Preface

The subject of this book, concentration compactness, is a method for establishing convergence, in functional spaces, of sequences that are not *a priori* located in a compact set. This situation occurs, in particular, in variational problems with functionals that are invariant under some non-compact group of operators, and therefore have non-compact level sets. The concentration compactness argument considers possible "dislocated" limits of the sequence, that is, limits under sequences of the "gauge" operators. The proof of convergence then can be based on elimination of the dislocated limits. Since a concentration compactness argument using blow-up sequences appeared in the paper of J. Sacks and K. Uhlenbeck [103] on harmonic maps and in the paper of H. Brézis and L. Nirenberg [24] on a semilinear elliptic equation with a critical nonlinearity, the term "concentration", rooted in the use of unbounded sequences of dilations, has become a common designation for all convergence arguments involving dislocated limits, whatever group of transformations is involved. This was the term adopted in the celebrated series of four papers [86], [87], [88] and [89] of P.-L. Lions, which laid the broad foundations of the method and outlined a wide scope of its applications. The book presents a function-analytic formulation of the concentration compactness, inspired by the connection between weak convergence under sequences of Euclidean shifts and convergence in L^p made in the paper [80] of E. Lieb, the celebrated improvement of the Fatou Lemma, known today as Brézis–Lieb lemma, [22], the use of the Brézis–Lieb lemma in P.-L. Lions' subadditivity reasoning, and the "multi-bump" expansions of M.Struwe [111], H.Brézis and J.-M. Coron [25], P.-L. Lions [90] and numerous later works. The function-analytic theory of concentration compactness follows the spirit of the work of P.-L. Lions in one important respect: it gives attention to convergence of arbitrary sequences before studying properties of sequences that originate in specific problems. The functional-analytic framework for concentration compactness is the dislocation space H, D, where H is a separable Hilbert space and D is a fixed group of unitary operators on H, satisfying certain compactness-related properties. The purpose of endowing a Hilbert space with a group D is to define an enhancement of the weak convergence: we say that a sequence u_k converges to zero D-weakly if for every sequence $g_k \in D$, $g_k u_k \rightharpoonup 0$. A refinement of the Banach–Alaoglou theorem (weak compactness of the unit balls) then can be stated in terms of such convergence: any bounded sequence has a convergent subsequence that, after subtraction of all dislocated weak limits (terms of the form $g_k w$, $g_k \in D$, $w \in H$), converges to zero D-weakly. If D is the group of all unitary operators, the D-weak convergence becomes convergence in norm, but the group is too large for the above decomposition to hold. On the other hand, the convergence result of Lieb ([80]) states that weak convergence in $H^1(\mathbb{R}^N)$ enhanced by the group of Euclidean shifts yields convergence in measure (which implies, together with the Sobolev imbedding, convergence in the correspondent space $L^p(\mathbb{R}^N)$).

We have selected the contents for the book in order to give an accessible, rather than technical, presentation of the concentration compactness. We have opted to present the topic in Hilbert space, rather than Banach space, and included three chapters with background material: Chapter 1 – a compilation of theorems from functional analysis, Chapter 2 – a compendium on Sobolev spaces with focus on $H^1(\Omega)$ and unbounded sets, and Chapters 7–8 on differentiable manifolds and Lie groups. The reader is expected to be familiar with basics of point-set topology, metric spaces and measure theory. The presentation of Sobolev spaces in Chapter 2 implicitly emphasizes the role of the conformal group of Euclidean space, an approach which is later generalized in the concentration compactness argument for a conformal group of a manifold in the treatment of subelliptic Sobolev spaces in Chapter 9. The functional-analytic grounds of the concentration compactness are presented in Chapter 3, followed by applications in Chapters 4, 5 and 6 to functions on Euclidean domains. Chapter 9 is an introduction of subelliptic Sobolev spaces on Lie groups, followed by some analogs of problems considered in the preceding chapters that involve subelliptic operators and "magnetic" Laplace–Beltrami operators on manifolds. Chapter 10 surveys several additional applications. The authors will use a follow-up web page *www.math.uu.se/~tintarev/cc.html* to provide additional materials, problems, corrections etc.

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