Chapter 1

Conjugacy Classes, Characters, and Clifford Theory

For the proof of the $k(GV)$ theorem many of the standard methods and techniques from ordinary and modular representation theory will be applied. In this section we describe the necessary concepts and tools from ordinary character theory. The reader is referred to [Isaacs, 1976] for the relevant background and some basic results used but not proved here. This book is cited as [I] in the text.

1.1. Class Functions and Characters

Fix a finite group X of order $|X|$ and exponent $\exp(X) = e$. Let $K = \mathbb{Q}(\varepsilon)$ for $\varepsilon = e^{2\pi i/e}$.

Each complex character χ of X has its values in K, even in its ring of integers $\mathbb{Z}[\varepsilon]$, because for each $x \in X$ there is a basis such that an underlying representation carries x to a diagonal matrix consisting of eth roots of unity, and $\chi(x)$ is the trace of this matrix. This χ is constant on conjugacy classes of X, a *class function*, and if $c = c_x$ is the conjugacy class of X containing x ($c_x = x^X = \{x^t = t^{-1}xt|\ t \in X\}$), we may write $\chi(c) = \chi(x)$. The distinct class sums $\widehat{c} = \sum_{y \in c} y$ form a basis of the centre $Z(\mathbb{C}X)$ of the semisimple group algebra $\mathbb{C}X$ (and of $Z(KX)$). It follows that the set $\mathrm{C}\ell(X)$ of conjugacy classes of X is in bijection with the ordinary (complex) irreducible characters of X, i.e.,

$$k(X) = |\mathrm{C}\ell(X)| = |\mathrm{Irr}(X)|. \tag{1.1a}$$

If χ, ψ are K-valued class functions on X, their inner product is denoted by $\langle \chi, \psi \rangle = \langle \chi, \psi \rangle_X = \frac{1}{|X|}\sum_{x \in X} \chi(x)\overline{\psi(x)} = \frac{1}{|X|}\sum_{c \in \mathrm{C}\ell(X)} |c| \cdot \chi(c)\overline{\psi(c)}$. One knows that $\mathrm{Irr}(X)$ is an orthonormal basis for the K-vector space of class functions on X [I, 2.17]. The (nonsingular) square matrix $\mathcal{X} = (\chi(c))_{\substack{\chi \in \mathrm{Irr}(X) \\ c \in \mathrm{C}\ell(X)}}$ (rows and columns somehow arranged) is the *character table* of X. For some simple groups (of small order) these tables can be found in [Conway *et al.*, 1985], which we usually refer to as the [Atlas].

Orthonormality of the irreducible characters may be expressed by the matrix equation $\mathcal{X} \cdot T \cdot \overline{\mathcal{X}}^t = I$, where T is the diagonal matrix with entries $\frac{1}{|C_X(x)|}$ for $x \in c$ and $C_X(x)$ is the centralizer ($|c| = |X : C_X(x)|$). This gives $\overline{\mathcal{X}}^t \cdot \mathcal{X} = \mathrm{T}^{-1}$ and hence the *second orthogonality relations*:

$$\text{(1.1b)} \qquad \textstyle\sum_{\chi \in \mathrm{Irr}(X)} \overline{\chi(x)}\chi(y) = \begin{cases} |C_X(x)| & \text{if } x^X = y^X \\ 0 & \text{otherwise} \end{cases} .$$

In particular $|X| = \sum_{\chi \in \mathrm{Irr}(X)} \chi(1)^2$ is the sum over the squares of the degrees of the irreducible characters.

A generalized character χ of X is a rational integer combination of irreducible characters of X ($\chi \in \mathbb{Z}[\mathrm{Irr}(X)]$). A subgroup E of X is called *p-elementary* if $E = P \times Z$ where P is a p-group for some prime p and Z is a cyclic p'-group, and it is elementary if it is p-elementary for some prime p. We have Brauer's characterization of characters:

Theorem 1.1c (Brauer). *Let χ be a complex valued class function on X. Then $\chi \in \mathbb{Z}[\mathrm{Irr}(X)]$ if and only if the restriction $\mathrm{Res}^X_E(\chi) \in \mathbb{Z}[\mathrm{Irr}(E)]$ for any elementary subgroup E of X. In fact, each generalized character of X is a $\mathbb{Z}$-linear combination of characters of X induced from linear characters of elementary subgroups.*

For a proof see [I, 8.10 and 8.12]. Induced characters will be discussed in Sec. 1.2 below. Since linear characters (of degree 1) can be realized over their value fields, using elementary properties of the Schur index we get:

Theorem 1.1d (Brauer). *K is a splitting field for X, that is, for all characters χ of X there is a matrix representation over K or, equivalently, a KX-module affording χ.*

It is immediate, then, that K is a splitting field for all subgroups of X. The *Schur index* will be briefly discussed in Sec. 1.3.

If V and W are (right) KX-modules affording the characters χ, ψ, then $V \oplus W$ affords $\chi + \psi$ (sum), and the KX-module $V \otimes_K W$ (diagonal X-action) affords $\chi\psi$ (product). $V \otimes_K V = \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V)$ decomposes into the *symmetric squares* and *alternating squares.*

1.2. Induced and Tensor-induced Modules

Suppose Y is a subgroup of X and W is a (right) KY-module affording the character θ. Let $n = |X : Y|$ be the index of Y in X, and let $Y\backslash X = \{Yt \,|\, t \in X\}$ denote the (transitive) X-set with respect to right multiplication $(Yt, x) \mapsto Ytx$. Then the *normal core* $N = \mathrm{Core}_X(Y) = \bigcap_{t\in X} Y^t$ is the kernel of this permutation representation. Hence $G = X/N$ is a transitive subgroup of the symmetric group S_n. If $Y \neq X$ $(n > 1)$, then $\bigcup_{t\in X} Y^t \neq X$ and there are at least $n-1$ permutations in G without fixed points. See also [Serre, 2003] for a recent discussion of this classical result by Jordan.

Theorem 1.2a (Frobenius). *There is an embedding of X into the wreath product $Y \mathrm{wr}\, \mathrm{S_n} = \mathrm{Y^{(n)}} : \mathrm{S_n}$, which is uniquely determined up to conjugacy.*

The wreath product is defined by letting S_n act on the nth direct power $Y^{(n)}$ of Y permuting the direct factors (so $(y_i)_i^\pi = (y_i)_{i\pi} = (y_{i\pi^{-1}})_i$, sending an entry in the ith position to the $i\pi$th position). The wreath product may be identified with the group of all monomial $n \times n$-matrices with entries in Y. “The” Frobenius embedding is obtained by choosing a right transversal $\{t_i\}_{i=1}^n$ to Y in X. Associate then to $x \in X$ the element $(x_i) \cdot \pi_x$ in the wreath product, where $\pi_x : i \mapsto ix$ in $G \subseteq S_n$ and $x_i \in Y$ are defined by $t_i x = x_i t_{ix}$. Replacing $\{t_i\}$ by $\{y_i t_i\}$ for certain $y_i \in Y$ leads to an embedding conjugate under $(y_i)^{-1}$.

