

Chapter 1

Bipartite and Multipartite Entanglement of Gaussian States

Gerardo Adesso and Fabrizio Illuminati

*Dipartimento di Fisica “E. R. Caianiello”, Università di Salerno;
CNR-Coherentia, Gruppo di Salerno; and INFN Sezione di Napoli-Gruppo
Collegato di Salerno, Via S. Allende, 84081 Baronissi (SA), Italy*

email: gerardo@sa.infn.it, illuminati@sa.infn.it

In this chapter we review the characterization of entanglement in Gaussian states of continuous variable systems. For two-mode Gaussian states, we discuss how their bipartite entanglement can be accurately quantified in terms of the global and local amounts of mixedness, and efficiently estimated by direct measurements of the associated purities. For multimode Gaussian states endowed with local symmetry with respect to a given bipartition, we show how the multimode block entanglement can be completely and reversibly localized onto a single pair of modes by local, unitary operations. We then analyze the distribution of entanglement among multiple parties in multimode Gaussian states. We introduce the continuous-variable tangle to quantify entanglement sharing in Gaussian states and we prove that it satisfies the Coffman-Kundu-Wootters monogamy inequality. Nevertheless, we show that pure, symmetric three-mode Gaussian states, at variance with their discrete-variable counterparts, allow a promiscuous sharing of quantum correlations, exhibiting both maximum tripartite residual entanglement and maximum couplewise entanglement between any pair of modes. Finally, we investigate the connection between multipartite entanglement and the optimal fidelity in a continuous-variable quantum teleportation network. We show how the fidelity can be maximized in terms of the best preparation of the shared entangled resources and, viceversa, that this optimal fidelity provides a clearcut operational interpretation of several measures of bipartite and multipartite entanglement, including the entanglement of formation, the localizable entanglement, and the continuous-variable tangle.

1. Introduction

One of the main challenges in fundamental quantum theory as well as in quantum information and computation sciences lies in the characterization and quantification of bipartite entanglement for mixed states, and in the definition and interpretation of multipartite entanglement both for pure states and in the presence of mixedness. While important insights have been gained on these issues in the context of qubit systems, a less satisfactory understanding has been achieved until recent times on higher-dimensional systems, as the structure of entangled states in Hilbert spaces of high dimensionality exhibits a formidable degree of complexity. However, and quite remarkably, in infinite-dimensional Hilbert spaces of continuous-variable systems, ongoing and coordinated efforts by different research groups have led to important progresses in the understanding of the entanglement properties of a restricted class of states, the so-called Gaussian states. These states, besides being of great importance both from a fundamental point of view and in practical applications, share peculiar features that make their structural properties amenable to accurate and detailed theoretical analysis. It is the aim of this chapter to review some of the most recent results on the characterization and quantification of bipartite and multipartite entanglement in Gaussian states of continuous variable systems, their relationships with standard measures of purity and mixedness, and their operational interpretations in practical applications such as quantum communication, information transfer, and quantum teleportation.

2. Gaussian States of Continuous Variable Systems

We consider a continuous variable (CV) system consisting of N canonical bosonic modes, associated to an infinite-dimensional Hilbert space \mathcal{H} and described by the vector $\hat{X} = \{\hat{x}_1, \hat{p}_1, \dots, \hat{x}_N, \hat{p}_N\}$ of the field quadrature (“position” and “momentum”) operators. The quadrature phase operators are connected to the annihilation \hat{a}_i and creation \hat{a}_i^\dagger operators of each mode, by the relations $\hat{x}_i = (\hat{a}_i + \hat{a}_i^\dagger)$ and $\hat{p}_i = (\hat{a}_i - \hat{a}_i^\dagger)/i$. The canonical commutation relations for the \hat{X}_i 's can be expressed in matrix form: $[\hat{X}_i, \hat{X}_j] = 2i\Omega_{ij}$, with the symplectic form $\Omega = \bigoplus_{i=1}^n \omega$ and $\omega = \delta_{ij-1} - \delta_{ij+1}$, $i, j = 1, 2$.

Quantum states of paramount importance in CV systems are the so-called Gaussian states, *i.e.* states with Gaussian characteristic functions and quasi-probability distributions.¹ The interest in this special class of states (important examples include vacua, coherent, squeezed, thermal, and squeezed-thermal states of the electromagnetic field) stems from the feasi-

bility to produce and control them with linear optical elements, and from the increasing number of efficient proposals and successful experimental implementations of CV quantum information and communication processes involving multimode Gaussian states (see Ref. 2 for recent reviews). By definition, a Gaussian state is completely characterized by first and second moments of the canonical operators. When addressing physical properties invariant under local unitary transformations, such as mixedness and entanglement, one can neglect first moments and completely characterize Gaussian states by the $2N \times 2N$ real covariance matrix (CM) σ , whose entries are $\sigma_{ij} = 1/2\langle\{\hat{X}_i, \hat{X}_j\}\rangle - \langle\hat{X}_i\rangle\langle\hat{X}_j\rangle$. Throughout this chapter, σ will be used indifferently to indicate the CM of a Gaussian state or the state itself. A real, symmetric matrix σ must fulfill the Robertson-Schrödinger uncertainty relation³

$$\sigma + i\Omega \geq 0, \quad (1)$$

to be a *bona fide* CM of a physical state. Symplectic operations (*i.e.* belonging to the group $Sp_{(2N, \mathcal{R})} = \{S \in SL(2N, \mathcal{R}) : S^T \Omega S = \Omega\}$) acting by congruence on CMs in phase space, amount to unitary operations on density matrices in Hilbert space. In phase space, any N -mode Gaussian state can be transformed by symplectic operations in its Williamson diagonal form⁴ ν , such that $\sigma = S^T \nu S$, with $\nu = \text{diag}\{\nu_1, \nu_1, \dots, \nu_N, \nu_N\}$. The set $\Sigma = \{\nu_i\}$ of the positive-defined eigenvalues of $|i\Omega\sigma|$ constitutes the symplectic spectrum of σ and its elements, the so-called symplectic eigenvalues, must fulfill the conditions $\nu_i \geq 1$, following from Eq. (1) and ensuring positivity of the density matrix associated to σ . We remark that the full saturation of the uncertainty principle can only be achieved by pure N -mode Gaussian states, for which $\nu_i = 1 \forall i = 1, \dots, N$. Instead, those mixed states such that $\nu_{i \leq k} = 1$ and $\nu_{i > k} > 1$, with $1 \leq k \leq N$, partially saturate the uncertainty principle, with partial saturation becoming weaker with decreasing k . The symplectic eigenvalues ν_i are determined by N symplectic invariants associated to the characteristic polynomial of the matrix $|i\Omega\sigma|$. Global invariants include the determinant $\text{Det } \sigma = \prod_i \nu_i^2$ and the quantity $\Delta = \sum_i \nu_i^2$, which is the sum of the determinants of all the 2×2 submatrices of σ related to each mode.⁵

