Lecture 1

Calculus on smooth manifolds

We introduce some basic structures on a finite dimensional real vector space with a metric: the metric on the exterior algebra, orientations, the volume element and the star isomorphism. We introduce smooth differential forms, de Rham cohomology groups, state the de Rham Theorem and discuss Weil's sheaf-theoretic approach to this theorem. We briefly discuss Riemannian metrics, orientations on manifolds and integration of top forms on oriented manifolds.

References for this lecture are [Warner, 1971], [Bott and Tu, 1986] and [Demailly, 1996].

Manifolds are assumed to be *connected* and to satisfy the second axiom of countability. One of the advantages of the second assumption is that it implies the existence of partitions of unity, an important tool for the study of smooth manifolds.

1.1 The Euclidean structure on the exterior algebra

Let V be an m-dimensional real vector space with inner product $\langle \, , \, \rangle$, i.e. $\langle \, , \, \rangle : V \times V \longrightarrow \mathbb{R}$ is a symmetric, positive definite bilinear form on V.

Let

$$\Lambda(V) = \bigoplus_{0 \leq p \leq m} \Lambda^p(V)$$

be the exterior algebra associated with V.

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If $\{e_1, \ldots, e_m\}$ is a basis for V, then the elements $e_I = e_{i_1} \wedge \ldots \wedge e_{i_p}$, where $I = (i_1, \ldots, i_p)$ ranges in the corresponding set of multi-indices with $1 \leq i_1 < \ldots < i_p \leq m$, form a basis for $\Lambda^p(V)$.

The elements of $\Lambda^p(V)$ can be seen as the alternating p-linear form on V^* as follows:

$$v_1 \wedge \ldots \wedge v_p = \sum_{\nu \in S_p} \epsilon(\nu)\, v_{\nu_1} \otimes \ldots \otimes v_{\nu_p},$$

where S_p is the symmetric group in p elements and $\epsilon(\nu)$ is the sign of the permutation ν.

When dealing with exterior algebras it is costumary to write summations such as $\sum_I u_I\, e_I$ or $\sum_{|I|=p} u_I\, e_I$ meaning that the summation is over the set of ordered multi-indices I as above and $u_I \in \mathbb{R}$.

There is a natural inner product on $\Lambda(V)$ defined by declaring any two distinct spaces $\Lambda^p(V)$ and $\Lambda^{p'}(V)$ mutually orthogonal and setting

$$\langle\, v_1 \wedge \ldots \wedge v_p\,,\, w_1 \wedge \ldots \wedge w_p \,\rangle := \det || \,\langle v_j, w_k \rangle\, ||, \qquad v_j, w_k \in V. \quad (1.1)$$

If the basis $\{e_1, \ldots, e_m\}$ for V is orthonormal, then so is the corresponding one for $\Lambda(V)$.

Exercise 1.1.1. Let $V = V' \oplus V''$ be an orthogonal direct sum decomposition, v', $w' \in \Lambda^{p'}(V')$ and v'', $w'' \in \Lambda^{p''}(V'')$. Show that

$$\langle\langle v' \wedge v''\,,\, w' \wedge w'' \rangle\rangle = \langle\langle v', w' \rangle\rangle\, \langle\langle v'', w'' \rangle\rangle.$$

1.2 The star isomorphism on $\Lambda(V)$

By definition, $\Lambda^0(V) = \mathbb{R}$. The positive half-line $\mathbb{R}^+ \subseteq \Lambda^0(V)$ is defined without ambiguity.

The real vector space $\Lambda^m(V)$ is one-dimensional and $\Lambda^m(V) \setminus \{0\}$ has two connected components, i.e. two half-lines.

However, unlike the case of $\Lambda^0(V)$, there is no canonical way to distinguish either of them.

A choice is required. This choice gives rise to an isometry $\Lambda^0(V) \simeq \Lambda^m(V)$.

The $\star$ operator is the operator that naturally arises when we want to complete the picture with linear isometries $\Lambda^p(V) \simeq \Lambda^{m-p}(V)$.

Definition 1.2.1. (**Orientations of** V) The choice of a connected component $\Lambda^m(V)^+$ of $\Lambda^m(V) \setminus \{0\}$ is called an *orientation* on V.

Let V be oriented and $\{e_i\}$ be an ordered, orthonormal basis such that $e_1 \wedge \ldots \wedge e_m \in \Lambda^m(V)^+$. This element is uniquely defined since any two bases as above are related by an orthogonal matrix with determinant $+1$.

Definition 1.2.2. (**The volume element**) The vector

$$\boxed{dV \;:=\; e_1 \wedge \ldots \wedge e_m \;\in\; \Lambda^m(V)^+} \tag{1.2}$$

is called *the volume element* associated with the oriented $(V, \langle\, ,\, \rangle, \Lambda^m(V)^+)$.