Now the *base group* $B = Y^{(n)}$ of the wreath product acts *diagonally* onto $W^{(n)} = W \oplus \cdots \oplus W$ (n direct summands) via $(w_i) \cdot (y_i) = (w_i y_i)$, and S_n through $(w_i)_i \cdot \pi = (w_{i\pi^{-1}})_i$. This makes $W^{(n)}$ into a $K[Y \mathrm{wr}\, \mathrm{S_n}]$-module. The induced KX-module $V = \mathrm{Ind}_Y^X(W)$ is obtained through “the” Frobenius embedding of X into $Y \mathrm{wr}\, \mathrm{S_n}$ (where conjugate embeddings yield isomorphic module structures). Fixing a transversal $\{t_i\}$ the character of X afforded by V is given by

$$\mathrm{Ind}_Y^X(\theta)(x) = \textstyle\sum_{i=1}^n \theta(t_i x t_i^{-1}), \tag{1.2b}$$

where we set $\theta(\cdot) = 0$ for elements outside Y.

Induction and restriction are maps on class functions *adjoint* to each other, namely related by the *Frobenius reciprocity*

$$\langle \mathrm{Ind}_Y^X(\theta), \chi\rangle_X = \langle \theta, \mathrm{Res}_Y^X(\chi)\rangle_Y. \tag{1.2c}$$

In terms of modules this says that if W is a KY-module, V a KX-module, every KY-homomorphism $f : W \to \mathrm{Res}_Y^X(V)$ extends uniquely to a KX-homomorphism $\mathrm{Ind}_Y^X(f) : \mathrm{Ind}_Y^X(W) \to V$.

We often will use *Mackey decomposition* for characters (and modules). Suppose Y and H are subgroups of X. Then H acts on $X\backslash Y$, and if $\{r_j\}$ is a set of representatives for the distinct H-orbits (or double cosets of X mod (Y, H)), for a character θ of Y we have

$$\text{(1.2d)} \qquad \mathrm{Res}_H^X(\mathrm{Ind}_Y^X(\theta)) = \textstyle\sum_j \mathrm{Ind}_{Y^{r_j}\cap H}^H(\mathrm{Res}_{Y^{r_j}\cap H}^{Y^{r_j}}(\theta^{r_j})).$$

Here for $x \in X$ we define the (conjugate) character θ^x of $Y^x = x^{-1}Yx$ by $\theta^x(y^x) = \theta(y)$ for $y \in Y$.

One can define $\mathrm{Ind}_Y^X(W) = W \otimes_{KY} KX$ (viewing KX as a left KY-module). Let now $W^{\otimes n} = W \otimes \cdots \otimes W$ (n factors, the tensors over K). Again $B = Y^{(n)}$ acts diagonally on $W^{(n)}$ via $(\otimes_i w_i)\cdot(y_i) = \otimes_i w_i y_i$, and S_n acts as $(\otimes_i w_i)\cdot\pi = \otimes_i w_{i\pi^{-1}}$. This makes $W^{\otimes n}$ into a $K[Y \,\mathrm{wr}\, \mathrm{S_n}]$-module. Then the *tensor-induced* KX-module $\widehat{V} = \mathrm{Ten}_Y^X(W) = W^{\otimes n}$ is obtained through "the" Frobenius embedding of X into the wreath product. Mackey decomposition (for modules) carries over to tensor induction in the obvious (multiplicative) way. So if $\{r_j\}$ is a set of representatives for the cycles (orbits) of an element $x \in X$ in its action on $Y\backslash X$, and if the jth cycle has size n_j (so that $r_j x^{n_j} r_j^{-1} \in Y$), then

$$\text{(1.2e)} \qquad \mathrm{Ten}_Y^X(\theta)(x) = \textstyle\prod_j \theta(t_j x^{n_j} t_j^{-1})$$

describes the character $\widehat{\chi}$ of X afforded by $\widehat{V} = \mathrm{Ten}_Y^X(W)$. In order to prove this it suffices to consider the case where $\widehat{V} = W \otimes Wx \otimes \cdots \otimes Wx^{n-1}$ (with $x^n \in Y$). If $\mathfrak{b}$ is a basis of W consisting of eigenvectors for x^n, then the $w_{i_0} \otimes w_{i_1}x \otimes \cdots \otimes w_{i_{n-1}}x^{n-1}$, with $w_{i_j} \in \mathfrak{b}$, form a basis of $\widehat{V}$ for which the matrix of x is monomial. Only basis vectors of the form $v_i = w_i \otimes w_i x \otimes \cdots \cdot \otimes w_i x^{n-1}$, with fixed $w_i \in \mathfrak{b}$, are eigenvectors of x on $\widehat{V}$, and if $w_i x^n = c_i w_i$ with $c_i \in K$, then $v_i x = c_i v_i$. Hence $\widehat{\chi}(x) = \theta(x^n)$, as desired.

1.3. Schur's Lemma

Suppose V and W are KX-modules affording the characters χ, ψ, respectively. Then

$$\langle \chi, \psi \rangle = \dim_{\mathrm{K}} \mathrm{Hom}_{\mathrm{KX}}(\mathrm{V}, \mathrm{W}).$$

If χ, ψ are irreducible, $\mathrm{Hom}_{KX}(V, W) = 0$ if $\chi \neq \psi$ and $\mathrm{End}_{KX}(V) \cong K$ if $V = W$ (Schur's lemma). Of course this yields the first orthogonality relations. In particular, if $\rho : X \to \mathrm{GL}_n(K)$ is a (matrix) representation of X affording $\chi \in \mathrm{Irr}(X)$, only the scalar matrices are permutable with all $\rho(x)$, $x \in X$. Extending ρ linearly to KX we get a K-algebra homomorphism into $M_n(K)$ and, by restriction to the centre, a homomorphism $\omega_\chi : Z(KX) \to K$, the *central character* associated to χ. If $c = c_x = x^X$ is the conjugacy class of $x \in X$,

$$\omega_\chi(\widehat{c}) = \tfrac{\chi(x)|X : C_X(x)|}{\chi(1)} = \tfrac{\chi(c)|c|}{\chi(1)}. \tag{1.3a}$$

The values of ω_χ are algebraic integers since the product of class sums is a (nonnegative) integer linear combination of class sums. This is important for block theory (Chapter 2). Writing $1 = \langle \chi, \chi \rangle = \frac{\chi(1)}{|G|} \sum_{c \in C\ell(X)} \omega_\chi(\widehat{c})\overline{\chi(c)}$ we see that $\frac{|G|}{\chi(1)}$ is an integer (being rational and an algebraic integer). Clifford theory will even yield the following (see also [I, 3.12 and 6.15]).

Theorem 1.3b (Itô)**.** *The degree $\chi(1)$ of an irreducible character χ of X divides $|X : V|$ for any abelian normal subgroup V of X.*

We used that irreducible representations over K are *absolutely* irreducible. Replacing K by the field $K_0 = \mathbb{Q}(\chi)$ generated by the values of χ, there is a unique irreducible K_0X-module V, up to isomorphism, whose character contains χ. Then V affords the character $m\chi$ where $m = m(\chi)$ is the *Schur index* of χ (over the rationals). $D = \mathrm{End}_{K_0X}(V)$ is a K_0-division algebra with centre K_0 and dimension m^2 [I, 9.21].

We give some further examples where Schur's lemma is involved. Let $x \in X$ and $\chi \in \mathrm{Irr}(X)$, and let $\bar{\chi}$ be the complex conjugate character. Then $\bar{\chi}(x) = \overline{\chi(x)} = \chi(x^{-1})$ (see Sec. 1.5). We assert that

$$|\chi(x)|^2 = (\chi\bar{\chi})(x) = \tfrac{\chi(1)}{|X|} \textstyle\sum_{y \in X} \chi([x, y]). \tag{1.3c}$$

Here $[x, y] = x^{-1}x^y$ denotes the commutator of x and y. Let ρ be a representation of X affording χ. The sum on the right is the trace of $\rho(x^{-1})\frac{\chi(1)}{|X|} \sum_{y \in X} \rho(x^y) = \rho(x^{-1})\frac{\chi(1)}{|X|}\omega_\chi(\widehat{c}_x)|C_X(x)| = \rho(x^{-1})\chi(x)$, and the assertion follows.