The degree of information about the preparation of a quantum state ρ can be characterized by its *purity* $\mu \equiv \text{Tr } \rho^2$, ranging from 0 (completely mixed states) to 1 (pure states). For a Gaussian state with CM σ one has⁶

$$\mu = 1/\sqrt{\text{Det } \sigma}. \quad (2)$$

As for the entanglement, we recall that positivity of the CM's partial transpose (PPT)⁷ is a necessary and sufficient condition of separability for $(M + N)$ -mode bisymmetric Gaussian states (see Sec. 4) with respect to the $M|N$ bipartition of the modes,⁸ as well as for $(M + N)$ -mode Gaussian states with fully degenerate symplectic spectrum.⁹ In the special, but important case $M = 1$, PPT is a necessary and sufficient condition for separability of all Gaussian states.^{11,10} For a general Gaussian state of any $M|N$ bipartition, the PPT criterion is replaced by another necessary and sufficient condition stating that a CM σ corresponds to a separable state if and only if there exists a pair of CMs σ_A and σ_B , relative to the subsystems A and B respectively, such that the following inequality holds:¹¹ $\sigma \geq \sigma_A \oplus \sigma_B$. This criterion is not very useful in practice. Alternatively, one can introduce an operational criterion based on a nonlinear map, that is independent of (and strictly stronger than) the PPT condition.¹²

In phase space, partial transposition amounts to a mirror reflection of one quadrature in the reduced CM of one of the parties. If $\{\tilde{\nu}_i\}$ is the symplectic spectrum of the partially transposed CM $\tilde{\sigma}$, then a $(1+N)$ -mode (or bisymmetric $(M + N)$ -mode) Gaussian state with CM σ is separable if and only if $\tilde{\nu}_i \geq 1 \forall i$. A proper measure of CV entanglement is the *logarithmic negativity*¹³ $E_{\mathcal{N}} \equiv \log \|\tilde{\rho}\|_1$, where $\|\cdot\|_1$ denotes the trace norm, which constitutes an upper bound to the *distillable entanglement* of the state ρ . It can be computed in terms of the symplectic spectrum $\tilde{\nu}_i$ of $\tilde{\sigma}$:

$$E_{\mathcal{N}} = \max \left\{ 0, -\sum_{i: \tilde{\nu}_i < 1} \log \tilde{\nu}_i \right\}. \quad (3)$$

$E_{\mathcal{N}}$ quantifies the extent to which the PPT condition $\tilde{\nu}_i \geq 1$ is violated.

3. Two-Mode Gaussian States: Entanglement and Mixedness

Two-mode Gaussian states represent the prototypical quantum states of CV systems, and constitute an ideal test-ground for the theoretical and experimental investigation of CV entanglement.¹⁴ Their CM can be written in the following block form

$$\sigma \equiv \begin{pmatrix} \alpha & \gamma \\ \gamma^T & \beta \end{pmatrix}, \quad (4)$$

where the three 2×2 matrices α , β , γ are, respectively, the CMs of the two reduced modes and the correlation matrix between them. It is well known¹⁰ that for any two-mode CM σ there exists a local symplectic operation

$S_l = S_1 \oplus S_2$ which takes σ to its standard form σ_{sf} , characterized by

$$\alpha = \text{diag}\{a, a\}, \quad \beta = \text{diag}\{b, b\}, \quad \gamma = \text{diag}\{c_+, c_-\}. \quad (5)$$

States whose standard form fulfills $a = b$ are said to be symmetric. Any pure state is symmetric and fulfills $c_+ = -c_- = \sqrt{a^2 - 1}$. The uncertainty principle Ineq. (1) can be recast as a constraint on the $Sp(4, \mathbb{R})$ invariants $\text{Det}\sigma$ and $\Delta(\sigma) = \text{Det}\alpha + \text{Det}\beta + 2\text{Det}\gamma$, yielding $\Delta(\sigma) \leq 1 + \text{Det}\sigma$. The standard form covariances a, b, c_+ , and c_- can be determined in terms of the two local symplectic invariants

$$\mu_1 = (\text{Det}\alpha)^{-1/2} = 1/a, \quad \mu_2 = (\text{Det}\beta)^{-1/2} = 1/b, \quad (6)$$

which are the marginal purities of the reduced single-mode states, and of the two global symplectic invariants

$$\mu = (\text{Det}\sigma)^{-1/2} = [(ab - c_+^2)(ab - c_-^2)]^{-1/2}, \quad \Delta = a^2 + b^2 + 2c_+c_-, \quad (7)$$

where μ is the global purity of the state. Eqs. (6-7) can be inverted to provide the following physical parametrization of two-mode states in terms of the four independent parameters μ_1, μ_2, μ , and Δ :¹⁵

$$a = \frac{1}{\mu_1}, \quad b = \frac{1}{\mu_2}, \quad c_{\pm} = \frac{\sqrt{\mu_1\mu_2}}{4} (\epsilon_- \pm \epsilon_+), \quad (8)$$

with $\epsilon_{\mp} \equiv \sqrt{[\Delta - (\mu_1 \mp \mu_2)^2 / (\mu_1^2 \mu_2^2)]^2 - 4/\mu^2}$. The uncertainty principle and the existence of the radicals appearing in Eq. (8) impose the following constraints on the four invariants in order to describe a physical state

$$\mu_1\mu_2 \leq \mu \leq \frac{\mu_1\mu_2}{\mu_1\mu_2 + |\mu_1 - \mu_2|}, \quad (9)$$

$$\frac{2}{\mu} + \frac{(\mu_1 - \mu_2)^2}{\mu_1^2 \mu_2^2} \leq \Delta \leq 1 + \frac{1}{\mu^2}. \quad (10)$$

The physical meaning of these constraints, and the role of the extremal states (*i.e.* states whose invariants saturate the upper or lower bounds of Eqs. (9-10)) in relation to the entanglement, will be investigated soon.

In terms of symplectic invariants, partial transposition corresponds to flipping the sign of $\text{Det}\gamma$, so that Δ turns into $\tilde{\Delta} = \Delta - 4\text{Det}\gamma = -\Delta + 2/\mu_1^2 + 2/\mu_2^2$. The symplectic eigenvalues of the CM σ and of its partial transpose $\tilde{\sigma}$ are promptly determined in terms of symplectic invariants

$$2\nu_{\mp}^2 = \Delta \mp \sqrt{\Delta^2 - 4/\mu^2}, \quad 2\tilde{\nu}_{\mp}^2 = \tilde{\Delta} \mp \sqrt{\tilde{\Delta}^2 - 4/\mu^2}, \quad (11)$$

where in our naming convention $\nu_- \leq \nu_+$ in general, and similarly for the $\tilde{\nu}_{\mp}$. The PPT criterion yields a state σ separable if and only if $\tilde{\nu}_- \geq 1$.

Since $\tilde{\nu}_+ > 1$ for all two-mode Gaussian states, the quantity $\tilde{\nu}_-$ also completely quantifies the entanglement, in fact the logarithmic negativity Eq. (3) is a monotonically decreasing and convex function of $\tilde{\nu}_-$, $E_N = \max\{0, -\log \tilde{\nu}_-\}$. In the special instance of symmetric Gaussian states, the *entanglement of formation*¹⁶ E_F is also computable¹⁷ but, being again a decreasing function of $\tilde{\nu}_-$, it provides the same characterization of entanglement and is thus fully equivalent to E_N in this subcase.