Definition 1.2.3. (**The $\star$ operator**) The $\star$ *operator* is the unique linear isomorphism

$$\star \,:\, \Lambda(V) \,\simeq\, \Lambda(V)$$

defined by the properties

$$\star \,:\, \Lambda^p(V) \,\simeq\, \Lambda^{m-p}(V),$$

$$\boxed{u \wedge \star v \;=\; \langle\, u, v \,\rangle\, dV, \qquad \forall\, u,\, v \,\in\, \Lambda^p(V),\; \forall p.}$$

The $\star$ operator depends on the inner product *and* on the chosen orientation.

Let us check that the operator $\star$ exists and is unique.

Consider the non-degenerate pairing

$$\Lambda^p(V) \times \Lambda^{m-p}(V) \longrightarrow \mathbb{R}, \qquad (u, w) \longrightarrow (u \wedge w)/dV.$$

More explicitly, let $\{e_i\}$ be an oriented orthonormal basis, $dV = e_1 \wedge \ldots \wedge e_m$ be the volume element, $u = \sum_I u_I \, e_I$, $w = \sum_J w_J \, e_J$. Then

$$u \wedge w \; = \; \sum_I \epsilon(I, CI) \, u_I \, w_{CI} \, dV,$$

where, if I is an ordered set of indices, then CI is the complementary ordered set of indices and $\epsilon(I, CI)$ is the sign of the permutation (I, CI) of the ordered set $\{1, \ldots, m\}$.

Let $v = \sum_I v_I \, e_I$ and define

$$\boxed{\star v \; := \; \textstyle\sum_I \epsilon(I, CI) \, v_I \, e_{CI}.} \tag{1.3}$$

We have that

$$u \wedge \star v \; = \; \sum_I u_I \, e_I \wedge \sum_{I'} \epsilon(I', CI') \, v_{I'} \, e_{CI'} \; =$$

$$= \sum_I u_I \, v_I \, \epsilon(I, CI) \, e_I \wedge e_{CI} \; = \; \langle u, v \rangle \, dV.$$

This shows the existence of $\star$, which seems to depend on the choice of the orthonormal basis.

Assume $\star'$ is another such operator. Then $u \wedge (\star - \star')(v) = (\langle u, v \rangle - \langle u, v \rangle) \, dV = 0$ for every u and v which implies that $\star = \star'$.

Exercise 1.2.4. Verify the following statements.

The $\star$ operator is a linear isometry.

Let $dV = e_1 \wedge \ldots \wedge e_m$ be the volume element for the fixed orientation.

$$\star(1) \; = \; dV, \quad \star(dV) \; = \; 1, \quad \star(e_1 \wedge \ldots \wedge e_p) = e_{p+1} \wedge \ldots \wedge e_m,$$

$$\star\star_{|\Lambda^p(V)} \; = \; (-1)^{p(m-p)} \, Id_{\Lambda^p(V)}, \tag{1.4}$$

$$\langle u, v \rangle \; = \; \star(v \wedge \star u) \; = \; \star(u \wedge \star v), \quad \forall \, u, \, v \; \in \; \Lambda^p(V), \quad \forall \, 0 \le p \le m.$$

1.3 The tangent and cotangent bundles of a smooth manifold

A smooth manifold M of dimension m comes equipped with natural smooth vector bundles.

Let $(U; x_1, \ldots, x_m)$ be a chart centered at a point $q \in M$.

- T_M the *tangent bundle* of M. The fiber $T_{M,q}$ can be identified with the linear span $\mathbb{R}\langle \partial_{x_1}, \ldots, \partial_{x_m} \rangle$.
- T^*_M the *cotangent bundle* of M. Let $\{dx_i\}$ be the dual basis of the basis $\{\partial_{x_i}\}$. The fiber $T^*_{M,q}$ can be identified with the span $\mathbb{R}\langle dx_1, \ldots, dx_m \rangle$.
- $\Lambda^p(T^*_M)$ the *p-th exterior bundle* of T^*_M. The fiber $\Lambda^p(T^*_M)_q = \Lambda^p(T^*_{M,q})$ can be identified with the linear span $\mathbb{R}\langle \{dx_I\}_{|I|=p} \rangle$.
- $\Lambda(T^*_M) := \oplus_{p=0}^m \Lambda^p(T^*_M)$ the *exterior algebra bundle* of M.

As it is custumary and by slight abuse of notation, one can use the symbols ∂_{x_i} and dx_j to denote the corresponding sections over U of the tangent and cotangent bundles.

In this case, the collections $\{\partial_{x_i}\}$ and $\{dx_i\}$ form *local frames* for the bundles in question.