We see that $|\chi(x)| = \chi(1)$ if and only if $\chi([x, y]) = \chi(1)$ for all $y \in X$. Note that $\mathrm{Ker}(\chi) = \{x \in X | \, \chi(x) = \chi(1)\}$ is the kernel of ρ, and $Z(\chi) = \{x \in X | \; |\chi(x)| = \chi(1)\}$ consists of those $x \in X$ for which $\rho(x)$ is a scalar matrix.

Suppose next that $X = \langle Y, x\rangle$ for some subgroup Y, and that $\theta = \mathrm{Res}^X_Y(\chi)$ is (still) irreducible. We assert that then

$$\text{(1.3d)} \qquad \textstyle\sum_{y\in Y} |\chi(xy)|^2 = |Y|.$$

The 1-character 1_X is contained in $\chi\bar{\chi}$ with multiplicity 1, because we have $\langle 1_X, \chi\bar{\chi}\rangle = \langle \chi,\chi\rangle = 1$. Similarly $\langle 1_Y, \mathrm{Res}^X_Y(\chi\bar{\chi})\rangle = \langle 1_Y, \theta\bar{\theta}\rangle = 1$. Clearly $\mathrm{Res}^X_Y(1_X) = 1_Y$. So if $\psi \neq 1_X$ is an irreducible constituent of $\chi\bar{\chi}$, $\mathrm{Res}^X_Y(\psi)$ does not contain 1_Y. Hence if W is a KX-module affording ψ and τ is an underlying representation, then $\sum_{y\in Y}\tau(y) = 0$ as

$$\mathrm{dim}_{\mathrm{K}}\mathrm{C_W(Y)} = \mathrm{dim}_{\mathrm{K}}\mathrm{Hom_{KY}(K,W)} = \langle 1_{\mathrm{Y}}, \psi\rangle_{\mathrm{Y}} = 0.$$

It follows that $\sum_{y\in Y}\psi(xy) = 0$ by considering the trace of $\tau(x)\sum_{y\in Y}\tau(y)$. We conclude that $\sum_{y\in Y}(\chi\bar{\chi})(xy) = \sum_{y\in Y} 1_X(xy) = |Y|$.

1.4. Brauer's Permutation Lemma

Suppose G is a finite group acting on the finite set Ω (from the right, by permutations). Then we write $\mathrm{C}\ell(G|\Omega) = \mathrm{orb}(G \text{ on } \Omega)$ for the set of orbits of G on Ω. So $\mathrm{C}\ell(X) = \mathrm{C}\ell(X|X)$ with X acting by conjugation. By the Cauchy–Frobenius fixed point formula

$$\text{(1.4a)} \qquad |\mathrm{C}\ell(G|\Omega)| = \tfrac{1}{|G|}\textstyle\sum_{g\in G}|C_\Omega(g)|.$$

This is sometimes also called Burnside's lemma; it is easily proved by means of the counting principle or using Frobenius reciprocity [Serre, 2003]. Each orbit is a (transitive) G-set and so isomorphic to $H\backslash G$ for some subgroup H, which is determined up to conjugacy in G (being a point stabilizer). The isomorphism type of the G-set Ω is determined by the "marks" $|C_\Omega(H)|$ for all (nonconjugate) subgroups H of G [I, 13.23]. We associate to the G-set Ω the permutation character π_Ω of G, counting the fixed points of each element.

Theorem 1.4b (Brauer)**.** *Suppose G is a finite group which acts on $\mathrm{Irr}(X)$ and on $\mathrm{C}\ell(X)$ such that $\chi^g(c^g) = \chi(c)$ for all $\chi \in \mathrm{Irr}(X)$, $c \in \mathrm{C}\ell(X)$ and $g \in G$. Then for each $g \in G$, the number $k_g(X) = |C_{\mathrm{C}\ell(X)}(g)|$ of g-invariant conjugacy classes agreas with the number $|C_{\mathrm{Irr}(X)}(g)|$ of g-invariant irreducible characters of X. In particular, the permutation characters $\pi_{\mathrm{C}\ell(X)} = \pi_{\mathrm{Irr}(X)}$ of G agree.*

The proof [I, 6.32] is based on the fact that the character matrix of X is nonsingular. It follows, in view of Eq. (1.4a), that G has the same number of orbits on $\mathrm{Irr}(X)$ and on $\mathrm{C}\ell(X)$. Of course, this does not mean that these sets are permutation isomorphic (unless G is cyclic).

1.5. Algebraic Conjugacy

Let $\Gamma = \mathrm{Gal}(K|\mathbb{Q})$ be the Galois group of K over the rationals. We have a natural action of Γ on $\mathrm{Irr}(X)$. We also have a permutation action on X as $\Gamma \cong (\mathbb{Z}/e\mathbb{Z})^\star$, where $x^\sigma = x^n$ if $\sigma \in \Gamma$ corresponds to the coset of n modulo $e = \exp(X)$. This preserves $\mathrm{C}\ell(X)$. Let $\chi \in \mathrm{Irr}(X)$. If $\varepsilon_i \in K$ are the eigenvalues of x appearing in a representation to χ, the eigenvalues for x^n are the ε_i^n. It follows that

$$\chi^\sigma(x) = \chi(x)^\sigma = \chi(x^n) = \chi(x^\sigma).$$

In order to apply Theorem 1.4b one has to alter the action of Γ on X (say) by assigning $(x,\sigma) \mapsto x^{\sigma^{-1}}$. This works since Γ is *abelian.* Notice that if σ is complex conjugation (restricted to K), $\chi^\sigma(x) = \bar{\chi}(x) = \overline{\chi(x)} = \chi(x^{-1})$.

So the number of real-valued irreducible characters of X is equal to the number of *real conjugacy classes* c of X, satisfying $c^{-1} = c$ (Burnside). X is called a *real group* if all its conjugacy classes are real, that is, if all $\chi \in \mathrm{Irr}(X)$ are real-valued.

Suppose X has odd order. Then $\{1\}$ is the unique real class of X. Since $\chi(1) = \bar{\chi}(1)$ is odd, we get $|X| = \sum_{\chi \in \mathrm{Irr}(X)} \chi(1)^2 = 1 + 2\sum_{i=1}^{\frac{k(X)-1}{2}} (1+2n_i)^2$ for certain integers $n_i \geq 1$. Consequently

$$|X| \equiv k(X) \pmod{16}, \tag{1.5a}$$

a well known result due to Burnside.

Recall that a character of X takes only values which are algebraic integers. If $\alpha \neq 0$ is such an algebraic integer with the distinct conjugates α_i over the rationals $(1 \leq i \leq n)$, then $\sum_{i=1}^n |\alpha_i| \geq n$, with equality only if α is a root of unity. For by the arithmetic–geometric mean inequality we have

$$\tfrac{1}{n} \textstyle\sum |\alpha_i| \geq \left(\prod |\alpha_i|\right)^{\frac{1}{n}} = |\mathrm{N}(\alpha)|^{\frac{1}{n}}, \tag{1.5b}$$

with equality only if all $|\alpha_i|$ are equal. But the *norm* $\mathrm{N}(\alpha) = \prod_i \alpha_i$ is a nonzero rational integer. Hence the assertion holds, with equality only if all $|\alpha_i|$ are equal and $\mathrm{N}(\alpha) = \pm 1$. In this case $|\alpha_i| = 1$ for all i, whence α is a root of unity (since only finitely many powers of α are distinct).