A first natural question that arises is whether there can exist two-mode Gaussian states of finite maximal entanglement at a given amount of mixedness of the global state. These states would be the analog of the maximally entangled mixed states (MEMS) that are known to exist for two-qubit systems.¹⁸ Unfortunately, it is easy to show that a similar question in the CV scenario is meaningless. Indeed, for any fixed, finite global purity μ there exist infinitely many Gaussian states which are infinitely entangled. However, we can ask whether there exist maximally entangled states at fixed global *and* local purities. While this question does not yet have a satisfactory answer for two-qubit systems, in the CV scenario it turns out to be quite interesting and nontrivial. In this respect, a crucial observation is that, at fixed μ , μ_1 and μ_2 , the lowest symplectic eigenvalue $\tilde{\nu}_-$ of the partially transposed CM is a monotonically increasing function of the global invariant Δ . Due to the existence of exact *a priori* lower and upper bounds on Δ at fixed purities (see Ineq. 10), this entails the existence of both maximally *and* minimally entangled Gaussian states. These classes of extremal states have been introduced in Ref. 19, and completely characterized (providing also schemes for their experimental production) in Ref. 15, where the relationship between entanglement and information has been extended considering generalized entropic measures to quantify the degrees of mixedness. In particular, there exist maximally and minimally entangled states also at fixed global and local generalized Tsallis p -entropies.¹⁵ In this short review chapter, we will discuss only the case in which the purities (or, equivalently, the linear entropies) are used to measure the degree of mixedness of a quantum state. In this instance, the Gaussian maximally entangled mixed states (GMEMS) are two-mode squeezed thermal states, characterized by a fully degenerate symplectic spectrum; on the other hand, the Gaussian least entangled mixed states (GLEMS) are states of partial minimum uncertainty (*i.e.* with the lowest symplectic eigenvalue of their CM being equal to 1). Studying the separability of the extremal states (via the PPT criterion), it is possible to classify the entanglement properties of

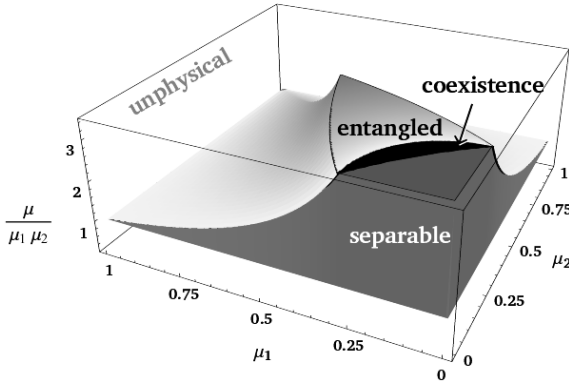


Fig. 1. Classification of the entanglement for two-mode Gaussian states in the space of marginal purities $\mu_{1,2}$ and normalized global purity $\mu/\mu_1\mu_2$. All physical states lie between the horizontal plane of product states $\mu = \mu_1\mu_2$, and the upper limiting surface representing GMEMMS. Separable states (dark grey area) and entangled states are well distinguished except for a narrow coexistence region (depicted in black). In the entangled region the average logarithmic negativity (see text) grows from white to medium grey. The expressions of the boundaries between all these regions are collected in Eq. (12).

all two-mode Gaussian states in the manifold spanned by the purities:

$$\begin{aligned}
 \mu_1\mu_2 \leq \mu \leq \frac{\mu_1\mu_2}{\mu_1+\mu_2-\mu_1\mu_2}, & \Rightarrow \text{separable;} \\
 \frac{\mu_1\mu_2}{\mu_1+\mu_2-\mu_1\mu_2} < \mu \leq \frac{\mu_1\mu_2}{\sqrt{\mu_1^2+\mu_2^2-\mu_1^2\mu_2^2}}, & \Rightarrow \text{coexistence;} \\
 \frac{\mu_1\mu_2}{\sqrt{\mu_1^2+\mu_2^2-\mu_1^2\mu_2^2}} < \mu \leq \frac{\mu_1\mu_2}{\mu_1\mu_2+|\mu_1-\mu_2|}, & \Rightarrow \text{entangled.}
 \end{aligned}
 \tag{12}$$

In particular, apart from a narrow “coexistence” region where both separable and entangled Gaussian states can be found, the separability of two-mode states at given values of the purities is completely characterized. For purities that saturate the upper bound in Ineq. (9), GMEMS and GLEMS coincide and μ have a unique class of states whose entanglement depends *only* on the marginal purities $\mu_{1,2}$. They are Gaussian maximally entangled states for fixed marginals (GMEMMS). The maximal entanglement of a Gaussian state decreases rapidly with increasing difference of marginal purities, in analogy with finite-dimensional systems.²⁰ For symmetric states ($\mu_1 = \mu_2$) the upper bound of Ineq. (9) reduces to the trivial bound $\mu \leq 1$ and GMEMMS reduce to pure two-mode states. Knowledge of the global and marginal purities thus accurately characterizes the entanglement of two-mode Gaussian states, providing strong sufficient conditions and exact, analytical lower and upper bounds. As we will now show, marginal and global purities allow as well an accurate quantification

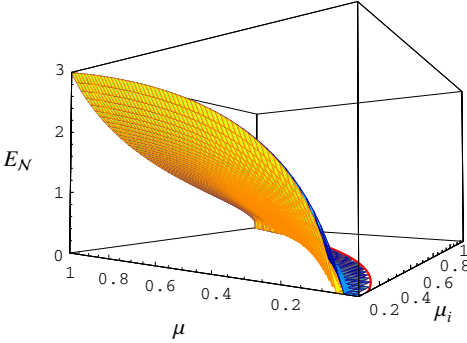


Fig. 2. Maximal and minimal logarithmic negativities as functions of the global and marginal purities of symmetric two-mode Gaussian states. The darker (lighter) surface represents GMEMS (GLEMS). In this space, a generic two-mode mixed symmetric state is represented by a dot lying inside the narrow gap between the two extremal surfaces.

of the entanglement. Outside the region of separability, GMEMS attain maximum logarithmic negativity $E_{\mathcal{N} \max}$ while, in the region of nonvanishing entanglement (see Eq. (12)), GLEMS acquire minimum logarithmic negativity $E_{\mathcal{N} \min}$. Knowledge of the global purity, of the two local purities, and of the global invariant Δ (*i.e.*, knowledge of the full covariance matrix) would allow for an exact quantification of the entanglement. However, we will now show that an estimate based only on the knowledge of the experimentally measurable global and marginal purities turns out to be quite accurate. We can in fact quantify the entanglement of Gaussian states with given global and marginal purities by the *average logarithmic negativity* $\bar{E}_{\mathcal{N}} \equiv (E_{\mathcal{N} \max} + E_{\mathcal{N} \min})/2$. We can then also define the relative error $\delta \bar{E}_{\mathcal{N}}$ on $\bar{E}_{\mathcal{N}}$ as $\delta \bar{E}_{\mathcal{N}}(\mu_{1,2}, \mu) \equiv (E_{\mathcal{N} \max} - E_{\mathcal{N} \min})/(E_{\mathcal{N} \max} + E_{\mathcal{N} \min})$. It is easy to see that this error decreases *exponentially* both with increasing global purity and decreasing marginal purities, *i.e.* with increasing entanglement, falling for instance below 5% for symmetric states ($\mu_1 = \mu_2 \equiv \mu_i$) and $\mu > \mu_i$. The reliable quantification of quantum correlations in genuinely entangled two-mode Gaussian states is thus always assured by the experimental determination of the purities, except at most for a small set of states with very weak entanglement (states with $E_{\mathcal{N}} \lesssim 1$). Moreover, the accuracy is even greater in the general non-symmetric case $\mu_1 \neq \mu_2$, because the maximal achievable entanglement decreases in such an instance. In Fig. 2, the surfaces of extremal logarithmic negativities are plotted versus μ_i and μ for symmetric states. In the case $\mu = 1$ the upper and lower bounds coincide, since for pure states the entanglement is completely quantified by the marginal purity. For mixed states this is not the case, but, as the plot shows, knowledge of the global and marginal purities strictly bounds the entanglement both from above and from below. This analysis shows that the

average logarithmic negativity $\bar{E}_{\mathcal{N}}$ is a reliable estimate of the logarithmic negativity $E_{\mathcal{N}}$, improving as the entanglement increases. We remark that the purities may be directly measured experimentally, without the need for a full tomographic reconstruction of the whole CM, by exploiting quantum networks techniques²¹ or single-photon detections without homodyning.²²