Exercise 1.3.1. Let M be a smooth manifold of dimension m. Take any definition in the literature for the tangent and cotangent bundles and verify the assertions that follow. Let $(U; x_1, \ldots, x_m)$ and $(U'; x'_1, \ldots, x'_m)$ be two local charts centered at $q \in M$. Show that the transition functions $\tau_{U'U}(x) : U \times \mathbb{R}^m \simeq U' \times \mathbb{R}^m$ ($\gamma_{U'U}(x)$, resp.) for the tangent bundle T_M (cotangent bundle T^*_M, resp.) are given, using the column notation for vectors in $\mathbb{R}^m$, by

$$\tau_{U'U}(x) = \left[J \begin{pmatrix} x' \\ x \end{pmatrix} (x) \right]^t, \qquad \gamma_{U'U}(x) = \left[J \begin{pmatrix} x' \\ x \end{pmatrix} (x) \right]^{-1},$$

where

$$J \begin{pmatrix} x' \\ x \end{pmatrix} (x) = \begin{pmatrix} \frac{\partial x'_1}{\partial x_1}(x) & \ldots & \frac{\partial x'_m}{\partial x_1}(x) \\ \vdots & \ldots & \vdots \\ \frac{\partial x'_1}{\partial x_m}(x) & \ldots & \frac{\partial x'_m}{\partial x_m}(x) \end{pmatrix}.$$

Exercise 1.3.2. Determine the tangent bundle of the sphere $S^n \subseteq \mathbb{R}^{n+1}$ given by the equation $\sum_{j=1}^{n+1} x_j^2 = 1$. Let $S^{n-1} \subseteq S^n$ be an "equatorial" embedding, i.e. obtained by intersecting S^n with the hyperplane $x_{n+1} = 0$. Study the "normal bundle" exact sequence

$$0 \longrightarrow T_{S^{n-1}} \longrightarrow (T_{S^n})_{|S^{n-1}} \longrightarrow N_{S^{n-1},S^n} \longrightarrow 0.$$

1.4 The de Rham cohomology groups

Definition 1.4.1. (**p-forms**) The elements of the real vector space

$$\boxed{E^p(M) := C^\infty(M, \Lambda^p(T^*_M))}$$

of smooth real-valued sections of the vector bundle $\Lambda^p(T^*_M)$ are called (smooth differential) *p-forms* on M.

Let $d : E^p(M) \longrightarrow E^{p+1}(M)$ denote the exterior derivation of differential forms.

Exercise 1.4.2. A p-form u on M can be written on U as $u = \sum_{|I|=p} u_I dx_I$. Show that exterior derivation of forms is well-defined. More precisely, show that if one defines, locally on the chart $(U; x_1, \ldots, x_m)$,

$$du := \sum_j \frac{\partial u_I}{\partial x_j} dx_j \wedge dx_I,$$

then du is independent of the choice of coordinates.

Definition 1.4.3. (**Complexes, cohomology of a complex**) A *complex* is a sequence of maps of vector spaces

$$\ldots \longrightarrow V^{i-1} \xrightarrow{\delta^{i-1}} V^i \xrightarrow{\delta^i} V^{i+1} \longrightarrow \ldots, \quad i \in \mathbb{Z},$$

also denoted by $(V^\bullet, \delta)$, such that $\delta^i \circ \delta^{i-1} = 0$ for every index i, i.e. such that $\operatorname{Im} \delta^{i-1} \subseteq \operatorname{Ker} \delta^i$.

The vector spaces $H^i(V^\bullet, \delta) =: \operatorname{Ker} \delta^i / \operatorname{Im} \delta^{i-1}$ are called the *cohomology groups* of the complex.

A complex is said to be *exact at i* if $\operatorname{Im} \delta^{i-1} = \operatorname{Ker} \delta^i$, i.e. if $H^i(V^\bullet, \delta) = 0$, and *exact* if it is exact for every $i \in \mathbb{Z}$.

Exercise 1.4.4. (**The de Rham complex**) Show that $d^2 = 0$ so that we get the so-called *de Rham* complex of vector spaces,

$$\boxed{0 \longrightarrow E^0(M) \xrightarrow{d} E^1(M) \xrightarrow{d} \ldots \xrightarrow{d} E^{m-1}(M) \xrightarrow{d} E^m(M) \longrightarrow 0.} \tag{1.5}$$

Definition 1.4.5. (**Closed/exact**) A p-form u is said to be *closed* if $du = 0$ and is said to be *exact* if there exists $v \in E^{p-1}(M)$ such that $dv = u$.

Exercise 1.4.6. Let u and v be closed forms. Show that $u \wedge v$ is closed. Assume, in addition, that v is exact and show that $u \wedge v$ is exact.

Exercise 1.4.7. Let

$$u = (2x + y \cos xy)\, dx + (x \cos xy)\, dy$$

on $\mathbb{R}^2$. Show that u is exact. What is the integral of u along any closed curve in $\mathbb{R}^2$?