Lemma 1.5c (Gallagher)**.** *Suppose $y \in X$ is such that $\chi(y) \neq 0$ for each $\chi \in \mathrm{Irr}(X)$. Let $N = [\langle y \rangle, X]$. Then $k(X) \leq |C_X(y)| - (|X/N| - k(X/N))$.*

Proof. We follow [Gallagher, 1962]. N is the (normal) subgroup of X generated by all commutators $[t, x]$, $t \in \langle y \rangle$, $x \in X$. By the second orthogonality relations (1.1b),

$$|C_X(y)| = \sum_{\chi \in \mathrm{Irr}(X)} |\chi(y)^2| = \Sigma_1 + \Sigma_2,$$

where the first sum is over those χ with $|\chi(y)| = \chi(1)$ and the second sum is over the others. Now $|\chi(y)| = \chi(1)$ if and only if $y \in Z(\chi)$ (as described in Sec. 1.3), and this happens if and only if N is in the kernel of χ and so χ may be viewed as a character of X/N. Thus $\Sigma_1 = |X/N|$, and the number of irreducible characters of X in Σ_1 is equal to $k(X/N) = |\mathrm{Irr}(G/N)|$.

For each σ in $\Gamma = \mathrm{Gal}(K|\mathbb{Q})$ we have $|\chi^2|^\sigma = (\chi \cdot \bar{\chi})^\sigma = \chi^\sigma \overline{\chi^\sigma} = |\chi^\sigma|^2$. Thus Σ_1 and Σ_2 are Galois stable. By hypothesis the average over the Galois class of $|\chi(y)^2|$ is ≥ 1 (and is equal to 1 only if $\chi(y)$ is a root of unity). Consequently $\Sigma_2 \geq k(X) - k(X/N)$, and the result follows. □

Theorem 1.5d. *Suppose X has an abelian normal Sylow p-subgroup, V, for some prime p. Then $X = GV$ for some p-complement G in X, uniquely determined up to conjugacy. For each $v \in V$,*

$$k(GV) \leq |C_G(v)| \cdot |V| - \big(|G| - k(G)\big).$$

In particular, if $C_G(v) = 1$ for some $v \in V$, then $k(GV) \leq |V|$ and equality only holds if G is abelian.

Proof. By a simple cohomological argument $X = GV$ is as claimed (Appendix A1; Schur–Zassenhaus theorem). Let $v \in V$. We assert that $\chi(v) \neq 0$ for each $\chi \in \mathrm{Irr}(X)$. By Theorem 1.3b, $\chi(1)$ is not divisible by p. Letting $\mathfrak{p}$ be a prime ideal above p in the ring of integers of K, we have $\chi(v) \equiv \chi(1) \pmod{\mathfrak{p}}$ (cf. Chapter 2). Hence the assertion. Now $N = [\langle v \rangle, X] = [v, G]$ is a normal subgroup of X contained in V, and

$C_X(v) = C_G(v)V$. By an elementary counting argument, carried out in Theorem 1.7a below, $k(X/N) \leq |V/N| \cdot k(G)$. Thus by the preceding lemma $k(X) \leq |C_G(v)| \cdot |V| - |V/N|\big(|G| - k(G)\big)$.

From $C_G(v) = 1$ it follows that $k(X) \leq |V|$, and then $k(X) = |V|$ only if $|G| = k(G)$, that is, if G is abelian. □

1.6. Coprime Actions

If G is a finite group and V is a finite G-module of order prime to $|G|$, then all (Tate) cohomology groups $\mathrm{H}^n(G, V)$ vanish (A1). For $n = 1, 2$ this leads to the Schur–Zassenhaus theorem (already mentioned). For $n = 0, -1$ this tells us that the fixed module $C_V(G)$ is the image of the trace map $v \mapsto \sum_{g \in G} vg$ on V and that the commutator module $[V, G] = [V, G, G]$ is its kernel. Then $C_V(G) \cap [V, G] = 0$ and so

$$V = C_V(G) \oplus [V, G]. \tag{1.6a}$$

$\mathrm{Irr}(V) = \mathrm{Hom}(V, \mathbb{C}^\star)$ is the character group of V, and $|C_{\mathrm{Irr}(V)}(G)| = |\mathrm{Irr}(V/[V, G])| = |V/[V, G]| = |C_V(G)|$. The corresponding holds for all subgroups of G. Hence we have the following.

Proposition 1.6b. *If V is a G-module where V and G have coprime order, then V and $\mathrm{Irr}(V)$ are isomorphic G-sets.*

The proposition is true without assuming that V is abelian. In fact, if G acts on X by automorphisms and $|G|$ and $|X|$ are coprime, then $\mathrm{Irr}(X)$ and $\mathrm{C\ell}(X)$ are isomorphic G-sets. This result is due to Isaacs and Dade. Its proof makes use of the Feit–Thompson theorem. So either G is solvable, in which case a proof can be found in [I, 13.24], or X is solvable, where a proof can be found in [Isaacs, 1973]. We do not need this result in this general form.

Theorem 1.6c (Glauberman)**.** *Suppose G is a cyclic group acting on X by automorphisms where $|G|$ and $|X|$ are coprime. Let ξ be a character of the semidirect product $GX = X : G$ for which $\chi = \mathrm{Res}_X(\xi)$ is irreducible. Then there is a unique irreducible constituent θ of the restriction to $Y = C_X(G)$ of χ, a unique linear character μ of G and a unique sign $\pm$ such that*

$$\xi(gy) = \pm\mu(g)\theta(y)$$

for all generators g of G and all $y \in Y$. If G is a p-group for some prime p, the sign is such that $\langle \chi, \theta \rangle_Y \equiv \pm 1 \pmod{p}$.

This is a special case of a more general character correspondence. For a proof we refer to [I, 13.6 and 13.14]. The character $\widehat{\xi} = \xi \cdot \bar{\mu}$ is the so-called *canonical extension* to GX of χ, determined by the fact that its determinantal character $\det(\widehat{\xi})$ has X in its kernel.

1.7. Invariant and Good Conjugacy Classes

Let N be a normal subgroup of X, and let $G = X/N$. Then G acts on $\mathrm{C}\ell(N)$ in the natural way, and $k_g(N) = |C_{\mathrm{C}\ell(N)}(g)|$ is the number of g-invariant conjugacy classes of N for each $g \in G$. The conjugacy class g^G of g is called "good for N" provided $C_X(x)N/N = C_G(g)$ for any (some) $x \in g$ (with $Nx = g$). This is well-defined. Suppose N is abelian. Then the class of g is good for N if $C_X(x)/N = C_G(g)$ for $x \in g$, that is, whenever a commutator $[x,y] \in N$ for some $y \in X$ then $[x,y] = 1$. In this case each conjugacy class of G is good for N if and only if N is central in X and no nontrivial element of N is a commutator in X.

Theorem 1.7a (Gallagher)**.** *Let Y be a subgroup of X, and let $N = \mathrm{Core}_X(Y)$ be its normal core. Let $G = X/N$.*

(i) *$k(Y) \le |X:Y| \cdot k(X)$ and $k(X) \le |X:Y| \cdot k(Y)$, the latter inequality being proper unless $Y = N$. Moreover,*

$$k(X) \le k(N) \cdot k(G), \tag{1.7b}$$

where equality holds if and only if each conjugacy class of G is good for N.