Finally, it is worth remarking that most of the results presented here (including the sufficient conditions for entanglement based on knowledge of the purities), being derived for CMs using the symplectic formalism in phase space, retain their validity for generic non Gaussian states of CV systems. For instance, any two-mode state with a CM equal to that of an entangled two-mode Gaussian state is entangled as well.²³ Our methods may thus serve to detect entanglement for a broader class of states in infinite-dimensional Hilbert spaces. The analysis briefly reviewed in this paragraph on the relationships between entanglement and mixedness, can be generalized to multimode Gaussian states endowed with special symmetry under mode permutations, as we will show in the next section.

4. Multimode Gaussian States: Unitarily Localizable Entanglement

We will now consider Gaussian states of CV systems with an arbitrary number of modes, and briefly discuss the simplest instances in which the techniques introduced for two-mode Gaussian states can be generalized and turn out to be useful for the quantification and the scaling analysis of CV multimode entanglement. We introduce the notion of *bisymmetric* states, defined as those $(M + N)$ -mode Gaussian states, of a generic bipartition $M|N$, that are invariant under local mode permutations on the M -mode and N -mode subsystems. The CM σ of a $(M + N)$ -mode bisymmetric Gaussian state results from a correlated combination of the fully symmetric blocks σ_{α^M} and σ_{β^N} :

$$\sigma = \begin{pmatrix} \sigma_{\alpha^M} & \mathbf{\Gamma} \\ \mathbf{\Gamma}^T & \sigma_{\beta^N} \end{pmatrix}, \quad (13)$$

where σ_{α^M} (σ_{β^N}) describes a M -mode (N -mode) reduced Gaussian state completely invariant under mode permutations, and $\mathbf{\Gamma}$ is a $2M \times 2N$ real matrix formed by identical 2×2 blocks γ . Clearly, $\mathbf{\Gamma}$ is responsible for the correlations existing between the M -mode and the N -mode parties. The identity of the submatrices γ is a consequence of the local invariance under mode exchange, internal to the M -mode and N -mode parties. A

first observation is that the symplectic spectrum of the CM σ Eq. (13) of a bisymmetric $(M + N)$ -mode Gaussian state includes two degenerate eigenvalues, with multiplicities $M - 1$ and $N - 1$. Such eigenvalues coincide, respectively, with the degenerate eigenvalue ν_{α}^{-} of the reduced CM $\sigma_{\alpha M}$ and the degenerate eigenvalue ν_{β}^{-} of the reduced CM $\sigma_{\beta N}$, with the same respective multiplicities. Equipped with this result, one can prove⁸ that σ can be brought, by means of a local unitary operation, with respect to the $M|N$ bipartition, to a tensor product of single-mode uncorrelated states and of a two-mode Gaussian state with CM σ^{eq} . Here we give an intuitive sketch of the proof (the detailed proof is given in Ref. 8). Let us focus on the N -mode block $\sigma_{\beta N}$. The matrices $i\Omega\sigma_{\beta N}$ and $i\Omega\sigma$ possess a set of $N - 1$ simultaneous eigenvectors, corresponding to the same (degenerate) eigenvalue. This fact suggests that the phase-space modes corresponding to such eigenvectors are the same for σ and for $\sigma_{\beta N}$. Then, bringing by means of a local symplectic operation the CM $\sigma_{\beta N}$ in Williamson form, any $(2N - 2) \times (2N - 2)$ submatrix of σ will be diagonalized because the normal modes are common to the global and local CMs. In other words, no correlations between the M -mode party with reduced CM $\sigma_{\alpha M}$ and such modes will be left: all the correlations between the M -mode and N -mode parties will be concentrated in the two conjugate quadratures of a single mode of the N -mode block. Going through the same argument for the M -mode block with CM $\sigma_{\alpha M}$ will prove the proposition and show that the whole entanglement between the two multimode blocks can always be concentrated in only two modes, one for each of the two multimode parties.

While, as mentioned, the detailed proof of this result can be found in Ref. 8 (extending the findings obtained in Ref. 24 for the case $M = 1$), here we will focus on its relevant physical consequences, the main one being that the bipartite $M \times N$ entanglement of bisymmetric $(M + N)$ -mode Gaussian states is *unitarily localizable*, *i.e.*, through local unitary operations, it can be fully concentrated on a single pair of modes, one belonging to party (block) M , the other belonging to party (block) N . The notion of “unitarily localizable entanglement” is different from that of “localizable entanglement” originally introduced by Verstraete, Popp, and Cirac for spin systems.²⁵ There, it was defined as the maximal entanglement concentrable on two chosen spins through local *measurements* on all the other spins. Here, the local operations that concentrate all the multimode entanglement on two modes are *unitary* and involve the two chosen modes as well, as parts of the respective blocks. Furthermore, the unitarily localizable entanglement (when computable) is always stronger than the localizable entanglement.

In fact, if we consider a generic bisymmetric multimode state of a $M|N$ bipartition, with each of the two target modes owned respectively by one of the two parties (blocks), then the ensemble of optimal local measurements on the remaining (“assisting”) $M + N - 2$ modes belongs to the set of local operations and classical communication (LOCC) with respect to the considered bipartition. By definition the entanglement cannot increase under LOCC, which implies that the localized entanglement (à la Verstraete, Popp, and Cirac) is always less or equal than the original $M \times N$ block entanglement. On the contrary, *all* of the same $M \times N$ original bipartite entanglement can be unitarily localized onto the two target modes, resulting in a reversible, of maximal efficiency, multimode/two-mode entanglement switch. This fact can have a remarkable impact in the context of quantum repeaters²⁶ for communications with continuous variables. The consequences of the unitary localizability are manifold. In particular, as already previously mentioned, one can prove that the PPT (positivity of the partial transpose) criterion is a necessary and sufficient condition for the separability of $(M + N)$ -mode bisymmetric Gaussian states.⁸ Therefore, the multimode block entanglement of bisymmetric (generally mixed) Gaussian states with CM σ , being equal to the bipartite entanglement of the equivalent two-mode localized state with CM σ^{eq} , can be determined and quantified by the logarithmic negativity in the general instance and, for all multimode states whose two-mode equivalent Gaussian state is symmetric, the entanglement of formation between the M -mode party and the N -mode party can be computed exactly as well.