Exercise 1.4.8. Let

$$u = \frac{1}{2\pi} \frac{x\, dy - y\, dx}{x^2 + y^2}$$

on $\mathbb{R}^2 \setminus \{0\}$. Show that u is closed. Compute the integral of u over the unit circle S^1. Is u exact? Is $u_{|S^1}$ exact?

Exercise 1.4.9. (a) Prove that every closed 1-form on S^2 is exact.
(b) Let

$$u = \frac{x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

on $\mathbb{R}^3 \setminus \{0\}$. Show that u is closed.
(c) Evaluate $\int_{S^2} u$. Conclude that u is not exact.
(d) Let

$$u = \frac{x_1\, dx_1 \ldots + x_n\, dx_n}{(x_1^2 + \ldots + x_n^2)^{n/2}}$$

on $R^n \setminus \{0\}$. Show that $\star u$ is closed.
(e) Evaluate

$$\int_{S^{n-1}} \star u.$$

Is $\star u$ exact?

The de Rham complex (1.5) is *not* exact and its deviation from exactness is an important invariant of M and is measured by the so-called de Rham cohomology groups of M; see Theorem 1.4.12.

Definition 1.4.10. (**The de Rham cohomology groups**) The real *de Rham cohomology groups* $H^\bullet_{dR}(M,\mathbb{R})$ of M are the cohomology groups of the complex (1.5), i.e.

$$\boxed{H^p_{dR}(M,\mathbb{R}) \simeq \frac{\text{closed } p\text{-forms on } M}{\text{exact } p\text{-forms on } M.}}$$

The de Rham complex is *locally exact* on M by virtue of the important:

Theorem 1.4.11. (**Poincaré Lemma**) *Let $p > 0$. A closed p-form u on M is locally exact, i.e. for every $q \in M$ there exists an open neighborhood U of q and $v \in E^{p-1}(U)$ such that*

$$\boxed{u_{|U} = dv.}$$

Proof. See [Bott and Tu, 1986] §4, [Warner, 1971], §4.18. □

The following result of de Rham's is fundamental. In what follows the algebra structures are given by the wedge and cup products, respectively.

Theorem 1.4.12. (**The de Rham Theorem**) *Let M be a not necessarily orientable smooth manifold. There is a canonical isomorphisms of $\mathbb{R}$-algebras*

$$\boxed{H^\bullet_{dR}(M,\mathbb{R}) \simeq H^\bullet(M,\mathbb{R}).}$$

Remark 1.4.13. Theorem 1.4.12 is obtained using integration over differentiable simplices and then using the various canonical identifications between real singular cohomology and real differentiable singular cohomology. Of course, one also has canonical isomorphisms with real Alexander-Spanier cohomology, sheaf cohomology with coefficients in the locally constant sheaf $\mathbb{R}_M$ and Čech cohomology with coefficients in $\mathbb{R}_M$. See [Warner, 1971], § 4.7, 4.17, 5.34 – 5.38, 5.43 – 5.45, and [Mumford, 1970], 5.23–5.28.

Remark 1.4.14. (**Sheaf-theoretic de Rham Theorem**) It is important to know that what above, and more, admits a sheaf-theoretic reformulation, due to A. Weil. Here is a sketch of this re-formulation. Let $\mathcal{E}^p_M$ be the sheaf of germs of smooth p-forms on M, $\mathcal{E}_M := \mathcal{E}^0_M$ be the sheaf of germs of smooth functions on M, $\mathbb{R}_M$ be the sheaf of germs of locally constant functions on M. Due to the existence of partitions of unity, the sheaf $\mathcal{E}_M$ is what one calls a *fine* sheaf (see [Griffiths and Harris, 1978], p. 42). Any sheaf of $\mathcal{E}_M$-modules is fine. In particular, the sheaves $\mathcal{E}^p(M)$ are fine. Fine sheaves have trivial higher sheaf cohomology groups. The operator d is defined locally and gives rise to maps of sheaves $d : \mathcal{E}^p_M \longrightarrow \mathcal{E}^{p+1}_M$. In this context, the Poincaré Lemma 1.4.11 implies that the complex of sheaves

$$0 \longrightarrow \mathbb{R}_M \hookrightarrow \mathcal{E}^0_M \xrightarrow{d} \mathcal{E}^1_M \xrightarrow{d} \dots \xrightarrow{d} \mathcal{E}^{m-1} \xrightarrow{d} \mathcal{E}^m_M \longrightarrow 0 \qquad (1.6)$$

is exact. In short, $\mathbb{R}_M \longrightarrow (\mathcal{E}^\bullet, d^\bullet)$ is a resolution of $\mathbb{R}_M$ by fine sheaves. The sheaf cohomology of a sheaf F on X is defined by considering a resolution $F \longrightarrow I^\bullet$ of F by injective sheaves and setting $H^p_{Sheaf}(X, F) := H^p(H^0(X, I^\bullet))$. Choosing another injective resolution gives canonically isomorphic sheaf cohomology groups. One can take a resolution of F by fine sheaves as well. Taking global sections in (1.6) and ignoring $\mathbb{R}_M$, we obtain the de Rham complex (1.5). It follows that there is a natural isomorphism between the sheaf cohomology of $\mathbb{R}_M$ and de Rham cohomology:

$$H^p_{Sheaf}(X, \mathbb{R}_M) \simeq H^p(E^\bullet, d) =: H^p_{dR}(X, \mathbb{R}).$$