(ii) *Let $g = Nx$ for some $x \in X$. The conjugacy class in $\mathrm{C}\ell(N)$ of an element $y \in N$ is fixed by g if and only if $C_g(y) \neq \varnothing$, and then $|C_g(y)| = |C_N(y)|$. The number of g-invariant conjugacy classes of N is*

$$k_g(N) = \tfrac{1}{|N|} \textstyle\sum_{y \in N} |C_N(xy)|. \tag{1.7c}$$

Proof. We follow [Gallagher, 1970]. By (1.4a) $k(X) = \frac{1}{|X|}\sum_{x \in X} |C_X(x)|$. The first inequality in (i) is immediate from $C_Y(x) \subseteq C_X(x)$. Using the inequality $|C_X(x)| \le |X:Y| \cdot |C_Y(x)|$ and the counting principle we have

$$\sum_{x \in X} |C_X(x)| \le |X:Y| \sum_{x \in X} |C_Y(x)| = |X:Y| \sum_{y \in Y} |C_X(y)|,$$

and this is at most equal to $|X : Y|^2 \sum_{y\in Y} |C_Y(y)|$. We have equality if and only if $C_X(x)Y = X$ for all $x \in X$, and in this case any two conjugate elements of X are Y-conjugate. Then Y is normal in X.

Before proving (1.7b) we settle (ii). Let $t \in N$. Then $tx \in C_g(y) \iff y^{-1}txy = tx \iff xyx^{-1} = t^{-1}yt \iff y^{x^{-1}} = y^t$. Hence $C_g(y) \neq \varnothing$ if and only if y^N is fixed by g^{-1} (or g), and then $C_g(y) = C_N(y)tx$ for some $t \in N$. So $|C_g(y)| = |C_N(y)|$ if $y^N \in C_{\mathrm{C}\ell(N)}(g)$ and $C_g(y) = \varnothing$ otherwise. We conclude that

$$k_g(N) = \sum_{y\in N: y^N \in C_{\mathrm{C}\ell(N)}(g)} 1/|N : C_N(y)| = \frac{1}{|N|}\sum_{y\in N} |C_g(y)|.$$

Counting the pairs $(tx, y) \in g \times N$ satisfying $(tx)y = y(tx)$ we see that $\sum_{t\in N} |C_N(tx)| = \sum_{y\in N} |C_g(y)|$. This proves Eq. (1.7c), and completes the proof of (ii).

For each $x \in X$ we have $C_X(x)/C_N(x) \cong C_X(x)N/N \subseteq C_G(Nx)$. Hence

$$\sum_{x\in X} |C_X(x)| \leq \sum_{x\in X} |C_G(Nx)| \cdot |C_N(x)| = \sum_{g\in G} |C_G(g)| \sum_{t\in g} |C_N(t)|,$$

and $\sum_{t\in g} |C_N(t)| = \sum_{y\in N} |C_g(y)| \leq \sum_{y\in N} |C_N(y)|$ by (ii). Hence

$$\sum_{x\in X} |C_X(x)| \leq \sum_{g\in G} |C_G(g)| \sum_{y\in N} |C_N(y)|,$$

where equality holds if and only if $C_X(Nx) = C_X(x)N$ for each $x \in X$ (and each X-class of N is an N-class). We are done. □

Theorem 1.7d (Keller)**.** *Let Y be a proper subgroup of X, $N = \mathrm{Core}_X(Y)$ and $G = X/N$. Let $\Omega = \mathrm{C}\ell(N|X)$ with N acting by conjugation, which is a G-set (with G acting by conjugation). Then we have a partition $\Omega = \biguplus_{g\in G} \Omega_g$ where Ω_g is the set of N-orbits contained in the coset g.*

(i) *For each $g \in G$ the centralizer $C_G(g)$ is the stabilizer in G of Ω_g, and $|\Omega_g| = k_g(N)$.*

(ii) *Let $g_1 = 1, g_2, \cdots, g_r, g_{r+1}, \cdots, g_{k(G)}$ be representatives for the distinct conjugacy classes of G, the first r classes being just those meeting $H = Y/N$. Then*

$$k(X) = \sum_{i=1}^{k(G)} |\mathrm{C}\ell(C_G(g_i)|\Omega_{g_i})| \leq k(Y) + (k(G) - r)\cdot M$$

where $M = \max\{k_g(N) |\ g \notin \bigcup_{t\in G} \mathrm{H}^t\}$.

Proof. This is a recent result due to [Keller, 2006]. Let $g \in G$. It is obvious that $C_G(g)$ is the stabilizer in G of Ω_g. By the Cauchy–Frobenius formula (1.4a) and part (ii) of the preceding theorem,

$$|\Omega_g| = \frac{1}{|N|}\sum_{y\in N}|C_g(y)| = \sum_{y\in N: y^N\in C_{\Omega_1}(g)} 1/|N:C_N(y)| = k_g(N).$$

This proves (i). Each G-orbit on Ω is of the form $(x^N)^G$ for some unique conjugacy class $x^X \in \mathrm{C}\ell(X)$, and determines the conjugacy g^G of G defined by $Nx = g$ or, equivalently, by $x^N \subseteq g$ ($x^N \in \Omega_g$). In particular, $k(X) = |\mathrm{C}\ell(G|\Omega)|$. For $h \in G$ we have $|\mathrm{C}\ell(C_G(g^h)|\Omega_{g^h})| = |\mathrm{C}\ell(C_G(g)|\Omega_g)|$. This yields the identity given in (ii). For $1 \le i \le r$ we may pick the representatives $g_i \in H$, belonging then to certain distinct conjugacy classes of H. Of course $r \le k(H)$, and $k(G) > r$ by Jordan's theorem. By what is already proved (applied to Y),

$$\sum_{i=1}^{r}|\mathrm{C}\ell(C_G(g_i)|\Omega_{g_i})| \le \sum_{i=1}^{r}|\mathrm{C}\ell(C_H(g_i)|\Omega_{g_i})| \le k(Y).$$

For the remaining $k(G) - r$ conjugacy classes g^G of G, for which g is not in $\bigcup_{t\in G} H^t$, we take the trivial estimate $|\mathrm{C}\ell(C_G(g)|\Omega_g)| \le |\Omega_g|$, and use the fact that $|\Omega_g| = k_g(N)$. □

1.8. Nonstable Clifford Theory

Let N be a normal subgroup of X. Then X acts on N via conjugation (as a group of automorphisms), and on $\mathrm{Irr}(N)$. We have induced actions of $G = X/N$ on $\mathrm{C}\ell(N)$ and on $\mathrm{Irr}(N)$, and Theorem 1.4b applies. Fix $\theta \in \mathrm{Irr}(N)$. The stabilizer of θ (in X) is called the *inertia group* $T = I_X(\theta)$, and $T/N = I_G(\theta)$. If $\chi \in \mathrm{Irr}(X|\theta)$ is an irreducible character of X *above* θ, that is, θ is a constituent of $\mathrm{Res}^X_N(\chi)$, there are just $s = |X : T|$ distinct X-conjugates $\theta = \theta_1, \cdots, \theta_s$ of θ and

$$\mathrm{Res}^X_N(\chi) = e_\chi \textstyle\sum_{i=1}^{s} \theta_i \qquad (1.8a)$$

for some integer $e_\chi \ge 1$, the *ramification index* of χ with respect to N.