For the sake of illustration, let us consider fully symmetric $2N$ -mode Gaussian states described by a $2N \times 2N$ CM $\sigma_{\beta^{2N}}$. These states are trivially bisymmetric under any bipartition of the modes, so that their block entanglement is always localizable by means of local symplectic operations. This class of states includes the pure, CV GHZ-type states (discussed in Refs. 27, 24) that, in the limit of infinite squeezing, reduce to the simultaneous eigenstates of the relative positions and the total momentum and coincide with the proper Greenberger-Horne-Zeilinger²⁸ (GHZ) states of CV systems.²⁷ The standard form CM $\sigma_{\beta^{2N}}^p$ of this particular class of pure, symmetric multimode Gaussian states depends only on the local mixedness parameter $b \equiv 1/\mu_\beta$, which is the inverse of the purity of any single-mode reduced block, and it is proportional to the single-mode squeezing. Exploiting our previous analysis, we can compute the entanglement between a block of K modes and the remaining $2N - K$ modes for pure states (in this

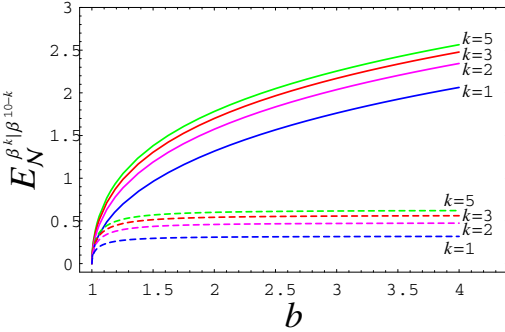


Fig. 3. Hierarchy of block entanglements of fully symmetric $2N$ -mode Gaussian states of $K \times (2N - K)$ bipartitions ($N = 10$) as a function of the single-mode mixedness b , for pure states (solid lines) and for mixed states obtained from $(2N + 4)$ -mode pure states by tracing out 4 modes (dashed lines).

case the block entanglement is simply the Von Neumann entropy of any of the reduced blocks) and, remarkably, for mixed states as well.

We can in fact consider a generic $2N$ -mode fully symmetric mixed state with CM $\sigma_{\beta^{2N}}^{p \setminus Q}$, obtained from a pure fully symmetric $(2N + Q)$ -mode state by tracing out Q modes. For any Q and any dimension N of the block ($K \leq N$), and for any nonzero squeezing (*i.e.* for any $b > 1$) one has that the state exhibits genuine multipartite entanglement, as first remarked in Ref. 27 for pure states: each K -mode party is entangled with the remaining $(2N - K)$ -mode block. Furthermore, the genuine multipartite nature of the entanglement can be precisely quantified by observing that the logarithmic negativity between the K -mode and the remaining $(2N - K)$ -mode block is an increasing function of the integer $K \leq N$, as shown in Fig. 3. The optimal splitting of the modes, which yields the maximal, unitarily localizable entanglement, corresponds to $K = N/2$ if N is even, and $K = (N - 1)/2$ if N is odd. The multimode entanglement of mixed states remains finite also in the limit of infinite squeezing, while the multimode entanglement of pure states diverges with respect to any bipartition, as shown in Fig. 3. For a fixed amount of local mixedness, the scaling structure of the multimode entanglement with the number of modes can be analyzed as well, giving rise to an interesting result.⁸ Let us consider, again for the sake of illustration, the class of fully symmetric $2N$ -mode Gaussian states, but now at fixed single-mode purity. It is immediate to see that the entanglement between any two modes decreases with N , while the $N|N$ entanglement increases (and diverges for pure states as $N \rightarrow \infty$): the quantum correlations become distributed among all the modes. This is a clear signature of genuine multipartite entanglement and suggests a detailed analysis of its sharing properties, that will be discussed in the next section. The scaling structure of multimode entanglement also elucidates the power of the unitary localizability as a strategy for entanglement purification, with its efficiency

improving with increasing number of modes. Finally, let us remark that the local symplectic operations needed for the unitary localization can be implemented by only using passive²⁹ and active linear optical elements such as beam splitters, phase shifters and squeezers, and that the original multimode entanglement can be estimated by the knowledge of the global and local purities of the equivalent, localized two-mode state (see Refs. 8, 24 for a thorough discussion), along the lines presented in Sec. 3 above.

5. Entanglement Sharing of Gaussian States

Here we address the problem of entanglement sharing among multiple parties, investigating the structure of multipartite entanglement.^{30,31} Our aim is to analyze the distribution of entanglement between different (partitions of) modes in CV systems. In Ref. 32 Coffman, Kundu and Wootters (CKW) proved for a three-qubit system ABC, and conjectured for N qubits (this conjecture has now been proven by Osborne and Verstraete³³) that the entanglement between, say, qubit A and the remaining two-qubits partition (BC) is never smaller than the sum of the A|B and A|C bipartite entanglements in the reduced states. This statement quantifies the so-called *monogamy* of quantum entanglement,³⁴ in opposition to the classical correlations which can be freely shared. One would expect a similar inequality to hold for three-mode Gaussian states, namely

$$E^{i|(jk)} - E^{i|j} - E^{i|k} \geq 0, \quad (14)$$

where E is a proper measure of CV entanglement and the indices $\{i, j, k\}$ label the three modes. However, an immediate computation on symmetric states shows that Ineq. (14) can be violated for small values of the single-mode mixedness b using either the logarithmic negativity $E_{\mathcal{N}}$ or the entanglement of formation E_F to quantify the bipartite entanglement. This is not a paradox;³¹ rather, it implies that none of these two measures is the proper candidate for approaching the task of quantifying entanglement sharing in CV systems. This situation is reminiscent of the case of qubit systems, for which the CKW inequality holds using the tangle τ ,³² but fails if one chooses equivalent measures of bipartite entanglement such as the concurrence³⁵ (*i.e.* the square root of the tangle) or the entanglement of formation itself. Related problems on inequivalent entanglement measures for the ordering of Gaussian states are discussed in Ref. 36.

We then wish to define a new measure of CV entanglement able to capture the entanglement distribution trade-off via the monogamy inequality (14). A rigorous treatment of this problem is presented in Ref. 30. Here

we briefly review the definition and main properties of the desired measure that quantifies entanglement sharing in CV systems. Because it can be regarded as the continuous-variable analogue of the tangle, we will name it, in short, the *contangle*.

For a pure state $|\psi\rangle$ of a $(1 + N)$ -mode CV system, we can formally define the contangle as

$$E_\tau(\psi) \equiv \log^2 \|\tilde{\varrho}\|_1, \quad \varrho = |\psi\rangle\langle\psi|. \quad (15)$$

$E_\tau(\psi)$ is a proper measure of bipartite entanglement, being a convex, increasing function of the logarithmic negativity $E_{\mathcal{N}}$, which is equivalent to the entropy of entanglement in all pure states. For pure Gaussian states $|\psi\rangle$ with CM σ^p , one has $E_\tau(\sigma^p) = \log^2(1/\mu_1 - \sqrt{1/\mu_1^2 - 1})$, where $\mu_1 = 1/\sqrt{\text{Det}\sigma_1}$ is the local purity of the reduced state of mode 1, described by a CM σ_1 (considering $1 \times N$ bipartitions). Definition (15) is extended to generic mixed states ϱ of $(N + 1)$ -mode CV systems through the convex-roof formalism, namely:

$$E_\tau(\varrho) \equiv \inf_{\{p_i, \psi_i\}} \sum_i p_i E_\tau(\psi_i), \quad (16)$$

where the infimum is taken over the decompositions of ϱ in terms of pure states $\{|\psi_i\rangle\}$. For infinite-dimensional Hilbert spaces the index i is continuous, the sum in Eq. (16) is replaced by an integral, and the probabilities $\{p_i\}$ by a distribution $\pi(\psi)$. All multimode mixed Gaussian states σ admit a decomposition in terms of an ensemble of pure Gaussian states. The infimum of the average contangle, taken over all pure Gaussian decompositions only, defines the *Gaussian contangle* G_τ , which is an upper bound to the true contangle E_τ , and an entanglement monotone under Gaussian local operations and classical communications (GLOCC).^{36,37} The Gaussian contangle, similarly to the Gaussian entanglement of formation,³⁷ acquires the simple form $G_\tau(\sigma) \equiv \inf_{\sigma^p \leq \sigma} E_\tau(\sigma^p)$, where the infimum runs over all pure Gaussian states with CM $\sigma^p \leq \sigma$.

