1.70}$$

Substitute for the δ functions $\delta(z) = \sum_{-\infty}^{\infty} e^{imz}/(2\pi)$, giving

$$D = \frac{1}{2T} \int \frac{dx_0 \cdots dx_T}{(2\pi)^{T+1}} S_T^2 \sum_{m_1} \cdots \sum_{m_T} e^{i \sum_1^T m_k(x_k - x_{k-1} - S_{k-1})}. \tag{1.71}$$

But

$$\sum_{1}^{T} m_k(x_k - x_{k-1} - S_{k-1}) \quad = \quad \sum_{k=1}^{T} m_k(x_k - x_{k-1})$$
$$-\epsilon[(m_2 + m_3 + \cdots m_T) sinx_1 + (m_3 + \cdots m_T) sinx_2 + \cdots m_T sinx_{T-1}] \tag{1.72}$$

Now use $e^{\pm iz sinx} = \sum_{-\infty}^{\infty} J_n(z) e^{\pm inx}$, and we obtain

$$\begin{aligned} D = \frac{1}{2T} \int \frac{dx_0 \cdots dx_T}{(2\pi)^{T+1}} S_T^2 \sum_{m_1} \cdots \sum_{m_T} e^{i \sum_1^T m_k(x_k - x_{k-1})} \\ \sum_{n_1} \cdots \sum_{n_{T-1}} J_{n_1}(\epsilon(m_2 + m_3 + \cdots + m_T)) e^{-in_1 x_1} \cdots \\ J_{n_p}(\epsilon(m_{p+1} + \cdots + m_T)) e^{-in_p x_p} \cdots \\ J_{n_{T-1}}(\epsilon m_T) e^{-in_{T-1} x_{T-1}}. \end{aligned} \tag{1.73}$$

Now for large ϵ use the fact that $J_n(\epsilon) \sim 1/\sqrt{\epsilon}$. Thus dominant contributions are terms in the sum with few Bessel functions different from $J_0(0)$. There is one term with all $m_k = 0$ and all $n_k = 0$ giving a contribution to D of

$$D_0 = \frac{1}{2T} \int \frac{dx_0 \cdots dx_T}{(2\pi)^{T+1}} S_T^2 = \epsilon^2/4, \tag{1.74}$$

which we recognize as the quasilinear value. The only terms which contain a single Bessel function with nonzero arguement are those arising from

$$\vec{m} = (0, 0, \cdots, 0, -1, 1, 0, \cdots, 0) \tag{1.75}$$

with the -1 appearing in the lth place, and there are $T - 1$ such terms, plus an additional $T - 1$ similar terms with -1,1 replaced by 1,-1, giving

$$D_1 = \frac{T-1}{T} \int \frac{dx_0 \cdots dx_T}{(2\pi)^{T+1}} S_T^2 e^{i(-2x_l + x_{l-1} + x_{l+1})} \sum_{n_l} J_{n_l}(\epsilon) e^{in_l x_l}. \tag{1.76}$$

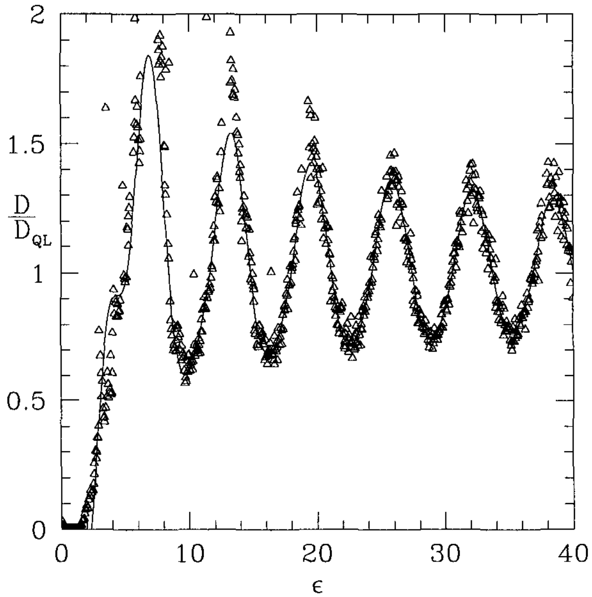

Fig. 1.8 Diffusion in the standard map.