The de Rham Theorem 1.4.12 follows from the natural identification of the standard singular cohomology with real coefficients with the sheaf cohomology of $\mathbb{R}_M$. See [Warner, 1971].

Surprisingly, while the left-hand side is a topological invariant, the right-hand side depends on the smooth structure. In other words: *the "number" of linearly independent smooth closed p-forms which are not exact is a topological invariant, independent of the smooth structure on M.* The advantage of this sheaf-theoretic approach to the de Rham Isomorphism Theorem is that it gives rise to a variety of isomorphisms in different contexts, as soon as one has Poincaré Lemma-type results. Another example is the sheaf-theoretic proof of the Dolbeault Isomorphism Theorem based on the Grothendieck-Dolbeault Lemma 3.7.4. See Remark 3.7.6.

Remark 1.4.15. It is important to know that the complexes (1.6) and (1.5) have counterparts in the theory of currents on a manifold:

$$0 \longrightarrow \mathbb{R}_M \hookrightarrow \mathcal{D}^0_M \xrightarrow{d} \mathcal{D}^1_M \xrightarrow{d} \dots \xrightarrow{d} \mathcal{D}^{m-1} \xrightarrow{d} \mathcal{D}^m_M \longrightarrow 0, \qquad (1.7)$$

$$0 \longrightarrow D^0(M) \xrightarrow{d} D^1(M) \xrightarrow{d} \dots \xrightarrow{d} D^{m-1}(M) \xrightarrow{d} D^m(M) \longrightarrow 0 \quad (1.8)$$

yielding results analogous to the ones of Remark 1.4.14. See [Griffiths and Harris, 1978], §3 for a quick introduction to currents.

The space of currents $D^p(X, \mathbb{R})$ is defined as the topological dual of the space of compactly supported smooth p-forms $E_c^{m-p}(M, \mathbb{R})$ endowed with the C^∞-topology.
The exterior derivation

$$d = D^p(M, \mathbb{R}) \longrightarrow D^{p+1}(M, \mathbb{R})$$

is defined by

$$d\,(T)(u) := (-1)^{p+1} T(du).$$

Currents have a local nature which allows to write them as differential forms with distribution coefficients. If M is oriented, then any differential p-form u is also a p-current via the assignment $v \mapsto \int_M u \wedge v$, $v \in E_c^{m-p}(M, \mathbb{R})$.

In this case, the complexes (1.6) and (1.7), and (1.5) and (1.8) are quasi-isomorphic, i.e. the natural injection induces isomorphisms at the level of cohomology.

By integration, a piecewise smooth oriented $(n-p)$-chain in M gives a p-current in $D^p(M)$.

Importantly, any closed analytic subvariety V of complex codimension d of a complex manifold X gives rise, via integration, to a $2d$-current $\int_V$ in $D^{2d}(X, \mathbb{R})$. By Stokes' Theorem for analytic varieties [Griffiths and Harris, 1978], p. 33, such a current is closed and therefore gives rise to a cohomology class $[\int_V]$ which coincides with the fundamental class $[V]$ of $V \subseteq X$.

Exercise 1.4.16. Let $X \to Y$ be the blowing up of a smooth complex surface at a point $y \in Y$. Show that

$$Rf_* \mathbb{R}_X \simeq Rf_* \mathcal{D}_X^\bullet \simeq f_* \mathcal{D}_X^\bullet.$$

Use the currents of integration

$$\int_X \qquad \text{and} \qquad \int_{f^{-1}(y)}$$

to construct an isomorphism

$$Rf_* \mathbb{R}_X \simeq \mathbb{R}_Y[0] \oplus \mathbb{R}_y[-2].$$

1.5 Riemannian metrics

A *Riemannian metric* g on a smooth manifold M is the datum of a smoothly-varying positive inner product $g(-,-)_q$ on the fibers of $T_{M,q}$ of the tangent bundle of M. This means that, using a chart $(U;x)$, the functions

$$g_{jk}(q) = g(\partial_{x_j}, \partial_{x_k})_q$$

are smooth on U.

A Riemannian metric induces an isomorphism of vector bundles $T_M \simeq_g T^*_M$ and we can naturally define a metric on T^*_M.

Exercise 1.5.1. (**The dual metric**) Verify the following assertions.