Equipped with these properties and definitions, one can prove several results.³⁰ In particular, the general (multimode) monogamy inequality $E^{i_m | (i_1 \dots i_{m-1} i_{m+1} \dots i_N)} - \sum_{l \neq m} E^{i_m | i_l} \geq 0$ is satisfied by all pure three-mode and all pure *symmetric* N -mode Gaussian states, using either E_τ or G_τ to quantify bipartite entanglement, and by all the corresponding mixed states using G_τ . Furthermore, there is numerical evidence supporting the conjecture that the general CKW inequality should hold for all *nonsymmetric*

ric N -mode Gaussian states as well.^a The sharing constraint (14) leads to the definition of the *residual contangle* as a tripartite entanglement quantifier. For nonsymmetric three-mode Gaussian states the residual contangle is partition-dependent. In this respect, a proper quantification of tripartite entanglement is provided by the *minimum* residual contangle

$$E_{\tau}^{i|j|k} \equiv \min_{(i,j,k)} \left[E_{\tau}^{i|(jk)} - E_{\tau}^{i|j} - E_{\tau}^{i|k} \right], \quad (17)$$

where (i, j, k) denotes all the permutations of the three mode indexes. This definition ensures that $E_{\tau}^{i|j|k}$ is invariant under mode permutations and is thus a genuine three-way property of any three-mode Gaussian state. We can adopt an analogous definition for the minimum residual Gaussian contangle $G_{\tau}^{i|j|k}$. One finds that the latter is a proper measure of genuine tripartite CV entanglement, because it is an entanglement monotone under tripartite GLOCC for pure three-mode Gaussian states.³⁰

Let us now analyze the sharing structure of multipartite CV entanglement, by taking the residual contangle as a measure of tripartite entanglement. We pose the problem of identifying the three-mode analogues of the two inequivalent classes of fully inseparable three-qubit states, the GHZ state²⁸ $|\psi_{\text{GHZ}}\rangle = (1/\sqrt{2}) [|000\rangle + |111\rangle]$, and the W state³⁸ $|\psi_W\rangle = (1/\sqrt{3}) [|001\rangle + |010\rangle + |100\rangle]$. These states are both pure and fully symmetric, but the GHZ state possesses maximal three-party tangle with no two-party quantum correlations, while the W state contains the maximal two-party entanglement between any pair of qubits and its tripartite residual tangle is consequently zero.

Surprisingly enough, in symmetric three-mode Gaussian states, if one aims at maximizing (at given single-mode squeezing b) either the two-mode contangle $E_{\tau}^{i|l}$ in any reduced state (*i.e.* aiming at the CV W -like state), or the genuine tripartite contangle (*i.e.* aiming at the CV GHZ-like state), one finds the same, unique family of pure symmetric three-mode squeezed states. These states, previously named “GHZ-type” states,²⁷ have been introduced for generic N -mode CV systems in the previous section, where their multimode entanglement scaling has been studied.^{8,24} The peculiar nature of entanglement sharing in this class of states, now baptized

^aVery recently, the conjectured monogamy inequality for all (pure or mixed) N -mode Gaussian states has been indeed proven by considering a slightly different version of the continuous-variable tangle, defined in terms of the (convex-roof extended) squared negativity instead of the squared logarithmic negativity. See T.Hiroshima, G. Adesso, and F. Illuminati, quant-ph/0605021.

CV GHZ/ W states, is further confirmed by the following observation. If one requires maximization of the 1×2 bipartite contangle $E_r^{z|(jk)}$ under the constraint of separability of all two-mode reductions, one finds a class of symmetric mixed states whose tripartite residual contangle is strictly smaller than the one of the GHZ/ W states, at fixed local squeezing.³⁹ Therefore, in symmetric three-mode Gaussian states, when there is no two-mode entanglement, the three-party one is not enhanced, but frustrated.

These results, unveiling a major difference between discrete-variable and CV systems, establish the *promiscuous* structure of entanglement sharing in symmetric Gaussian states. Being associated with degrees of freedom with continuous spectra, states of CV systems need not saturate the CKW inequality to achieve maximum couplewise correlations. In fact, without violating the monogamy inequality (14), pure symmetric three-mode Gaussian states are maximally three-way entangled and, at the same time, maximally robust against the loss of one of the modes due, for instance, to decoherence, as demonstrated in full detail in Ref. 39. This fact may promote these states, experimentally realizable with the current technology,⁴⁰ as candidates for reliable CV quantum communication. Exploiting a three-mode CV GHZ/ W state as a quantum channel can ensure for instance a tripartite quantum information protocol like a teleportation network or quantum secret sharing; or a standard, highly entangled two-mode channel, after a unitary (reversible) localization has been performed through a single beam splitter; or, as well, a two-party quantum protocol with better-than-classical efficiency, even if one of the modes is lost due to decoherence. We will next focus on a relevant applicative setting of CV multipartite entanglement, in which various of its properties discussed so far will come in a natural relation.

6. Exploiting Multipartite Entanglement: Optimal Fidelity of Continuous Variable Teleportation

In this section we analyze an interesting application of multipartite CV entanglement: a quantum teleportation-network protocol, involving N users who share a genuine N -partite entangled Gaussian resource, completely symmetric under permutations of the modes. In the standard multiuser protocol, proposed by Van Loock and Braunstein,⁴¹ two parties are randomly chosen as sender (Alice) and receiver (Bob), but, in order to accomplish teleportation of an unknown coherent state, Bob needs the results of $N - 2$ momentum detections performed by the other cooperating parties.