Theorem 1.8b (Clifford). *Let* $T = I_X(\theta)$. *The map* $\psi \mapsto \chi = \mathrm{Ind}^X_T(\psi)$ *is a bijection from* $\mathrm{Irr}(T|\theta)$ *onto* $\mathrm{Irr}(X|\theta)$. *The ramification indices* $e_\chi = e_\psi$ *are divisors of* $|T/N|$.

For a proof we refer to [I, 6.11 and 11.29].

1.9. Stable Clifford Theory

Let again N be a normal subgroup of X, and let $G = X/N$. Suppose that $\theta \in \mathrm{Irr}(N)$ is G-invariant ($I_G(\theta) = G$). This is a necessary condition for the existence of a character χ of X extending θ. If such a χ exists then $\mathrm{Irr}(X|\theta) = \{\chi\lambda = \chi \otimes \lambda |\ \lambda \in \mathrm{Irr}(G)\}$, and this has cardinality $k(G)$ [I, 6.17]. Moreover, then $\{\chi\lambda |\, \lambda \in \mathrm{Irr}(G), \lambda(1) = 1\}$ is the set of all characters of X extending θ, and this has cardinality $|G/G'|$. Here $G' = [G,G]$ is the commutator subgroup of G, the kernel of the characters of G of degree 1.

In general we proceed as follows. Let $K_0 = \mathbb{Q}(\theta)$ be the field generated by the values of θ, and let W be an irreducible K_0N-module affording $m\theta$, where $m = m(\theta)$ is the Schur index. Then $D = \mathrm{End}_{K_0N}(W)$ is a centrally simple K_0-algebra with $\dim_{\mathrm{K}_0} D = m^2$ (1.3). Since N is normal in X and θ is stable under G, each conjugate module $Wg = W \otimes g$ (affording θ^g) is a K_0N-module isomorphic to W ($g \in G$). Choose K_0N-isomorphisms $\tau_g : Wg \to W$ (with $\tau_1 = id_W$). We have $\mathrm{Res}^X_N\big(\mathrm{Ind}^X_N(W)\big) = \bigoplus_{g\in G} Wg$, and by Frobenius reciprocity (1.2c) the τ_g extend uniquely to units in the G-graded ring $\mathrm{End}_{K_0X}\big(\mathrm{Ind}^X_N(W)\big) = \bigoplus_{g\in G} D\tau_g$. Then $\tau_g^{-1}D\tau_g = D$ for all $g \in G$. By the Skolem–Noether theorem [Bourbaki, 1958, Chap. 8, §10] we may choose the τ_g such that they centralize D (via conjugation). Then $\tau_{gh}^{-1}\tau_g\tau_h = \tau(g,h) \cdot id_W$ for some nonzero scalar $\tau(g,h) \in K_0$.

We have a *projective representation* of G with 2-cocycle $\tau \in Z^2(G, K_0^\star)$, where the multiplicative group $K_0^\star = (K_0 \smallsetminus \{0\}, \cdot)$ is viewed as a trivial G-module. The cohomology class of τ depends only on θ, K_0 and the group extension $N \rightarrowtail X \twoheadrightarrow G$; it is written $\mu_{K_0G}(\theta)$. This “Clifford obstruction” is functorial in that it maps onto the corresponding cohomology class when replacing K_0 by an extension field.

Proposition 1.9a. *The Clifford obstruction $\mu_{K_0G}(\theta) \in \mathrm{H}^2(G, K_0^\star)$ vanishes if and only if there is a character χ of X extending θ and satisfying $K_0(\chi) = K_0$. The order of $\mu_{K_0G}(\theta)$ is a divisor of the number of $|G|$th roots of unity in K_0. There is a distinguished central group extension $Z \rightarrowtail G(\theta) \twoheadrightarrow G$, where Z is a cyclic group of order* $\exp(N)$*, whose cohomology class maps onto $\mu_{KG}(\theta)$ through an (appropriate) embedding of Z into $K^\star$. The exponent of $G(\theta)$ is a divisor of* $e = \exp(X)$.

Proof. If $\mu_{K_0G}(\theta)$ vanishes, by definition one can give $W = \widehat{W}$ the structure of a K_0X-module satisfying $\mathrm{End}_{K_0X}(\widehat{W}) = D$. This $\widehat{W}$ affords $m\chi$ where $\chi \in \mathrm{Irr}(X)$ extends θ, with $m(\chi) = m = m(\theta)$. It follows that

$K_0 = \mathbb{Q}(\chi)$. The converse is proved similarly. The order of $\mu_{K_0G}(\theta)$ divides $|G|$ by an elementary property of cohomology groups (Appendix A1). It also divides the number of roots of unity in K_0 [Dade, 1974] which, however, will not be used here. We briefly discuss the further (basic) statements.

Replacing K_0 by the complex number field we are just concerned with Schur's theory of lifting projective representations. The *Schur multiplier* $\mathrm{M}(G) = \mathrm{H}_2(G, \mathbb{Z})$ of G fits into the natural *universal coefficient* exact sequence (Appendix A5)

$$0 \to \mathrm{Ext}(G/G', K_0^\star) \to \mathrm{H}^2(G, K_0^\star) \to \mathrm{Hom}(\mathrm{M}(G), K_0^\star) \to 0.$$

Passing to the complex number field, and noting that $\mathbb{C}^\star$ is divisible, we see that $\mathrm{H}^2(G, \mathbb{C}^\star)$ is nothing but the dual of $\mathrm{M}(G)$. So there is a (complex) character χ of X extending θ if $\mathrm{M}(G) = 1$. Of course, then $\mathbb{Q}(\chi) \subseteq K$ and so $\mu_{KG}(\theta)$ vanishes. Let $K_1 = \mathbb{Q}(\varepsilon_1)$ where ε_1 is a primitive $\exp(N)$th root of unity, and let $Z = \langle \varepsilon_1 \rangle$. By Theorem 1.1d this K_1 is a splitting field for N. Let W be a K_1N-module affording θ, and let $\tau(g,h) = \tau_{gh}^{-1}\tau_g\tau_h$ be a 2-cocycle with class $\mu_{K_1G}(\theta)$. We wish to show that there is a unique element in $\mathrm{H}^2(G, Z)$ mapping onto this cohomology class.

Consider the long exact cohomology sequence to $Z \rightarrowtail K_1^\star \twoheadrightarrow K_1^\star/Z$:

$$\mathrm{H}^1(G, K_1^\star/Z) \xrightarrow{\delta} \mathrm{H}^2(G, Z) \to \mathrm{H}^2(G, K_1^\star) \to \mathrm{H}^2(G, K_1^\star/Z).$$

Either $K_1^\star/Z$ is torsion-free or $|Z|$ is odd and $(-1)Z$ is its unique torsion element. At any rate, δ is the zero map, and it suffices to show that $\mu_{K_1G}(\theta)$ has trivial image in $\mathrm{H}^2(G, K_1^\star/Z)$. In order to prove this, as well as for the proof of the final statement, we may assume that G is a p-group for some prime p (A4). The construction of $G(\theta)$ will show that its exponent divides $\exp(X)$. Arguing by induction on $|X|$ we may also assume that θ is faithful, because $\mathrm{Ker}(\theta)$ is normal in X, and that there is no proper subgroup X_0 of X covering G such that $N_0 = X_0 \cap N$ has a $G \cong X_0/N_0$-invariant irreducible character θ_0 satisfying $\mu_{K_1G}(\theta_0) = \mu_{K_1G}(\theta)$ in $\mathrm{H}^2(G, K_1^\star)$. This reduction will lead us, in the case where G is a p-group, to $X = G(\theta)$.