Let E be a real, rank r vector bundle on M, $(U;x)$ and $(V;y)$ be two local charts around $q \in M$, $\tau_U : E_{|U} \longrightarrow U \times \mathbb{R}^r \longleftarrow V \times \mathbb{R}^r : \tau_V$ be two trivializations of E on the charts and $\tau_{VU}(x) = \tau_V \circ \tau_U^{-1} : (U \cap V) \times \mathbb{R}^r \longrightarrow (V \cap U) \times \mathbb{R}^r$ be the transition functions which we view as an invertible $r \times r$ matrix of real-valued functions of $x \in U \cap V$.

A metric g on E is a smoothly-varying inner product on the fibers of E. It can be viewed as a symmetric bilinear map of vector bundles $E \times E \longrightarrow \mathbb{R}_M$. Any bilinear map $b : E \times E \longrightarrow \mathbb{R}_M$ can be written, on a chart (U, x), as a $r \times r$ matrix b_U subject to the relation

$$b_V \; = \; (\tau_{VU}^{-1})^t \, b_U \, \tau_{VU}^{-1}.$$

If b is non-degenerate, then its local representations b_U are non-degenerate and we can define a non-degenerate bilinear form on the dual vector bundle E, $b^* : E^* \times E^* \longrightarrow \mathbb{R}_M$ by setting

$$b^*_U \; := \; b_U^{-1},$$

where it is understood that we are using as bases to represent b^* the dual bases to the ones employed to represent b on U.

If b is symmetric, then so is b^*. If b is positive definite, then so is b^*. If $\{e_1, \dots, e_m\}$ is an orthonormal frame for E, with respect to g, then the dual frame $\{e^*_1, \dots, e^*_m\}$ is orthonormal for g^*.

Conclude that if (M, g) is a Riemannian manifold, then there is a unique metric g^* on T^*_M such that the isomorphism induced by the metric (every bilinear form $V \times V \longrightarrow \mathbb{R}$, induces a linear map $V \longrightarrow V^*$) $T_M \simeq_g T^*_M$ is an isometry on every fiber.

1.6 Partitions of unity

Recall that a (smooth) *partition of unity* on M is a collection $\{\rho_\alpha\}$ of non-negative smooth functions on M such that the sum $\sum_\alpha \rho_\alpha$ is locally finite on M and adds-up to the value 1. This means that for every $q \in M$ there is a neighborhood U of q in M such that $\rho_{\alpha|U} \equiv 0$ for all but finitely many indices α so that the sum $\sum_\alpha \rho_\alpha(q)$ is finite and adds to 1.

Definition 1.6.1. (**Partition of unity**) Let $\{U_\alpha\}_{\alpha\in A}$ be an open covering of M. A *partition of unity subordinate to the covering* $\{U_\alpha\}_{\alpha\in A}$ is a partition of unity $\{\rho_\alpha\}$ such that the support of each ρ_α is contained in U_α.

Note that, on a non-compact manifold, it is not possible in general to have a partition of unity subordinate to a given covering and such that the functions ρ_α have compact support, e.g. the covering of $\mathbb{R}$ given by the single open set $\mathbb{R}$.

Theorem 1.6.2. *(***Existence of partitions of unity***) Let* $\{U_\alpha\}_{\alpha\in A}$ *be an open covering of* M*. Then there are*

a) a partition of unity subordinate to $\{U_\alpha\}$ *and*

b) a partition of unity $\{\rho_j\}_{j\in J}$*, where* $J \neq A$ *in general, such that (i) the support of every* ρ_j *is compact and (ii) for every index* j *there is an index* α *such that supp* $(\rho_j) \subseteq U_\alpha$*.*

Proof. See [Warner, 1971], p. 10. □

Exercise 1.6.3. Prove that every smooth manifold admits Riemannian metrics on it. (Hint: use partitions of unity. See [Bott and Tu, 1986], p. 42.)

1.7 Orientation and integration

Given any smooth manifold M, the space $\Lambda^m(T^*_M) \setminus M$, where M is embedded in the total space of the line bundle $\Lambda^m(T^*_M)$ as the zero section, has at most two connected components. The reader should verify this.

Definition 1.7.1. (**Orientation**) A smooth manifold M is said to be *orientable* if $\Lambda^m(T^*_M) \setminus M$ has two connected components, *non-orientable* otherwise.

If M is orientable, then the choice of a connected component of $\Lambda^m(T^*_M)\backslash M$ is called an *orientation* of M which is then said to be *oriented.*

Example 1.7.2. The *standard orientation* on $\mathbb{R}^m = \{(x_1, \ldots, x_m) \mid x_j \in \mathbb{R}\}$ is the one associated with

$$dx_1 \wedge \ldots \wedge dx_m.$$

Similarly, the torus $\mathbb{R}^m/\mathbb{Z}^m \simeq (S^1)^m$ is oriented using the form above descended to the torus.