A nonclassical teleportation fidelity (i.e. $\mathcal{F} > \mathcal{F}^{cl}$) between *any* pair of parties is sufficient for the presence of genuine N -partite entanglement in the shared resource, while in general the converse is false (see *e.g.* Fig. 1 of Ref. 41). The *fidelity*, which quantifies the success of a teleportation experiment, is defined as $\mathcal{F} \equiv \langle \psi^{in} | \varrho^{out} | \psi^{in} \rangle$, where “in” and “out” denote the input and the output state. \mathcal{F} reaches unity only for a perfect state transfer, $\varrho^{out} = |\psi^{in}\rangle\langle\psi^{in}|$, while without entanglement in the resource, by purely classical communication, an average fidelity of $\mathcal{F}_{cl} = 1/2$ is the best that can be achieved if the alphabet of input states includes all coherent states with even weight.⁴² This teleportation network has been recently demonstrated experimentally⁴³ by exploiting three-mode squeezed Gaussian states,⁴⁰ yielding a best fidelity of $\mathcal{F} = 0.64 \pm 0.02$, an index of genuine tripartite entanglement. Our aim is to determine the optimal multi-user teleportation fidelity, and to extract from it a quantitative information on the multipartite entanglement in the shared resources. By “optimal” here we mean maximization of the fidelity over all local single-mode unitary operations, at fixed amounts of noise and entanglement in the shared resource. We consider realistically mixed N -mode Gaussian resources, obtained by combining a mixed momentum-squeezed state (with squeezing parameter r_1) and $N - 1$ mixed position-squeezed states (with squeezing parameter $r_2 \neq r_1$ and in principle a different noise factor) into an N -splitter⁴¹ (a sequence of $N - 1$ suitably tuned beam splitters). The resulting state is a completely symmetric mixed Gaussian state of a N -mode CV system. For a given thermal noise in the individual modes (comprising the unavoidable experimental imperfections), all the states with equal average squeezing $\bar{r} \equiv (r_1 + r_2)/2$ are equivalent up to local single-mode unitary operations and possess, by definition, the same amount of multipartite entanglement with respect to any partition. The teleportation efficiency, instead, depends separately on the different single-mode squeezings. We have then the freedom of unbalancing the local squeezings r_1 and r_2 without changing the total entanglement in the resource, in order to single out the optimal form of the resource state, which enables a teleportation network with maximal fidelity. This analysis is straightforward (see Ref. 44 for details), but it yields several surprising side results. In particular, one finds that the optimal form of the shared N -mode symmetric Gaussian states, for $N > 2$, is neither unbiased in the x_i and p_i quadratures (like the states discussed in Ref. 45 for $N = 3$), nor constructed by N equal squeezers ($r_1 = r_2 = \bar{r}$). This latter case, which has been implemented experimentally⁴³ for $N = 3$, is clearly not optimal, yielding fidelities lower than $1/2$ for $N \geq 30$ and \bar{r} falling in

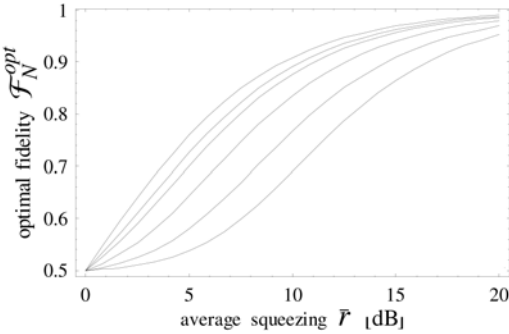


Fig. 4. Plot of the optimal fidelity for teleporting an arbitrary coherent state from any sender to any receiver chosen from N ($N = 2, \dots, 50$) parties, exploiting N -party entangled, pure symmetric Gaussian states as resources. A nonclassical fidelity $\mathcal{F}_N^{opt} > 0.5$ is always assured for any N , if the shared entangled resources are prepared in their optimal form.

a certain interval.⁴¹ According to the authors of Ref. 41, the explanation of this paradoxical behavior should lie in the fact that their teleportation scheme might not be optimal. However, a closer analysis shows that the problem does not lie in the choice of the protocol, but rather in the choice of the resource states. If the shared N -mode squeezed states are prepared, by local unitary operations, in the optimal form (described in detail in Ref. 44), the teleportation fidelity \mathcal{F}^{opt} is guaranteed to be nonclassical (see Fig. 4) as soon as $\bar{r} > 0$ for any N , in which case the considered class of pure states is genuinely multipartite entangled, as we have shown in the previous sections. In fact, one can show⁴⁴ that this nonclassical optimal fidelity is *necessary and sufficient* for the presence of multipartite entanglement in any multimode symmetric Gaussian state used as a shared resource for CV teleportation. These findings yield quite naturally a direct operative way to quantify multipartite entanglement in N -mode (mixed) symmetric Gaussian states, in terms of the so-called *Entanglement of Teleportation*,⁴⁴ defined as the normalized optimal fidelity

$$E_T \equiv \max \{0, (\mathcal{F}_N^{opt} - \mathcal{F}_{cl}) / (1 - \mathcal{F}_{cl})\}, \quad (18)$$

going from 0 (separable states) to 1 (CV GHZ/ W state). Moreover, one finds that the optimal shared entanglement that allows for the maximal fidelity is *exactly* the CV counterpart of the localizable entanglement, originally introduced for spin systems by Verstraete, Popp, and Cirac.²⁵ The CV localizable entanglement (not to be confused with the unitarily localizable entanglement introduced in Sec. 4) thus acquires a suggestive operational meaning in terms of teleportation processes. In fact, the localizable entanglement of formation (computed by finding the optimal set of local measurements — unitary transformations and nonunitary momentum

detections — performed on the assisting modes to concentrate the highest possible entanglement onto Alice and Bob pair of modes) is a monotonically increasing function of E_T : $E_F^{loc} = f[(1 - E_T)/(1 + E_T)]$, with $f(x) \equiv \frac{(1+x)^2}{4x} \log \frac{(1+x)^2}{4x} - \frac{(1-x)^2}{4x} \log \frac{(1-x)^2}{4x}$. For $N = 2$ (standard two-user teleportation⁴⁶) the state is already localized and $E_F^{loc} = E_F$, so that E_T is equivalent to the entanglement of formation E_F of two-mode Gaussian states. Remarkably, for $N = 3$, *i.e.* for three-mode pure Gaussian resource states, the residual contangle $E_\tau^{i|j|k}$ introduced in Sec. 5 (see Eq. (17)) turns out to be itself a monotonically increasing function of E_T :

$$E_\tau^{i|j|k} = \log^2 \frac{2\sqrt{2}E_T - (E_T + 1)\sqrt{E_T^2 + 1}}{(E_T - 1)\sqrt{E_T(E_T + 4)} + 1} - \frac{1}{2} \log^2 \frac{E_T^2 + 1}{E_T(E_T + 4) + 1}. \quad (19)$$

The quantity E_T thus represents another *equivalent* quantification of genuine tripartite CV entanglement and provides the latter with an operational interpretation associated to the success of a three-party teleportation network. This suggests a possible experimental test of the promiscuous sharing of CV entanglement, consisting in the successful (with nonclassical optimal fidelity) implementation of both a three-user teleportation network exploiting pure symmetric Gaussian resources, and of two-user standard teleportation exploiting any reduced two-mode channel obtained discarding a mode from the original resource.

Besides their theoretical aspects, the results reviewed in this section are of direct practical interest, as they answer the experimental need for the best preparation recipe of an entangled squeezed resource, in order to implement quantum teleportation and in general CV communication schemes with the highest possible efficiency.

7. Conclusions and Outlook

We have reviewed some recent results on the entanglement of Gaussian states of CV systems. For two-mode Gaussian states we have shown how bipartite entanglement can be qualified and quantified via the global and local degrees of purity. Suitable generalizations of the techniques introduced for two-mode Gaussian states allow to analyze various aspects of entanglement in multimode CV systems, and we have discussed recent findings on the scaling, localization, and sharing properties of multipartite entanglement in symmetric, bisymmetric, and generic multimode Gaussian states. Finally, we have shown that many of these properties acquire a clear and simple operational meaning in the context of CV quantum communication

and teleportation networks. Generalizations and extensions of these results appear at hand, and we may expect further progress along these lines in the near future, both for Gaussian and non Gaussian states.

A good portion of the material reported in this chapter originates from joint work with our friend and colleague Alessio Serafini, whom we warmly thank for the joy of collaborating together. It is as well a pleasure to acknowledge stimulating exchanges over the last two years with Nicolas Cerf, Ignacio Cirac, Silvio De Siena, Jens Eisert, Marie Ericsson, Rosario Fazio, Jaromír Fiurášek, Tohya Hiroshima, Ole Krüger, Gerd Leuchs, Norbert Lütkenhaus, Chiara Macchiavello, Klaus Mølmer, Tobias Osborne, Matteo Paris, Martin Plenio, Eugene Polzik, Mario Rasetti, Gustavo Rigolin, Anna Sanpera, Peter van Loock, Frank Verstraete, David Vitali, Reinhard Werner, Michael Wolf, Bill Wootters, and Paolo Zanardi.