Only the term with $n_l = 2$ survives. Expanding S_T and the exponential we find for large T

$$D_1 = -\epsilon^2 \frac{J_2(\epsilon)}{2}. \tag{1.77}$$

The terms containing two Bessel functions can also be found. They are given by $\vec{m} = (0, 0, \cdots, 0, -1, 0, 1, 0, \cdots, 0)$, $\vec{m} = (0, 0, \cdots, 0, 1, -2, 1, 0, \cdots, 0)$, and $\vec{m} = (0, 0, \cdots, 0, -1, 1, 1, -1, 0, \cdots, 0)$. The final result to order J^2 is

$$\frac{D}{D_{QL}} = 1 - 2J_2(\epsilon) - 2J_1^2(\epsilon) + 2J_3^2(\epsilon) + 2J_2^2(\epsilon). \tag{1.78}$$

In Fig 1.8 is shown a numerical evaluation of the diffusion constant versus ϵ, and a plot of the analytic result given above. The Chirikov-Taylor map, because of its symmetries, possesses stronger correlations than many other maps, but the phenomenon of ringing above stochastic threshold is observed also in other maps.

For some values of ϵ very close to multiples of π there exist, even for large values of ϵ, small islands in phase space which give rise to nondiffusive motion for initial conditions within the islands, and no matter how small the islands are these trajectories dominate any evaluation of transport for large time, resulting in nondiffusive motion. See Karney *et al.* (1982).

Also of interest is the behavior near threshold, which exhibits a universal character, valid for a large class of Hamiltonian systems. It was found, both numerically (Rechester, Rosenbluth, White, 1981) and analytically (Mackay, Meiss, and Percival, 1984) that near threshold the diffusion has the form

$$D \simeq a(\epsilon - \epsilon_c)^p \tag{1.79}$$

with $p \simeq 3$, and ϵ_c the threshold value. Furthermore, the critical exponent $p \simeq 3$ appears to be independent of the particular map.

Near threshold trajectories undergo long time sticking in complex structures consisting of nested chains of islands with narrow stochastic bands between them. This phenomenon leads to nondiffusive transport $x^2 \sim t^p$, with both hyperdiffusion, $p > 1$ and subdiffusion $p < 1$. For a description of this phenomenon see White *et al.* (1998).

1.9 Problems

1. In a reversed field pinch the field relaxes to a Taylor state, with $\vec{j} = \mu\vec{B}$, μ a constant. Assume cylindrical symmetry, so that $\psi_p = \psi_p(r)$ and $\nabla\phi = \hat{\phi}/R$ with $R \gg r$. Find explicitly the covariant and contravariant representations of $\vec{B}$.

2. For any map $x' = f(x,y)$, $y' = g(x,y)$, the matrix describing the map of the differentials dx, dy is called the tangent space map. From the tangent space map derived from the standard map calculate the internal $q_I(\epsilon)$ associated with the O-point, and the eigenvalues at the X-point. Find the value of ϵ at which the O-point bifurcates. Find the values of q_I at Chirikov overlap and at stochastic threshold, $\epsilon \simeq 1$.

3. Consider the Fermi model for the acceleration of cosmic rays, consisting of a particle bouncing elastically between two walls oscillating with velocity $v_W = v_0 sin\omega t$. Let $\psi = \omega t$ and describe the motion by a map giving successive values of the particle velocity u and collision phase ϕ. Show that linearization in u results in the standard map.

4. Consider a straight stellarator with three pairs of helical windings. The vacuum field is $\vec{B} = B_0\hat{z} + \lambda B_0\nabla[r^3 cos(3\theta - kz)]$.

Assume $kr \ll 1$. *i.e.* work close to the axis. Find the magnetic surfaces ψ = constant, with $\vec{B}\cdot\nabla\psi = 0$. Find a condition on λ such that the flux surfaces are topologically circles (no X-point) inside radius r. Use this bound on λ. Find the rotational transform

$$\left\langle \frac{\Delta\theta}{k\Delta z} \right\rangle$$

where <> means an average over z, since the transform is not a constant in z. Note that the equations for the field lines depend only on r and $\zeta = 3\theta - kz$. (Rosenbluth lecture notes)

5. A dipole magnetic field can be written as $\vec{B} = \nabla(\vec{M}\cdot\vec{r}/r^3)$, with $\vec{M}$ the dipole moment. Find the poloidal flux function $\psi_p(r,\theta)$ with $\vec{M}\cdot\vec{r} = Mrcos\theta$ such that

$$\vec{B} = \nabla\phi \times \nabla\psi_p.$$

Find the equations for the field lines.

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