Exercise 1.7.3. Let M be a differentiable manifold of dimension m. Show that the following three statements are equivalent. See [Warner, 1971], §4.2 and [Bott and Tu, 1986], §I.3.

(a) M is orientable.

(b) There is a collection of coordinate system $\{U^\alpha, x^\alpha\}$ such that

$$\det || \frac{\partial x_i^\beta}{\partial x_j^\alpha}(x^\alpha)|| > 0, \qquad \text{on } U^\alpha \cap U^\beta.$$

(c) There is a nowhere vanishing m-form on M.

Remark 1.7.4. A collection of charts with the property specified in Exercise 1.7.3.(b) is called *orientation-preserving.* Such a collection determines uniquely an orientation preserving collection of coordinate systems containing it which is maximal with respect to inclusion. Such a maximal collection is called an *orientation preserving atlas* for the orientable M. Note that composing with the automorphism $x \to -x$ we obtain a different orientation preserving atlas. Given an orientation preserving atlas, we can orient M by choosing the connected component of $\Lambda^m(T^*_M) \setminus M$ containing the vectors $\tau^*_\alpha(dx_1 \wedge \ldots \wedge dx_m)(q)$, where $\tau_\alpha : U_\alpha \longrightarrow \mathbb{R}^m$ is a chart in the atlas, $q \in U_\alpha$ and $dx_1 \wedge \ldots \wedge dx_n$ is the canonical orientation of $\mathbb{R}^m$. Of course, one could choose the opposite one as well. However, if M is oriented, then we consider the orientation preserving atlas and coordinate systems that agree with the orientation in the sense mentioned above.

Exercise 1.7.5. Let $f : \mathbb{R}^{m+1} \longrightarrow \mathbb{R}$ be a smooth function such that $df(q) \neq 0$ for every $q \in \mathbb{R}^{m+1}$ such that $f(q) = 0$.

Show that the equation $f = 0$ defines a possibly empty collection of smooth and connected m-dimensional submanifolds of $\mathbb{R}^{m+1}$.

Let M be a connected component of the locus $(f = 0)$.

Let

$$T := \{ (q, x) \mid x \cdot df(q) = 0 \} \subseteq M \times R^{m+1}.$$

Show that

$$T_M \simeq T$$

as smooth manifolds and as vector bundles over M.

Show that if M is unique, then $\mathbb{R}^{m+1} \setminus M$ has two connected components A^+ and A^- determined by the sign of the values of f so that $A := A^- \coprod M$ is an m-manifold with boundary.

Show that M carries a natural orientation associated with A : let $q \in M$, $\{v_1, \ldots, v_m\}$ be vectors in $T^*_{\mathbb{R}^{m+1},q}$ such that $(df(q), v_1, \ldots, v_m)$ is an oriented basis for $\mathbb{R}^{m+1}$ at q. Check that one indeed gets an orientation for M by taking the vectors v restricted to $T^*_{M,q}$.

Compare the above with the notion of induced orientation on the boundary of a manifold with boundary in [Bott and Tu, 1986], p. 31. They coincide.

Compare the above with [Warner, 1971], Exercise 4.1 which asks to prove that a codimension one submanifold of $\mathbb{R}^{m+1}$ is orientable if and only if there is a nowhere-vanishing smooth normal vector field (i.e. a section ν of $T_{\mathbb{R}^{m+1}}$ defined over $M \subseteq \mathbb{R}^{m+1}$, smooth over M, such that $\nu(q) \perp T_{M,q}$ for every $q \in M$).

Compute the volume form associated with the natural orientation on M and the Riemannian metric on M induced by the Euclidean metric on $\mathbb{R}^{m+1}$. The answer is that it is the restriction to M of the contraction of $dx_1 \wedge \ldots \wedge dx_{m+1}$ with the oriented unit normal vector field along M. See [Warner, 1971], Exercise 20.a. See [Warner, 1971], p. 61 and [Demailly, 1996], p. 22 for the definition, properties and explicit form of the contraction operation.

Make all the above explicit in the case when $M = S^m$ is defined by the equation $f := \sum_{j=1}^{m+1} x_j^2 = 1$ in the Euclidean space $\mathbb{R}^{m+1}$.

Compute everything using the usual spherical coordinates of calculus books.

Solve Exercise 4.20.(b) of [Warner, 1971] which gives the volume form for surfaces $(x, y, \varphi(x, y))$ in $\mathbb{R}^3$.

Exercise 1.7.6. Prove that the Möbius strip, the Klein bottle and $\mathbb{RP}^2$ are non-orientable.

Prove that $\mathbb{RP}^m$ is orientable iff m is odd. (Hint: the antipodal map $S^m \longrightarrow S^m$ is orientation preserving iff m is odd.)