References

1. *Quantum Information Theory with Continuous Variables*, S. L. Braunstein and A. K. Pati Eds. (Kluwer, Dordrecht, 2002).
2. S. L. Braunstein and P. van Loock, *Rev. Mod. Phys.* **77**, 513 (2005); J. Eisert and M. B. Plenio, *Int. J. Quant. Inf.* **1**, 479 (2003).
3. R. Simon, E. C. G. Sudarshan, and N. Mukunda, *Phys. Rev. A* **36**, 3868 (1987).
4. J. Williamson, *Am. J. Math.* **58**, 141 (1936); see also V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, New York, 1978).
5. A. Serafini, F. Illuminati, and S. De Siena, *J. Phys. B* **37**, L21 (2004).
6. M. G. A. Paris, F. Illuminati, A. Serafini, and S. De Siena, *Phys. Rev. A* **68**, 012314 (2003).
7. A. Peres, *Phys. Rev. Lett.* **77**, 1413 (1996); R. Horodecki, P. Horodecki, and M. Horodecki, *Phys. Lett. A* **210**, 377 (1996).
8. A. Serafini, G. Adesso, and F. Illuminati, *Phys. Rev. A* **71**, 032349 (2005).
9. A. Botero and B. Reznik, *Phys. Rev. A* **67**, 052311 (2003); G. Giedke, J. Eisert, J. I. Cirac, and M. B. Plenio, *Quant. Inf. Comp.* **3**, 211 (2003).
10. R. Simon, *Phys. Rev. Lett.* **84**, 2726 (2000); L.-M. Duan, G. Giedke, I. Cirac, and P. Zoller, *ibid.* **84**, 2722 (2000).
11. R. F. Werner and M. M. Wolf, *Phys. Rev. Lett.* **86**, 3658 (2001).
12. G. Giedke, B. Kraus, M. Lewenstein, and J. I. Cirac, *Phys. Rev. Lett.* **87**, 167904 (2001).
13. G. Vidal and R. F. Werner, *Phys. Rev. A* **65**, 032314 (2002); K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, *Phys. Rev. A* **58**, 883 (1998); J. Eisert, *PhD Thesis* (University of Potsdam, Potsdam, 2001); M. B. Plenio, *Phys. Rev. Lett.* **95**, 090503 (2005).
14. J. Laurat, G. Keller, J.-A. Oliveira-Huguenin, C. Fabre, T. Coudreau, A. Serafini, G. Adesso, and F. Illuminati, *J. Opt. B* **7**, S577 (2005).
15. G. Adesso, A. Serafini, and F. Illuminati, *Phys. Rev. A* **70**, 022318 (2004).

16. C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, *Phys. Rev. A* **54**, 3824 (1996).
17. G. Giedke, M. M. Wolf, O. Krüger, R. F. Werner, and J. I. Cirac, *Phys. Rev. Lett.* **91**, 107901 (2003).
18. S. Ishizaka and T. Hiroshima, *Phys. Rev. A* **62**, 022310 (2000); F. Verstraete, K. Audenaert, and B. De Moor, *ibid.* **64**, 012316 (2001); W. J. Munro, D. F. V. James, A. G. White, and P. G. Kwiat, *ibid.* **64**, 030302 (2001).
19. G. Adesso, A. Serafini, and F. Illuminati, *Phys. Rev. Lett.* **92**, 087901 (2004).
20. G. Adesso, F. Illuminati, and S. De Siena, *Phys. Rev. A* **68**, 062318 (2003).
21. A. K. Ekert, C. M. Alves, D. K. L. Oi, M. Horodecki, P. Horodecki, and L. C. Kwek, *Phys. Rev. Lett.* **88**, 217901 (2002); R. Filip, *Phys. Rev. A* **65**, 062320 (2002).
22. J. Fiurášek and N. J. Cerf, *Phys. Rev. Lett.* **93**, 063601 (2004); J. Wenger, J. Fiurášek, R. Tualle-Broui, N. J. Cerf, and P. Grangier, *Phys. Rev. A* **70**, 053812 (2004).
23. P. van Loock, *Fortschr. Phys.* **50**, 12 1177 (2002).
24. G. Adesso, A. Serafini, and F. Illuminati, *Phys. Rev. Lett.* **93**, 220504 (2004).
25. F. Verstraete, M. Popp, and J. I. Cirac, *Phys. Rev. Lett.* **92**, 027901 (2004).
26. H.-J. Briegel, W. Dür, J. I. Cirac, and P. Zoller, *Phys. Rev. Lett.* **81**, 5932 (1998).
27. P. van Loock and A. Furusawa, *Phys. Rev. A* **67**, 052315 (2003).
28. D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, *Am. J. Phys.* **58**, 1131 (1990).
29. M. M. Wolf, J. Eisert, and M. B. Plenio, *Phys. Rev. Lett.* **90**, 047904 (2003).
30. G. Adesso and F. Illuminati, *New J. Phys.* **8**, 15 (2006).
31. G. Adesso and F. Illuminati, quant-ph/0506213, *Int. J. Quant. Inf.* (2006), in press.
32. V. Coffman, J. Kundu, and W. K. Wootters, *Phys. Rev. A* **61**, 052306 (2000).
33. T. J. Osborne and F. Verstraete, *Phys. Rev. Lett.* **96**, 220503 (2006).
34. B. M. Terhal, *IBM J. Res. & Dev.* **48**, 71 (2004), and quant-ph/0307120.
35. W. K. Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998).
36. G. Adesso and F. Illuminati, *Phys. Rev. A* **72**, 032334 (2005).
37. M. M. Wolf, G. Giedke, O. Krüger, R. F. Werner, and J. I. Cirac, *Phys. Rev. A* **69**, 052320 (2004).
38. W. Dür, G. Vidal, and J. I. Cirac, *Phys. Rev. A* **62**, 062314 (2000).
39. G. Adesso, A. Serafini, and F. Illuminati, *Phys. Rev. A* **73**, 032345 (2006).
40. T. Aoki, N. Takei, H. Yonezawa, K. Wakui, T. Hiraoka, A. Furusawa, and P. van Loock, *Phys. Rev. Lett.* **91**, 080404 (2003).
41. P. van Loock and S. L. Braunstein, *Phys. Rev. Lett.* **84**, 3482 (2000).
42. S. L. Braunstein, C. A. Fuchs, and H. J. Kimble, *J. Mod. Opt.* **47**, 267 (2000); K. Hammerer, M. M. Wolf, E. S. Polzik, and J. I. Cirac, *Phys. Rev. Lett.* **94**, 150503 (2005).
43. H. Yonezawa, T. Aoki, and A. Furusawa, *Nature* **431**, 430 (2004).
44. G. Adesso and F. Illuminati, *Phys. Rev. Lett.* **95**, 150503 (2005).
45. W. P. Bowen, P. K. Lam, and T. C. Ralph, *J. Mod. Opt.* **50**, 801 (2003).
46. S. L. Braunstein and H. J. Kimble, *Phys. Rev. Lett.* **80**, 869 (1998).