On the basis of Theorem 1.1c, we find a p-elementary subgroup X_0 of X covering G and an X_0-invariant irreducible character θ_0 of $N_0 = X_0 \cap N$ such that $\langle \theta_0, \theta \rangle_{N_0}$ is not divisible by p [I, 8.24]. But this implies that $\mu_{K_1G}(\theta_0) = \mu_{K_1G}(\theta)$. In order to see this, let U be a K_1N_0-module affording θ_0, and consider the K_1-space $\mathcal{H} = \mathrm{Hom}_{K_1N_0}(U, W)$. By Frobenius reciprocity each $f \in \mathcal{H}$ extends uniquely to a K_1X_0-morphism $\mathrm{Ind}_{N_0}^{X_0}(U) \to \mathrm{Ind}_N^X(W)$, preserving the G-gradings. Since the G-graded ring $\mathrm{End}_{K_1X_0}\big(\mathrm{Ind}_{N_0}^{X_0}(U)\big) = \bigoplus_{g\in G} K_1\sigma_g$ is a crosssed product, the maps $f \mapsto \sigma_g^{-1} f \tau_g$ may be considered as elements $\psi_g \in \mathrm{GL}(\mathcal{H})$. We obtain that

$$\psi_{gh}^{-1}\psi_g\psi_h = \sigma(g,h)^{-1}\tau(g,h) \cdot id_{\mathcal{H}}$$

where $\sigma(g,h) = \sigma_{gh}^{-1}\sigma_g\sigma_h \in K_1^\star$ is a factor set with class $\mu_{K_1G}(\theta_0)$. Taking determinants we get that σ^d and τ^d agree modulo the coboundary obtained from $g \mapsto \det(\psi_g)$. Here $d = \dim_{\mathrm{K}_1}\mathcal{H} = \langle\theta_0,\theta\rangle_{\mathrm{N}_0}$. Since G is a p-group, hence so is $\mathrm{H}^2(G, K_1^\star)$, and since p does not divide d, the cohomology classes of σ and τ agree. Thus by our choice $X = X_0$ is p-elementary.

Let next M be a G-invariant abelian subgroup of N of maximal order. Since X is p-elementary and X/N a p-group, it follows that $C_N(M) = M$. Let $\lambda \in \mathrm{Irr}(M)$ be an irreducible (linear) constituent of $\mathrm{Res}_M^N(\theta)$ and $X_1 = I_X(\lambda)$, $N_1 = I_N(\lambda) = X_0 \cap N$. Let $\theta_1 \in \mathrm{Irr}(N_1|\lambda)$ be the unique character satisfying $\mathrm{Ind}_{N_1}^N(\theta_1) = \theta$ (Theorem 1.8b). By a Frattini argument X_1 covers G and, by the same argument as before, $\mu_{K_1G}(\theta_1) = \mu_{K_1G}(\theta)$ since $\langle\theta_1,\theta\rangle_{N_1} = 1$. Thus $X = X_1$ and $N_1 = N$. It follows that $\mathrm{Res}_M^N(\theta) = \theta(1)\lambda$ and that $M \subseteq Z(N)$ as θ is faithful. But $C_N(M) = M$. Hence $N = M$ is central in X and $\theta = \lambda$ is linear.

Now $\mu_{K_1G}(\theta)$ is the image of the cohomology class of the central extension $N \rightarrowtail X \twoheadrightarrow G$ under the map induced by $\bar\theta = \theta^{-1} : N \to K_1^\star$. Indeed, choose $\tau_g : w \otimes t_g \mapsto w$ for some transversal $\{t_g\}_{g\in G}$ to N in X. Letting $t(g,h) = t_{gh}^{-1}t_gt_h$ be the corresponding factor set, $\mu_{K_1G}(\theta)$ is the class of the factor set $(g,h) \mapsto \theta(t(g,h)^{-1})$ of G, which has its values in Z. □

Definition 1.9b. The group $G(\theta)$ in the preceding proposition is called the *representation group* of θ (with respect to G). The *extended representation group* is defined as the "fibre-product" (pull-back; "diagonal group" in the terminology of the Atlas)

$$X(\theta) = G(\theta)\Delta_G X$$

of $G(\theta)$ and X amalgamating $G = X/N$. Letting $\tau(g,h) = \tau_{gh}^{-1}\tau_g\tau_h$ be a 2-cocycle with values in Z (viewed as a group of scalar multiplications) and class $\mu_{KG}(\theta)$ in $\mathrm{H}^2(G, K^\star)$, we may write $G(\theta)$ as the group consisting of all pairs $(g,z) \in G\times Z$ with multiplication $(g,z)(h,z') = (gh, zz'\tau(g,h))$. Then $X(\theta)$ consists of all elements $((g,z),x)$ for which $Nx = g$. By Proposition

1.9a, $X(\theta)$ has the same exponent as X. Hence K is a splitting field for $X(\theta)$ by Theorem 1.1d.

Suppose W is a KN-module affording θ. By construction $W = \widehat{W}$ gets the structure of a $KX(\theta)$-module through $w((g,z),x) = (zwx)\tau_g = (zw)\tau_g x$ for $w \in W \subseteq \mathrm{Ind}_N^X(W)$, $g \in G$, $z \in Z$ and $x \in X$ (with $Nx = g$). For $x \in N$ we have $v((1,1),x) = vx$. Thus $\widehat{W}$ is an extension of W when viewed as a module for $\mathrm{Ker}(X(\theta) \twoheadrightarrow G(\theta)) \cong N$.

We may replace K by any subfield which is a splitting field for θ. In this manner we find a character $\widehat{\theta}$ of $X(\theta)$ extending θ, when viewed as a character of $\mathrm{Ker}(X(\theta) \twoheadrightarrow G(\theta))$, which can be written in this same field. This applies in particular when the Schur index $m(\theta) = 1$ (which is true in prime characteristic).

Theorem 1.9c (Clifford)**.** *Let N be a normal subgroup of X, let $G = X/N$ and let $\theta \in \mathrm{Irr}(N)$ be stable in X. Let $\widehat{\theta} \in \mathrm{Irr}(X(\theta))$ extend θ in the above sense, and let $\widetilde{\theta}^{-1}$ be the unique irreducible (linear) constituent of $\widehat{\theta}$ on $\mathrm{Ker}(X(\theta) \twoheadrightarrow X) \cong Z$. Then $\zeta \leftrightarrow \chi = \widehat{\theta} \otimes \zeta$ is a 1-1 correspondence between $\mathrm{Irr}(G(\theta)|\widetilde{\theta})$ and $\mathrm{Irr}(X|\theta)$. Moreover $|\mathrm{Irr}(X|\theta)| = |\mathrm{Irr}(G(\theta)|\widetilde{\theta})| \leq k(G)$.*

Proof. Let $\chi \in \mathrm{Irr}(X|\theta)$, and view χ as a character of $X(\theta)$ by inflation. Since $\widehat{\theta}$ extends θ when viewed as a character of $\mathrm{Ker}(X(\theta) \twoheadrightarrow G(\theta)) \cong N$, there is a unique (irreducible) character ζ of $G(\theta)$ such that $\chi = \widehat{\theta} \otimes \zeta$ (see above). But $\mathrm{Ker}(X(\theta) \twoheadrightarrow X) \cong Z$ is in the kernel of χ. It follows that $\zeta \in \mathrm{Irr}(G(\theta)|\widetilde{\theta})$ where $\widetilde{\theta}$ is as described. Conversely, every $\zeta \in \mathrm{Irr}(G(\theta)|\widetilde{\theta})$ gives rise to an irreducible character $\chi = \widehat{\theta} \otimes \zeta$ in $\mathrm{Irr}(X|\theta)$.