Let M be an oriented manifold of dimension m. In particular, M admits an orientation preserving atlas. It is using this atlas and partitions of unity that we can define the operation of integrating m-forms with compact support on M and verify Stokes' Theorem 1.7.8.

Let us discuss how integration is defined. See [Bott and Tu, 1986], §I.3 (complemented by Remark 1.7.4 above) or [Warner, 1971], §4.8.

Let σ be an m-form on M with compact support contained in a chart $(U; x)$ with trivialization $\tau_U : U \longrightarrow \mathbb{R}^m$.

The m-form $(\tau_U^{-1})^*\sigma = f(x)\, dx_1 \wedge \ldots \wedge dx_m$, for a unique compactly supported real-valued function $f(x)$ on $\mathbb{R}^m$.

Define

$$\int_M^U \sigma := \int_{\mathbb{R}^m} f(x)\, dx,$$

where dx denotes the Lebesgue measure on $\mathbb{R}^m$ and the left-hand side is the Riemann-Stijlties-Lebesgue integral of $f(x)$.

The trouble with this definition is that it depends on the chosen chart. If (V, y) is another such chart and $T : y \to x$ is the patching function, then

$$(\tau_V^{-1})^*\sigma = g(y)\, dy_1 \wedge \ldots \wedge dy_m = f(T(y))\, J(T)(y)\, dy_1 \wedge \ldots \wedge dy_m$$

so that

$$\int_M^V \sigma = \int_{\mathbb{R}^m} g(y)\, dy = \int_{\mathbb{R}^m} f(T(y))\, J(T)(y)\, dy.$$

On the other hand, the change of variable formula for the Riemann integral gives

$$\int_M^U \sigma := \int_{\mathbb{R}^m} f(x)\, dx = \int_{\mathbb{R}^m} f(T(y))\, |J(T)(y)|\, dy.$$

It follows that the two definitions agree iff $J(T)(y) > 0$ on the support of σ.

This suggests that we can define the integral of m-forms only in the presence of an orientation-preserving atlas.

Let ω be a smooth m-form with compact support on M.

Let $\{U_\alpha\}$ be an orientation preserving collection of coordinate systems (see Remark 1.7.4) with orientation preserving trivializations $\tau_\alpha : U_\alpha \simeq \mathbb{R}^m$ and $\{\rho_\alpha\}$ be a partition of unity subordinate to the covering $\{U_\alpha\}$.

The forms $\rho_\alpha \omega$ have compact support contained in U_α.

Define

$$\boxed{\int_M \omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega.} \tag{1.9}$$

One checks that it is well-defined by a simple partition of unity argument.

Changing the orientation simply changes the signs of the integrals.

Exercise 1.7.7. Check that $\int_M \omega$, as defined in (1.9), is well-defined, i.e. that the definition does not depend on the covering and partition of unity chosen.

Note that one can integrate compactly supported functions f on any Riemannian manifold so that, if the manifold is also oriented, then the integral coincides with the integral of the top-form $\star f$. See [Warner, 1971], p. 150.

Complex manifolds are always orientable and are usually oriented using a standard orientation; see Proposition 3.4.1. Using the standard orientation, if ω is the $(1,1)$-form associated with the Fubini-Study metric of the complex projective space $\mathbb{P}^n$ (see Exercise 5.1.4), then $\int_{\mathbb{P}^n} \omega^n = 1$.

Theorem 1.7.8. (**Stokes' Theorem** *(simple version)). Let M be an oriented manifold of dimension M and u be an $(m-1)$-form on M with compact support. Then*

$$\boxed{\int_M du = 0.}$$

Proof. See [Bott andt Tu, 1986], §I.3.5, [Warner, 1971], p. 148. □

We have stated a weak version of Stokes' Theorem. See the references above for the complete version. See also [Warner, 1971], pp. 150-151 and Exercise 4.4 for its re-formulation as the Divergence Theorem.

If a Riemannian metric g on the oriented manifold M is given, then we have the notion of *Riemannian volume element* associated with (M, g) and with the orientation. It is the unique m-form dV on M such that, for every $q \in M$, dV_q is the volume element of Definition 1.2.2 for the dual metric g_q^* on $T^*_{M,q}$.

If the integral

$$\int_M dV$$

converges, then its value is positive and it is called the *volume* of the oriented Riemannian manifold.

Exercise 1.7.9. Compute the volume of the unit sphere $S^m \subseteq \mathbb{R}^{m+1}$ with respect to the metric induced by the Euclidean metric and the induced orientation.

Do the same for the tori $\mathbb{R}^m/\mathbb{Z}^m$, where the lattice $\mathbb{Z}^m \subseteq \mathbb{R}^m$ is generated by the vectors $(0, \ldots, r_j, \ldots, 0)$, $r_j \in \mathbb{R}^+$, $1 \leq j \leq m$.