For the final statement we may assume that $X = G(\theta)$, $N = Z$ is central in X and $\widetilde{\theta} = \theta$. By Frobenius reciprocity (1.2c) we then have $\mathrm{Ind}_N^X(\theta) = \sum_{\chi \in \mathrm{Irr}(X|\theta)} \chi(1)\chi$, and this vanishes outside N and agrees with $|G|\theta$ on N. For $\chi \in \mathrm{Irr}(X|\theta)$ we have $|\chi(x)|^2 = \frac{\chi(1)}{|X|} \sum_{y \in X} \chi([x,y])$ by (1.3c) and, of course, $|X| = \sum_{x \in X} |\chi(x)|^2$. Hence

$$|\mathrm{Irr}(X|\theta)| = \frac{1}{|X|} \sum_{x \in X} \sum_{\chi \in \mathrm{Irr}(X|\theta)} |\chi(x)|^2 = \frac{|G|}{|X|^2} \sum_{\substack{x,y \in X \\ [x,y] \in N}} \theta([x,y]).$$

This is at most equal to $\frac{|G|}{|X|^2} \sum_{x \in X} \sum_{y \in C_X(Nx)} 1 = \frac{1}{|G|} \sum_{\bar{x} \in G} |C_G(\bar{x})| = k(G)$. □

We now shall discuss Clifford theory of tensor induction. Suppose N is a nonabelian normal subgroup of X which is the central product of the X-conjugates of some (proper) subgroup N_0. Let $X_0 = N_X(N_0)$ be the normalizer, and let $G = X/N$ and $G_0 = X_0/N_0$. Assume $\theta \in \mathrm{Irr}(N)$ is stable in X, and let $\theta_0 \in \mathrm{Irr}(N_0)$ be the unique irreducible constituent of θ on N_0. As θ_0 is absolutely irreducible, θ is the (tensor) product of the $|X : X_0|$ distinct X-conjugates of θ_0 [I, 4.21].

Define $X(\theta)$ and $X_0(\theta_0)$ as before, with the same Z, and let $\widehat{Z} = \mathrm{Ind}_{X_0}^{X}(Z)$ be the (induced) permutation module.

Theorem 1.9d. *Keeping these assumptions, let $\widehat{\theta}_0$ be an irreducible character of $X_0(\theta_0)$ extending θ_0 (as above). Then there is a group extension $\widehat{Z} \rightarrowtail \widehat{X} \twoheadrightarrow X$ mapping onto $X(\theta)$ such that $\widehat{\theta} = \mathrm{Ten}_{\widehat{X}_0}^{\widehat{X}}(\widehat{\theta}_0)$ is a character of $X(\theta)$ extending θ, $\widehat{X}_0$ being the inverse image in $\widehat{X}$ of X_0.*

Proof. Let $\rho_0 : N_0 \to \mathrm{GL}(W_0)$ be a K-representation affording θ_0 (Theorem 1.1d). Let $\{t_i\}_{i=1}^n$ be a right transversal to X_0 in X. Let $h = \prod_i h_i^{t_i}$ be an element in N (with all $h_i \in N_0$). Then $\rho(h) = \otimes_{i=1}^n \rho_0(h_i^{t_i})$ is a K-representation of N on $W = \bigotimes_{i=1}^n W_0 t_i$ affording θ. Since θ_0 is stable in X_0, we may extend ρ_0 to a projective representation $\widehat{\rho}_0 : X_0 \to \mathrm{GL}(W_0)$. We may choose $\widehat{\rho}_0$ such that its factor set $\tau_0 \in Z^2(X_0, K^\star)$, being inflated from G_0, has order dividing $|Z| = \exp(N_0)$. Let $x \in X$, and let $t_i x = x_i t_{ix}$ be as in Sec. 1.2 (with $x_i \in X_0$). Then $\widehat{\rho}(x) = \otimes_{ix=1}^n \widehat{\rho}_0(x_i)$ defines a projective representation of X *tensor induced* from $\widehat{\rho}_0$ which extends ρ and has factor set

$$\widehat{\tau}_0(x, y) = \prod_{i=1}^n \tau_0(x_i, y_{ix}).$$

This $\widehat{\tau}_0$ is *co-induced* from τ_0 (and inflated from G). We have $h^x = \prod_i (h_i^{x_i})^{t_{ix}}$ and so

$$\rho(h^x) = \otimes_{ix=1}^n \rho_0(h_i^{x_i}) = \widehat{\rho}(x)^{-1}\rho(h)\widehat{\rho}(x).$$

Thus the class of $\tau = \widehat{\tau}_0$ in $\mathrm{H}^2(G, K^\star)$ is nothing but $\mu_{KG}(\theta)$, and $\widehat{\rho}$ lifts to an ordinary representation of $X(\theta)$, say affording $\widehat{\theta}$.

The group extension $\widehat{X}$ represents the cohomology class obtained from $X_0(\theta_0)$ under the natural isomorphism $\mathrm{H}^2(X_0, Z) \cong \mathrm{H}^2(X, \widehat{Z})$ underlying Shapiro's lemma (A3). Here the group $X(\theta)$ is the image of $\widehat{X}$ under the map $\widehat{Z} \to Z$ sending $(z_i)_{i=1}^n$ to $\prod_{i=1}^n z_i$ $(z_i \in Z)$. □

1.10. Good Conjugacy Classes and Extendible Characters

Let N be a normal subgroup of X, and let $G = X/N$. Let $\theta \in \mathrm{Irr}(N)$ be G-invariant. The conjugacy class of an element $g = Nx$ in G is called “good for θ” provided θ can be extended to $\langle N, x, y\rangle$ for all $y \in X$ satisfying $[x, y] \in N$ [Gallagher, 1970]. By virtue of Theorem 1.9c this may be studied by passing to $G(\theta)$. Hence we may assume that $N = Z$ is cyclic and central in X and that $\theta = \widetilde{\theta}$ is linear. Assume also that θ is faithful. If there is a (linear) character λ of $Y = \langle N, x, y\rangle$ extending θ, then Y/N is abelian (as $[x, y] \in N$), $Y/\mathrm{Ker}(\lambda)$ is abelian and $N \cap \mathrm{Ker}(\lambda) = \mathrm{Ker}(\theta) = 1$. Hence Y is abelian. Conversely, if Y is abelian, then θ can be extended to Y. We have proved the following.

Theorem 1.10a. *Suppose* $\theta \in \mathrm{Irr}(N)$ *is stable under* $G = X/N$. *Then* $|\mathrm{Irr}(X|\theta)|$ *is the number of conjugacy classes of* G *which are good for* θ.

Combining this with Theorem 1.8b we obtain the *Clifford–Gallagher formula*

$$(1.10\mathrm{b}) \qquad k(X) = \textstyle\sum_{\theta \in \mathrm{Irr}(N)} k_\theta\big(I_G(\theta)\big)/|G : I_G(\theta)|,$$

where $k_\theta(I_G(\theta))$ is the number of conjugacy classes of $I_G(\theta)$ which are good for θ. Observe that $k_\theta(I_G(\theta)) = k(I_G(\theta))$ whenever θ can be extended to $I_X(\theta)$, that is, whenever $\mu_{KI_G(\theta)}(\theta)$ vanishes. This happens for instance if all Sylow subgroups of $I_G(\theta)$ are cyclic, because then its Schur multiplier is trivial by (A4).